Dynamical Systems and Nonlinear ODEs SS2024
Exercise Sheet

Exercise 1 *Let $f : X \to X$ and $g : Y \to Y$ be two continuous mappings that are conjugate via the homeomorphism ψ: $\psi \circ f = g \circ \psi$. Show that*
a) *ψ maps (pre)periodic points of f to (pre)periodic points of g;*
b) *ψ maps omega-limit sets of f to omega-limit sets of g;*
c) *ψ maps attracting fixed points of f to attracting fixed points of g.*

Exercise 2 *Let $Q_a(x) = ax(1-x)$, $a \in [0,4]$ be the quadratic family.*
Which are the fixed points and for which values of a are they stable?
b) *Find the period two points. For which values of a do they exist?*
c) *For which values of a is the period 2 orbit stable?*

Exercise 3 *Given are the one-parameter quadratic families $f_c : z \mapsto z^2 + c$ and $Q_a : x \to ax(1-x)$.*
a) *Show that for $c \in [-2, \frac{1}{4}]$ there is an $a \in [1,4]$ such that f_c is conjugate to Q_a.*
b) *Find the parameter regions in $\mathbb{R}$ where f_c has a stable fixed point resp. stable period 2 point.*
c) *Name the bifurcations that take place at parameters $c = \frac{1}{4}, c = -\frac{3}{4}, c = -\frac{5}{4}$.*

Exercise 4 *Find the phase portraits and exact solutions of initial value problems*

$$\dot{x} = x^2 \text{ with } x(0) = 0, \qquad \text{and} \qquad \dot{x} = -x^3 \text{ with } x(0) = 0,$$

for $x \in \mathbb{R}$. Discuss the (exponential?) stability of the equilibria.

Exercise 5 a) *Consider the initial value problem*

$$\dot{x} = \cos x + 1 \text{ with } x(0) = 0$$

for $x \in \mathbb{R}$. Find the equilibria and indicate if they are hyperbolic. Hence compute the ω-limit and α-limit of the given initial point.
b) *Give the definition of an ω-limit set and calculate the ω-limit set for the initial value problem*

$$\dot{x} = x^2 + x^3, \qquad x(0) = -\frac{1}{2}.$$

Exercise 6 *Consider the initial value problem*

$$\dot{x} = f(x) := x^3 - 4x + c, \quad x(0) = x_0,$$

for some parameter $c \in \mathbb{R}$.
a) *First take $c = 0$. Draw the phase portrait and determine whether the stationary points are stable/unstable.*
b) *Still for $c = 0$, solve the ODE (separation of variables, partial fractions).*
c) *As c varies over $\mathbb{R}$, at what values of c do which types of bifurcation occur?*

Exercise 7 *Let $f : \mathbb{R} \to \mathbb{R}$ be a C^2 map with fixed point x_0 such that $f'(x_0) = 1$. Show (by example) that x_0 can be stable or unstable, but also prove that it cannot be exponentially stable.*

Exercise 8 *Prove or give a counter-example among circle maps $f : \mathbb{S}^1 \to \mathbb{S}^1$:*

1. *Exponentially stable $\Rightarrow$ asymptotically stable.*
2. *Asymptotically stable $\Rightarrow$ exponentially stable.*
3. *Exponentially stable $\Rightarrow$ Lyapunov stable.*
4. *Asymptotically stable $\Rightarrow$ Lyapunov stable.*

Exercise 9 *Find the complete solution to the differential equation*

$$\dot{x} = ax(1-x), \qquad x(0) = x_0.$$

Assuming $a > 0$, which are the stationary points and are they (asymptotically) stable? Are they exponentially (un)stable? What happens with the forward orbit of a initial poistion $x_0 < 0$, and with the backward orbit for $x_0 > 1$?

Exercise 10 *Let $\dot{x} = f(x)$ have an equilibrium at $x = 0$ and $f'(0) = 0$. Show that 0 cannot be exponentially stable.*

Exercise 11 *The map $T : [0,1] \to [0,1]$, $T(x) = \min\{2x, 2(1-x)\}$ is called the tent-map (or full tent-map, because it is onto $[0,1]$).*
a) *Compute the multiplier of each periodic point. Compute the Lyapunov exponents of arbitrary points. Which points $x \in [0,1]$ do not have a Lyapunov exponent?*
b) *Let $Q : [0,1] \to [0,1]$ and $\psi : [0,1] \to [0,1]$ be defined by $Q(x) = 4x(1-x)$ and $\psi : [0,1] \to [0,1]$ and $\psi(x) = \frac{1}{2}(1 - \cos \pi x)$. Show that $Q \circ \psi = \psi \circ T$.*
c) *Conclude that every n-periodic point $p \neq 0$ of Q has multiplier $|(Q^n)'(p)| = 2^n$. Why doesn't this argument apply also to $p = 0$?*
d) *What is the Lyapunov exponent of points $x \in [0,1]$ w.r.t. Q? Is this Lyapunov exponent defined for all x?*

Exercise 12 *Consider the two ODEs on $\mathbb{R}$:*

$$\dot{x} = -x \qquad \text{and} \qquad \dot{y} = -2y.$$

a) *Show that the corresponding flows, say $\varphi^t(x)$ and $\psi^t(y)$ are conjugate, i.e., find a homeomorphism such that $\varphi^t(h(x)) = h(\psi^t(x))$. Is your solution h a diffeomorphism? Is it unique*
b) *A function $h : \mathbb{R} \to \mathbb{R}$ is called* **Hölder continuous** *with exponent $\alpha \in (0,1]$ if there is a constant K such that*

$$\sup_{x \neq y} \frac{|h(x) - h(y)|}{|x - y|^\alpha} \leq K.$$

(So Hölder continuous with exponent $\alpha = 1$ is the same as Lipschitz continuous.) Show that $h(x) = |x|^\alpha$ is indeed Hölder continuous with exponent $\alpha \in (0,1]$. Check that your solution in a) is a Hölder conjugacy, i.e., both h and h^{-1} are Hölder continuous in a neighbourhood of 0.
c) *Consider two ODEs*

$$\dot{x} = f(x) \qquad \text{and} \qquad \dot{y} = g(y).$$

with $f(0) = g(0) = 0$ and $f'(0) < g'(0) < 0$. Prove that their flows can be Hölder conjugate, but not with an exponent $> g'(0)/f'(0)$.

