Dynamical Systems and Nonlinear ODEs – PS 250088
Exercises   Summer Semester 2018

Exercise 1 **a)** *Consider the initial value problem*

$$\dot{x} = \cos x + 1 \text{ with } x(0) = 0$$

for $x \in \mathbb{R}$. *Find the equilibria and indicate if they are hyperbolic. Hence compute the* ω*-limit and* α*-limit of the given initial point.*
b) *Give the definition of an* ω*-limit set and calculate the* ω*-limit set for the initial value problem*

$$\dot{x} = x^2 + x^3, \quad x(0) = -\frac{1}{2}.$$

Exercise 2 *Find the complete solution to the differential equation*

$$\dot{x} = ax(1-x), \qquad x(0) = x_0.$$

Assuming $a > 0$, *which are the stationary point and are they (asymptotically) stable? Are they exponentially (un)stable?*

Exercise 3 *Let* $f : X \to X$ *and* $g : Y \to Y$ *be two continuous mappings that are conjugate via the homeomorphism* ψ: $\psi \circ f = g \circ \psi$. *Show that*
a) ψ *maps (pre)periodic points of* f *to (pre)periodic points of* g;
b) ψ *maps omega-limit sets of* f *to omega-limit sets of* g;
c) ψ *maps attracting fixed points of* f *to attracting fixed points of* g.

Exercise 4 *Let* $Q_a(x) = ax(1-x)$, $a \in [0, 4]$ *be the quadratic family.*
Which are the fixed points and for which values of a *are they stable?*
b) *Find the period two points. For which values of* a *do they exist?*
c) *For which values of* a *is the period* 2 *orbit stable?*

Exercise 5 **a)** *Find the solutions and draw the phase portraits for the following systems of ODEs* $\dot{x} = Ax$:

$$(i) \quad A = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} \qquad\qquad (i) \quad A = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

b) Construct explicit conjugacies $h : \mathbb{R}^2 \to \mathbb{R}^2$ between the two-dimensional system $\dot{x} = -x$ and

$$(i) \quad \dot{y} = -2y \qquad\qquad (ii) \quad \dot{y} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} y.$$

Exercise 6 **a)** *Given is the ODE*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y - x \\ x(y-2) \end{pmatrix}.$$

Indicate the equilibria and their type (saddle, sink, source, center). Hence show that there is exactly one invariant horizontal line $\{y = L\}$. Compute the solutions on this line.
b) *Consider the ODE*

$$\dot{x} = 3x + xy$$
$$\dot{y} = y + x(y - x)$$

and find a near identity transformation $u = x + axy$ $v = y + bx^2 + cxy$, that removes the quadratic terms.
c) *Consider the ODE*

$$\dot{x} = 4x + y^2 - 3xy$$
$$\dot{y} = -y + y(x - y^2)$$

and find a near identity transformation $u = x + ay^2 + bxy$, $v = y + cxy$ that removes the quadratic terms. Is the equilibrium at zero structurally stable? (Justify your answer!)

Exercise 7 *Let $T_s(x) = \min\{sx, s(1-x)\}$ be the tent map with $s = 3$. Describe the set C of points $x \in \mathbb{R}$ such that $T_s^n(x) \in [0,1]$ for all n. What happens with points $x \in \mathbb{R} \setminus C$?*

Exercise 8 *Let $T : [0,1] \to [0,1]$, $T(x) = \min\{2x, 2(1-x)\}$ be the tent-map.*
a) *Argue that q is n-periodic if and only if the graph of T^n intersects the diagonal $\{y = x\}$ at $x = q$.*
b) *How many n-periodic points does T have? How many where n is the smallest period?*
c) *Prove Fermat's little theorem: If p is prime, then $p|2^p - 2$ and more generally, if $2 \leq a \in \mathbb{N}$ and p is prime, then $p|a^p - a$*

