Hamiltonian Dynamics

in general coordinates q = position
p = usually momentum

$$\begin{pmatrix} \dot q \\ \dot p \end{pmatrix} = X_H \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial p} \\ -\frac{\partial H}{\partial q} \end{pmatrix} = \underbrace{\begin{pmatrix} \nabla_p H \\ -\nabla_q H \end{pmatrix}}_{\text{vector notation}}$$

for the Hamiltonian function $H : \mathbb{R}^{2n} \to \mathbb{R}$.

Throughout we assume that H is C^2.

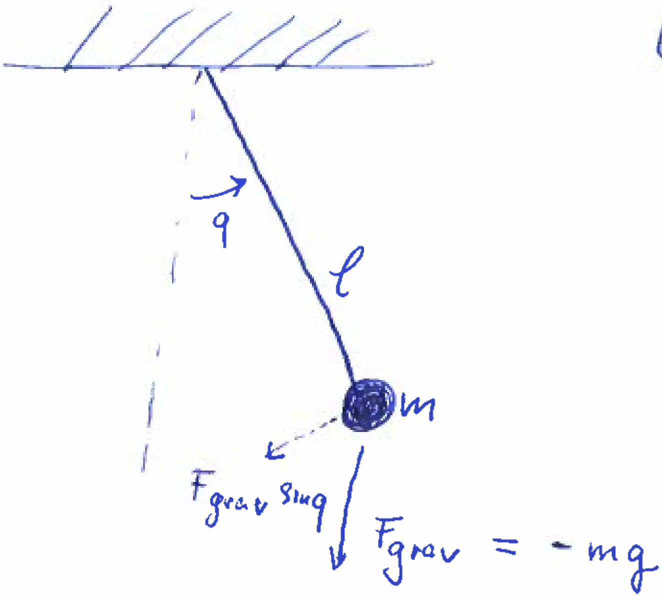
H is preserved along orbits:

$$\dot H = \nabla_q H \circ \dot q + \nabla_p H \circ \dot q$$

$$= \nabla_q H \, \nabla_p H - \nabla_p H \, \nabla_q H = 0$$

Usually H represents the total energy,
but other preserved quantities (momentum,
angular momentum, ...) are possible too.

Example of pendulum



Newton equation of motion

$$\boxed{\ddot{q} + \frac{g}{\ell}\sin q = 0}$$

$g \approx 9.8 \frac{m}{sec^2}$ = gravitational constant

q = angle

$p = m\ell^2\dot{q}$ ← See page 8 for the reason of this choice of p. Roughly, since $q = \text{position}/\ell$, "momentum" p has extra factor ℓ.

$$E_{kin} = \frac{m(\ell\dot{q})^2}{2} = \frac{p^2}{2m\ell^2} \qquad E_{pot} = -mg\ell\cos q$$

$$H(p,q) = E_{kin} + E_{pot} = \frac{p^2}{2m\ell^2} - mg\ell\cos q$$

$$\frac{\partial H}{\partial p} = \frac{p}{m\ell^2} = \dot{q}$$

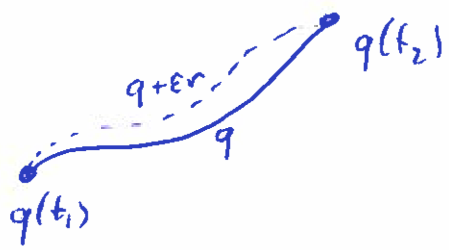
$$-\frac{\partial H}{\partial q} = -mg\ell\sin q \overset{\text{Newton eq.}}{=} \frac{m}{\ell^2}\ddot{q} = \dot{p}$$

Lagrangian Dynamics

in general coordinates $(v, q) \in \mathbb{R}^{2n}$ $\quad q$ position, $v = \dot{q}$

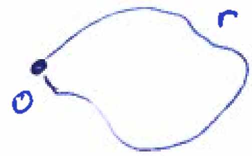
and Lagrangian $L : \mathbb{R}^{2n} \to \mathbb{R}$ twice differentiable

The task is to minimize the action of the Lagrangian over all C^1 paths, i.e.



minimize

$$I(q) = \int_{t_1}^{t_2} L(\dot{q}(t), q(t))\, dt$$



Fix $r(t_1) = r(t_2) = 0$ ($q(t_1)$ and $q(t_2)$ are fixed !)

and compute the Gateaux derivative

$$\frac{d}{d\varepsilon} I(q+\varepsilon r)\Big|_{\varepsilon=0} = \int_{t_1}^{t_2} \nabla_q L(\dot{q}+\varepsilon\dot{r}, q+\varepsilon r) \circ r + \nabla_v L(\dot{q}+\varepsilon\dot{r}, q+\varepsilon r) \circ \dot{r}\, dt \Big|_{\varepsilon=0}$$

$$\underset{\text{by parts}}{\overset{\text{Integrate}}{=}} \int_{t_1}^{t_2} \left(\nabla_q L(\dot{q}, q) - \frac{d}{dt} \nabla_v L(\dot{q}, q) \right) \circ r\, dt$$

NB: constant terms disappear because $r(t_1) = r(t_2) = 0$.