Exercise 13 *Consider the two ODEs on* $\mathbb{R}$*:*

$$\dot{x} = -x \qquad \text{and} \qquad \dot{y} = -2y.$$

a) *Show that the corresponding flows, say* $\varphi^t(x)$ *and* $\psi^t(y)$ *are conjugate, i.e., find a homeomorphism such that* $\varphi^t(h(x)) = h(\psi^t(x))$*. Is your solution* h *a diffeomorphism? Is it unique*
b) *A function* $h : \mathbb{R} \to \mathbb{R}$ *is called* **Hölder continuous** *with exponent* $\alpha \in (0, 1]$ *if there is a constant* K *such that*

$$\sup_{x \neq y} \frac{|h(x) - h(y)|}{|x - y|^\alpha} \leq K.$$

(So Hölder continuous with exponent $\alpha = 1$ *is the same as Lipschitz continuous.) Show that* $h(x) = |x|^\alpha$ *is indeed Hölder continuous with exponent* $\alpha \in (0, 1]$*. Check that your solution in a) is a Hölder conjugacy, i.e., both* h *and* h^{-1} *are Hölder continuous ina neighbourhood of* 0*.*
c) *Consider two ODEs*

$$\dot{x} = f(x) \qquad \text{and} \qquad \dot{y} = g(y).$$

with $f(0) = g(0) = 0$ *and* $f'(0) < g'(0) < 0$*. Prove that their flows can be Hölder conjugate, but not with an exponent* $> g'(0)/f'(0)$*.*

Exercise 14 a) *Find the solutions and draw the phase portraits for the following systems of ODEs* $\dot{x} = Ax$*:*

$$(i) \quad A = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} \qquad\qquad (i) \quad A = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

b) *Construct explicit conjugacies* $h : \mathbb{R}^2 \to \mathbb{R}^2$ *between the two-dimensional system* $\dot{x} = -x$ *(so* $x = (x_1, x_2)^T$*) and*

$$(i) \quad \dot{y} = -2y, \quad y = (y_1, y_2)^T \qquad\qquad (ii) \quad \dot{z} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} z, \quad z = (z_1, z_2).$$

Exercise 15 a) *Given is the ODE*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y - x \\ x(y - 2) \end{pmatrix}.$$

Indicate the equilibria and their type (saddle, sink, source, center). Hence show that there is exactly one invariant horizontal line $\{y = L\}$*. Compute the solutions on this line.*
b) *Consider the ODE*

$$\begin{aligned} \dot{x} &= 3x + xy \\ \dot{y} &= y + x(y - x) \end{aligned}$$

and find a near identity transformation $u = x + axy$ $v = y + bx^2 + cxy$*, that removes the quadratic terms.*
c) *Consider the ODE*

$$\begin{aligned} \dot{x} &= 4x + y^2 - 3xy \\ \dot{y} &= -y + y(x - y^2) \end{aligned}$$

and find a near identity transformation $u = x + ay^2 + bxy$*,* $v = y + cxy$ *that removes the quadratic terms. Is the equilibrium at zero structurally stable? (Justify your answer!)*

Exercise 16 *Consider the circle map* $f_c : \mathbb{S}^1 \to \mathbb{S}^1$, $x \mapsto 2x + c \pmod 1$.
a) Show that this map is chaotic in the sense of Devaney.
b) A pair of points (x, y) *is called Li-Yorke if*

$$\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0 \qquad \text{and} \qquad \liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0.$$

Show that $f_c : \mathbb{S}^1 \to \mathbb{S}^1$ *has a Li-Yorke pair. Find a set* $\{x, y, z\}$ *such that every two of them form a Li-Yorke pair. (Such set is called a* **scrambeled set**. *A map if Li-Yorke chaotic if there exists an uncountable scrambeled set.)*

Exercise 17 *Which of the following dynamical systems has (i) a dense set of periodic orbits, (ii) a dense orbit, (iii) sensitive dependence on initial conditions?*

1. *a circle rotation;*
2. *the tent-map* $T(x) = \min\{2x, 2(1-x)\}$ *on* $[0, 1]$*;*
3. *the twist map on the torus* $\mathbb{T}^2$ *defined by* $T(x, y) = (x, x + y \bmod 1)$*;*
4. *the pendulum* $\ddot{x} + \sin x = 0$*;*
5. *The cat-map on the torus* $\mathbb{T}^2$ *defined as* $T(x, y) = (2x + y \bmod 1, x + y \bmod 1)$. *Hint: locally the cat-map is linear, so it helps to consider the stable and unstable directions at each point.*

Exercise 18 *Suppose* T *is a continuous map on an* X *is an infinite space. If* T *has a dense set of periodic orbits as well as a dense orbit, then* T *has sensitive dependence on initial conditions.*

Exercise 19 *Consider the map* $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x/2$. *Show that this map is* C^1 *structurally stable, but not* C^0 *structurally stable.*

Exercise 20 *Show that the map* $f : \mathbb{R} \to R$, $x \mapsto x^3 + x/2$ *is* C^1 *structurally stable, i.e., all maps that are* C^1 *close to* f *are conjugate to* f.
Steps in the proof: 1) find a conjugacy h *between the fixed points. 2) Check that this preserves* $\alpha(x)$ *and* $\omega(x)$ *for non-fixed points. 3) Identify "fundamental domains" between the fixed points. These are intervals such that every non-fixed point has a unique forward or backward iterate inside one of these iterates. 4) Let* h *map fundamental domains to fundamental domains. 5) Extend* h *to the rest of* $\mathbb{R}$.

