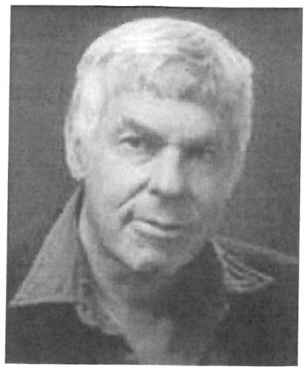
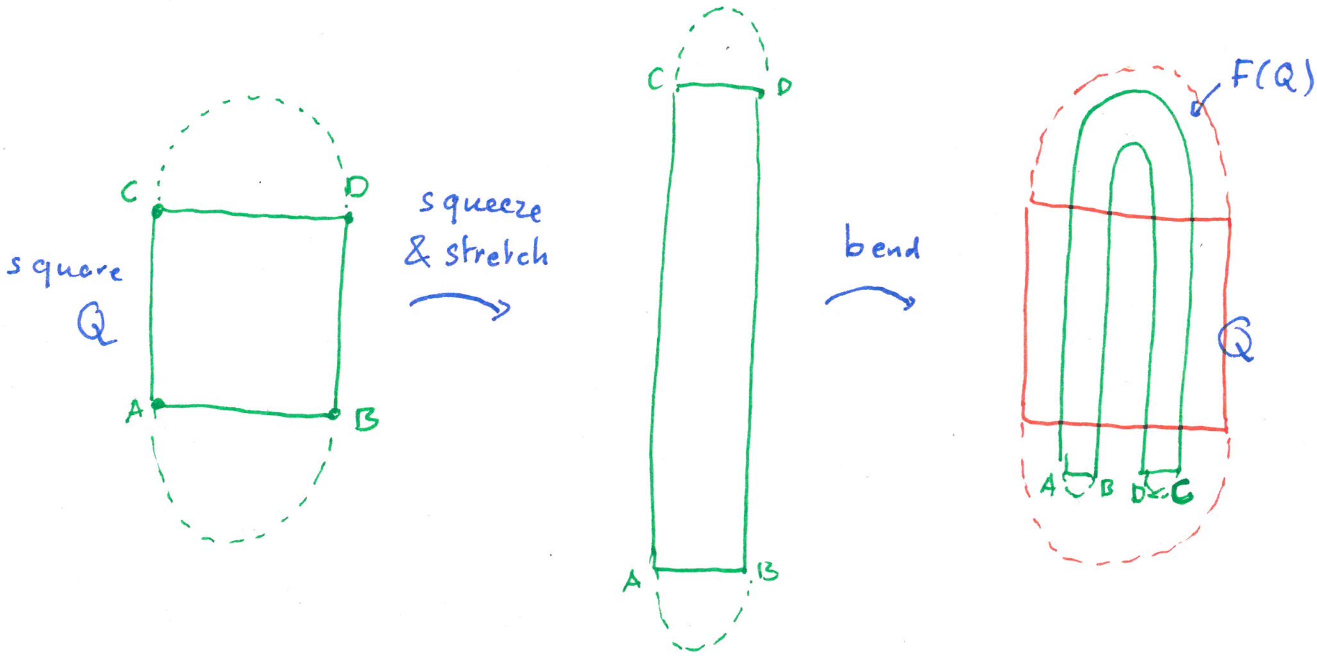
Smale's horseshoe:

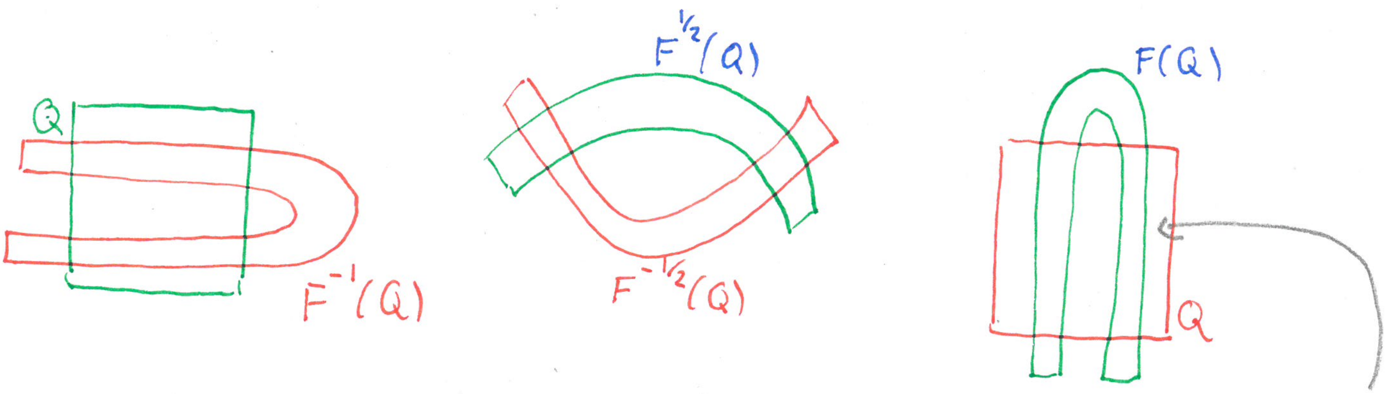
An ingredient of non-linear maps (and via the Poincaré map also of non-linear flows) that leads to chaos, e.g. in the sense of Devaney.



