

1 Bass Exercise 2.1: Example of a Monotone Class That Is Not a σ -Algebra

We aim to find a set X and a collection $\mathcal{M} \subseteq \mathcal{P}(X)$ such that:

- $\emptyset \in \mathcal{M}$,
- $X \in \mathcal{M}$,
- $\mathcal{M}$ is a **monotone class**,
- $\mathcal{M}$ is **not** a σ -algebra.

Definitions

- A **monotone class** $\mathcal{M}$ is a collection of subsets of X satisfying:
 - If $A_1 \subseteq A_2 \subseteq \dots$ and each $A_n \in \mathcal{M}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$,
 - If $A_1 \supseteq A_2 \supseteq \dots$ and each $A_n \in \mathcal{M}$, then $\bigcap_{n=1}^{\infty} A_n \in \mathcal{M}$.
- A **σ -algebra** must be closed under:
 - Complements,
 - Countable unions.

Example

Let:

$$X = \mathbb{N}, \quad \mathcal{M} = \{A \subseteq \mathbb{N} \mid A \text{ is finite or } \mathbb{N} \setminus A \text{ is finite}\}$$

This collection $\mathcal{M}$ is known as the algebra of *finite/cofinite sets*.

Verification

- $\emptyset \in \mathcal{M}$ since $\emptyset$ is finite.
- $\mathbb{N} \in \mathcal{M}$ since $\mathbb{N} \setminus \mathbb{N} = \emptyset$ is finite.
- $\mathcal{M}$ is closed under monotone (increasing and decreasing) limits:
 - Increasing unions: If $A_1 \subseteq A_2 \subseteq \dots$ and all $A_n \in \mathcal{M}$, then:
 - * If all A_n are finite, their union is at most countably infinite. If the union is finite or cofinite, it belongs to $\mathcal{M}$.
 - * If at some point A_n becomes cofinite, then the union remains cofinite.
 - Decreasing intersections: Similar reasoning applies.
- However, $\mathcal{M}$ is **not** a σ -algebra. For instance, consider the set of even numbers:

$$A = \{2, 4, 6, \dots\}$$

Then $A \subseteq \mathbb{N}$ is infinite, and its complement (the odd numbers) is also infinite. Hence:

$$A \notin \mathcal{M}$$

But A can be expressed as a countable union of singleton sets:

$$A = \bigcup_{n=1}^{\infty} \{2n\}, \quad \text{each } \{2n\} \in \mathcal{M}$$

Since $\mathcal{M}$ is not closed under arbitrary countable unions, it is not a σ -algebra.

Conclusion:

The collection

$$\mathcal{M} = \{A \subseteq \mathbb{N} : A \text{ is finite or cofinite}\}$$

is a monotone class that contains $\emptyset$ and $\mathbb{N}$, but it is not a σ -algebra.

2 Example: Union of Two σ -Algebras Need Not Be a σ -Algebra

We are asked to find an example of:

- A set X ,
- Two σ -algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ on X , such that:

$$\mathcal{A}_1 \cup \mathcal{A}_2 \text{ is not a } \sigma\text{-algebra.}$$

Definitions

- A σ -algebra $\mathcal{A} \subseteq \mathcal{P}(X)$ satisfies:
 - $\emptyset \in \mathcal{A}$,
 - If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$,
 - If $A_1, A_2, \dots \in \mathcal{A}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

Construction of the Example

Let $X = \{1, 2\}$. Define two σ -algebras:

$$\begin{aligned}\mathcal{A}_1 &= \{\emptyset, \{1\}, \{2\}, X\} \\ \mathcal{A}_2 &= \{\emptyset, \{1\}, X\}\end{aligned}$$

Clearly, both $\mathcal{A}_1$ and $\mathcal{A}_2$ are σ -algebras on X .

Now consider the union:

$$\mathcal{A}_1 \cup \mathcal{A}_2 = \{\emptyset, \{1\}, \{2\}, X\}$$

At first glance, this looks promising. But let's take a slightly different pair of σ -algebras to demonstrate the failure more clearly.

