
IMéRA
Aix-Marseille Université
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CIRM
INSTITUT de MATHÉMATIQUES de MARSEILLE
Mathematical
Feelings
of SPACE
IMéRA, Marseille 10-11 October 2017
Confirmed Participants:
Edmund Harriss
Pierre Arnoux
Peter Davidson
Paul Prudence
Arnaud Chéritat
The notion of space is a broad and complex one, in mathematics alone the word is used in many different ways. It can be regarded as one of the central motivating concepts, for the subject.
The aim of this workshop is to bring together mathematicians with artists and scientists to explore the different ways we feel and understand space, how the mathematical understanding allows us to express precisely many different common intuitions about space and also how logical exploration of mathematical concepts allows us to experiment with new intuitions never seen or felt before.

Chaire Jean Morlet

Shigeki Akiyama, Chaire Morlet Professor at CIRM
School on tilings November 20-24, 2017
Conference Tilings and recurrence December 4-8, 2017
http ://akiyama-arnoux.weebly.com/school.html

Continued fractions as induction on dynamical systems : S-adic systems

Pierre Arnoux
Joint work with Valérie Berthé, Milton Minervino, Wolfgang Steiner, Jörg Thuswaldner

September 28, 2017
Vienne

Relying on results from :

Geometry, Dynamics, and Arithmetic of S-adic shifts

By Valérie Berthé, Wolfgang Steiner, Jörg Thuswaldner

https ://arxiv.org/abs/1410.0331v3

Part 1

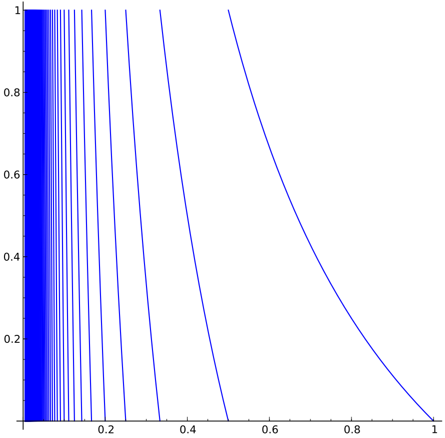
The usual continued fraction :

Rotations, symbolic dynamics, tilings, and the geodesic flow on the modular surface

Usual CF : the projective map

- For some reason, in ancient times, people like to write numbers $x \in (0,1)$ as
$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\dots}}}$$
.
- Hence $\frac{1}{x} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\dots}}$.
- Define $T(x) = x - \left[\frac{1}{x}\right] = \left\{\frac{1}{x}\right\}$ and $F(x) = \left[\frac{1}{x}\right]$.
- Then $a_n = F(T^{n-1}(x))$.
- T is the *Gauss map*.

The Gauss map



Usual CF : the linear map

- a more primitive view : Euclid's algorithm.
- A piecewise linear map on a positive cone.
- $0 < x < y \mapsto y - [\frac{y}{x}]x, x$.
- Projective version by normalizing y to 1.
- $(x, 1) \mapsto (1 - [\frac{1}{x}]x, x) \approx (\frac{1}{x} - [\frac{1}{x}], 1)$.
- We recover the Gauss map.
- The linear version has interesting properties.

An additive version

- Linear algorithm : start with a vector $V_0 = (x_0, y_0)$.
- The symmetric algorithm defines a sequence of vectors $V_n = (x_n, y_n)$.
- Define $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.
- $V_{n+1} = A^{-1} V_n$ or $V_{n+1} = B^{-1} V_n$.
- It is more convenient to write $V_n = AV_{n+1}$ or $V_n = BV_{n+1}$.
- Hence $V_0 = M_0 \dots M_{n-1} V_n$.
- $V_0 = M_{[0,n)} V_n$: an image of the positive cone.
- $M_{[0,n)} \in SL(2, \mathbb{N})$.

Induction of a dynamical system

- Dynamical system $T : X \to X$.
- $A \subset X$.
- The induced map of T on A is
- $T_{|A} : A \to A$.
- $T_{|A}(x) = T^{n_x}(x)$, where $n_x = \inf\{n > 0 | T^n(x) \in A\}$.
- Also called first-return map of T to A.

Rotations

- $R_\alpha : [0,1) \to [0,1)$, $x \mapsto x + \alpha \mod 1$.
- A model for the rotation of angle α on a circle of length 1.
- Two continuity intervals : $I_0 = [0, 1-\alpha)$ and $I_1 = [1-\alpha, 1)$.
- Consider the induced map of R_α on the image J of the longest interval ($[0,\alpha)$ or $[\alpha, 1)$).
- This is the rotation $R_{\alpha|J} = R_{F(\alpha)}$ up to renormalization, where F is the symmetric Farey map.
- The dynamics of the rotation (recurrence, best approximation) is directed by the Gauss/Farey map.

Rotation sequences

- Coding map $\nu : [0,1) \to \{0,1\}$, $\nu(x) = a$ if $x \in I_a$.
- The itinerary, or symbolic sequence, of x for R_α is defined by $u_n = \nu(R_\alpha^n(x))$.
- This is a *rotation sequence*.
- Many geometric definitions : cutting sequences, billiard sequences...
- It appears in all periodic phenomena with 2 independent periods.

Sturmian sequences

- Definition
  The complexity of u is the function $P_u(n) = \# L_n(u)$.
  - If $P_u(n) = P_U(n+1)$, u is ultimately periodic.
  - For a nonperiodic sequence, $p_u(n) \geq n+1$.
- Definition
  A sequence u is Sturmian if $P_u(n) = n+1$.