Exercise 21 *Let* $T_s(x) = \min\{sx, s(1-x)\}$ *be the tent map with* $s = 3$. *Describe the set* C *of points* $x \in \mathbb{R}$ *such that* $T_s^n(x) \in [0, 1]$ *for all* n. *How is the set* C *called? What happens with points* $x \in \mathbb{R} \setminus C$*?*

Exercise 22 *Let* $T : [0, 1] \to [0, 1]$, $T(x) = \min\{2x, 2(1-x)\}$ *be the tent-map.*
a) *Argue that* q *is* n*-periodic if and only if the graph of* T^n *intersects the diagonal* $\{y = x\}$ *at* $x = q$.
b) *How many* n*-periodic points does* T *have? How many where* n *is the smallest period?*
c) *Prove Fermat's little theorem: If* p *is prime, then* $p | 2^p - 2$ *and more generally, if* $2 \le a \in \mathbb{N}$ *and* p *is prime, then* $p | a^p - a$

Exercise 23 *The quadratic map* $Q(x) = 4x(1-x)$ *is also called the Chebyshev polynomial, and* $T : [0,1] \to [0,1]$, $T(x) \min\{2x, 2(1-x)\}$ *is called the tent map. Let* $\psi : [0,1] \to [0,1]$ *be defined by* $\psi(x) = \frac{1}{2}(1 - \cos \pi x)$.
a) *Show that* $Q \circ \psi = \psi \circ T$.
b) *Show that if* p *is an* n*-periodic point of* T*, then* $\psi(p)$ *is an* n*-periodic point of* Q.
c) *Conclude that every* n*-periodic point* $p \neq 0$ *of* Q *has multiplier* $|(Q^n)'(p)| = 2^n$*. Why doesn't this argument apply also to* $p = 0$*?*

Exercise 24 *Which of the following maps* $f : \mathbb{R} \to \mathbb{R}$ *are conjugate? If so, can the conjugacy be chosen to be differentiable?*

(a) $f(x) = x/2$;

(b) $f(x) = 2x$;

(c) $f(x) = -2x$;

(d) $f(x) = 3x$;

(e) $f(x) = x^3$.

Exercise 25 *Suppose* $f, g : [0,1] \to [0,1]$ *are two* C^1 *maps that are conjugate via* $h : [0,1] \to [0,1]$*, i.e.,* $h \circ f = g \circ h$.
a) Show that f *is chaotic in the sense of Devaney if and only if* g *is chaotic in the sense of Devaney.*
b) Assume in addition that h *is a* C^1 *diffeomorphism. Show that if* p *is periodic for* f*, then* $q := h(p)$ *is periodic for* g*, with the same period and* **multiplier**.
c) For general (i.e., not necessarily periodic) points, do x *and* $y = h(x)$ *have the same Lyapunov exponent?*

Exercise 26 *Consider the map*

$$f : [0,1] \to [0,1], \quad x \mapsto \begin{cases} \frac{x}{1-x} & \text{if } x \in [0, \frac{1}{2}]; \\ \frac{2x-1}{x} & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

a) Show that every $x \in \mathbb{Q} \cap (0,1]$ *is eventually mapped to* 1.
b) Show that x *and* $f(x)$ *have the same Lyapunov exponent. Find the Lyapunov exponent* $\lambda(x)$ *of the fixed points and the period 2 points of* f.
c) For which $\Lambda \in \mathbb{R}$ *do you think there are points* $x \in [0,1]$ *such that its Lyapunov exponent* $\lambda(x) = \Lambda$*? Does every point* x *have a well-defined Lyapunov exponent?*

Exercise 27 *Consider the "normal form" of the cusp bifurcation* $\dot{x} = r + kx + x^3$.
(a) *Find the bifurcation curve(s) in the parameter plane.*
(b) *Fix* $k = -3$*. Describe the nature of the equilibria and the bifurcations that take place when* r *increases (say from* -3 *to* 3*).*
(c) *Replace* $+x^3$ *by* $-x^3$ *in the normal form, so* $\dot{x} = r + kx - x^3$*. Repeat part (b) for* $k = +3$.

Exercise 28 *Consider the logistic family $Q_a(x) = ax(1-x)$, where $a \in [0,4]$ is such that the critical point $c = \frac{1}{2}$ is periodic of period 3. We abbreviate $c_k = Q_a^k(c)$; the* **core** *$[c_2, c_1]$ is an invariant set for Q_a. A partition $\{I_i\}_{i=1}^N$ of $[c_2, c_1]$ is a* **Markov partition** *if Q_a maps each interval I_i homeomorphically onto a union of interval I_j. (We allow ourselves some sloppiness, and don't care about overlap at the boundary points of the I_is.)*

(a) *Show that the intervals $[c_2, c]$ and $[c,c_1]$ form a Markov partition of $[c_2, c_1]$.*

(b) *Define a* **transition matrix** *$A = (a_{i,j})_{i,j=1}^2$ where $a_{i,j} = 1$ if $Q_a(I_i) \supset I_j$ and $a_{i,j} = 0$ otherwise. Write down the transition matrix for item (a).*

(c) *Argue that the number of n-periodic points (not necessarily prime period) of $Q_a|_{[c_2,c_1]}$ equals the trace $tr(A^n)$. How many periodic points of prime period 11 does Q_a have?*