For $\frac{d}{d\varepsilon} I(q+\varepsilon r)\Big|_{\varepsilon=0} = 0$ for all paths r we need

Euler-Lagrange equation $\quad \boxed{\nabla_q L(\dot{q}, q) = \frac{d}{dt} \nabla_v L(\dot{q}, q)}$

Example pendulum continued

Lagrangian $L(v, q) = L(\dot{q}, q) = E_{kin} - E_{pot}$

$$= \frac{m \ell^2 \dot{q}^2}{2} + m g \ell \cos q$$

Euler-Lagrange:

$$0 = \nabla_q L - \frac{d}{dt} \nabla_v L = - m g \ell \sin q - \frac{d}{dt} m \ell^2 \dot{q}$$

$$\Leftrightarrow \quad 0 = \frac{g}{\ell} \sin q + \ddot{q}$$

which is Newton's equation of motion.

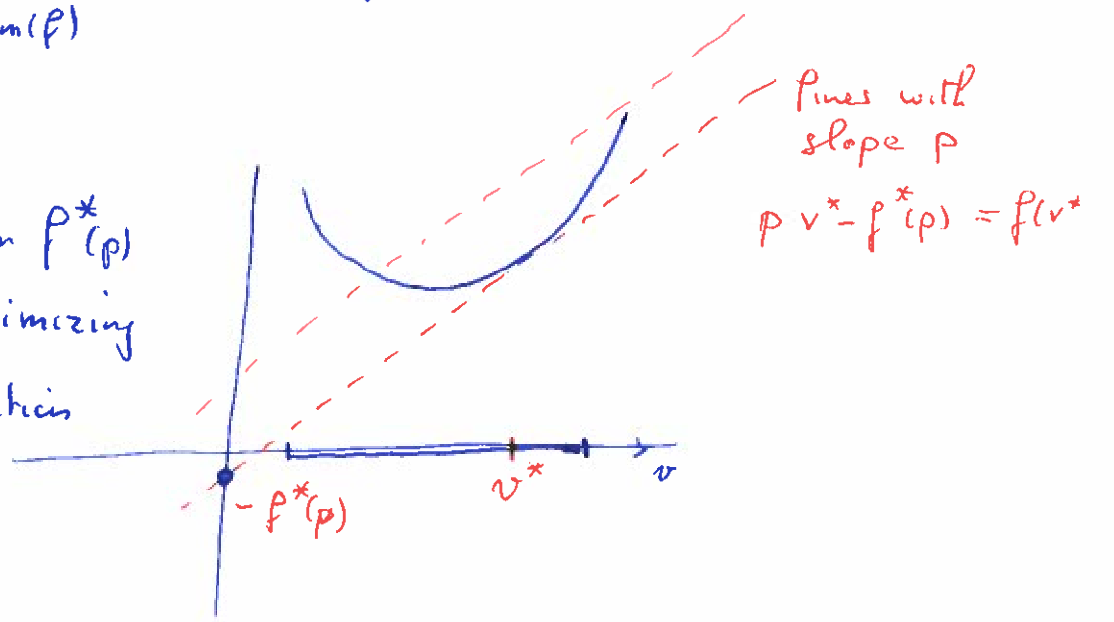
Legendre Transform

Let $f : \mathrm{Dom}(f) \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be C^2 & strictly convex.

Define the Legendre transform

$$f^*(p) = \sup_{v \in \mathrm{Dom}(f)} p \cdot v - f(v)$$

The supremum in $f^*(p)$ translates to minimizing $-f^*(p)$ = intersection of lines of slope p with vertical axis provided the line intersects the graph of f.



This is achieved at the tangent line of slope p.

