# Cusp bifurcation & hysteresis

As an example we look at a model of the population of the spruce budworm, a type of moth that harms Canadian coniferous forest by eating away needles. A small increase of the food supply can cause a sudden jump in the budworm population.

$N$ = population size [budworms]

$\mathbb{R} \ni \tau$ = time [days]

$0 < R$ = growth rate [1/days]

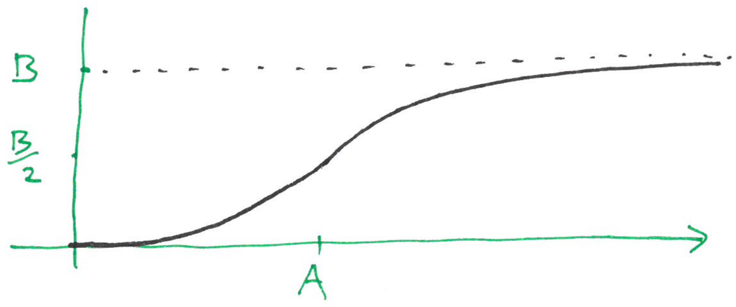
$0 < K$ = carrying capacity of forest [budworms]

$0 < A$ = critical detection constant [budworms]

$0 < B$ = predator eating capacity [budworms/day]

$$\frac{dN}{d\tau} = \underbrace{RN\left(1 - \frac{N}{K}\right)}_{\text{logistic population ODE}} - \underbrace{\frac{BN^2}{A^2+N^2}}_{\text{effect of predating birds}}$$

exp. growth near equil. $N=0$



A change in the coordinates $N$ and $\tau$ eliminates two of the four parameters:

$$u = \frac{N(t \cdot A/B)}{A} \qquad t = \tau \frac{B}{A}$$

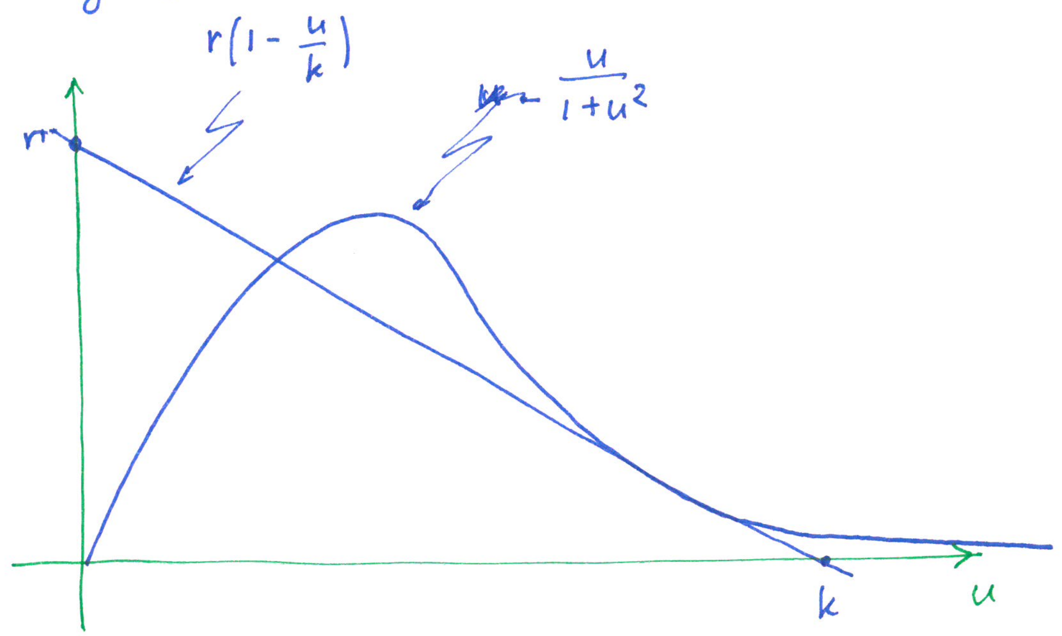
$$\dot{u} = \frac{1}{A} N' \cdot \frac{d\tau}{dt} = \frac{1}{B} N' = \frac{1}{B}\left( RN\left(1 - \frac{N}{K}\right) - \frac{BN^2}{A^2+N^2} \right)$$

$$= \underbrace{\frac{RA}{B}}_{=r} u \left(1 - \underbrace{\frac{A}{K}}_{=\frac{1}{k}} u\right) - \frac{A^2u^2}{A^2 + A^2u^2}$$

gives

$$\dot{u} = r u \left(1 - \frac{u}{k}\right) - \frac{u^2}{1+u^2}$$

Stationary points for $u = 0$ and when $r\left(1 - \frac{u}{k}\right) = \frac{u}{1+u^2}$