Instead, let us use this standard counterexample. Let: $X = \{1, 2, 3\}$. Define:

$$\begin{aligned}\mathcal{A}_1 &= \{\emptyset, \{1\}, \{2, 3\}, X\} \\ \mathcal{A}_2 &= \{\emptyset, \{2\}, \{1, 3\}, X\}\end{aligned}$$

Each of these is a σ -algebra on X :

- $\mathcal{A}_1$ is closed under complements and countable unions:
 - $\{1\}^c = \{2, 3\} \in \mathcal{A}_1$,
 - $\{2, 3\}^c = \{1\} \in \mathcal{A}_1$, etc.
- Similarly, $\mathcal{A}_2$ is a σ -algebra.

Now consider:

$$\mathcal{A}_1 \cup \mathcal{A}_2 = \{\emptyset, \{1\}, \{2\}, \{2, 3\}, \{1, 3\}, X\}$$

Let's test whether this union is a σ -algebra. We have:

$$\{1\}, \{2\} \in \mathcal{A}_1 \cup \mathcal{A}_2$$

But their union:

$$\{1\} \cup \{2\} = \{1, 2\}$$

is **not** in $\mathcal{A}_1 \cup \mathcal{A}_2$. Therefore, $\mathcal{A}_1 \cup \mathcal{A}_2$ is not closed under finite (hence not under countable) unions, and thus:

$$\mathcal{A}_1 \cup \mathcal{A}_2 \text{ is not a } \sigma\text{-algebra.}$$

Conclusion:

We have found:

- $X = \{1, 2, 3\}$,
- $\mathcal{A}_1 = \{\emptyset, \{1\}, \{2, 3\}, X\}$,
- $\mathcal{A}_2 = \{\emptyset, \{2\}, \{1, 3\}, X\}$,

such that both $\mathcal{A}_1$ and $\mathcal{A}_2$ are σ -algebras, but their union is not.

3 Bass Exercise 2.2: Question

Suppose $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \dots$ are σ -algebras consisting of subsets of a set X . Is $\bigcup_{i=1}^{\infty} \mathcal{A}_i$ necessarily a σ -algebra?

Answer: No, not necessarily.

The countable union of increasing σ -algebras is not necessarily a σ -algebra.

Counterexample

Let $X = \mathbb{R}$, and define $\mathcal{A}_n$ as the σ -algebra generated by the interval $(-\infty, n]$, for each $n \in \mathbb{N}$. That is:

$$\mathcal{A}_n = \sigma(\{(-\infty, n]\})$$

Each $\mathcal{A}_n$ is the smallest σ -algebra that contains $(-\infty, n]$, and hence contains all intervals of the form $(-\infty, a]$ with $a \leq n$.

Thus, we have:

$$\mathcal{A}_1 \subset \mathcal{A}_2 \subset \dots$$

Now consider the union:

$$\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$$

This union contains all sets of the form $(-\infty, a]$ for rational a , but it may **not** contain arbitrary Borel sets such as $(0, 1)$, which is not in any $\mathcal{A}_n$, since these σ -algebras are too small.

More concretely, let's build a simpler counterexample with finite sets.

Simpler Finite Counterexample

Let $X = \mathbb{N}$, the natural numbers. Define:

$$\mathcal{A}_n = \sigma(\{\{1\}, \{2\}, \dots, \{n\}\})$$

This is the σ -algebra generated by the finite subsets of $\{1, 2, \dots, n\}$, and it consists of all subsets of X that are determined by the behavior on the first n elements. Specifically, each $\mathcal{A}_n$ consists of subsets of $\mathbb{N}$ that depend only on whether each of $1, 2, \dots, n$ is included or not, and whether the rest of the set is included as a block or not.

Then:

$$\bigcup_{n=1}^{\infty} \mathcal{A}_n$$

contains many subsets of $\mathbb{N}$, but it is **not** closed under countable unions. For instance, the singleton sets $\{n\} \in \mathcal{A}_n \subset \mathcal{A}_{n+1}$, so each $\{n\} \in \bigcup_n \mathcal{A}_n$, but:

$$\bigcup_{n=1}^{\infty} \{n\} = \mathbb{N}$$

is **not** in any $\mathcal{A}_n$, because each $\mathcal{A}_n$ only sees the first n elements. Therefore, the union is not closed under countable unions and hence is not a σ -algebra.