Steve Smale (1930–)

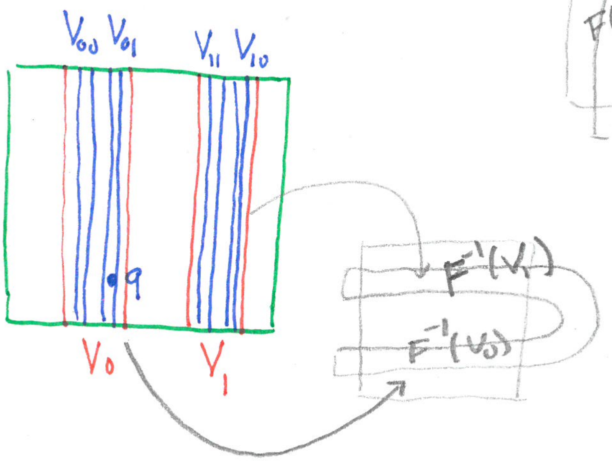


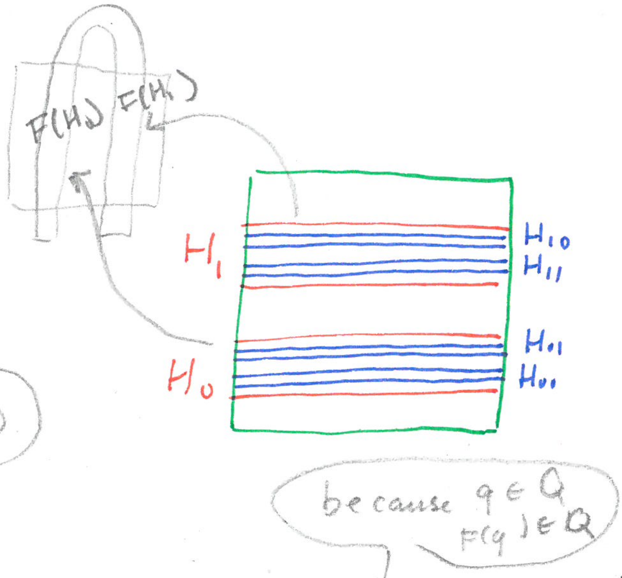
Inverse F^{-1} is also a horseshoe



when trying to bend the horse-shoe straight, the square Q bends to a horizontal horse shoe $F^{-1}(Q)$

$$\Lambda := \{ q \in Q : F^n(q) \in Q \quad \forall n \in \mathbb{Z} \}$$





$q \in V_0 \cup V_1 \iff q \in Q \cap F(Q)$

$q \in H_0 \cup H_1 \iff q \in Q \cap F^{-1}(Q)$

$q \in V_{00} \cup V_{01} \cup V_{10} \cup V_{11} \iff$
$q \in Q \cap F(Q) \cap F^2(Q)$

$q \in H_{00} \cup H_{01} \cup H_{10} \cup H_{11} \iff$
$q \in Q \cap F^{-1}(Q) \cap F^{-2}(Q)$

all backward iterates of q belong to Q

$q \in \bigcap_{n \geq 0} F^n(Q)$ for a Cantor set of vertical lines V_∞

all forward iterates of q belong to Q

$q \in \bigcap_{n \geq 0} F^{-n}(Q)$ for a Cantor set of horizontal lines H_∞

$$\Lambda = H_\infty \cap V_\infty = \text{Cantor set.}$$

Proposition $F : \Lambda \to \Lambda$ is chaotic in the sense of Devaney.

