The Lebesgue Density Theorem

Throughout this note, let m stand for Lebesgue measure (although the results also hold for Lebesgue-Stieltjes measures in general).

Definition 1. *Let $E \subset \mathbb{R}$ be a measurable set and $x \in E$. The* **density** *of E at the point x is*

$$d_E(x) := \lim_{h \to 0} \frac{m(E \cap [x-h, x+h])}{m([x-h, x+h])}$$

if the limit exists. The point x is called a **density point** *if $d_E(x) = 1$. Let $D(E)$ be the set of density points of E.*

Remark 1. *It is not necessary for a density point x to belong to E. For example, 0 is a density point of $E = \mathbb{R} \setminus \{x\}$.*

Example 1. *The point $x = 0$ has density $0, 1, \frac{1}{2}$ for the set $E = [1, 2]$, $E = [-1, 1]$ and $E = [0, 1]$, respectively. Other densities are possible too. For example $x = 0$ has density $\frac{1}{4}$ w.r.t. the set $E = \bigcup_{n \geq 1} [\frac{1}{2n+1}, \frac{1}{2n}]$.*

Proposition 1. *The following properties hold for densities:*

1. *$d_\emptyset(x) = 0$ and $d_\mathbb{R}(x) = 1$;*
2. *$d_{E^c}(x) = 1 - d_E(x)$;*
3. *If $A \subset B$, then $d_A(x) \leq d_B(x)$ if these densities exist. Hence $D(A) \subset D(B)$.*
4. *If $m(A \triangle B) = 0$, then $d_A(x) = d_B(x)$ for all $x \in \mathbb{R}$, so $D(A) = D(B)$; ($A \triangle B$ stands for the symmetric difference: $A \triangle B = (A \setminus B) \cup (B \setminus A)$.)*
5. *$D(A \cap B) = D(A) \cap D(B)$.*

Proof. We only prove property 5. The others are straightforward.

Property 3. gives $D(A \cap B) \subset D(A) \cap D(B)$. For the other inclusion, let I be any interval. We have $I \setminus (A \cap B) = (I \setminus A) \cup (I \setminus B)$, so

$$m(I) - m(A \cap B \cap I) \leq m(I) - m(A \cap I) + m(I) - m(B \cap I),$$

and

$$\frac{m(A \cap I))}{m(I)} + \frac{m(B \cap I)}{m(I)} - 1 \leq \frac{m(A \cap B \cap I)}{m(I)}.$$

Now take $I = [x-h, x+h]$ and let $h \to 0$. If the RHS < 1, so $x \notin D(A \cap B)$, then one of the first terms on the LHS < 1 as well, so $x \notin D(A)$ or $x \notin D(B)$. Hence $D(A \cap B)^c \subset D(A)^c \cup D(B)^c$ and thus $D(A \cap B) \supset D(A) \cap D(B)$. □

Theorem 1 (Lebesgue Density Theorem)**.** *For every Lebesgue measurable set $E \subset \mathbb{R}$, $m(E \triangle D(E)) = 0$.*

Remark 2. *The analogous statement holds in higher dimensional Euclidean space $\mathbb{R}^N$, as it does for Lebesgue-Stieltjes measure. In particular, for the extreme case that the Lebesgue-Stieltjes is Dirac measure δ_p, then every $E \subset \mathbb{R}$ has at most one density point, namely p if $p \in E$.*

Proof. We will prove that $m(E \setminus D(E)) = 0$. This will suffice because of Proposition 1, proerty 2. and 3., and the fact that $D(E) \setminus E \subset E^c \setminus D(E^c)$.

Without loss of generality, we can assume that E is bounded. Write $E \setminus D(E) = \bigcup_{n \geq 1} A_n$, where

$$A_n = \Big\{ x \in E : \liminf_{h \to 0} \frac{m(E \cap [x-h, x+h])}{m([x-h, x+h])} < 1 - \frac{1}{n} \Big\}.$$

It suffices to prove that $m(A_n) = 0$ for all $n \geq 1$.

Assume by contradiction that $n \geq 1$ and $A := A_n$ are such that $m^*(A) > 0$ (outer measure!)

Thus there is a bounded open set G such that $A \subset G$ and $m(G) < \frac{n}{n-1} m^*(A)$.

Let $\mathcal{C}$ be the collection of all closed intervals $I \subset G$ such that $m(E \cap I) \leq \frac{n}{n-1} m(I)$. Then

(i) Every $x \in A$ is the center of arbitrarily small intervals in $\mathcal{C}$.

(ii) Whenever $\{I_k\} \subset \mathcal{C}$ are pairwise disjoint, then $m^*(A \setminus \bigcup_k I_k\}) > 0$.

Property (i) follows by definition of A. Property (ii) follows because

$$m^*(A \cap \bigcup_k I_k) \leq \sum_k m^*(A \cap I_k) \leq \sum_k m(E \cap I_k) \leq \frac{n-1}{n} \sum_k m(I_k) \leq \frac{n-1}{n} m(G) < m^*(A).$$

Take $I_1 \in \mathcal{C}$ arbitrary, and if $I_1, \ldots, I_k$ have been selected, set

$$\mathcal{C}_k = \{ I \in \mathcal{C} : I \text{ is disjoint from } I_1 \cup \cdots \cup I_k \}.$$

By properties (i) and (ii), $\mathcal{C}_k$ is nonempty, so we can always find a next interval in $\mathcal{C}$ disjoint from the previous ones. Let

$$s_k = \sup\{ m(I) : I \in \mathcal{C}_k \} > 0$$

and pick I_{k+1} so that $m(I_{k+1}) > s_k/2$.

Set $B = A \setminus \cup_{k=1}^{\infty} I_k$. By property (ii), $m^*(B) > 0$. So there exists $K \in \mathbb{N}$ so that

$$\sum_{k > K} m(I_k) < m^*(B)/3. \tag{1}$$

Let J_k be the interval concentric with I_k so that $m(J_k) = 3m(I_k)$.

By (1), $\bigcup_{k>K} J_k \not\supset B$, so there is some $x \in B \setminus \bigcup_{k>K} J_k$. By property (i), there is an $I \in \mathcal{C}_K$ centered at x. If $I \cap I_k = \emptyset$ for all $k > K$, then $m(I) \leq s_k < 2m(I_{k+1})$ for all $k > K$, contradicting that $\sum_{k>K} m(I_k) \leq m(G) < \infty$.

Thus there is $k > K$ such that $I \cap I_k \neq \emptyset$, and therefore $x \in I \subset J_k$. But this contradicts that $x \notin \bigcup_{k>K} J_k$. $\square$