Not only energy can supply the Hamiltonian, there are often other preserved quantities, e.g. momentum, angular momentum, ...

These can often be related to a <u>continuous symmetry</u> in the system. This is the content of Noether's Theorem.

Emmy Noether
German 1882-1935

Let $Q(s,q)$ be a continuous family of motions in $\mathbb{R}^n$.
For instance, rotations in $\mathbb{R}^2$

$$Q: (s,q) \longmapsto \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

Formally $Q(0,q) = q \quad \forall q \in \mathbb{R}^n$, $Q(s+t,q) = Q(s, Q(t,q)) \quad \forall s,t \quad \forall q \in \mathbb{R}^n$
so we can describe $Q(s,q)$ as the flow of a vector field, say f:

$$\begin{cases} \frac{d}{ds} Q(s,q) = f(Q(s,q)) \\ Q(0,q) = q \end{cases}$$

The action on tangent vectors $v = \dot{q}$

$$v \longmapsto D_q Q(s,q)\, v$$

$$v \in T_q \mathbb{R}^n \xrightarrow{DQ(s,q)} T_{Q(s,q)} \mathbb{R}^n$$

with s-derivative at $s=0$:

$$q \in \mathbb{R}^n \xrightarrow[Q(s,q)]{} \mathbb{R}^n$$

$$\boxed{\frac{d}{ds} D_q Q(s,q)\Big|_{s=0} = D_q \frac{d}{ds} Q(s,q)\Big|_{s=0} = D_q f(Q(s,q))\Big|_{s=0} = D_q f(q).} \quad (\dagger)$$

Def A vector field f generates a symmetry of a Lagrangian system if $L(v,q) = L(D_q Q(s,q)v, Q(s,q))$ (*)
for all $s \in \mathbb{R}$, $q \in \mathbb{R}^n$, $v \in T_q\mathbb{R}^n$, $\frac{d}{ds}Q(s,q) = f(Q(s,q))$

Thm (Noether) If f generates a symmetry, then

$$I(v,q) := \nabla_v L(v,q) \cdot f(q)$$

is a first integral (= other word for preserved quantity)

Proof Differentiate (*) w.r.t s at $s=0$

$$0 = \frac{d}{ds} L(D_q Q(s,q)v, Q(s,q))\Big|_{s=0}$$

$$= \nabla_v L \cdot \frac{d}{ds} D_q Q(s,q)v + \nabla_q L \cdot \underbrace{\frac{d}{ds}Q(s,q)}_{f(q)}\Big|_{s=0} \quad (***)$$

Hence $$\dot{I}(v,q) = \frac{d}{dt}\nabla_v L(v,q)\cdot f + \nabla_v L(v,q)\cdot \underbrace{\Big(\nabla_q f \cdot \underbrace{\frac{d}{dt}q}_{v}\Big)}_{\frac{d}{ds}D_q Q(s,q)v\big|_{s=0}}$$

by (***) by (‡) on page 11.

$$= \frac{d}{dt}\nabla_v L\cdot f(q) - \nabla_q L \cdot f(q)$$

$$\overset{\text{Euler-Lagrange}}{=} \left(\frac{d}{dt}\nabla_v L - \nabla_q L\right)\cdot f = 0. \qquad \blacksquare$$

13

Example Assume $q \in \mathbb{R}^n$ and $L(v,q) = E_{kin}(v) - E_{pot}(q)$ does not depend on coordinate q_j

Take $Q(s,q) = q + s e_j$

so $f(q) = e_j$ e_j is j-th basis vector.

$D_q Q(s,q) = Id$

$$I(v,q) = \nabla_v L(v,q) \cdot f(q) = \nabla_v (\underbrace{E_{kin}}_{\frac{mv^2}{2}} - \underbrace{E_{pot}(q)}_{\text{indep. of } q_j}) \cdot f(q)$$

$$= mv \cdot e_j = mv_j = p_j$$

That is: the j-th component of the momentum is preserved.

Example Rotational symmetries $q \mapsto Q(s,q) = A(s)\, q$ for a family of skew-symmetric matrices,

e.g. $A(s) = \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix}$,

lead to the preservation of angular momentum (or components thereof)

$$I(v,q) = m v^T A q \Big|_{s=0}$$

Def When a Hamiltonian system in $\mathbb{R}^{2n}$ has n first integrals, $I_j(q,p)$ for $j=1,\dots,n$, then the system is <u>integrable</u>.

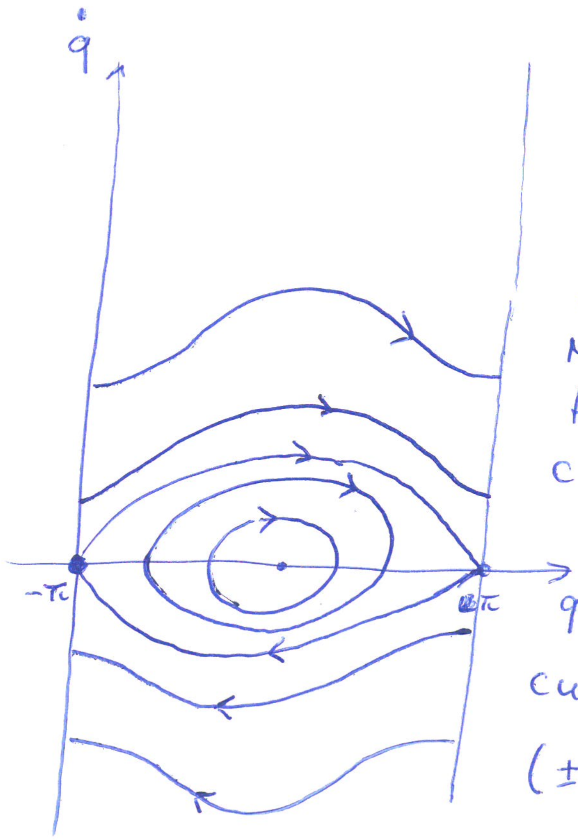
This <u>doesn't</u> mean that you can solve the system explicitly (in general), but motions are confined to the intersection of level sets

$$\bigcap_{j=1}^{n} \{ I_j(q,p) = a_j \}.$$

When these level sets are compact, then almost surely they are n-dimensional tori

Pendulum example

$n=1$, $H = E_{kin} + E_{pot}$ is a first integral, so the system is integrable.



blue curves are level sets. Most of them are circles, except for the curve through $(\pm\pi, 0)$

Example (Kepler = two body problem)

This is the motion of two particles (e.g. Sun & Earth) in 3-dim space under influence of their mutual gravitation (no other forces).

Here $n=6$ (3 space coordinates of the Sun, 3 of the Earth) and there (more then) 6 first integrals namely:

- the momentum vector p^* (3 components)
- the angular momentum vector $\ell^* = \sum_{i=1}^{2} \vec{p}_i \times \vec{r}_i$ (3 components)

Therefore all compact motions are confined to 6-dimensional tori (integrable).

There are, however, more first integrals:

- from Galilei transformations $(v, q) \mapsto (v - v_0, q - t v_0)$ (allows us to put $p^* = 0$)
- Energy $E_{kin} + E_{pot}$
- Laplace - Runge-Lenz vector $p^* \times \ell^* - m k \, \vec{r}/|r|$.
  Only! because gravitational force is inverse proportional to the square of the distance

This reduces the motion to 1-dim tori; in fact ellipses (or parabolas or hyperbolas for non-compact motion).