  .
  - It is not trivial that there exists Sturmian sequences.
  - One such sequence is given as limit of $U_0 = 0,\ U_1 = 1$, $U_{n+2} = U_{n+1} U_n$.

Balanced sequences

- A sequence u is balanced if, for any two factors $U, V \in L_n(u)$, one has $|U|a - |V|_a \leq 1$.
- This has a geometric interpretation :
- To a sequence u we associate a sequence of point in $\mathbb{R}^2$ (*stepped line*) by abelianization of the prefixes.
- All finite paths in this sequence have approximately same slope.
- The complete path is contained in a corridor.

A balanced staircases

- This can be seen geometrically.

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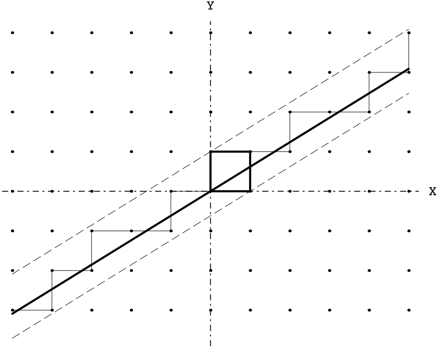

Main theorem

- Theorem
  The three conditions are equivalent for a sequence u :
    1. *u is Sturmian*
    2. *u is balanced and aperiodic*
    3. *u is a rotation sequence with irrational angle* α
  - This will allow a combinatorial continued fraction.

A combinatorial continued fraction

- Suppose u sturmian ; then $P_u(2) = 3$.
- One of $00, 01, 10, 11$ does not occur. But $01, 10$ must occur
- Suppose that 11 does not occur : every 1 is followed by 0.
- u can be decomposed in 10 and 0.
- Define $\sigma_0 : 0 \mapsto 0, 1 \mapsto 10$.
- $u = \sigma_0(v)$.
- One can prove that v is sturmian.
- Hence $u = \sigma_{i_0}\sigma_{i_1} \dots \sigma_{i_{n-1}}(v_n)$: a Farey model !.

The general case : S-adic system

- In general, the infinite *directive sequence* of substitution completely defines the symbolic system.
- This symbolic system is renormalizable, but to another system.
- This is what is called an S-adic system : a system defined by an infinite sequence of substitutions in a set S.
- The corresponding rotation can be induced on the good interval to another rotation.

Sturmian staircases

- This can also be seen as renormalisation of sequences.

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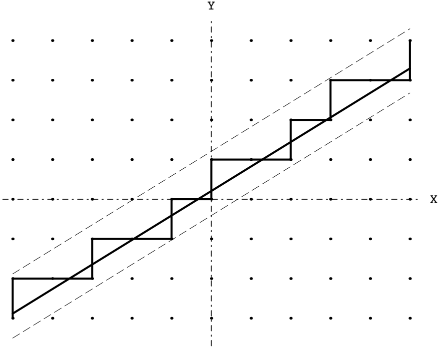

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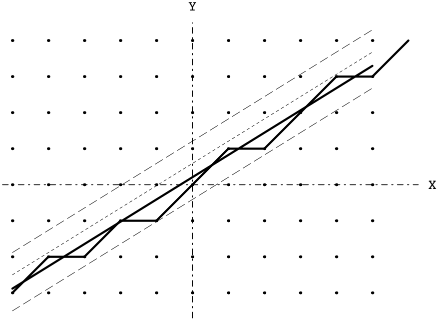

Sturmian tilings

- If we use a second direction, we obtain a Sturmian line tiling.
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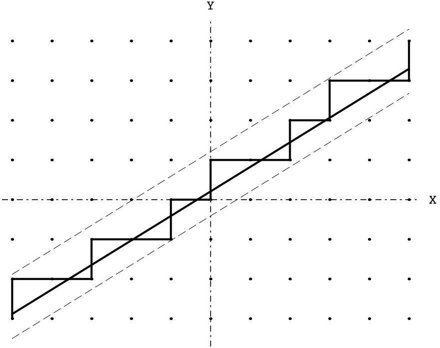

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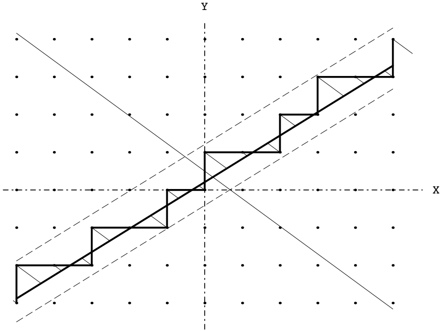

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Sturmian tilings : the dual viewpoint

- Replace $\mathbb{Z}^2$ by a lattice, take the prefered directions as axis

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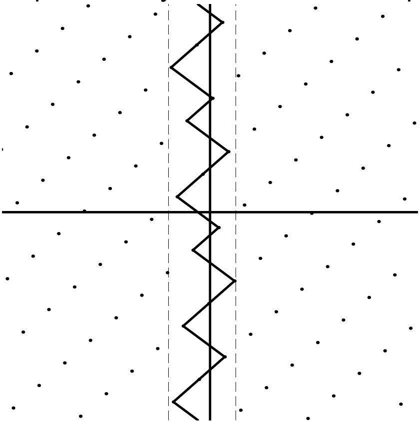