Conclusion

The increasing union of σ -algebras is not necessarily a σ -algebra. The union may fail to be closed under countable unions, so:

$$\bigcup_{i=1}^{\infty} \mathcal{A}_i$$

is not necessarily a σ -algebra.

4 Bass Exercise 2.3: Question

Suppose

$$\mathcal{M}_1 \subset \mathcal{M}_2 \subset \dots$$

are monotone classes, and let

$$\mathcal{M} = \bigcup_{n \geq 1} \mathcal{M}_n.$$

Suppose $A_j \uparrow A$ and each $A_j \in \mathcal{M}$. Is A necessarily in $\mathcal{M}$? If not, give a counterexample.

Answer: No, not necessarily.

Counterexample: Let

$$\Omega = \{0, 1\}^{\mathbb{N}}, \quad \omega = (\omega_1, \omega_2, \dots).$$

For each $n \geq 1$, let $\mathcal{M}_n$ be the σ -algebra (hence a monotone class) generated by the first n coordinate maps $\omega \mapsto \omega_1, \dots, \omega_n$. Define

$$\mathcal{M} = \bigcup_{n \geq 1} \mathcal{M}_n.$$

Each $\mathcal{M}_n$ is a monotone class and $\mathcal{M}_1 \subset \mathcal{M}_2 \subset \dots$.

Now, for $j \geq 1$, set

$$A_j = \{\omega \in \Omega : \omega_i = 1 \text{ for some } i \leq j\}.$$

Clearly $A_j \in \mathcal{M}_j \subset \mathcal{M}$, and

$$A_j \uparrow A := \bigcup_{j \geq 1} A_j = \{\omega : \exists i \ \omega_i = 1\},$$

i.e. A is the set of all sequences except the all-zero sequence

$$0^\infty = (0, 0, 0, \dots).$$

We claim $A \notin \mathcal{M}$. Suppose otherwise. Then $A \in \mathcal{M}_n$ for some n . But every set in $\mathcal{M}_n$ depends only on the first n coordinates: if two sequences agree on the first n coordinates, then they are either both in the set or both outside it.

Consider the two sequences:

$$\omega' = (0, 0, \dots, 0, 1, 0, 0, \dots) \quad (\text{with a 1 at position } n+1),$$

and

$$0^\infty = (0, 0, 0, \dots).$$

They agree on the first n coordinates, but $\omega' \in A$ while $0^\infty \notin A$. Thus $A \notin \mathcal{M}_n$ for any n , hence $A \notin \mathcal{M}$.

$$\therefore \quad A_j \uparrow A, \quad A_j \in \mathcal{M}, \quad \text{but } A \notin \mathcal{M}.$$

This shows that the union of an increasing sequence of monotone classes need not be closed under increasing limits.

5 Bass Exercise 2.6: Question

Suppose X is non-empty. Suppose $\mathcal{A}$ is a σ -algebra with the property that whenever $A \in \mathcal{A}$ is non-empty, there exist $B, C \in \mathcal{A}$ with $B \cap C = \emptyset$, $B \cup C = A$, and neither B nor C is empty. Prove that $\mathcal{A}$ is uncountable.

Proof. We shall produce a countably infinite family of pairwise disjoint nonempty sets in $\mathcal{A}$. From that it will follow immediately that $\mathcal{A}$ contains at least $2^{\aleph_0}$ distinct sets (all possible unions of those countably many disjoint sets), hence $\mathcal{A}$ is uncountable.

Define sets $C_0, C_1, \dots$ and $B_1, B_2, \dots$ recursively as follows. Put $C_0 := X$ (which is nonempty by assumption). Given $C_{n-1} \in \mathcal{A}$ and $C_{n-1} \neq \emptyset$, the hypothesis yields $B_n, C_n \in \mathcal{A}$ with

$$B_n \cap C_n = \emptyset, \quad B_n \cup C_n = C_{n-1},$$

and $B_n \neq \emptyset$, $C_n \neq \emptyset$. This defines B_n and C_n for every $n \geq 1$.