Proof By two-sided symbolic dynamics

$$\underline{i}(q) = \cdots i_{-2}\, i_{-1} \,.\, i_0\, i_1\, i_2 \cdots \in \{0,1\}^{\mathbb{Z}} =: \Sigma$$

(itinerary map)

where

$$i_n(q) = \begin{cases} 0 & \text{if } F^n(q) \in V_0 \quad (\Leftrightarrow F^{n-1}(q) \in H_0) \\ 1 & \text{if } F^n(q) \in V_1 \quad (\Leftrightarrow F^{n-1}(q) \in H_1) \end{cases}$$

This makes the diagram

$$\begin{array}{ccc} \Lambda & \xrightarrow{F} & \Lambda \\ i \downarrow & & \downarrow i \\ \Sigma & \xrightarrow{\sigma} & \Sigma \end{array}$$

commute. (σ is left-shift)

Since F is uniformly expanding in the vertical direction and uniformly contracting in the horizontal direction, $i : \Lambda \to \Sigma$ is in fact a homeomorphism.

- Take ($q \in \Lambda$ with itinerary $\underline{i}(q) = \cdots i_{-2}\, i_{-1} . i_0\, i_1\, i_2 \cdots$) $\varepsilon > 0$ arbitrary and $n \in \mathbb{N}$ so large that

$$J := H_{i_0 \cdots i_n}(q) \cap V_{i_0 \cdots i_{-n}}(q) \subset B_\varepsilon(q)$$

So $i_{-n}(p) \cdots i_0(p) \cdots i_n(p)$ is constant on J.

Take $s = (i_{-n} \cdots i_0 \cdots i_n)^\infty$ and $p \in \underline{i}^{-1}(s)$.

Then p is a $(2n+1)$-periodic point in $B_\varepsilon(q)$

$\Longrightarrow$ There is a dense set of periodic points in Λ

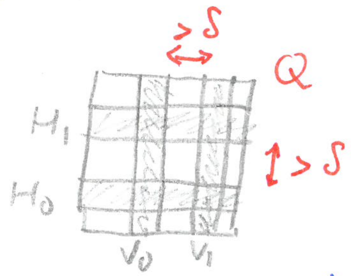
- Let s be a sequence containing all 0-1-words:

$s = \ldots .01\ 00\ 1\ 0 . 0\ 1\ 00\ 01\ 10\ 11\ 000\ 001 \ldots .$

Then $\mathrm{orb}_\sigma(s)$ is dense in $\Sigma = \{0,1\}^{\mathbb{Z}}$

and $q = \underline{i}^{-1}(s)$ has a dense F-orbit in Λ

- Take $\delta < \min\{ d(V_0, V_1), d(H_0, H_1)\}$



Take $q \in \Lambda$ arbitrary

Take $\varepsilon > 0$ arbitrary

Take $m \in \mathbb{N}$ s.t. $H_{i_0 \cdots i_{-m}} \cap V_{i_0 \cdots i_m} \subset B_\varepsilon(q)$

Take $n > m$ (or $n < -m$) arbitrary and $p \in B_\varepsilon(q)$

such that $i_n(p) \neq i_n(q)$.

Then $F^n(p)$ and $F^n(q)$ lie in different strips

(either V_0 and V_1 if $n > 0$ or H_0 and H_1 if $n < 0$)

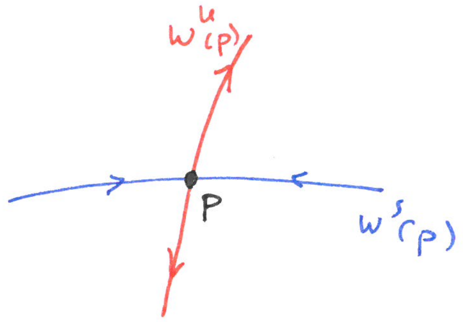
Thus $d(F^n(p), F^n(q)) > \delta$

This proves sensitive dependence on initial conditions

∎

Let p be a saddle point of an invertible planar map $T : \mathbb{R}^2 \to \mathbb{R}^2$

The stable / unstable manifolds of p are:

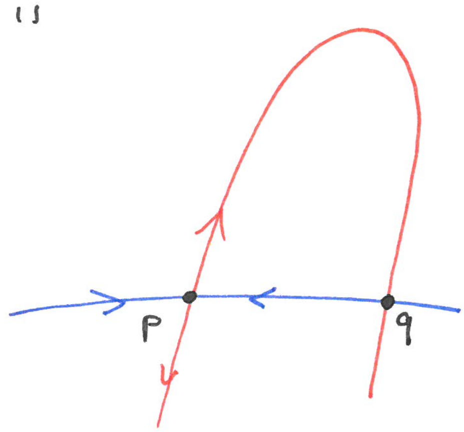


$W^s(p) = \{ z \in \mathbb{R}^2 : T^n(z) \to p \quad n \to \infty \}$

$W^u(p) = \{ z \in \mathbb{R}^2 : T^n(z) \to p \quad n \to -\infty \}$

It can happen that there is

$p \neq q \in W^s(p) \cap W^u(p)$

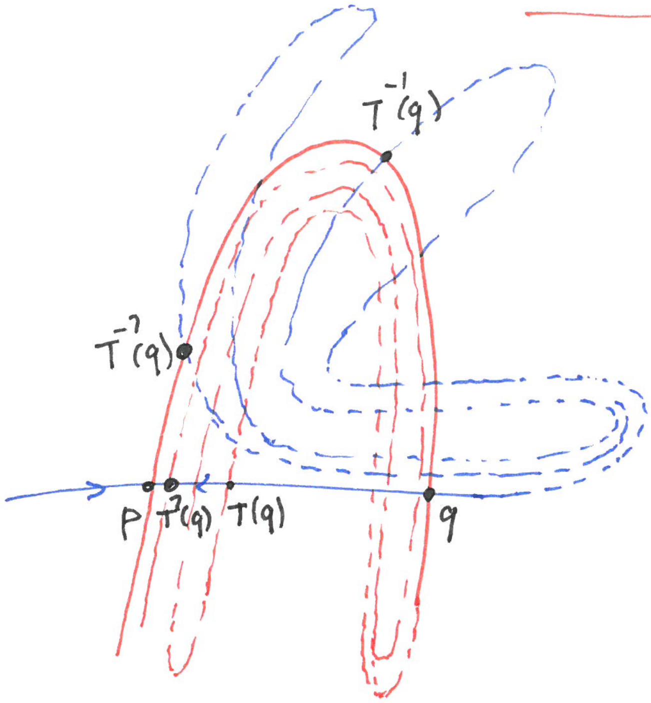


This point q is a homoclinic point to p because $T^n(q) \to p$ for both $n \to +\infty$ and $n \to -\infty$

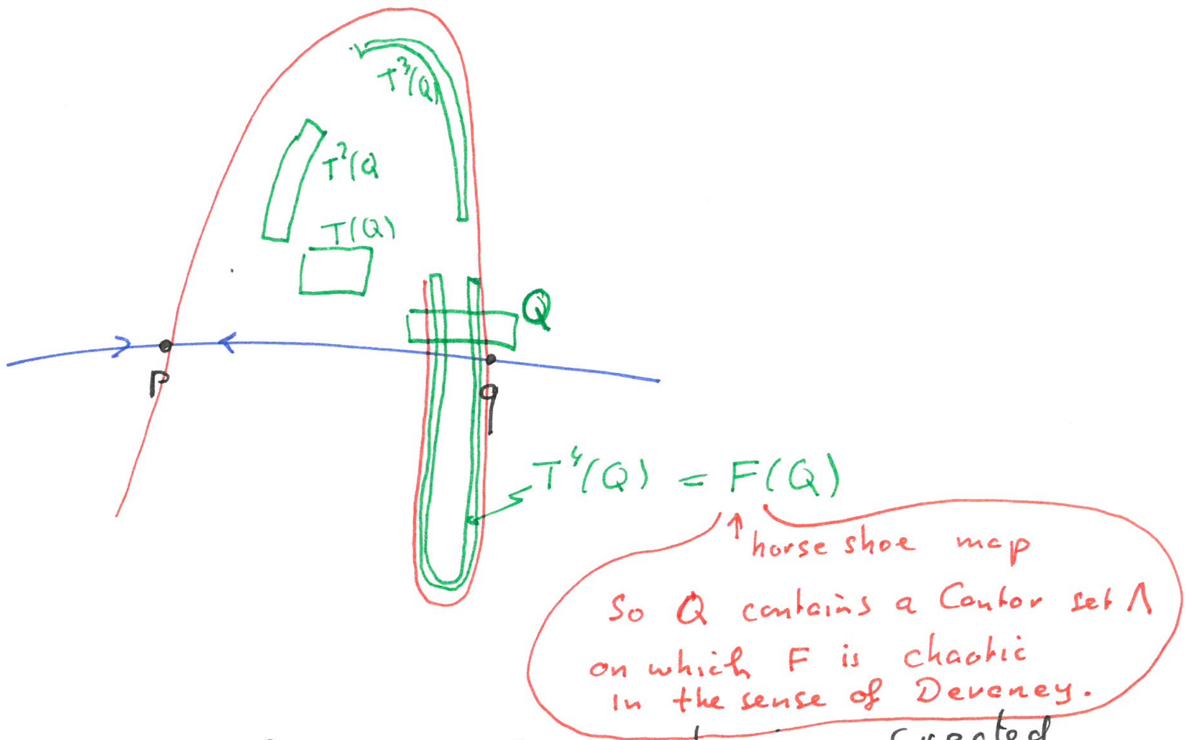
But then also $T(q), T^2(q), T^3(q), \ldots$ and $T^{-1}(q), T^{-2}(q), T^{-3}(q), \ldots$ are homoclinic.