Exercise 9 *The quadratic map $Q(x) = 4x(1-x)$ is also called the Chebyshev polynomial, and $T : [0,1] \to [0,1]$, $T(x) \min\{2x, 2(1-x)\}$ is called the tent map. Let $\psi : [0,1] \to [0,1]$ be defined by $\psi(x) = \frac{1}{2}(1 - \cos \pi x)$.*
a) *Show that $Q \circ \psi = \psi \circ T$.*
b) *Show that if p is an n-periodic point of T, then $\psi(p)$ is an n-periodic point of Q.*
c) *Conclude that every n-periodic point $p \neq 0$ of Q has multiplier $|(Q^n)'(p)| = 2^n$. Why doesn't this argument apply also to $p = 0$?*

Exercise 10 *Which of the following map $f : \mathbb{R} \to \mathbb{R}$ are conjugate? Which are differentiably conjugate?*

(a) $f(x) = x/2$;

(b) $f(x) = 2x$;

(c) $f(x) = -2x$;

(d) $f(x) = 3x$;

(e) $f(x) = x^3$.

Exercise 11 *Show that the map $f : \mathbb{R} \to R$, $x \mapsto x^3 + x/2$ is C^1 structurally stable, i.e., all maps that are C^1 close to f are conjugate to f.*

Exercise 12 *Consider the "normal form" of the cusp bifurcation* $\dot{x} = r + kx + x^3$.
(a) *Find the bifurcation curve(s) in the parameter plane.*
(b) *Fix* $k = -3$. *Describe the nature of the equilibria and the bifurcations that take place when* r *increases (say from* -3 *to* 3*).*
(c) *Replace* $+x^3$ *by* $-x^3$ *in the normal form, so* $\dot{x} = r + kx - x^3$. *Repeat part (b) for* $k = +3$.

Exercise 13 *Consider the logistic family* $Q_a(x) = ax(1-x)$, *where* $a \in [0,4]$ *is such that the critical point* $c = \frac{1}{2}$ *is periodic of period* 3. *We abbreviate* $c_k = Q_a^k(c)$; *the* **core** $[c_2, c_1]$ *is an invariant set for* Q_a. *A partition* $\{I_i\}_{i=1}^N$ *of* $[c_2, c_1]$ *is a* **Markov partition** *if* Q_a *maps each interval* I_i *homeomorphically onto a union of interval* I_j. *(We allow ourselves some sloppiness, and don't care about overlap at the boundary points of the* I_i*s.)*

- **(a)** *Show that the intervals* $[c_2, c]$ *and* $[c, c_1]$ *form a Markov partition of* $[c_2, c_1]$.
- **(b)** *Define a* **transition matrix** $A = (a_{i,j})_{i,j=1}^2$ *where* $a_{i,j} = 1$ *if* $Q_a(I_i) \supset I_j$ *and* $a_{i,j} = 0$ *otherwise. Write down the transition matrix for item (a).*
- **(c)** *Argue that the number of* n*-periodic points (not necessarily prime period) of* $Q_a|_{[c_2,c_1]}$ *equals the trace* $tr(A^n)$. *How many periodic points of prime period* 11 *does* Q_a *have?*
- **(d)** *Repeat the construction for the case that parameter* a *is such that* $c_2 < c_3 < c_4 < \cdots < c_n = c < c_1$. *What is the exponential growth rate of the number of* n*-periodic points?*

Exercise 14 *Give a* C^3*-function* $f : \mathbb{R} \to \mathbb{R}$, *define the* **Schwarzian derivative** *of* f *as*

$$Sf = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 \qquad \text{if } f' \neq 0.$$

a) *Show that Möbius transformations* $g(x) = \frac{ax+b}{cx+d}$ *(with* $ad - bc = \pm 1$*) have* $Sg = 0$, *but* $SQ_a < 0$ *for* $Q_a(x) = ax(1-x)$.
b) *Show that* $S(f \circ g) = (Sf) \circ g \cdot (g')^2 + Sg$. *Conclude that* $S(Q_a^n) < 0$ *for all* $n \geq 1$.
c) *Suppose that* C^3*-function* $f : \mathbb{R} \to \mathbb{R}$ *has* $Sf < 0$. *Then* f' *cannot have a positive local minimum or a negative local maximum.*
d) *Suppose* $Sf < 0$ *and* p *is an attracting fixed point. Show that there must be a critical point* c *(i.e., a point* c *where* $f'(c) = 0$*) such that* $[p, c]$ *contains no other fixed point of* f. *Therefore* $f^n(c) \to p$.
e) *Conclude that* Q_a *can have at most one attracting periodic orbit.*