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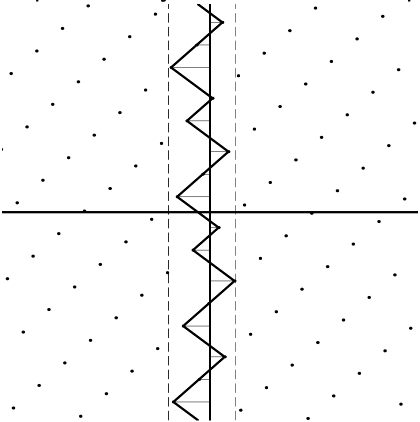

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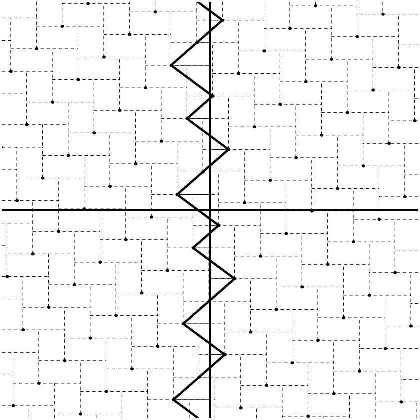

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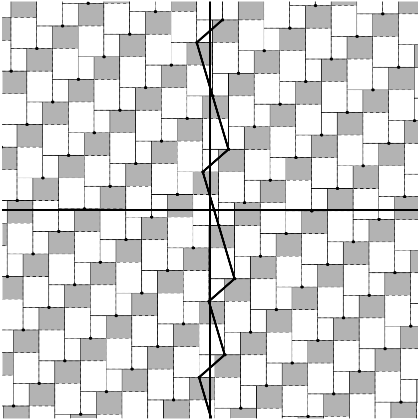

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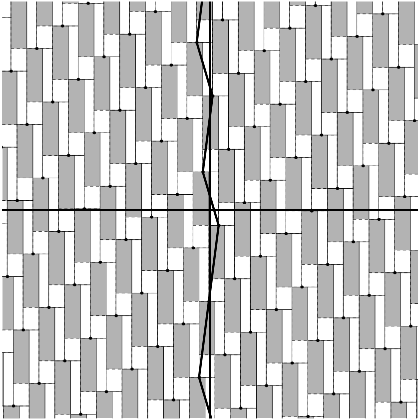

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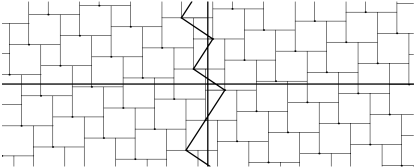
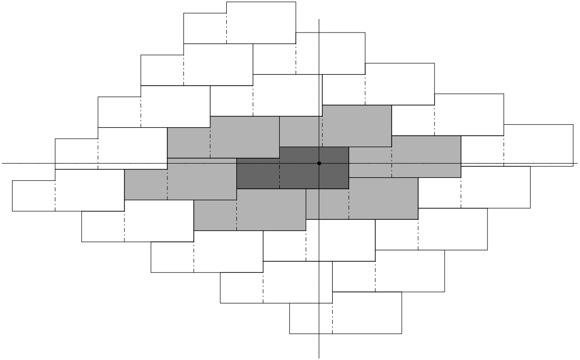


Coordinates on the space of tilings

- A basis (a, c), (b, d) of a tiling of size 1 gives a matrice $\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in SL(2, \mathbb{R})$.
- Any two basis of a given lattice differ by an element of $SL(2, \mathbb{Z})$.
- The space of lattices is $SL(2, \mathbb{Z}) \backslash SL(2\mathbb{R})$.
- For any tiling, we can find a L-shaped fundamental domain made of two rectangles $a \times d$ and $b \times c$, with basis (a, c) and $(-b, d)$, with $ad + bc = 1$.

L-shaped fundamental domain



The geodesic flow on the modular surface

- $SL(2,\mathbb{R})$ acts by isometry on the hyperbolic plane $\mathbb{H}$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}.z = \frac{az+b}{cz+d}$
- It acts transitively on $T^1\mathbb{H}$, the kernel is $\pm Id$; hence $PSL(2,\mathbb{R})$ can be identified (in a non-canonical way) with $T1\mathbb{H}$.
- The diagonal flow, acting on the right, is easily seen to be the geodesic flow on $T1\mathbb{H}$.
- The group $SL(2\mathbb{Z})$ acts discretely on $\mathbb{H}$; the quotient is the modular surface.
- The flow we have considered can be identified with the geodesic flow on the modular surface.

Down from the geodesic flow to the Gauss map

- $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})$ can be seen as the space of lattices : basis in $\mathbb{R}^2$ up to an equivalent relation. It has dimension 3.
- For each lattice, we can chose a unique special basis $(a,c),(-b,d)$ with $0 \leq a < 1 \leq b$, $0 \leq d \leq c$ **or** $0 \leq b < 1 \leq a$, $c \leq d$.
- We can define a section Σ by $\max(a,b) = 1$. It has dimension 2, and it is transverse to the geodesic flow.
- The first return map on Σ is given by $\hat{T} : (a,d) \mapsto (\{\frac{1}{a}\}, a - a^2 d)$.
- This map factors on the interval map (Gauss map) T by projection on the first coordinate.

