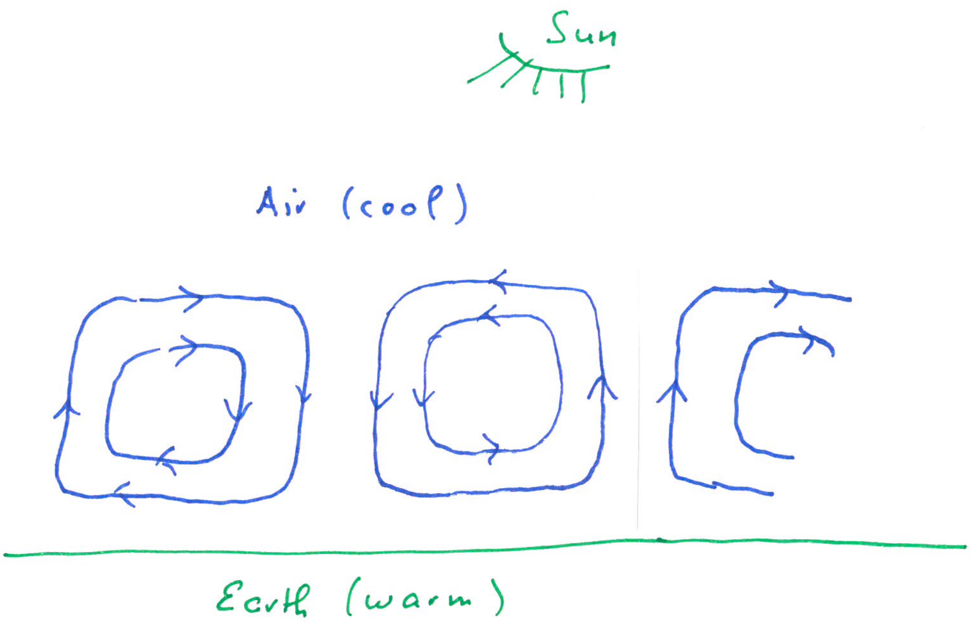
Lorenz Attractors

In the 1960s, the MIT meteorologist Edward Lorenz studied a model for convection rolls in the atmosphere with his computer



This is a model with PDEs, but by translating the problem to Fourier series and truncating to the first three Fourier modes of the solution (Galerkin approximation) he reduced the problem to an non-linear! ODE in $\mathbb{R}^3$, which he then fed to the computer