Exercise 15 *Recall the Sharkovskiy order*

$$\begin{array}{rl} 3 \succ & 5 \succ 7 \succ 9 \succ 11 \succ \dots \\ & 2\cdot 3 \succ 2\cdot 5 \succ 2\cdot 7 \succ 2\cdot 9 \succ 2\cdot 11 \succ \dots \\ & 4\cdot 3 \succ 4\cdot 5 \succ 4\cdot 7 \succ 4\cdot 9 \succ 4\cdot 11 \succ \dots \\ & \quad \vdots \qquad\qquad \vdots \\ \succ & \dots\dots \succ 8 \succ 4 \succ 2 \succ 1. \end{array}$$

A tail S *is any set of integers such that if* $s \in S$, *then also* $t \in S$ *for all* $s \succ t$. *Therefore the tail of* 3 *is* $\mathbb{N} \setminus \{0\}$, *the tail of* 6 *are all even numbers, and there is a single tail* $\{1, 2, 4, 8, 16, \dots\}$

having no Sharkovskiy maximum. Show that for every S there is a parameter $a \in [0, 4]$ such that $\{p \in \mathbb{N} : Q_a \text{ has a p-periodic point}\} = S$.
Hint: *The off-shot of Exercise 14 is that at every parameter value $a \in [0, 4]$ only one bifurcation can take place, since the orbit of $c = \frac{1}{2}$ converges to the stable periodic orbit emerging in the bifurcation.*

Exercise 16 **a)** *Let $T : M \to M$ be a continuous map on a compact manifold. Show that every omega-limit set is closed and T-invariant ($T(\omega(x)) = \omega(x)$).*
b) *If φ^t is a flow on a compact manifold, show that $\omega(x)$ is connected.*

Exercise 17 *Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 vector field, and assume that the ODE $\dot{x} = F(x)$ has a limit cycle Γ. Show that the bounded component U of $\mathbb{R}^2 \setminus \Gamma$ contains an equilibrium point.*

Exercise 18 *Given is the differential equation*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \beta & -\alpha \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \sqrt{x^2 + y^2} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{1}$$

for parameters $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$. **a)** *Rewrite (1) in polar coordinates.*
b) *Describe the bifurcation that takes place if β goes through zero.*
c) *Give the definition of ω-limit and α-limit set. Hence describe the ω-limit and α-limit sets of the system (1) for parameters $\beta = 1$ and $\alpha = 0.3$.*

Exercise 19 **a)** *Consider the ODE*

$$\frac{d^2x}{dt^2} + \mu \frac{dx}{dt} = f(x)$$

with parameter $\mu \in \mathbb{R}$ and $x \in \mathbb{R}$. Write this as a system of two first order equations. Show that if $f(x^*) = 0$, $\mu > 0$ and $f'(x^*) < -\mu^2/4$ then there is an equilibrium that is a stable spiral.
b) *Sketch a bifurcation diagram showing the location of all equilibria of the ODE*

$$\dot{x} = x^4 - \mu x$$

with $x \in \mathbb{R}$, on varying the parameter $\mu \in \mathbb{R}$. Indicate the stability of equilibria and the location and type of all bifurcation points. Which type of bifurcation takes place?
c) *Consider the ODE*

$$\dot{x} = (x^2 - 2)^2 + \mu x,$$

with parameter $\mu \in \mathbb{R}$. Name a describe in detail the bifurcations that take place when μ increases from -1 to 1.