Let $v^*(p)$ the value where the supremum is assumed. Then

$$\begin{cases} p = \nabla_v f(v^*(p)) & (*) \\ f^*(p) = p \cdot v^*(p) - f(v^*(p)) & (**) \end{cases}$$

Thm a) $f^{**} = f$

b) The Legendre transform of a C^2 strictly convex function is C^2 and strictly convex.

Proof a)

$$\nabla_p f^*(p) \overset{(**)}{=} \nabla_p \left(p \cdot v^*(p) - f(v^*(p)) \right)$$

$$= v^*(p) + p \cdot \nabla_p v^*(p) - \nabla_v f \, \nabla_p v^*(p)$$

$= \nabla_v f(v^*(p))$ by (*)

$$= v^*(p),$$

so (*) holds for f^* instead of f. But then (**) follows as well, and hence $f^{**} = f$.

(1) so $\nabla_p f^*(p) = v$
$\nabla_v f(v) = p$ so
$\nabla_v f$ and $\nabla_p f^*$ are each others inverse

Now f is C^2 & strictly convex
iff the Jacobian $J(\nabla f)$ is positive definite
iff $J(\nabla f)^{-1} = J(\nabla f^*)$ is positive definite

∇f and ∇f^* are each others inverse by (1)

iff f^* is C^2 and strictly convex.

$\boxed{//}$

Thm Let (for fixed q) $p = \nabla_v L(v^*, q)$ and

$$H(p,q) = L^*(v,q) = p \cdot v^* - L(v^*, q)$$

That is, the Hamiltonian is the Legendre transform of the Lagrangian.
Then the Euler-Lagrange equations are equivalent to the Hamiltonian equations.

Proof Since also $L = H^*$, using ⊛ with $f = H$ and v and p swapped, we find

$$\dot{q} = v = \nabla_p H$$

Also

$$\nabla_q H = \nabla_q \left(p \cdot v^*(p,q) - L(v^*(p,q), q) \right)$$

$$= p \cdot \nabla_q v^* - \nabla_v L \cdot \nabla_q v^* - \nabla_q L$$

$$\overset{\circledast \text{ with } f=L}{=} p \cdot \nabla_q v^* - p \cdot \nabla_q v^* - \nabla_q L$$

$$= - \nabla_q L$$

$$\overset{\text{Euler-Lagrange}}{=} - \frac{d}{dt} \nabla_v L$$

$$\overset{\circledast \text{ with } f=L}{=} - \frac{d}{dt} p = \dot{p}$$

$\blacksquare$

Example pendulum continued

Take $v = \dot{q}$ and $L(v, q) = \frac{m\ell^2 v^2}{2} + mg\ell \cos q$

Take the Legendre transform to obtain the Hamiltonian:

$$H(p,q) = \sup_{v \in \mathbb{R}} \; p \cdot v - \left(\frac{m\ell^2 v^2}{2} + mg\ell \cos q \right).$$

Set derivative $\frac{d}{dv} = 0$: $\quad p - m\ell^2 v = 0 \Leftrightarrow v = \frac{p}{m\ell^2}$

Insert $\quad H(p,q) = \frac{p^2}{m\ell^2} - \frac{m\ell^2 p^2}{2m^2\ell^4} - mg\ell \cos q$

$$= \frac{p^2}{2m\ell^2} - mg\ell \cos q .$$

$$= E_{kin} + E_{pot} .$$

Note that we get from this computation that:

$$p = m\ell^2 v = m\ell^2 \dot{q}$$

Same example as before, but now with Einstein's formula of kinetic energy:

$$E_{kin} = m(v)\, c^2 \qquad m(v) = m_0 \sqrt{1 + \frac{|v|^2}{c^2}}$$

$$E_{pot} = E_{pot}(q)$$

m_0: rest mass; c: speed of light

Lagrangian $\quad L(v,q) = E_{kin}(v) - E_{pot}(q)$

Euler-Lagrange $\quad \frac{d}{dq} E_{pot}(q) \mp \frac{d}{dt} \frac{m_0 c\, \dot q}{\sqrt{c^2 + \dot q^2}} \quad , \quad \dot q = v$

Hamiltonian via Legendre transform

$$H(p,q) = L^*(v,q) = \sup_{v \in \mathbb{R}} v \cdot p - \left(m_0 c^2 \sqrt{1 + \frac{|v|^2}{c^2}} - E_{pot}(q) \right)$$

Solve $\frac{d}{dv} = 0$: $\quad p - \frac{m_0 c\, v}{\sqrt{c^2 + |v|^2}} = 0$

gives $\quad p^2 (c^2 + v^2) = m_0^2 c^2 v^2 \Leftrightarrow v^2 = \frac{c^2 p^2}{c^2 m_0^2 - p^2}$.