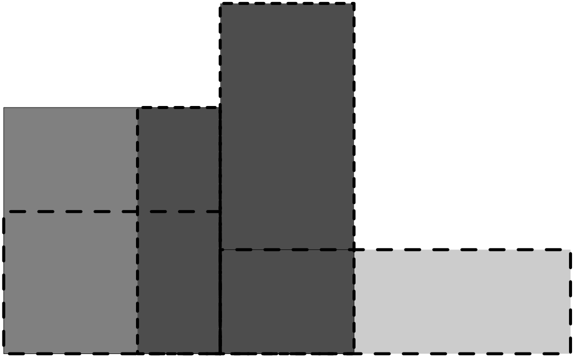
Up from the Gauss map to the geodesic flow

- We can go in the reverse direction.
- Start with the Gauss map on $[0,1]$, $T(x) = \{\frac{1}{x}\}$.
- Define the natural extension $\hat{T}$ on $\Sigma = \{(x,y) | 0 \le x \le 1, 0 \le y \le \frac{1}{1+x}\}$.

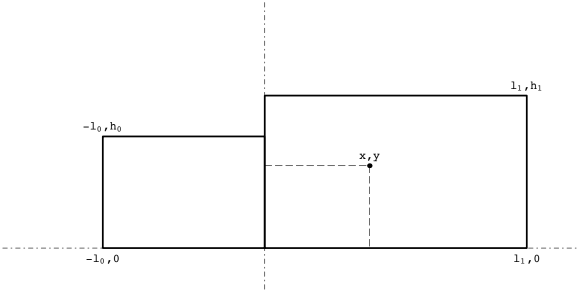
$$\begin{array}{ccc} \Sigma & \stackrel{\hat{T}}{\rightarrow} & \Sigma \\ \pi\downarrow & & \downarrow\pi \\ ([0,1] & \stackrel{T}{\rightarrow} & [0,1] \end{array}$$

- Take a suspension of $\hat{T}$ with time $-\log x$.
- We recover the geodesic flow on the modular surface.
- We must think of x as a parameter for the dynamical system $R_x : [0,1] \rightarrow [0,1]$.
- Can we study orbits of individual points in this system ?

The geodesic flow



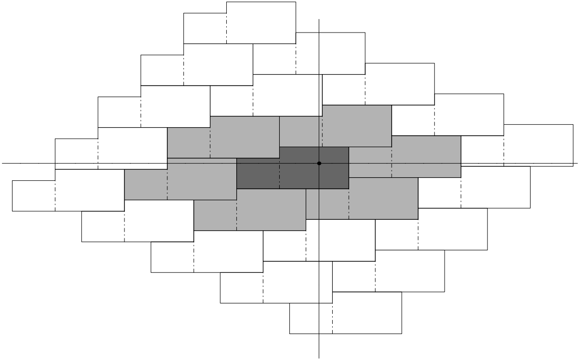
The scenery flow



Remarks

- To one point in the domain of the Gauss map, there corresponds many sturmian sequences, with the same language,
- corresponding to orbits of different points under the same rotation.
- To one point in the domain of the geodesic flow, there correspond many tilings of the line, with the same set of patterns and the same lengths of tiles, but different place of the origin and different combinatorics.
- The Gauss map determines a system, not a particular orbit (even if we can decide to consider a special one).
- There is a way to put all these dynamics in one set.

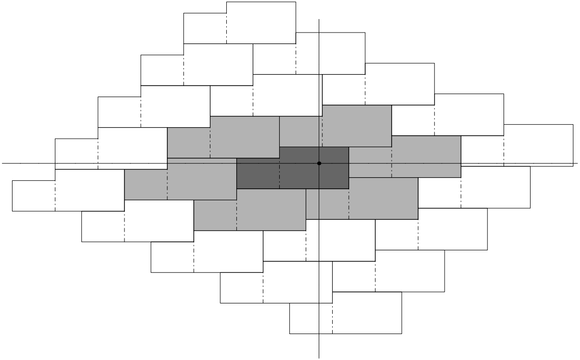
The scenery flow



Families of systems as a fiber bundle

- One can put together all the dynamical systems in one set :
- Given a lattice L, consider cosets $L + U$ of this lattice.
- $SL(2, \mathbb{R})$ acts on the cosets.
- $\mathbb{R}^2$ acts on the cosets.
- We can unify these actions as the affine group, whose elements are maps $\mathbb{R}^2 \to \mathbb{R}^2$ given as $U \mapsto AU + V$, where $A \in SL(2, \mathbb{R})$ and $V \in \mathbb{R}^2$.
- There is a group $SA(2, \mathbb{Z})$, and we can consider $SA(2, \mathbb{Z})\backslash SA(2, \mathbb{R})$.
- Fiber bundle over $SL(2, \mathbb{Z})\backslash SL(2, \mathbb{R})$ with torus fibers.
- Several flows acting on the right (geodesic, horocyclic, vertical, horizontal) : we now act on *cosets* of lattices.