(d) *Repeat the construction for the case that parameter a is such that $c_2 < c_3 < c_4 < \cdots < c_n = c < c_1$. What is the exponential growth rate of the number of n-periodic points?*

Exercise 29 *Give a C^3-function $f : \mathbb{R} \to \mathbb{R}$, define the* **Schwarzian derivative** *of f as*

$$Sf = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 \qquad \text{if } f' \neq 0.$$

a) *Show that Möbius transformations $g(x) = \frac{ax+b}{cx+d}$ (with $ad - bc = \pm 1$) have $Sg = 0$, but $SQ_a < 0$ for $Q_a(x) = ax(1-x)$.*
b) *Show that $S(f \circ g) = (Sf) \circ g \cdot (g')^2 + Sg$. Conclude that $S(Q_a^n) < 0$ for all $n \geq 1$.*
c) *Suppose that C^3-function $f : \mathbb{R} \to \mathbb{R}$ has $Sf < 0$. Then f' cannot have a positive local minimum or a negative local maximum. (Draw the possible shapes of the graph of f between critical points.) Conclude that f cannot undergo a pitchfork bifurcation making the middle fixed point stable.*
d) *Suppose $Sf < 0$ and p is an attracting fixed point. Show that there must be a critical point c (i.e., a point c where $f'(c) = 0$) such that $[p, c]$ contains no other fixed point of f. Therefore $f^n(c) \to p$.*
e) *Conclude that Q_a can have at most one attracting periodic orbit.*

Exercise 30 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map with periodic orbit $x_0 < x_1 < \cdots < x_{n-1}$ with $f(x_i) = x_{i+1 \bmod n}$ and $n \geq 3$. Use (the method of) Sharkovskiy's Theorem to show that f has periodic points of all period.*

Exercise 31 *Recall the Sharkovskiy order*

$$\begin{array}{rl} 3 \succ & 5 \succ 7 \succ 9 \succ 11 \succ \dots \\ & 2\cdot 3 \succ 2\cdot 5 \succ 2\cdot 7 \succ 2\cdot 9 \succ 2\cdot 11 \succ \dots \\ & 4\cdot 3 \succ 4\cdot 5 \succ 4\cdot 7 \succ 4\cdot 9 \succ 4\cdot 11 \succ \dots \\ & \quad\vdots \qquad\qquad \vdots \\ \succ & \dots\dots \succ 8 \succ 4 \succ 2 \succ 1. \end{array}$$

A tail S is any set of integers such that if $s \in S$, then also $t \in S$ for all $s \succ t$. Therefore the tail of 3 is $\mathbb{N} \setminus \{0\}$, the tail of 6 are all even positive integers, and there is a single tail

$\{1, 2, 4, 8, 16, \dots\}$ *having no Sharkovskiy maximum. Show that for every* S *there is a parameter* $a \in [0, 4]$ *such that* $\{p \in \mathbb{N} : Q_a$ *has a* p*-periodic point*$\} = S$.

Hint: *The off-shot of Exercise 29 is that at every parameter value* $a \in [0, 4]$ *only one bifurcation can take place, since the orbit of* $c = \frac{1}{2}$ *converges to the stable periodic orbit emerging in the bifurcation.*

Exercise 32 **a)** *Let* $T : M \to M$ *be a continuous map on a compact manifold. Show that every omega-limit set is closed and* T*-invariant* ($T(\omega(x)) = \omega(x)$).

b) *If* φ^t *is a flow on a compact manifold, show that* $\omega(x)$ *is connected.*

Exercise 33 *Let a one-parameter family of interval maps be given by*

$$f_\mu(x) = \mu - x^2.$$

a) *Find the smallest* $\mu_0 \in \mathbb{R}$ *such that* f_μ *has a fixed point. Describe the bifurcation that takes place at* μ_0.

b) *There is a smallest* $\mu_1 > \mu_0$ *such that* f_μ *undergoes a period doubling bifurcation. Let* p *be the rightmost fixed point of* f_{μ_1}. *What is* $f'_{\mu_1}(p)$*? Compute* μ_1.

c) *Let*

$$\mu_2 = \inf\{\mu \in \mathbb{R} \ : \ f_\mu \text{ has a periodic point of period } 6\}.$$

Argue which bifurcation takes place at μ_2.

Exercise 34 *Let* $F : \mathbb{R}^2 \to \mathbb{R}^2$ *be a* C^1 *vector field, and assume that the ODE* $\dot{x} = F(x)$ *has a limit cycle* Γ. *Show that the bounded component* U *of* $\mathbb{R}^2 \setminus \Gamma$ *contains an equilibrium point.*

Exercise 35 *Given is the differential equation*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \beta & -\alpha \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \sqrt{x^2 + y^2} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{1}$$

for parameters $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$.

a) *Rewrite* (1) *in polar coordinates.*

b) *Describe the bifurcation that takes place if* β *goes through zero.*

c) *Give the definition of* ω*-limit and* α*-limit set. Hence describe the* ω*-limit and* α*-limit sets of the system* (1) *for parameters* $\beta = 1$ *and* $\alpha = 0.3$.

Exercise 36 **a)** *Consider the ODE*

$$\frac{d^2x}{dt^2} + \mu \frac{dx}{dt} = f(x)$$

with parameter $\mu \in \mathbb{R}$ *and* $x \in \mathbb{R}$. *Write this as a system of two first order equations. Show that if* $f(x^*) = 0$, $\mu > 0$ *and* $f'(x^*) < -\mu^2/4$ *then there is an equilibrium that is a stable spiral.*

b) *Sketch a bifurcation diagram showing the location of all equilibria of the ODE*

$$\dot{x} = x^4 - \mu x$$

with $x \in \mathbb{R}$, *on varying the parameter* $\mu \in \mathbb{R}$. *Indicate the stability of equilibria and the location and type of all bifurcation points. Which type of bifurcation takes place?*

c) *Consider the ODE*

$$\dot{x} = (x^2 - 2)^2 + \mu x,$$

with parameter $\mu \in \mathbb{R}$. *Name a describe in detail the bifurcations that take place when* μ *increases from* -1 *to* 1.