Insert in $H(p,q)$:

$$H(p,q) = \frac{c\, p^2}{\sqrt{c^2 m_0^2 - p^2}} - m_0 c^2 \sqrt{1 + \frac{p^2}{c^2 m_0^2 - p^2}} + E_{pot}(q)$$

$$= -c \sqrt{c^2 m_0^2 - p^2} + E_{pot}(q).$$

<u>Thm</u> The Hamiltonian flow preserves volume in $\mathbb{R}^{2n}$.

<u>Remark</u> This "Volume" is sometimes called <u>Liouville measure</u> in this context.

<u>Proof</u> Recall $X_H = \begin{pmatrix} \nabla_p H \\ -\nabla_q H \end{pmatrix}$, the Hamiltonian vector field. Let $V(t) = \int_{\varphi_H^t(\Omega)} d\mathrm{Vol}$, φ_H^t Hamiltonian flow.

Earlier (Thm is in the Schmeiser notes) we have seen

$$\dot{V}(t) = \int_{\varphi_H^t(\Omega)} \operatorname{div} X_H \, d\mathrm{Vol},$$

but

$$\operatorname{div} X_H = \sum_i \frac{\partial}{\partial q_i}\left(\frac{\partial}{\partial p_i} H\right) + \sum_i \frac{\partial}{\partial p_i}\left(-\frac{\partial}{\partial q_i} H\right) = 0.$$

Therefore $\dot{V}(t) = 0$, $V(t)$ is constant $\blacksquare$

Not only energy can supply the Hamiltonian, there are often other preserved quantities, e.g. momentum, angular momentum, ...

These can often be related to a <u>continuous symmetry</u> in the system. This is the content of Noether's Theorem.

Let $Q(s,q)$ be a continuous family of motions in $\mathbb{R}^n$.
For instance, rotations in $\mathbb{R}^2$

$$(s,q) \longmapsto \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

Formally $Q(0,q) = q \quad \forall q \in \mathbb{R}^n$, $Q(s+t,q) = Q(s, Q(t,q)) \quad \forall s,t \ \forall q \in \mathbb{R}^n$

so we can describe $Q(s,q)$ as the flow of a vector field, say f:

$$\begin{cases} \frac{d}{ds} Q(s,q) = f(Q(s,q)) \\ Q(0,q) = q \end{cases}$$

The action on tangent vectors $v = \dot{q}$

$$v \longmapsto D_q Q(s,q)\, v \qquad v \in T_q\mathbb{R}^n \xrightarrow{DQ(s,q)} T_{Q(s,q)}\mathbb{R}^n$$

with s-derivative at $s=0$:

$$\boxed{\frac{d}{ds} D_q Q(s,q)\Big|_{s=0} = D_q \frac{d}{ds} Q(s,q)\Big|_{s=0} = D_q f(Q(s,q))\Big|_{s=0} = D_q f(q).} \quad (\ddagger)$$

$$q \in \mathbb{R}^n \xrightarrow{Q(s,q)} \mathbb{R}^n$$

Def A vector field f generates a symmetry of a Lagrangian system if $L(v,q) = L(D_q Q(s,q)v, Q(s,q))$ (*) for all $s \in \mathbb{R}$, $q \in \mathbb{R}^n$, $v \in T_q\mathbb{R}^n$.

Thm (Noether) If f generates a symmetry, then

$$I(v,q) := \nabla_v L(v,q) \cdot f(q)$$

is a first integral

Proof Differentiate (*) w.r.t s at $s=0$

$$0 = \frac{d}{ds} L(D_q Q(s,q)v, Q(s,q))\Big|_{s=0}$$

$$= \nabla_v L \cdot \frac{d}{ds} D_q Q(s,q) v + \nabla_q L \cdot \underbrace{\frac{d}{ds} Q(s,q)}_{f(q)}\Big|_{s=0} \qquad (**\!*)$$

Hence

$$\dot{I}(v,q) = \frac{d}{dt}\nabla_v L(v,q) \cdot f + \nabla_v L(v,q) \cdot \underbrace{\Big(\nabla_q f \cdot \underbrace{\frac{d}{dt} q}_{v}\Big)}_{\frac{d}{ds} D_q Q(s,q) v \big|_{s=0}}$$

by (†) on page 11.

$$\overset{\text{by } (**\!*)}{=} \frac{d}{dt}\nabla_v L \cdot f - \nabla_q L \cdot f$$

$$\overset{\text{Euler-Lagrange}}{=} \left(\frac{d}{dt}\nabla_v L - \nabla_q L\right) \cdot f = 0.$$

$\blacksquare$

Example Assume $q \in \mathbb{R}^n$ and $L(v,q)$ does not depend on coordinate q_j

Take $Q(s,q) = q + s e_j$

so $f(q) = e_j$ e_j is j-th basis vector.

