The hyperspace of noncut subcontinua of a hairy dendrite

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36th Summer Topology Conference  
Vienna, Austria, July 19, 2022
Joint work with Jorge Vega
Dendrites

A **dendrite** is a locally connected metric continuum that does not contain simple closed curves.
Menger-Urysohn order

Let $X$ be a dendrite and $p \in X$. The **Menger-Urysohn order** of $p$ in $X$ is the number of components of $X \setminus \{p\}$ and is denoted by $\text{ord}(p, X)$. 

$E(X) = \{ p \in X : \text{ord}(p, X) = 1 \}$

$O(X) = \{ p \in X : \text{ord}(p, X) = 2 \}$

$R(X) = \{ p \in X : \text{ord}(p, X) \geq 3 \}$

**Important:** For dendrites, $\text{ord}(p, X)$ can be any finite number or $\omega$. 
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Example: points by their order
“Hairy” dendrites

Lemma (J. Chatatonik, W. Charatonik and J. Prajs, 1994)

For a dendrite $X$, the following are equivalent.

1. $E(X)$ is dense,
2. $R(X)$ is dense, and
3. if $\alpha$ is an arc in $X$, then $\alpha \cap R(X)$ is dense.
Non-cut subcontinua

Let $X$ be a metric continuum; then

$$2^X = \{ A \subset X : A \text{ is closed and nonempty} \},$$

$$C(X) = \{ A \in 2^X : A \text{ is a continuum} \}.$$
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It is known that $2^X$ and $C(X)$ are non-degenerate metric continua if $X$ is non-degenerate.

$$NC^*(X) = \{A \in C(X) : X \setminus A \text{ is connected}\}.$$
Theorem (Jorge Martinez-Montejano, Verónica Martinez-de-la-Vega and Jorge Vega)

If $X$ is a dendrite where $R(X)$ is dense, then $NC^*(X)$ is totally disconnected.
Total disconnected and … ?

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If $X$ is a dendrite where $R(X)$ is dense, then $NC^*(X)$ is totally disconnected.

Theorem (HG and Vega)

If $X$ is a dendrite where $R(X)$ is dense, then $NC^*(X) \approx \mathbb{R} \setminus \mathbb{Q}$. 
Characterization of $\mathbb{R} \setminus \mathbb{Q}$

Theorem (Alexandroff and Urysohn, 1928)

For a separable metrizable space $X$ the following are equivalent:

- $X \cong \mathbb{R} \setminus \mathbb{Q}$, and
- $X$ is Polish, zero dimensional and nowhere locally compact.
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Theorem (Krupski and Samulewicz, 2017)

If $X$ is a locally connected continuum, then the family $S(X)$ of all compacta that separate $X$ is an $F_\sigma$-subset of $2^X$. 
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$$NC^*(X) = C(X) \setminus S(X)$$
Elements of $NC^*(X)$ when $X$ is a dendrite

Theorem (Martinez-Montejano, Martinez-de-la-Vega and Vega)

Let $X$ be a dendrite and let $A \in C(X)$. Then $A \in NC^*(X)$ if and only if one of the following holds:

1. $A = X$,
2. $A = \{ e \}$ for some $e \in E(X)$, or
3. $A = X \setminus C$, where $C$ is a component of $X \setminus \{ p \}$ with $p \in X \setminus E(X)$. 


Example of $A \in NC^*(X)$.

$$A = X \setminus C$$

Notice: $bd_X(A) = \{p\}$. 
Some closed discrete sets

Let $A \in NC^{*}(X)$.
Some closed discrete sets

Let $A \in NC^*(X)$. Choose $B \in NC^*(X)$ close to $A$ such that $q \in R(X)$. 

\[ e \quad q \in R(X) \quad p \quad B \quad A \]
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Then $\{B_n : n \in \mathbb{N}\}$ is closed and discrete in $NC^*(X)$. 
Clopen sets in $NC^*(X)$

Let $q, r \in [ab \setminus \{a, b\}] \cap R(X)$.

$$\mathcal{B}(q, r) = \{B_x : x \in qr \setminus \{q\}\}$$
Clopen sets in $NC^*(X)$

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Open questions

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In general, what space is $NC^*(X)$ when $X$ is a dendroid?
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Question

If $X$ is the Mohler-Nikiel universal smooth dendroid, is $NC^*(X)$ totally disconnected and not zero-dimensional?
Thank you

Preprint available at:
https://arxiv.org/abs/2108.06020

Figure: The Julia set of $z \mapsto z^2 + i$ is homeomorphic to $D_3$.
https://sciencedemos.org.uk/julia.php