Exercise 37 *Consider the following ODE on the first (on-negative) quadrant of* $\mathbb{R}^2$*:*

$$\begin{cases} \dot{x} = a_1 x - a_2 xy \\ \dot{y} = a_2 xy - a_3 y \end{cases} \qquad a_1, a_2, a_3 > 0. \tag{2}$$

a) *Find the equilibrium points and their types (sink, saddle, source, center) of* (2).
b) *Show that*

$$L(x,y) = a_2(x+y) - a_1 - a_3 - a_3 \log \frac{a_2 x}{a_3} - a_1 \log \frac{a_2 y}{a_1}$$

is a Lyapunov function (but never strict). Hence sketch the phase portrait of (2).
c) *Using the change of coordinates* $u = \log \frac{a_2 y}{a_1}$, $v = \log \frac{a_2 x}{a_3}$, *show that* (2) *is in fact a Hamiltonian system.*

Exercise 38 *a) Given is a general Lotka-Voterra equation:*

$$\begin{cases} \dot{x} = (A - By)x, \\ \dot{y} = (Cx - D)y, \end{cases} \quad A, B, C, D > 0.$$

Find changes of coordinates that bring this equation into the form

$$\begin{cases} \dot{x} = (1 - y)x, \\ \dot{y} = \alpha(x - 1)y, \end{cases} \quad \alpha > 0.$$

b) Consider the following variation of the Lotka Volterra equations:

$$\begin{cases} \dot{x} = (1 - y - \lambda(x-1))x, \\ \dot{y} = \alpha(x - 1 + \lambda(1-y))y, \end{cases} \quad 1 \geq \alpha > \lambda > 0.$$

Find the stationary points and their type. Use a Lyapunov function if linearization at the stationary point is not sufficient to draw a conclusion.

Exercise 39 *Consider the standard Van der Pol equation:*

$$\ddot{x} + x = \varepsilon(1 - x^2)\dot{x}, \qquad \varepsilon > 0. \tag{3}$$

a) *Write this system as a first order ODE in* $\mathbb{R}^2$*, and then write the first order ODE in polar coordinates.*
b) *Assume that there is a periodic solution* $R(\phi)$*. Argue that by "averaging over* ϕ*", this solution should satisfy*

$$\dot{R} = \frac{-\varepsilon R}{8}(R^2 - 4), \text{ with some initial condition } R(0) = R_0.$$

and show that its solution is $R(t) = \frac{2}{\sqrt{1 + (4/R_0^2 - 1)e^{-\varepsilon t}}}$.
c) *Analyse what happens in* (3) *if* $\varepsilon < 0$*: compare this case with the case* $\varepsilon > 0$.

Exercise 40 *Let $\mathcal{A}$ be a finite alphabet and $\Sigma = \mathcal{A}^{\mathbb{N}}$, equipped with product topology.*
a) *Show that Σ is a Cantor set, i.e., it is compact, totally disconnected ($\forall x, y \in \Sigma\ \exists U, V \subset \Sigma$ open, $x \in U, y \in V, U \cap V = \emptyset, U \cup V = \Sigma$) and without isolated points.*
b) *Show that the metric*

$$d_\Sigma(x,y) = \begin{cases} 2^{-\max\{k\ :\ x_i = y_i\ \forall |i| < k\}} & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

induces the product topology.
c) *Another metric is*

$$d'_\Sigma(x,y) = \begin{cases} \frac{1}{1+\max\{k\ :\ x_i = y_i\ \forall |i| < k\}} & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Two metrics d and d' are **equivalent** *if*

$$\exists C > 0\ \forall x, y \quad \frac{1}{C} d(x,y) \leq d'(x,y) \leq C d(x,y). \tag{4}$$

Show that d_Σ and d'_Σ are not equivalent in the sense of (4), but that the identity map $x \in (\Sigma, d_\Sigma) \mapsto x \in (\Sigma, d'_\Sigma)$ is a homeomorphism. Conclude that d_Σ and d'_Σ induce the same topology.

Exercise 41 *Let $T : \mathbb{S}^1 \to \mathbb{S}^1$, $x \mapsto 2x \bmod 1$ be the doubling map. Take $a \in [0, \frac{1}{4}]$ and let $J_0 = (a, a + \frac{1}{2})$ and $J_1 = \mathbb{S}^1 \setminus J_0$ represent a partition of the circle $\mathbb{S}^1$. Let us call this partition* **generating** *if every two points $x, y \in \mathbb{S}^1$ whose orbits do not contain a or $a + \frac{1}{2}$, have distinct symbolic itineraries: $\boldsymbol{i}(x) \neq \boldsymbol{i}(y)$.*
a) *Show that for $a = 0$, the partition $\{J_0, J_1\}$ is generating.*
b) *Let $S(x) = 1 - x$. Show that $T \circ S = S \circ T$. Use this to show that for $a = \frac{1}{4}$, the partition is $\{J_0, J_1\}$ not generating. In fact, $\boldsymbol{i} : \mathbb{S}^1 \to \{0,1\}^{\mathbb{N}}$ is two-to-one.*
c) *For which $a \in [0, \frac{1}{4}]$ is the partition $\{J_0, J_1\}$ generating?*