Depending on the values of $r$ & $k$

$$r\left(1-\frac{u}{k}\right) = \frac{u}{1+u^2}$$

has one, two or three solutions, and the two/three solutions emerge in saddle node bifurcations when the line $u \longmapsto r\left(1-\frac{u}{k}\right)$ is tangent to the graph of $u \longmapsto \frac{u^2}{1+u^2}$

We trace these saddle node bifurcations in the $(r,k)$-plane.

$u$ not divided out.

Stationary points $\quad ru\left(1-\frac{u}{k}\right) = \frac{u^2}{1+u^2} \qquad (1)$

Both graphs are tangent: $\quad r\left(1-\frac{2u}{k}\right) = \frac{2u}{(1+u^2)^2} \qquad (2)$

The joint solution of (1) & (2) has the form of a parametrised curve

$$r = \frac{2u^3}{(1+u^2)^2} \qquad k = \frac{2u^3}{u^2-1} \qquad u > 1$$

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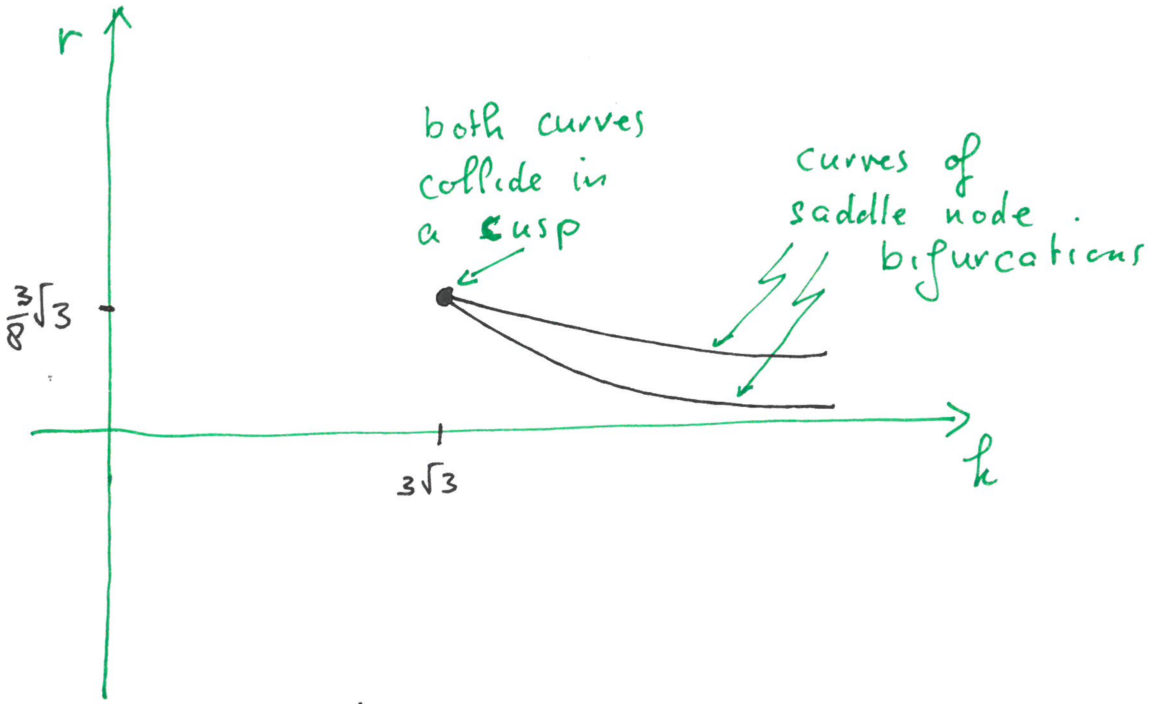
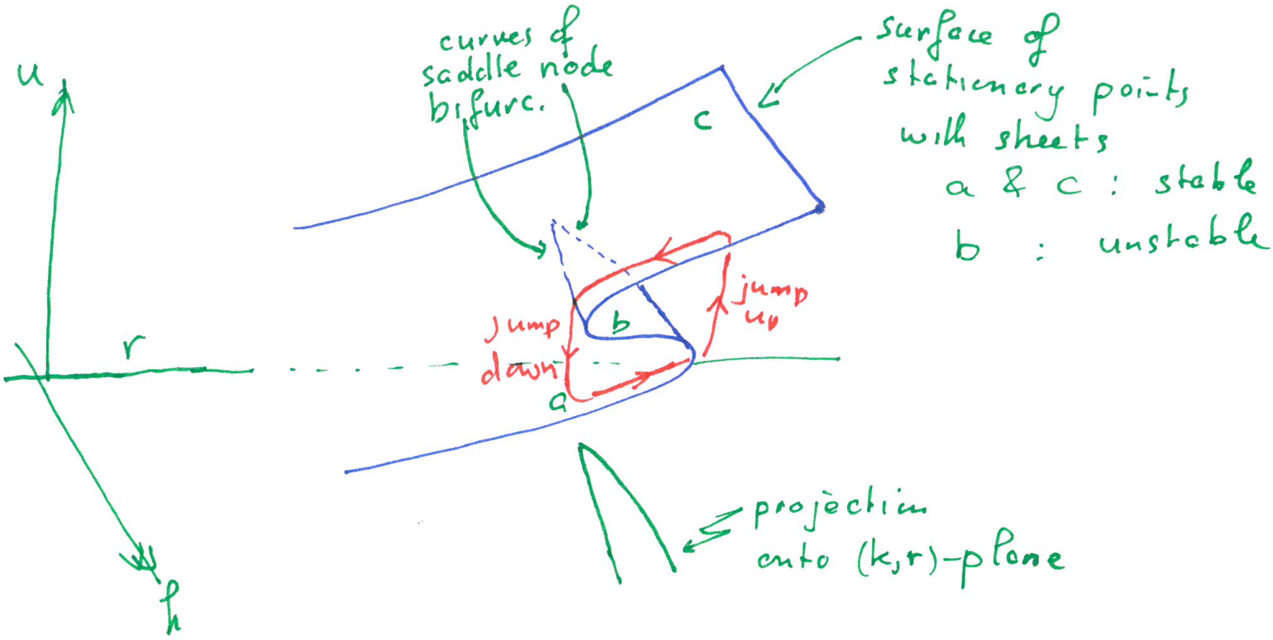