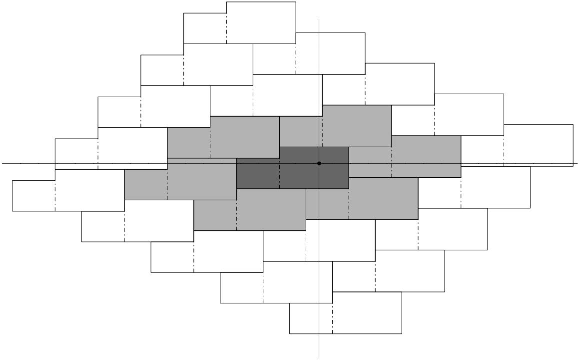
The scenery flow

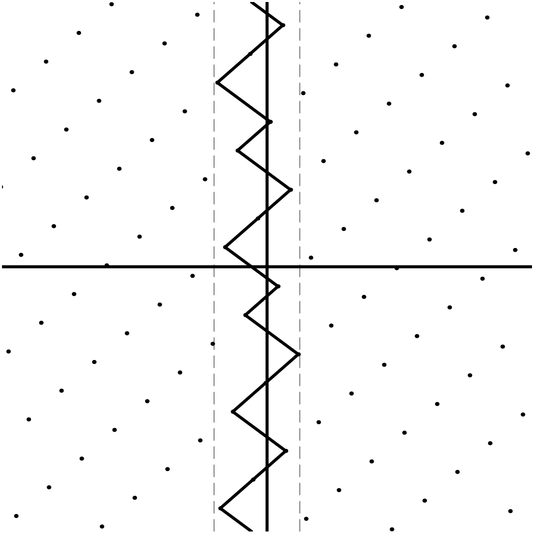


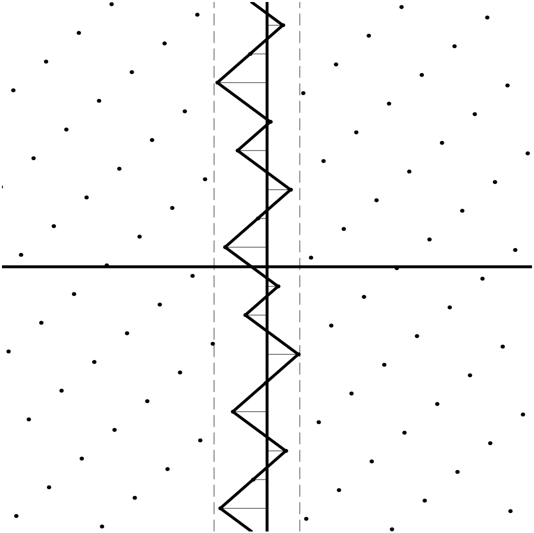
Many models of the scenery flow

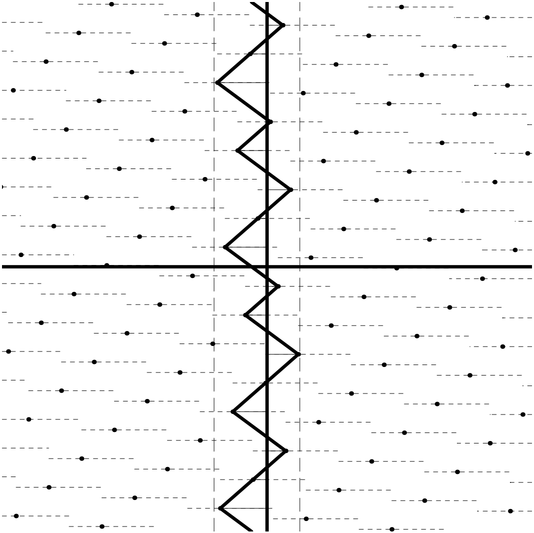
- An element of $SA(2,\mathbb{Z})\backslash SA(2,\mathbb{R})$ can be seen as a coset of a lattice.
- Or as a point in a fundamental domain of this particular lattice.
- Or as a tiling of the vertical line, with an origin point :
- The length of the tiles give the heights of the rectangle
- The origin of the tiling gives the height of the point.
- The frequencies of the tiles give the width of the rectangle.
- The combinatorics of the tiling give the abscissa of the point.

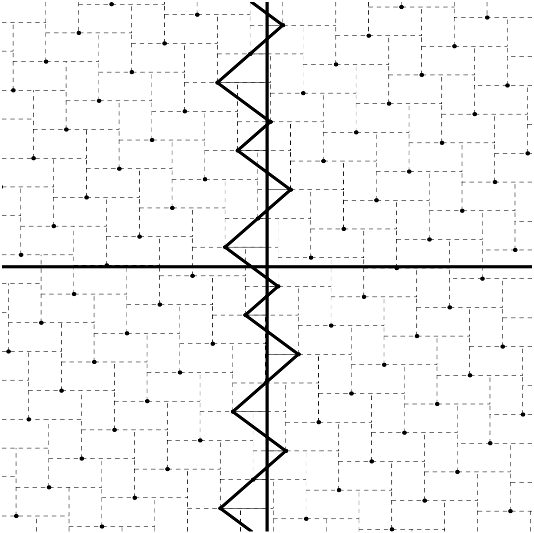
Many models of the scenery flow







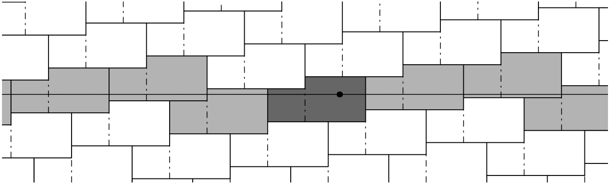




Many models of the scenery flow

- Local (continuous) structure gives the vertical coordinates.
- Global (combinatorial, discrete) structure gives the horizontal coordinates.
- Of course, we can also consider the tiling of the horizontal line induced by the lattice of the plane and its fundamental domain.
- We exchange then the role of local and global.
- All this can obviously be rephrased in the language of model sets, or quasi-crystals.
- Remark : all this is dimension 5 ; 3 for the linear part, 2 for the translation part.

The scenery flow



Translation flows

- The translation flows project to trivial flows on $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})$, hence they preserve the torus fibers (2 dimensional).
- The vertical translation flow is a collection of linear flows on invariant tori (2-dimensional).
- If we take as section the base of the L-domain, the return map is a rotation (1-dimensional).
- All possible rotations appear in this return map.
- Two elements in the same vertical translation orbit converge exponentially fast under the geodesic flow.