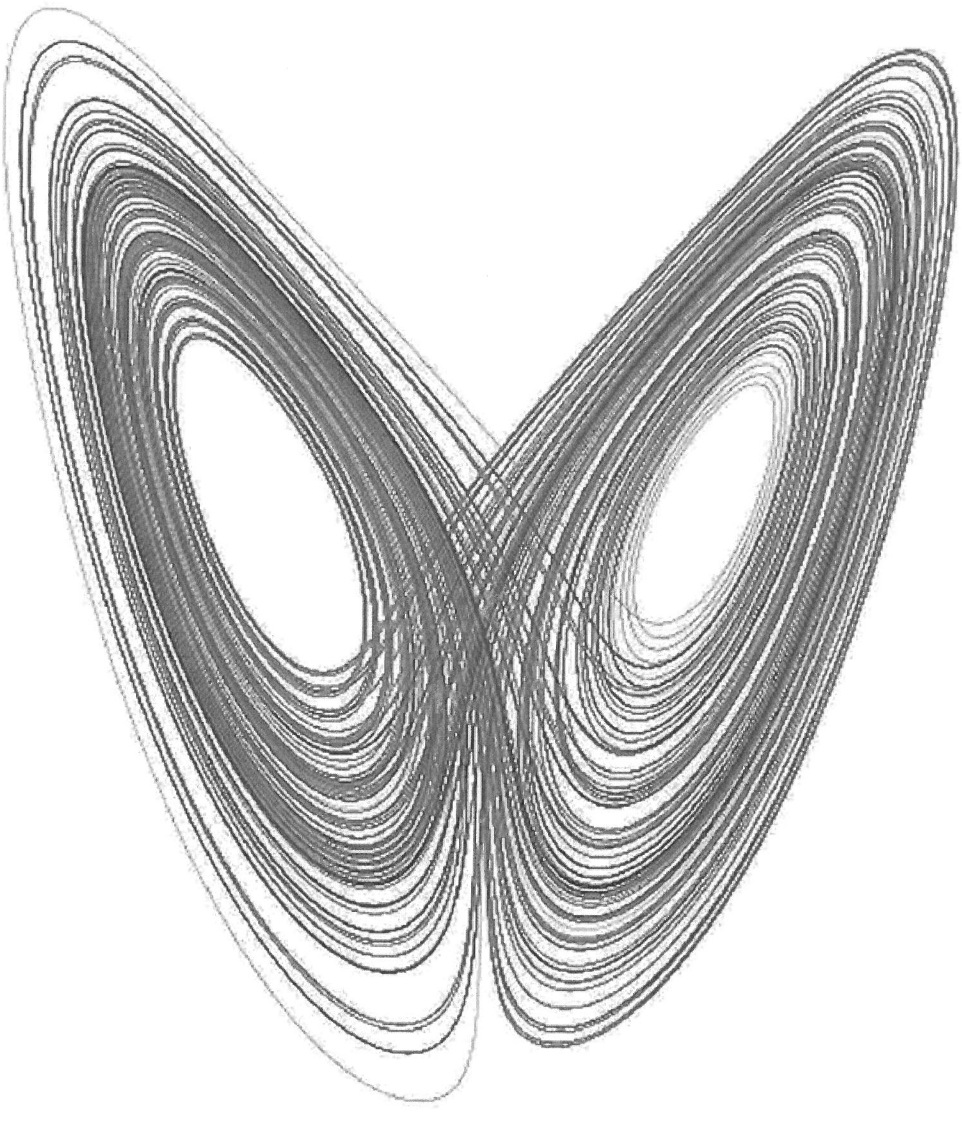


FIGURE 2. The Lorenz attractor

The Lorenz equations

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \sigma(y-x) \\ rx - y - \underline{xz} \\ -bz + \underline{xy} \end{pmatrix}$$

non-linear terms.

$\sigma = 10$ Prandtl number

$b = 8/3$ Rayleigh number

$r = 28$ $\longleftarrow$ we will vary this parameter

Note: The symmetry $(x,y,z) \mapsto (-x,-y,z)$ (reflection in z-axis) sends solutions to solutions.
The z-axis itself consists of solutions.

There is a stationary point at the origin

$$DF\Big|_{(0,0,0)} = \begin{pmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix}\Bigg|_{(0,0,0)} = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}$$

with eigenvalues $-b$ and $-\frac{\sigma+1}{2} \pm \sqrt{\left(\frac{\sigma+1}{2}\right)^2 + \sigma(r-1)}$

$$\Rightarrow \begin{cases} \text{sink for } r < 1 \\ \text{saddle for } r > 1 \end{cases}$$

with two attracting directions and one repelling direction

Further stationary points when

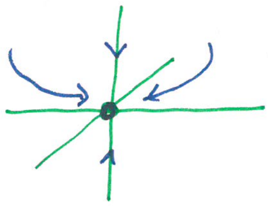
$$\begin{cases} x = y \\ r - \frac{y}{x} = z \\ bz = xy \end{cases} \Rightarrow \begin{cases} x = y \\ r-1 = z \\ x = \pm\sqrt{b(r-1)} \end{cases}$$

i.e. $P_\pm = \left(\pm\sqrt{b(r-1)},\ \pm\sqrt{b(r-1)},\ r-1 \right)$ provided $r \geq 1$.

Bifurcation Analysis

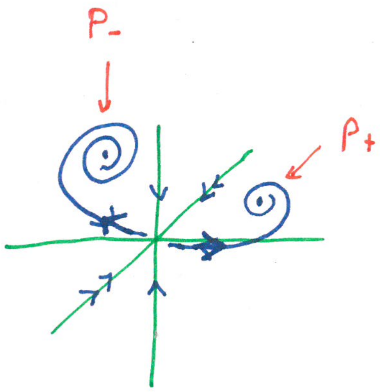
(i) $0 \leq r < 1$

Sink at $(0,0,0)$
Attracts every orbit



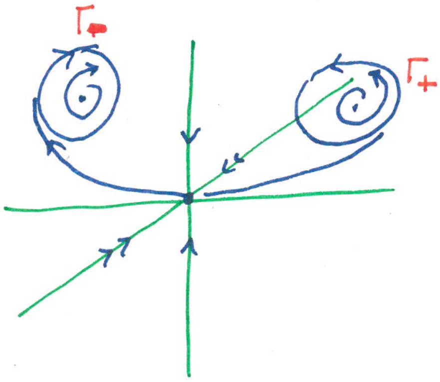
(ii) $r = 1$

Pitchfork bifurcation
$(0,0,0)$ becomes saddle
Two new stationary points emerge



(iii) $r \approx \frac{470}{19}$

$P_\pm$ undergo Hopf bifurcations, creating two attracting periodic orbits $\Gamma_\pm$



(iv) $r \approx 28$

$\Gamma_\pm$ "merge", creating a two-winged "strange attractor"