Exercise 42 *Let A be an $N \times N$ transition matrix, and (Σ_A, σ) is the corresponding subshift of finite type.*
a) *Prove that trace(A^n) gives the number of periodic sequences $s \in \Sigma_A$ of period n (although this need not be the minimal period).*
b) *Assume that there is $m \geq 1$ such that A^m has only positive entries. Show that (Σ_A, σ) is chaotic in the sense of Devaney.*
c) *Show that (Σ_A, σ) is chaotic in the sense of Li-Yorke.*

Exercise 43 *Define the annulus $\mathcal{A} = \mathbb{S}^1 \times [0,1]$ (where $\mathbb{S}^1 = [0,1]/0 \sim 1$ is the interval with endpoints identified. Define the map T on $\mathcal{A}$ as*

$$f(x,y) = (3x \bmod 1, (2x+y)/3).$$

a) *Show that f has a horseshoe, and that its invariant set Λ is a Cantor set. Hence show that is Devaney chaotic on Λ.*

b) *The map f has a lift $F : \mathbb{R} \times [0,1] \to \mathbb{R} \times [0,1]$ satisfying $F(x+1,y) = F(x,y) + (3,0)$, and you can define rotation numbers just as in the case of circle homeomorphisms:*

$$\rho_f(p) = \lim_{n\to\infty} \frac{\|F^n(p) - p\|}{n}.$$

Show that the limit **does** *depend on p: for each $c \in [0,2]$ there are points p with $\rho_f(p) = c$, and there are also points where the limit does not exist.*

Exercise 44 *The Hénon map $H_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2$ is given by*

$$H_{a,b}(x,y) = (1 + y - ax^2, bx).$$

a) *Show that the Hénon map has a horseshoe for $a = 3$ and $b = \frac{1}{5}$. Hint: draw and investigate what happens to (the boundary of the) rectangle $R = [-\frac{5}{6}, \frac{5}{6}] \times [-\frac{1}{6}, \frac{1}{6}]$ if H is applied.*
b) *Suppose $a > 2$. Show that the Hénon map has a horseshoe provided $|b|$ is sufficiently small.*

Exercise 45 *Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be an orientation preserving homeomorphism. Show that the rotation number $\rho(f) \in \mathbb{Q}$ if and only if f has a periodic orbit.*

Exercise 46 *Let $R_\alpha : S^1 \to \mathbb{S}^1$, $x \mapsto x + \alpha \bmod 1$ be a circle rotation.*
a) Show that (i) $\alpha \in \mathbb{Q}$ if and only if every point is periodic, and $\alpha \notin \mathbb{Q}$ if and only if every point has a dense orbit.
b) Compute the Lyapunov exponent of every point.
c) If $\alpha \neq \pm\beta \bmod 1$, show that R_α and R_β are not conjugate.

Exercise 47 *Let $F : \mathbb{R} \to \mathbb{R}$ be a* lift *of an orientation preserving circle homeomorphism $f : \mathbb{S}^1 \to \mathbb{S}^1$, i.e., F is continuous and $F(x+1) = F(x) + 1$ and $F(x) \bmod 1 = f(x \bmod 1)$ for all $x \in \mathbb{R}$. Recall that the* rotation number *of f is defined as*

$$\rho(f) = \lim_{n\to\infty} \frac{F^n(x) - x}{n} \bmod 1.$$

a) *Verify that $\rho(f)$ doesn't depend on the choice of the point x.*
b) *Verify that $\rho(f)$ doesn't depend on the choice of the lift F.*
c) *Let $f_\varepsilon(x) = f(x) + \varepsilon$ and $F_\varepsilon(x) = F(x) + \varepsilon$. Show that $\varepsilon \mapsto \rho(f_\varepsilon)$ is non-decreasing.*
d) *Show that $\rho(f) = \frac{p}{q} \in \mathbb{Q}$ (in lowest terms) if and only if f has a q-periodic point.*

Exercise 48 *Consider the Arnol'd family $f_\varepsilon : \mathbb{S}^1 \to S^1$, $x \mapsto x + \alpha + \varepsilon \sin(2\pi x)$.*
a) *For which $\varepsilon \geq 0$ is f_ε a homeomorphism (diffeomorphism)?*
b) *Compute the region of (α, ε) where f_ε has resonance of period 1 (i.e., f_ε has a fixed point).*
c) *Fix $\varepsilon > 0$ small and let I_ε be the set of $\alpha \in S^1$ where the rotation number $\rho(f_\varepsilon) \notin \mathbb{Q}$. Given is that there is $C > 0$ such that the width of any resonance tongue of period q is $\leq C\varepsilon^q$. Show that $\overline{I_\varepsilon}$ is a Cantor set of positive Lebesgue measure.*

Exercise 49 *In this example, we make the Denjoy example of a circle homeomorphism without dense orbits more concrete. Let $R_\alpha : \mathbb{S}^1 \to S^1$ be a circle rotation with irrational α. Let $R_\alpha^n(0)$ for $n \in \mathbb{Z}$. Let $I_n = [a_n, b_n]$ be intervals of length $|I_n| = \frac{1}{1+n^2}$.*

a) *Define*

$$\psi_n : [a_n, b_n] \to [a_{n+1}, b_{n+1}, \qquad x \mapsto a_{n+1} + \int_{a_n}^{x} 1 + 6\frac{|I_{n+1}| - |I_n|}{|I_n|^3}(b_n - t)(t - a_n)\, dt.$$

Show that $\psi_n : I_n \to I_{n+1}$ is a C^2 diffeomorphism. In particular, show that ψ' is bounded with $\psi_n'(a_n) = \psi_n'(b_n) = 1$. Also compute that $\psi''(\frac{a_n+b_n}{2}) = 0$.