Observe that the sets B_n are pairwise disjoint: if $m < n$ then

$$B_n \subseteq C_{n-1} \subseteq C_m \quad \text{and} \quad B_m \subseteq C_{m-1} \setminus C_m,$$

so $B_m \cap B_n = \emptyset$. (Equivalently, each B_n lies in the difference $C_{n-1} \setminus C_n$, and these differences are disjoint for different n .) By construction each B_n is nonempty and belongs to $\mathcal{A}$.

Now consider the map

$$\Phi : \mathcal{P}(\mathbb{N}) \longrightarrow \mathcal{A}, \quad I \mapsto \bigcup_{n \in I} B_n.$$

Every $I \subseteq \mathbb{N}$ is at most countable, so the union $\bigcup_{n \in I} B_n$ is a countable union of elements of $\mathcal{A}$ and therefore lies in $\mathcal{A}$ (since $\mathcal{A}$ is a σ -algebra). Because the B_n are pairwise disjoint, the map Φ is injective: if $I \neq J$ let k be the least index in the symmetric difference $I \triangle J$; then B_k is contained in exactly one of $\Phi(I)$ or $\Phi(J)$, hence $\Phi(I) \neq \Phi(J)$.

Thus $\mathcal{A}$ contains at least as many elements as $\mathcal{P}(\mathbb{N})$, so

$$|\mathcal{A}| \geq |\mathcal{P}(\mathbb{N})| = 2^{\aleph_0},$$

in particular $\mathcal{A}$ is uncountable. □

6 Bass Exercise 2.8: Countable σ -algebra?

Does there exist a σ -algebra which has countably many elements, but not finitely many?

Answer

No. A σ -algebra is either finite or of cardinality at least $2^{\aleph_0}$ (in particular uncountable). In particular there is no countably infinite σ -algebra. Below is a standard proof.

Proof. Let $(X, \mathcal{A})$ be a measurable space. If $X = \emptyset$ then $\mathcal{A} = \{\emptyset\}$ is finite, so there is nothing to show. Assume $X \neq \emptyset$.

Call a nonempty set $A \in \mathcal{A}$ an *atom* if there is no $B \in \mathcal{A}$ with $\emptyset \subsetneq B \subsetneq A$ (i.e. A has no nontrivial measurable subset). The atoms of $\mathcal{A}$ are pairwise disjoint: if A and A' are distinct atoms then $A \cap A'$ is a measurable subset of A , so by minimality it must be either $\emptyset$ or A ; since $A' \neq A$ we get $A \cap A' = \emptyset$.

Let $\mathcal{A}_{\text{atom}}$ be the (possibly empty or infinite) collection of atoms. Two cases arise.

Case 1: $\mathcal{A}_{\text{atom}}$ is finite. Show that every measurable set is a union of atoms. Indeed, let

$$N := X \setminus \bigcup_{A \in \mathcal{A}_{\text{atom}}} A$$

be the part of X not covered by atoms. If $N \neq \emptyset$ then N contains no atom, so every nonempty measurable subset of N admits a proper nonempty measurable subset; iterating this (or using Zorn's lemma on the family of measurable subsets of N ordered by inclusion) contradicts that atoms were maximal minimal nonempty sets. Thus $N = \emptyset$, i.e. the atoms partition X . Consequently every $E \in \mathcal{A}$ is a union of some subcollection of the finitely many atoms, hence $|\mathcal{A}| \leq 2^{|\mathcal{A}_{\text{atom}}|} < \infty$. So $\mathcal{A}$ is finite.

Case 2: $\mathcal{A}_{\text{atom}}$ is infinite. Then the atoms provide an infinite sequence of pairwise disjoint nonempty measurable sets $\{A_n : n \in \mathbb{N}\}$. For each subset $I \subset \mathbb{N}$ the union $\bigcup_{n \in I} A_n$ is measurable (countable unions are allowed), and different I give different unions because the A_n are disjoint. Hence there is an injection

$$\mathcal{P}(\mathbb{N}) \hookrightarrow \mathcal{A}, \quad I \mapsto \bigcup_{n \in I} A_n,$$

so $|\mathcal{A}| \geq |\mathcal{P}(\mathbb{N})| = 2^{\aleph_0}$. In particular $\mathcal{A}$ is uncountable.

Combining the two cases we see that a σ -algebra is either finite or has cardinality at least $2^{\aleph_0}$. Therefore there is no σ -algebra with countably infinitely many elements.