For (i) $\boxed{r<1}$

Try Lyapunov function $L = \frac{1}{2}(r x^2 + \sigma y^2 + \sigma z^2)$

Then $\dot{L} = \nabla L \bullet F = \begin{pmatrix} r x \\ \sigma y \\ \sigma z \end{pmatrix} \bullet \begin{pmatrix} \sigma(y-x) \\ r x - y - x z \\ -b z + x y \end{pmatrix}$

$$= r\sigma(xy - x^2) + r\sigma xy - \sigma y^2 - \sigma x y z$$
$$- \sigma b z^2 + \sigma x y z$$
$$= -\sigma\left(r(x-y)^2 + (1-r)y^2 + b z^2\right) \leqslant 0.$$

So L is indeed a strict Lyapunov function, and every orbit is asymptotic to the sink at $(0,0,0)$, provided $r<1$.

In general, so also $\boxed{r \geq 1}$, we can try

$$L = \frac{1}{2}\left(r x^2 + \sigma y^2 + \sigma(z-2r)^2\right)$$

Then $\dot{L} = \nabla L \bullet F = \begin{pmatrix} r x \\ \sigma y \\ \sigma(z-2r) \end{pmatrix} \bullet \begin{pmatrix} \sigma(y-x) \\ r x - y - x z \\ -b z + x y \end{pmatrix}$

$$= -\sigma\left(r x^2 + y^2 + b(z+r)^2 - b r^2\right)$$

and this is negative <u>outside</u> the filled ellipsoid

$E: \quad r x^2 + y^2 + b(z+r)^2 \leq b r^2$

Hence all orbit converge to and are then confined to this ellipsoid.

Over time this ellipsoid decreases in volume because

$$\frac{d}{dt} \mathrm{Vol}(\varphi^t(E)) = \int_{\varphi^t(E)} \mathrm{div}\, F \, d\mathrm{Vol}$$

flow

and for the vector field F of the Lorenz equations $\mathrm{div}\, F = -\sigma - 1 - b < 0$,

so $\mathrm{Vol}(\varphi^t(E)) = \mathrm{Vol}(E)\, e^{-(\sigma+1+b)}$.

It follows that the <u>Lorenz attractor</u>

$$A = \bigcap_{t \geq 0} \varphi^t(E)$$

has zero Lebesgue measure.

However A is much more complicated than (the union of) stationary points and limit cycles.

for large values of r

For (iii), the Hopf bifurcation.

By the symmetry $(x,y,z) \mapsto (-x,-y,z)$ it suffices to look only at the stationary point $P_+ = (x, x, r-1)$

$x = \sqrt{b(r-1)}$

The characteristic equation of

$$DF\Big|_{P_+} = \left.\begin{pmatrix} -\sigma & \sigma & 0 \\ r-z & -1-\lambda & -x \\ x & x & -b-\lambda \end{pmatrix}\right|_{P_+} = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -x \\ x & x & -b \end{pmatrix}$$

is

$$\lambda^3 + \lambda^2(\sigma+1+b) + \lambda b(\sigma+r) + 2b\sigma(r-1) = 0$$

At a Hopf bifurcation, there are purely imaginary roots, so we try $\lambda = \pm i\omega$.

$$\begin{cases} 0 = \text{real part} = -\omega^2(\sigma+1+b) + 2b\sigma(r-1) \\ 0 = \text{imaginary part} = -\omega^3 + \omega b(\sigma+r) \end{cases}$$