b) *We construct a sequence of maps $(f_N)_{N\geq 0}$ as follows. To create f_0, replace 0 with an interval I_0 and map $f_0(x) = R_\alpha(0)$ for every $x \in I_n$, and $f_0(x) = R_\alpha(x)$ for every $x \notin I_0$.*

Once f_{N-1} is constructed, construct f_N by replacing $R_\alpha^N(0)$ by an interval I_N and replacing $R_\alpha^{-N}(0)$ interval I_{-N}. Also define f_N on I_{N-1} as ψ_{N-1} and on I_{-N} as ψ_{-N} and on I_N as constant $R_\alpha^{N+1}(0)$. Show that f_N is a C^1 map, except at ∂I_N.

c) *Let $f = \lim_N f_N$. Show that it is a C^1 diffeomorphism. Is it C^2?*

Exercise 50 *An approximation of the Poincaré map on the section $\{z = 0\}$ is given by the map $P : \Sigma \to \Sigma$ for $\Sigma = [-1, 1] \times [0, 1]$:*

$$P(x, y) = \begin{cases} (2x + 1, \frac{2-xy}{3}) & x < 0 \\ (2x - 1, \frac{xy}{3}) & x < 0 \end{cases}.$$

a) *Describe the set $\Lambda = \bigcup_{n\geq 0} P^n(\Sigma)$ topologically. Is it a Cantor set of arcs?*
b) *Show that P is chaotic in the sense of Devaney on the attracting set*

Exercise 51 *Solve the Lorenz equations*

$$\begin{aligned} \dot{x} &= -\sigma(x - y) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz \end{aligned}$$

for parameters $\sigma = 0$, $b = 1$ and $r > 0$.

Exercise 52 **a)** *Consider the $3 : 1$ subharmonically forced Duffing equation*

$$\ddot{x} + x + \varepsilon x^3 = \cos \Omega t,$$

where $0 < \varepsilon \ll 1$ and $\Omega \approx 3$. By setting $\Omega^2 = 9(1 + \varepsilon\delta)$, choosing a slow time $T = \epsilon t$,

$$x(t, T) = x_0(t, T) + \epsilon x_1(t, T) + \epsilon^2 x_2(t, T) + \cdots$$

and writing $\ddot{x} + x = \ddot{x} + (\Omega^2/9 - \varepsilon\delta)x$, show that the 0-th order equation can be written

$$\partial_t^2 x_0 + \frac{\Omega^2}{9} x_0 = \cos \Omega t$$

while the first order equation can be written

$$\partial_t^2 x_1 + \frac{\Omega^2}{9} x_1 = -2\partial_t \partial_T x_0 + \delta x_0 - x_0^3,$$

where ∂_t and ∂_T represent the derivatives with respect to the fast and slow times.
b) *Show that the zeroth order equation has solution*

$$x_0(t,T) = C(T)e^{i(\Omega t/3)} + \gamma e^{i\Omega t} + c.c.$$

where c.c. denotes the complex conjugate, γ is a constant that depends on Ω and that you should find and $C(T)$ is a function of slow time that is discussed in part c).
c) *It is possible to show* **[NB You are not asked to do this!]** *that the first order equation has a solution if*

$$\frac{2i\Omega}{3}\partial_T C = C(\delta - 6\gamma^2 - 3|C|^2) + 3\gamma C^{*2}.$$

where γ is as in b) and C^* is the complex conjugate of C. Use this, and write $C = re^{i\theta}$, to show that for small ε and some values of δ there is more that one periodic response to the forcing.

Exercise 53 *Consider the second order ODE*

$$\ddot{x} + 2\epsilon\ddot{x}x + x = 0.$$

for $\varepsilon \geq 0$ with initial conditions $x(0) = A$, $\dot{x}(0) = 0$.
a) *Write the differential equation as a system of equations and find all equilibria. Calculate the linearization at the equilibria. How do the eigenvalues depend on the parameter ε. For $\varepsilon = 0$ solve the differential equation explicitly.*
b) *Expand the solution of the ODE in terms of the parameter ε, i.e., write*

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3).$$

Write down the zeroth and first order equation and show that the solution is given by

$$x(t) = \varepsilon A^2 + \left(A - \frac{2A^2}{3}\varepsilon\right)\cos(t) - \frac{A^2}{3}\varepsilon\cos(2t) + O(\varepsilon^2).$$

Hint: $\cos(2\phi) = 2\cos^2(\phi) - 1$.
c) *Show that for $0 < \varepsilon \ll 1$ the approximation does not approximate the true solutions well $A < 0$ and $|A| \gg 0$.* Hint: Look for discontinuities of the vector field.

Exercise 54 *Given is the differential equation*

$$\dot{x} = C, \qquad x \in \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2,\ C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \in \mathbb{R}^2.$$

Show that:

- *the return map $F : \mathbb{S}^1 \to S^1$ of the flow φ^t to the circle $\mathbb{S}^1 = \{x \in \mathbb{T}^2 : x_1 = 0\}$ is a rotation over c_1/c_2.*
- *hence all orbits of F are dense if and only if $c_1/c_2 \notin \mathbb{Q}$.*
- *hence all orbit of φ^t are dense if and only if $c_1/c_2 \notin \mathbb{Q}$.*

What does this say about the Poincaré-Bendixson theorem on the torus $\mathbb{T}^2$?

Exercise 55 *The harmonic oscillator with damping is given by the ODEs*

$$\ddot{x} + r\dot{x} + \omega^2 x = 0. \qquad r > 0$$

Depending on the size of the damping parameter r, there is moderate damping, overdamping (when the solution is no longer oscillatory) and critical damping in between. Find the critical damping parameter $r = r_c$, and find the solution of ODE at critical damping.