Periodic orbits of the geodesic flow

- Restrict to a periodic orbit (1-dimensional) for the geodesic flow on $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})$, given by $M \in SL(2,\mathbb{N})$.
- Its preimage in $SA(2,\mathbb{Z})\backslash SA(2,\mathbb{R})$ is a fiber bundle, with torus fibers, over a circle (3 dimensional).
- The geodesic flow on $SA(2,\mathbb{Z})\backslash SA(2,\mathbb{R})$ preserves this bundle.
- Take a section (torus) over a point on the circle ; the first return map of the geodesic flow is given by the matrix M.
- We have built Markov partitions for these automorphisms (done first by Adler-Weiss, 1965).
- If λ is the dominant eigenvalue of the matrix, we build a natural extension of $x \mapsto \lambda x \mod 1$, followed by a suspension.
- Same process as above, replacing T by $x \mapsto \lambda x \mod 1$.

Our goal

- The goal for the next part is to generalize this to other continued fraction.
- Get a family of parametrized dynamical systems such that the given continued fraction appears as an induction.
- One famous example : Rauzy induction and the Teichmüller flow, *via* the interval exchanges.
- We want to find other.
- But many reasons imply that any reasonable domain in dimension > 1 should have fractal boundary.
- We can have no direct geometric contruction.
- We will need to use symbolic dynamics,
- and to recover the system by projection.

Part 2

Generalized continued fractions :

Symbolic dynamics, tilings, toral translations and the Weyl chamber flow

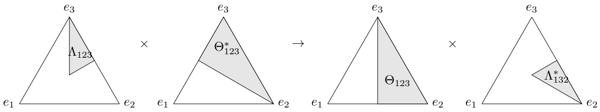
Continued fraction

- Generalized unimodular continued fraction :
- A piecewise projective map T on a cone $\Lambda \subset \mathbb{R}^d$
- It is associated with a map $M : \Lambda \to GL(d, \mathbb{Z})$.
- $T(x) = M(x)^{-1}x$.
- In many cases, we can build a natural extension
- $\tilde{T}(x, y) = (M(x)^{-1}x, {}^tM(x)y)$.

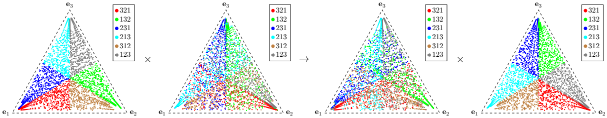
A simple example : Brun Continued Fractions

- Given 3 positive real numbers x,y,z
- Subtract the second biggest from the biggest, and iterate
- It is associated with the 6 elementary matrices on the 6 sorted subcones of the positive cone.
- It satisfies a Markov condition.

Natural extension of Brun Continued Fractions



Natural extension of Brun Continued Fractions



A dynamic model for continued fractions

- We want to find a family of systems R_α.
- where α is in the domain of T.
- and a family of subsets A_α.
- such that $R_{\alpha|A_\alpha}$ is conjugate to $R_{T\alpha}$.
- the first example : continued fraction, rotations and the geodesic flow on the modular surface.
- Another example : interval exchange maps, Rauzy induction and the Teichmüller flow.
- this has been hugely successful (but highly technical)

A dynamic model for continued fractions

- Extend this to higher dimensional continued fractions ?
- with given algorithm ; problem : find the family of dynamical systems (translations on compact groups).
- this is difficult in dimension > 1.
- the set A_α has special properties :
- It should be a bounded remainder set (not easy to find)
- It should give symbolic dynamics with linear complexity
- consider the periodic case :
- It needs to have fractal boundaries (Markov partition).

Dynamic models for continued fractions

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Symbolic dynamics

- With a given point, we associate a sequence M_n of matrices.
- With each matrix M_n, we associate a substitution σ_n having this matrix.
- This infinite sequence of substitution defines an infinite limit sequence of letters (S-adic sequence).
- Think of a generalized Sturmian sequence, with its infinite sequence of renormalizations.
- This associates a symbolic system (Ω_x, S) to the point x.
- We could take this as an answer, but we want a geometric model.
- We consider the stepped line obtained by abelianization of $u \in \Omega_x$: we want it to be a model set.

Some properties of symbolic systems

- We want (Ω_x, S) to be minimal.
- We want (Ω_x, S) to be uniquely ergodic : asymptotic direction for the stepped line.
- We want this asymptotic direction to be totally irrational.
- We want the stepped line to be *balanced*, and remain within bounded distance of the asymptotic direction.
- In that case, the stepped line is (almost) a model set.
- We can project it on the diagonal plane to obtain a compact set with nonempty interior which gives a locally finite covering by action of the diagonal subgroup of $\mathbb{Z}^3$.
- If this covering is a tiling, the stepped line is a true model set.
- We can ensure all this by a generalized Pisot property on the sequence of matrices, plus some combinatorics.

Our plan

1. Study some properties of sequence of positive matrices.
2. Define the corresponding S-adic system.
3. Prove that the asymptotic properties of sequence of matrices imply properties of the S-adic system.
4. Define the dynamical geometric model for this sequence of matrices : Rauzy fractals and Rauzy boxes.
5. Prove that these properties are almost everywhere defined for some algorithms, using Oseledets and computations by Avila-Delecroix.