Hence $\omega^2 = b(\sigma+r) = \dfrac{2b\sigma(r-1)}{\sigma+1+b}$

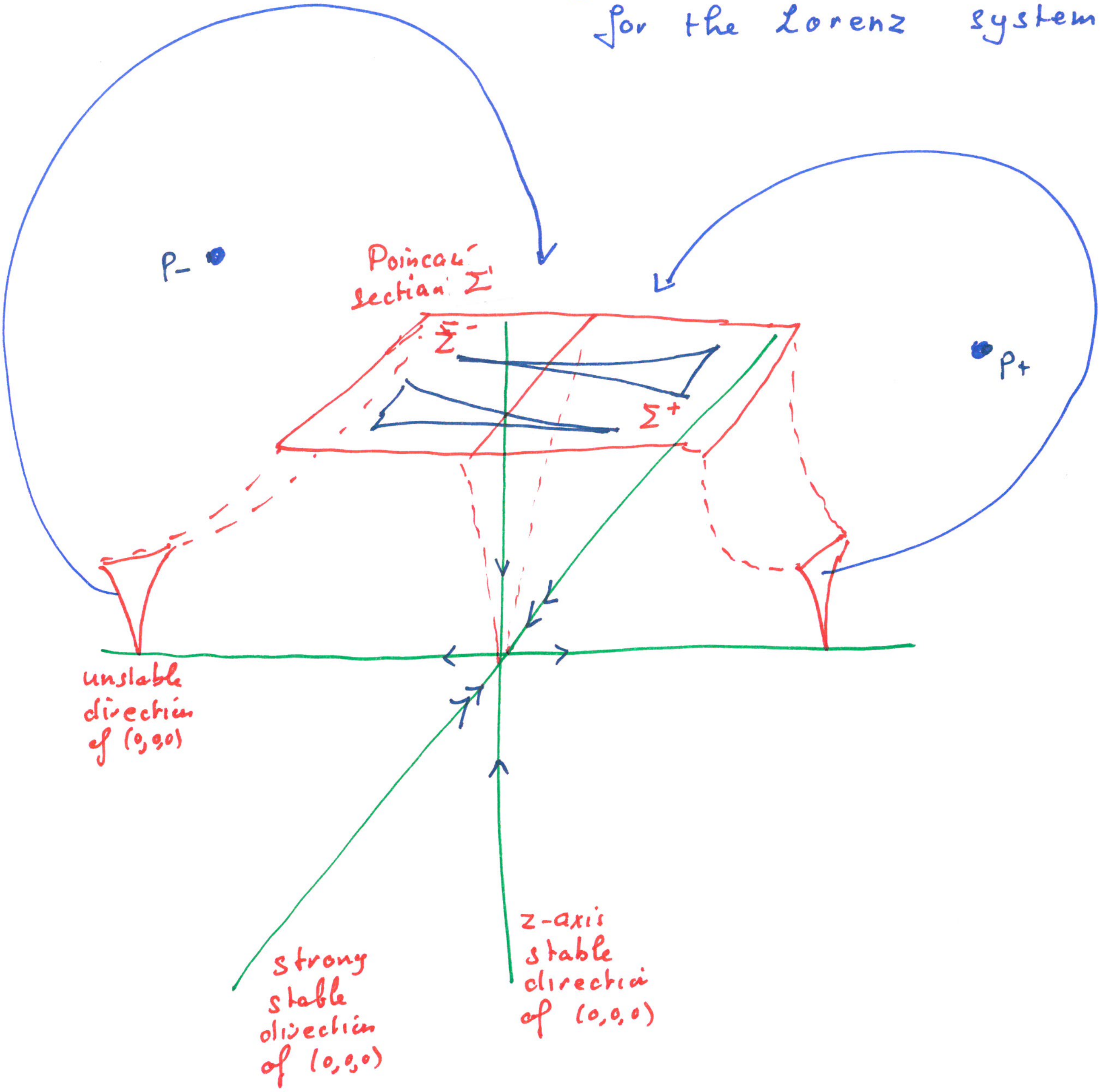
make r subject

$$r = \frac{\sigma(3+b+\sigma)}{\sigma-1-b} > 0$$

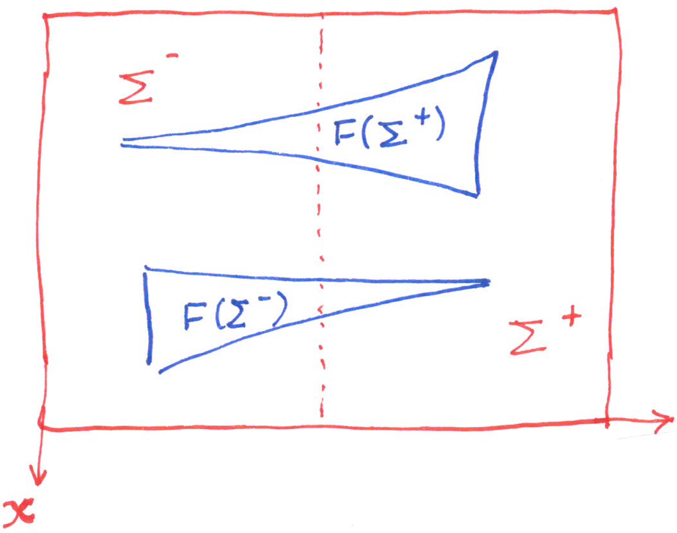
$\sigma = 10$
$b = 8/3$

$$\boxed{r_c = \frac{470}{19}}$$

Guckenheimer - Williams model for the Lorenz system.



Poincaré map $F : \Sigma \to \Sigma$



contracting in x-direction

repelling in y-direction: The y-direction is responsible for the sensitive dependence on initial condition, i.e. the chaos