$W^u(p)$ and $W^s(p)$ have to be continued in such a way that they both contain all these points

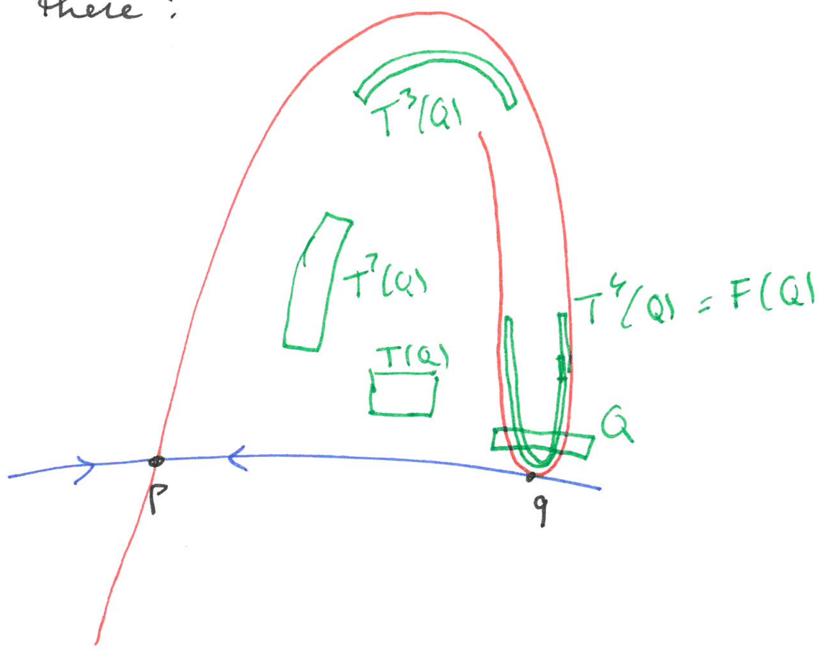
This gives rise to a <u>homoclinic tangle</u>



Homoclinic point force the occurrence of horse shoes.



Even if a homoclinic point is being created (in a homoclinic bifurcation) the horse shoe is already there:



The Poincaré Map

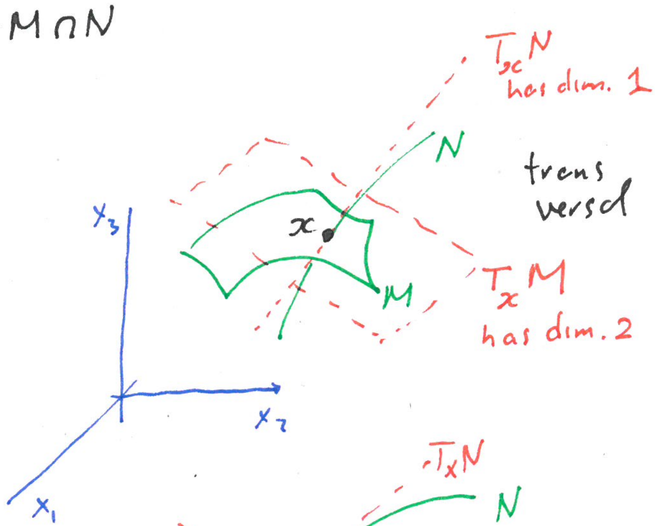
Invertible maps in 2D can be very chaotic, due to horse shoes, but flows in $\mathbb{R}^2$ are quite tame, due to the Poincaré-Bendixson Theorem. But 2D maps can be a good way of understanding flows in 3D due to Poincaré maps.