Sequences of matrices : Notations

- We consider a sequence $\mathbf{M} = (M_n)_{n\in\mathbb{Z}}$ with value in $GL(d,\mathbb{N})$.
- It means nonnegative, integer coefficients, invertible (in $GL(d,\mathbb{Z})$), determinant ± 1.
- We are interested in products $M_n M_{n+1} \dots M_{n+k}$.
- Useful notation :

$$M_{[n,m)} = M_n M_{n+1} \dots M_{m-1}$$

Sequences of matrices : Primitivity

- We say that **M** is *primitive* if
- For any n, there exists $m > n$ such that $M_{[n,m)}$ is positive.
- This implies that we can apply Perron-Frobenius theorem to all $M_{[n,p)}$ for $p \geq m$.
- It also implies that the coefficients of $M_{[0,n)}$ tend to infinity.
- and the dominant eigenvalue tends to infinity.
- STANDING ASSUMPTION :
  The sequence **M** is primitive.

Sequences of matrices : Algebraic irreducibility

- We say that **M** is *algebraically irreducible* if
- For any n, there exists $m > n$ such that, for all $p \geq m$, the characteristic polynomial of $M_{[n,p)}$ is $\mathbb{Z}$-irreducible.
- This implies that all eigenvalues are simple, and $M_{[n,p)}$ is diagonalizable over $\mathbb{C}$.

Sequences of matrices : Recurrence

- We say that **M** is *recurrent* if
- There exists n such that, for all $k > n$, the sequence $M_n \dots M_k$ occurs another time (and hence infinitely often) in **M**.
- With primitivity, this implies that the same positive matrix A occurs many times in $M_{[0,n)}$ for large n.
- By a standard argument (Furstenberg), this implies that the intersection of the images of the positive cones, $M_{[0,n)}\mathbb{R}_+^d$, is reduced to a line $\mathbb{R}_+ w$ for some $w \in \mathbb{R}_+^d$.

Sequences of matrices : Generalized eigenvector

- We say that **M** *admits a generalized eigenvector* $\mathbf{u} \in \mathbb{R}^d_{\geq 0}$ if
- $$\bigcap_{n \in \mathbb{N}} M_{[0,n)} \mathbb{R}^d_+ = \mathbb{R}_+ \mathbf{u}$$
- Recurrence implies existence of a generalized eigenvector.
- This replaces the PF dominant eigenvector.
- Remark that $M_{[0,n)}\mathbf{u}$ is a generalized eigenvector for the shifting sequence.
- This is a *nonstationary* eigenvector.

Sequences of matrices : weak, strong and exponential convergence

- We say that $\mathbf{M}$ converges to the eigenvector :
- *weakly* if, for all $a \in \mathcal{A}$, $M_{[0,n)}\mathbf{e}_a$ converges to $\mathbf{u}$ in projective space.
- *strongly* if, for all $a \in \mathcal{A}$, $M_{[0,n)}\mathbf{e}_a$ converges to $\mathbb{R}_+\mathbf{u}$ in $\mathbb{R}^d$.
- *exponentially* if, for all $a \in \mathcal{A}$, $M_{[0,n)}\mathbf{e}_a$ converges exponentially fast to $\mathbb{R}_+\mathbf{u}$ in $\mathbb{R}^d$.
- What we really want is that the series $\sum d(M_{[0,n)}\mathbf{e}_a, \mathbb{R}_+\mathbf{u})$ converges, which is implied by exponential convergence.
- Recurrence implies weak convergence, not strong convergence.

Sequences of matrices : Lyapounov exponents

- For a matrix A, we can define singular values as square roots of eigenvalues of A^*A.
- The image of the unit sphere by A is an ellipsoid : the singular values $\delta_i(A)$ are the lengths of the (orthogonal) axis.
- For the sequence $M_{[0,n)}$, we can define $\delta_i(n)$.

- Definition

  The i-th Lyapunov exponent (if it exists) is $\lim_{n\to\infty} \frac{1}{n}\log(\delta_i(n))$.

  .

- Lyapunov exponents are **not** (logarithmes of) eigenvalues !
- But there are relations, in particular for a sequence M^n.

Sequences of matrices : The Generalized Pisot condition

- We say that **M** satisfies the *Generalized Pisot Condition* if
- It admits a generalized eigenvector, and Lyapunov exponents such that

$$\theta_1 > 0 > \theta_2$$

- For a periodic sequence $M_{[0,n)} = M^n$ with $M \in GL(d, \mathbb{N})$, this means that the dominant eigenvalue is a Pisot unit : an algebraic unit > 1 whose conjugates all have modulus < 1.

The generalized Pisot condition : consequences

- This hypothesis implies that the dominant eigenvalue of $M_{[0,n)}$ increases faster than $e^{\alpha n}$, for all $\alpha < \theta_1$.
- It implies that all the other eigenvalues decrease exponentially.
- It implies algebraic irreducibility.
- It implies exponential convergence (up to small technical details).

The generalized Pisot condition : consequences

- It also implies that the generalized eigenvector $\mathbf{u}$ is totally irrational :
- Suppose $\mathbf{u}$ satisfies a rational condition.
- It is contained in the kernel of a linear form α with integer coefficient : $\langle \mathbf{u}, \alpha \rangle = 0$.
- Hence all $\langle M_{[0,n)}\mathbf{e}_a, \alpha \rangle$ are uniformly bounded.
- Hence all $\langle \mathbf{e}_a, M^*_{[0,n)}\alpha \rangle$ are uniformly bounded.
- But $M^*_{[0,n)}\alpha$ take value in $\mathbb{Z}^n$. Since they are bounded, they take infinitely often the same value.
- $M_{[n,k)}$ has 1 as eigenvalue for infinitely many k : this contradicts irreducibility.