Exercise 56 *The harmonic oscillator with* **parametric driving** *is given by the non-autonomous ODEs*

$$\ddot{x} + r(t)\dot{x} + \omega^2(t)x = 0.$$

a) Show that you can eliminate the linear term using the change of coordinates $q(t) = e^{\frac{1}{2}\int^t r(s)\,ds}x(t)$. The result should be

$$\ddot{q} + \Omega^2(t)q = 0,$$

for $\Omega^2(t) = \omega^2(t) - \frac{1}{2}\dot{r}(t) - \frac{1}{4}r^2(t)$.

b) Assume now that $r(t)$ and $\omega^2(t)$ are functions that oscillate mildly with the same frequency around some fixed value. That is

$$r(t) = \omega_0(b + O(\varepsilon)) \qquad \omega^2(t) = \omega_0^2(1 + O(\varepsilon))$$

where the $O(\varepsilon)$ stand for oscillating functions of fixed frequency ω_1 and small amplitude $\approx \varepsilon$. Show that this reduces the ODE to

$$\ddot{q} + \omega_0^2(1 - \frac{b^2}{4})(1 + \varepsilon f(t))q = 0,$$

where f is periodic with frequency $2\omega_2$ for some ω_2.

c) Assume $f(t) = f_0 \sin 2\omega_2 t$. Use the change of coordinates $q(t) = A(t)\cos(\omega_2 t) + B(t)\sin(\omega_2 t)$ to come to an ODEs

$$\begin{cases} 2\omega_2\dot{A} = \frac{f_0}{2}\omega_0^2 A - (\omega_2^2 - \omega_0^2)\,B, \\ 2\omega_2\dot{B} = -\frac{f_0}{2}\omega_0^2 B + (\omega_2^2 - \omega_0^2)\,A. \end{cases}$$

d) Approximate the solutions of this latter ODE using the Ansatz $A(t) = p(t)\cos\theta(t)$ and $B(t) = p(t)\sin\theta(t)$. This should lead to

$$\begin{cases} \dot{p} = p_{max}\cos(2\theta(t))\,p(t) \\ \dot{\theta} = -p_{\max}\,(\sin 2\theta - \sin 2\theta_{eq}) \end{cases} \qquad \begin{aligned} p_{\max} &= \tfrac{f_0\omega_0^2}{4\omega_2} \\ \sin 2\theta_{eq} &= \tfrac{2(\omega_2^2 - \omega_0^2)}{f_0\omega_0^2} \end{aligned}$$

e) The equation for $\theta(t)$ is independent of $p(t)$, and is close to a linear equation. Its solution decays exponentially fast to the constant solution $\theta(t) \equiv \theta_{eq}$. Use this solution to solve the equation for $p(t)$.

f) What conclusion can you draw for the original variable $x(t) = q(t)e^{-\frac{1}{2}\int^t r(s)\,ds}$? Specifically, is the equilibrium solution $x(t) \equiv 0$ stable?

Exercise 57 *Show that if the Hamiltonian $H = E_{kin}(p) + E_{pot}(q)$ and $E_{kin} = \frac{p^2}{2m}$, then the Lagrangian is $L = E_{kin}(p) - E_{pot}(q)$.*

Exercise 58 *Assume that X_H is a Hamiltonian vector field in $\mathbb{R}^2$:*

- *Show that equilibria of X_H can only be centers or saddles.*
- *Which bifurcations (of the ones we treated in class) can occur in a family of Hamiltonian vector fields?*
- *Find a family of Hamiltonians $H_\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ such that at $\varepsilon = 0$, a saddle becomes a center.*

Exercise 59 *A Lagrangian system in $\mathbb{R}^3$ has the Lagrangian*

$$L(v, q) = \frac{v_1^2 + v_2^2 + v_3^2}{2} - \frac{q_1^2 + q_2^2 + q_3^3}{2}.$$

Use Noether's Theorem to find first integrals. Is the system integrable?

Exercise 60 *We have a Hamiltonian system in coordinates $(x, y) \in \mathbb{R}^2$ where the Hamiltonian has the form*

$$H(x, y) = \frac{y^2}{2} + V(x), \qquad V \text{ is } C^2\text{-smooth,}$$

and assume that $V(x) = V(-x)$ has $V''(0) > 0$. This means that $(0, 0)$ is

(a) *Show that $(0, 0)$ is a center, with periodic motion around it.*

(b) *Let $T(a)$ be the period of the orbit starting at $(a, 0)$. Show that*

$$T(a) = \int_0^a \frac{4}{\sqrt{2(V(a) - V(x))}} dx.$$

Hint: Integrate $T(a) = \int_{t_1}^{t_2}$ a quarter of the periodic orbit and invert $t = t(x)$ (instead of $x = x(t)$) to rewrite the integral.

- *Show that $T(a) = 2\pi$ is constant for $V(x) = \frac{x^2}{2}$ (harmonic oscillator).*
- *Show that $T(a)$ is increasing if $V(x) = -\cos x$ (pendulum), and find $\lim_{a \searrow 0} T(a)$ and $\lim_{a \nearrow 0} T(a)$.*

Exercise 61 *Consider the following sets in $\mathbb{R}^2$.*

$$(i)\ x(y - 2) = 0 \quad (ii)\ x^2 + y^2 = 1 \quad (iii)\ (x - 1)^2 + (y - 1)^2 = 1.$$

a) *Which of these is a manifolds and which one has co-dimension 1?*
b) *Which of the sets is transversely to the vector field $f(x, y) = (1, 0)$ at the points $(1, 0)$ and $(0, 1)$?*
c) *Compute the Poincaré map of the flow generated by the vector field $g(x, y) = (y, -x)$ for the set (iii) above as Poincaré section.*