Henri Poincaré (1854 – 1912)

Def Two manifolds M and N embedded in $\mathbb{R}^n$ are <u>transversal</u> (notation $M \pitchfork N$) if $\forall x \in M \cap N$

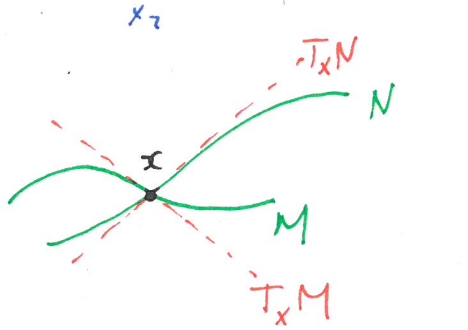
$$T_x M \oplus T_x N = \mathbb{R}^n$$

tangent space of M & N at x



A manifold M and a vector field $X : \mathbb{R}^n \to T\mathbb{R}^n \simeq \mathbb{R}^n$ are <u>transversal</u> if $\forall x \in M$

$$T_x M \oplus \operatorname{span}(X(x)) = \mathbb{R}^n$$



not transversal in $\mathbb{R}^3$

NB: $\dim(M) + \dim(N) \geq n$ is necessary for transversality

Given a flow φ^t on $\mathbb{R}^n$ that solves an ODE $\dot{x} = X(x)$, take a manifold Σ transversal (and of dimension $n-1$) to the vector field X. We call Σ a **Poincaré section**.

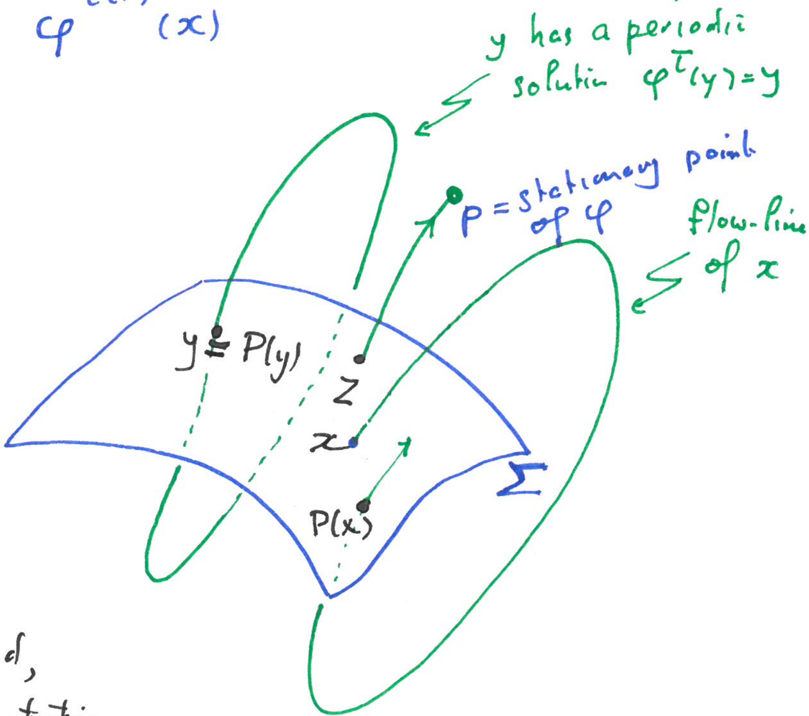
For $x \in \Sigma$ let

$$\tau(x) = \min\{t > 0 : \varphi^t(x) \in \Sigma\}$$

be the first return time to Σ.

