Exercise sheet Advanced Measure Theory 2025/26

Given a set $E \subset \mathbb{R}^n$, let $N_\varepsilon(E)$ be the minimal number of ε-balls necessary to cover E. Then the **upper** and **lower box-dimensions** of E are

$$\overline{\dim}_B(E) = \limsup_{\varepsilon\to 0} \frac{\log N_\varepsilon(E)}{-\log\varepsilon} \qquad \text{and} \qquad \underline{\dim}_B(E) = \liminf_{\varepsilon\to 0} \frac{\log N_\varepsilon(E)}{-\log\varepsilon}.$$

If $\overline{\dim}_B(E) = \underline{\dim}_B(E)$, then we speak of the box-dimension of E, and write is as $\dim_B(E)$

1. Let $\eta \in (0,1)$ be given and let $E_\eta \subset [0,1]$ be the middle-η Cantor set (constructed as the middle third Cantor set, but instead of a third, we remove the middle interval of relative length η of the remaining intervals at step k. Compute the upper/lower box-dimension of E_η.

2. For a sequence $(\eta_k)_{k\geq 1}$, we can define the middle-(η_k) Cantor set $E_{(\eta_k)}$ as above, by at step k removing the proportion η_k from the remaining intervals. Find a sequence $(\eta_k)_{k\geq 1}$ such that $\underline{\dim}_B(E_{(\eta_k)}) < \overline{\dim}_B(E_{(\eta_k)})$.

3. Let $E = \{\frac{1}{n} : n \geq 1\} \subset [0,1]$. Compute the box-dimension of E. For each $\alpha \in [0,1]$, find a countable set E such that $\dim_B(E) = \alpha$.

4. Look up (on the internet) what the Koch curve/snow-flake, the Sierpinski gasket and the Sierpinski carpet are, and compute their box-dimensions.

5. Let $E \subset \mathbb{R}^n$ be a compact set. Show that if $\underline{\dim}_B(E) < n$, then the n-dimensional Lebesgue measure of E is zero. Find a subset of $E \subset [0,1]$ such that $\overline{\dim}_B(E) = 1$ but the Lebesgue measure of E is zero.

6. Let $E, F \subset \mathbb{R}^n$ and $G \subset \mathbb{R}^k$. Show that

$$\overline{\dim}_B(E \cup F) = \max\{\overline{\dim}_B(E), \overline{\dim}_B(F)\}$$

and

$$\overline{\dim}_B(E \times G) = \overline{\dim}_B(E) + \overline{\dim}_B(G),$$

and the same relations hold for $\underline{\dim}_B$.

7. For a bounded set $E \subset \mathbb{R}^n$, show that $\overline{\dim}_B(E) = \overline{\dim}_B(\overline{E})$, where $\overline{E}$ stands for the closure in Euclidean topology. Does $\underline{\dim}_B(E) = \underline{\dim}_B(\overline{E})$ hold as well?

8. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a bi-Lipschitz map, i.e., a Lipschitz continuous map with a Lipschitz-continuous inverse. For a compact set $E \subset \mathbb{R}^n$, show that $\overline{\dim}_B(E) = \overline{\dim}_B(f(E))$ and $\underline{\dim}_B(E) = \underline{\dim}_B(f(E))$. Do bi-Hölder maps have this same property?

9. Let A_k, $k \in \mathbb{N}$, be Borel sets in a metric space (X,d). Show that $\dim_H(\cup_k A_k) = \sup_k \dim_H(A_k)$. Hence compute the Hausdorff dimension of a countable set.

10. Find a set $A \subset [0,1]$ such that $\dim_H(A) < \underline{\dim}_B(A) < \overline{\dim}_B(A)$.

11. Let S be the spiral in $\mathbb{R}^2$ given in polar coordinates by $r = 1/\log\varphi$, $e \leq \varphi < \infty$. Compute $s = \dim_H(S)$ and $\mathcal{H}^s(S)$.

    How would the answer be if the parametrization is $r = e^{-\varphi}$?

12. Let $F : X \to Y$ be a bi-Lipschitz map between metric spaces., that is F is Lipschitz continuous and F^{-1} is well-defined and Lipschitz continuous. Let $A \subset X$ and $B \subset Y$ be Borel sets such that $B = F(A)$. Show that $\dim_H(A) = \dim_H(B)$. Is also $\mathcal{H}^s(A) = \mathcal{H}^s(B)$ for $s = \dim_H(A)$?

13. Let $C \subset [0,1]$ be the middle third Cantor set. Compute the Hausdorff dimension of $\{\sqrt{x} : x \in C\}$.