$$\begin{cases} \dfrac{dr}{du} = \dfrac{-2u^4+6u^2}{(1+u^2)^3} = 0 \\[2ex] \dfrac{dk}{du} = \dfrac{2u^4-6u^2}{u^2-1} = 0 \end{cases}$$

has a simultaneous solution for $u = \sqrt{3}$,

$$\underbrace{r(\sqrt{3}) = \tfrac{3}{8}\sqrt{3}}_{\text{max for } r}, \quad \underbrace{k(\sqrt{3}) = 3\sqrt{3}}_{\text{min for } k}$$

In the $(r, k, u)$-space




As $r$ grows (as effect of increasing tree = food supply), you go through a saddle node, and the equilibrium jumps to sheet c

Decreasing $r$ again, we remain on sheet c; you have to decrease $r$ <u>much further</u> to jump back to the original sheet a

Hence, reversing the parameter doesn't have the reverse effect on the behaviour.
This phenomenon is called <u>hysteresis</u>

# Normal Form for the Cusp Bifurcation

$$\boxed{\dot{u} = r + ku + u^3}$$

Stationary points $\quad \dot{u} = r + ku + u^3 = 0 \qquad (1)$

Saddle node bifurcations when the derivative of the RHS $\quad k + 3u^2 = 0 \qquad (2)$

$(1) \Rightarrow \quad u = \pm\sqrt{\frac{-k}{3}}$ so $k \leq 0$ required

$(2)+(1) \Rightarrow \quad r = \mp \frac{2}{3} k \sqrt{\frac{-k}{3}}$

Both saddle node bifurcations occur <u>at the same time</u> in a <u>cusp bifurcation</u> when $k = r = 0$