Exercise 20 *Consider the following ODE on the first (on-negative) quadrant of $\mathbb{R}^2$:*

$$\begin{cases} \dot{x} = a_1 x - -a_2 xy \\ \dot{y} = a_2 xy - a_3 y \end{cases} \qquad a_1, a_2, a_3 > 0. \tag{2}$$

a) *Find the equilibrium points and their types (sink, saddle, source, center) of (2).*
b) *Show that*

$$L(x, y) = a_2(x + y) - a_1 - a_3 - a_3 \log \frac{a_2 x}{a_3} - a_1 \log \frac{a_2 y}{a_1}$$

is a Lyapunov function (but never strict). Hence sketch the phase portrait of (2).
c) *Using the change of coordinates $u = \log \frac{a_2 y}{a_1}$, $v = \log \frac{a_2 x}{a_3}$, show that (2) is in fact a Hamiltonian system.*

Exercise 21 *Let a one-parameter family of interval maps be given by*

$$f_\mu(x) = \mu - x^2.$$

a) *Find the smallest* $\mu_0 \in \mathbb{R}$ *such that* f_μ *has a fixed point. Describe the bifurcation that takes place at* μ_0.
b) *There is a smallest* $\mu_1 > \mu_0$ *such that* f_μ *undergoes a period doubling bifurcation. Let* p *be the rightmost fixed point of* f_{μ_1}. *What is* $f'_{\mu_1}(p)$*? Compute* μ_1.
c) *Let*

$$\mu_2 = \inf\{\mu \in \mathbb{R} \ : \ f_\mu \text{ has a periodic point of period } 6\}.$$

Argue which bifurcation takes place at μ_2.

Exercise 22 *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous map with periodic orbit* $x_0 < x_1 < \cdots < x_{n-1}$ *with* $f(x_i) = x_{i+1 \bmod n}$ *and* $n \geq 3$. *Use (the method of) Sharkovskiy's Theorem to show that* f *has periodic points of all period.*

Exercise 23 *Let* $f : \mathbb{S}^1 \to \mathbb{S}^1$ *be an orientation preserving homeomorphism. Show that the rotation number* $\rho(f) \in \mathbb{Q}$ *if and only if* f *has a periodic orbit.*

Exercise 24 *Let* $F : \mathbb{R} \to \mathbb{R}$ *be a* lift *of an orientation preserving circle homeomorphism* $f : \mathbb{S}^1 \to \mathbb{S}^1$, *i.e.,* $F(x+1) = F(x)+1$ *and* $F(x) \bmod 1 = f(x \bmod 1)$ *for all* $x \in \mathbb{R}$. *Recall that the* rotation number *of* f *is defined as*

$$\rho(f) = \lim_{n\to\infty} \frac{F^n(x) - x}{n} \bmod 1.$$

a) *Verify that* $\rho(f)$ *doesn't depend on the choice of the point* x.
b) *Verify that* $\rho(f)$ *doesn't depend on the choice of the lift* F.

Exercise 25 *Consider the Arnol'd family* $f_\varepsilon : \mathbb{S}^1 \to S^1$, $x \mapsto x + \alpha + \varepsilon \sin(2\pi x)$.
a) *for which* $\varepsilon \geq 0$ *is* f_ε *a homeomorphism (diffeomorphism)?*
b) *Compute the region of* (α, ε) *where* f_ε *has resonance of period* 1 *(i.e.,* f_ε *has a fixed point).*
c) *Fix* $\varepsilon > 0$ *small and let* I_ε *be the set of* $\alpha \in S^1$ *where the rotation number* $\rho(f_\varepsilon) \notin \mathbb{Q}$. *Given is that there is* $C > 0$ *such that the width of any resonance tongue of period* q *is* $\leq C\varepsilon^q$. *Show that* $\overline{I_\varepsilon}$ *is a Cantor set of positive Lebesgue measure.*