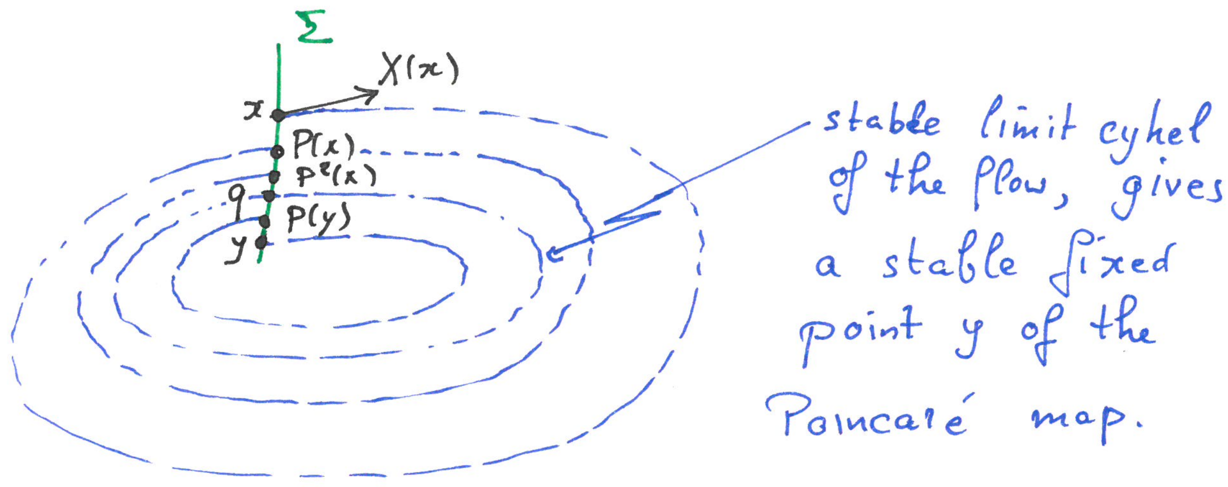
The **Poincaré map** (or first return map) is defined as

$$P : \Sigma \to \Sigma \qquad x \mapsto \varphi^{\tau(x)}(x)$$



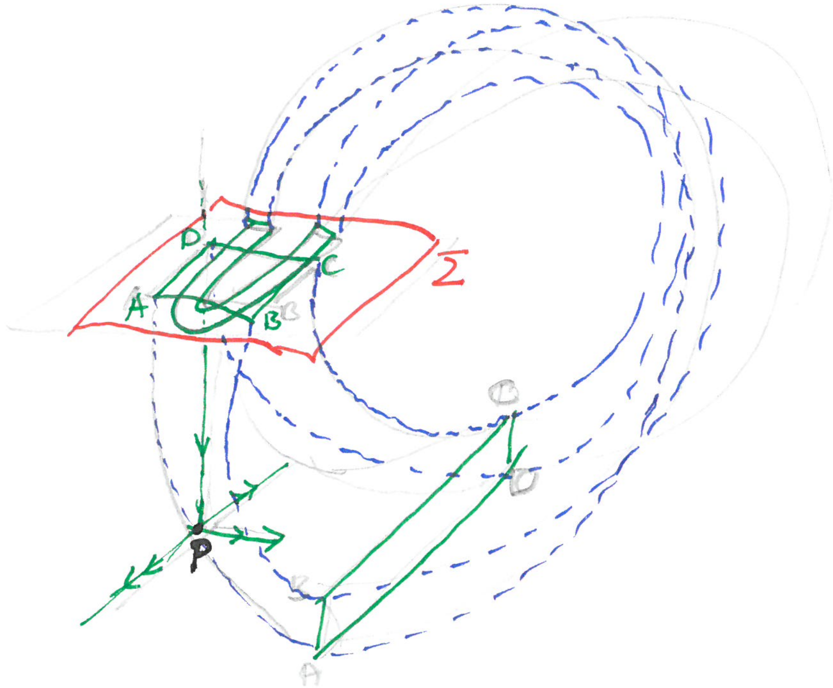
- Periodic points of the flow are periodic under the Poincaré map (stability/unstability is carried over)
- P need not be defined, e.g. if $\varphi^t(z)$ is asymptotic to a stationary point $p \notin \Sigma$
- P can be discontinuous

An example of a 1-D Poincaré map



Poincaré maps:

- reduce dimension by 1;
- turn a continuous time to a discrete time dynamical system;
- but time-scales need not be comparable
  - one iterate of the Poincaré map may correspond to unbounded flow-time
  - a bounded amount of flow-time may correspond to an unbounded number of iterates of the Poincaré map.



Example of a 3D flow and Poincaré map exhibiting a horse-shoe.

Henri Poincaré discovered horse-shoes in his studies of the differential equations describing the movement of planets in the solar systems, and thus came to the conclusion that there is no way of proving whether the solar system is stable for ever.