$D_q Q(s,q) = Id$

$$\mathcal{I}(v,q) = \nabla_v L(v,q) \cdot f(q) = \nabla_v (\underbrace{E_{kin}}_{\frac{mv^2}{2}} - \underbrace{E_{pot}(q)}_{\text{indep. of } q_j}) \cdot f(q)$$

$$= mv \cdot e_j = mv_j = p_j$$

That is: the j-th component of the momentum is preserved.

Example Rotational symmetries $q \mapsto Q(s,q) = A(s)\, q$

for a family of skew-symmetric matrices,

e.g. $A(s) = \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix}$,

lead to the preservation of angular momentum (or components thereof)

$$\mathcal{I}(v,q) = m v^T A q \Big|_{s=0}$$

Def When a Hamiltonian system in $\mathbb{R}^{2n}$ has n first integrals, $I_j(q,p)$ for $j=1,\dots,n$, then the system is <u>integrable</u>.

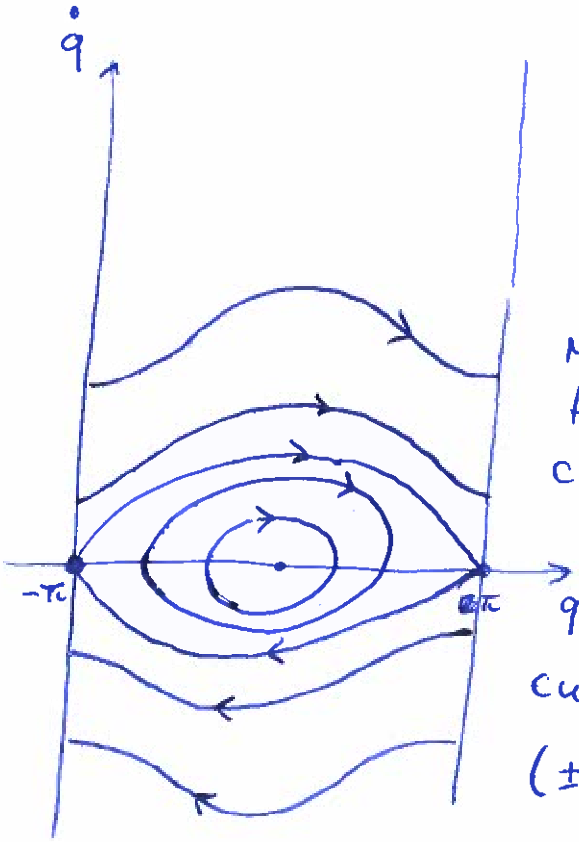
This doesn't mean that you can solve the system explicitly (in general), but motions are confined to the intersection of level sets

$$\bigcap_{j=1}^{n} \{ I_j(q,p) = a_j \} .$$

When these level sets are compact, then almost surely they are n-dimensional tori

Pendulum example

$n=1$, $H = E_{kin} + E_{pot}$ is a first integral, so the system is integrable.



blue curves are level sets. Most of them are circles, except for the curve through $(\pm\pi, 0)$

Example (Kepler = two body problem)

This is the motion of two particles (e.g. Sun & Earth) in 3-dim space under influence of their mutual gravitation (no other forces).

Here $n=6$ (3 space coordinates of the Sun, 3 of the Earth) and there (more than) 6 first integrals namely:
- the momentum vector p^* (3 components)
- the angular momentum vector $\ell^* = \sum_{i=1}^{2} \vec{p}_i \times \vec{r}_i$ (3 components.)

Therefore all compact motions are confined to 6-dimensional tori (Integrable).

There are, however, more first integrals:
- from Galilei transformations $(v, q) \mapsto (v - v_0, q - t v_0)$ (allows us to put $p^* = 0$)
- Energy $E_{kin} + E_{pot}$
- Laplace - Runge-Lenz vector $p^* \times \ell^* - m k \vec{r}/|r|$.
  Only! because gravitational force is inverse proportional to the square of the distance

This reduces the motion to 1-dim tori, in fact ellipses (or parabolas or hyperbolas for non-compact motion).