Exercise 26 *Which of the following dynamical systems has (i) a dense set of periodic orbits, (ii) a dense orbit, (iii) sensitive dependence on initial conditions?*

1. *a circle rotation;*
2. *the tent-map* $T(x) = \min\{2x, 2(1-x)\}$ *on* $[0,1]$*;*
3. *the twist map on the torus* $\mathbb{T}^2$ *defined by* $T(x,y) = (x, x+y \bmod 1)$*;*
4. *the pendulum* $\ddot{x} + \sin x = 0$*;*
5. *The cat-map on the torus* $\mathbb{T}^2$ *defined as* $T(x,y) = (2x+y \bmod 1, x+y \bmod 1)$. *Hint: locally the cat-map is linear, so it helps to consider the stable and unstable directions at each point.*

Exercise 27 *Suppose T is a continuous map on an X is an infinite space. If T has a dense set of periodic orbits as well as a dense orbit, then T has sensitive dependence on initial conditions.*

Exercise 28 *Define the annulus $\mathbb{A} = \mathbb{S}^1 \times [0,1]$ (where $\mathbb{S}^1 = [0,1]/0 \sim 1$ is the interval with endpoints identified. Define the map T on $\mathbb{A}$ as*

$$f(x,y) = (3x \bmod 1, (2x+y)/3).$$

a) *Show that f has a horseshoe, and that its invariant set Λ is a Cantor set. Hence show that is Devaney chaotic on Λ.*
b) *The map f has a lift $F : \mathbb{R} \times [0,1] \to \mathbb{R} \times [0,1]$ satisfying $F(x+1,y) = F(x,y)+1$, and you can define rotation numbers just as in the case of circle homeomorphisms:*

$$\rho_f(p) = \lim_{n\to\infty} \frac{\|F^n(p) - p\|}{n}.$$

Show that the limit **does** *depend on p: for each $c \in [0,2]$ there are points p with $\rho_f(p) = c$, and there are also points where the limit does not exist.*

Exercise 29 *The Hénon map $H_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2$ is given by*

$$H_{a,b}(x,y) = (1 + y - ax^2, bx).$$

a) *Show that the Hénon map has a horseshoe for $a = 3$ and $b = \frac{1}{5}$. Hint: draw and investigate what happens to (the boundary of the) rectangle $R = [-\frac{5}{6}, \frac{5}{6}] \times [-\frac{1}{6}, \frac{1}{6}]$ if H is applied.*
b) *Suppose $a > 2$. Show that the Hénon map has a horseshoe provided $|b|$ is sufficiently small.*

Exercise 30 *Consider the following sets in $\mathbb{R}^2$.*

$$(i)\ x(y-2) = 0 \quad (ii)\ x^2 + y^2 = 1 \quad (iii)\ (x-1)^2 + (y-1)^2 = 1.$$

a) *Which of these is a manifolds and which one has co-dimension 1?*
b) *Which of the sets is transversely to the vector field $f(x,y) = (1,0)$ at the points $(1,0)$ and $(0,1)$?*
c) *Compute the Poincaré map of the flow generated by the vector field $g(x,y) = (y,-x)$ for the set (iii) above as Poincaré section.*

Exercise 31 *An approximation of the Poincaré map on the section $\{z = 0\}$ is given by the map $P : \Sigma \to \Sigma$ for $\Sigma = [-1,1] \times [0,1]$:*

$$P(x,y) = \begin{cases} (2x+1, \frac{2-xy}{3}) & x < 0 \\ (2x-1, \frac{xy}{3}) & x > 0 \end{cases}.$$

a) *Describe the set $\Lambda = \bigcup_{n\geq 0} P^n(\Sigma)$ topologically. Is it a Cantor set of arcs?*
b) *Show that P is chaotic in the sense of Devaney on the attracting set*

Exercise 32 *Solve the Lorenz equations*

$$\begin{aligned} \dot{x} &= -\sigma(x-y) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz \end{aligned}$$

for parameters $\sigma = 0$, $b = 1$ and $r > 0$.