Sequences of substitutions

- For each matrix $M \in GL(d, \mathbb{N})$ we choose a substitution σ_M in a finite or countable set S, acting on the alphabet $\mathcal{A} = \{1, \ldots, d\}$.
- We associate in this way an infinite sequence $\boldsymbol{\sigma} = (\sigma_n) \in S^{\mathbb{N}}$.
- $\sigma_{[m,n)} = \sigma_m \ldots \sigma_{n-1}$.
- We transfer to sequences of substitutions the properties of sequence of matrices : primitivity, recurrence
- STANDING ASSUMPTION :
  The sequence $\boldsymbol{\sigma}$ is primitive.

Limit words for primitive sequences of substitutions

- Let $\boldsymbol{\sigma} = (\sigma_n) \in S^{\mathbb{N}}$ be primitive.
- A limit word of $\boldsymbol{\sigma}$ is the first word of a sequence of words $w^{(n)} \in \mathcal{A}^{\mathbb{Z}}$ such that $w^{(n)} = \sigma_n(w^{(n+1)})$ (and $w_{-1}^{(n)} w_0^{(n)}$ has other occurences in $w^{(n)}$) .
- Such words always exist for primitive sequences, because we can always find a sequence (a_n) such that a_n is the first letter of $\sigma_n(a_{n+1})$ (and the same for last letters)
- A primitive sequence of substitutions has at most d^2 limit words.

S-adic systems

- Let $\boldsymbol{\sigma} = (\sigma_n) \in S^{\mathbb{N}}$ be primitive.
- It has a finite number of limit words.
- They all have the same language :
- if $\mathbf{u}, \mathbf{v}$ are two limit words, and U occurs in $\mathbf{u}$, it occurs in some $\sigma_{[0,n)}(a)$, hence in all $\sigma_{[0,n+k)}(c)$ for all $c \in \mathcal{A}$ and k large enough
- Hence U occurs in $\mathbf{v}$.
- Hence they all define the same symbolic system
- It is the S-adic system associated with $\boldsymbol{\sigma}$.

S-adic systems : minimality

- This proof gives more :
- the S-adic system associated with a primitive sequence $\boldsymbol{\sigma}$ is minimal
- Because every factor occurs with bounded gaps.

S-adic systems : unique ergodicity

- One can prove that recurrence implies unique ergodicity
- The generalized eigenvector is the frequency vector.
- This idea is at the heart of the proof of unique ergodicity for almost all interval exchange transformations (Keane's conjecture).

Dumont-Thomas decomposition

- Let U be a finite word occuring in a limit word $u^{(0)}$.
- We can write $U = S_0.\sigma_0(V).P_0$, where S_0 (resp. P_0) is a prefix (resp. suffix) of some $\sigma_0(a)$.
- We can restart this with V, and iterate.
- We obtain the combinatorial Dumont-Thomas decomposition :
- $U = S_0.\sigma_0(S_1).\sigma_{[0,2)}(S_2)\dots\sigma_{[0,k)}(S_kP_k)\dots\sigma_{[0,2)}(P2).\sigma_0(P1).P_0.$
- If our set S of substitutions is finite, there is only a finite number of possibilities for the S_i, P_i.

Exponential convergence and balance

- We are interested by the distance of $I(U)$ to the generalized eigenline.
- This distance is bounded by the sum of the distances of the $I(\sigma_{[0,k)}(S_k P_k))$ to the line.
- Exponential convergence implies that this sum is bounded.
- Hence exponential convergence implies balance.
- This is a weak form of model set.
- We would like to find the window.

Pisot condition and Rauzy fractals

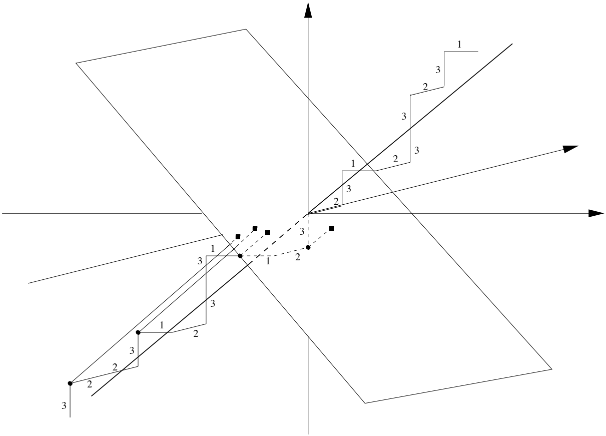
- Consider a Pisot algorithm with a Gauss map and a natural extension $\tilde{T}$.
- For almost every point (u, v) is the domain of $\tilde{T}$, there is an associate sequence of substitutions $(\sigma_n)_{n\in\mathbb{Z}}$
- which is primitive, recurrent, Pisot
- u is totally irrational, and is the frequency vector of the limit word of (σ_n).
- The limit word is balanced ; we can project it on $v^\perp$ along u and take the closure.
- We obtain a set $\mathcal{R}$ which is compact, the closure of its interior, and cover the plane by translation along the projection of the diagonal group (Rauzy fractal) .

Coincidence condition

- We suppose that the set S satisfy the coincidence condition : all limit words have a common letter, with prefixes of same abelianization.
- This condition is not known to be true in general.
- But easy to prove for some algorithms.
- For exemple if they all have the same first letter.
- This implies that the Rauzy fractal splits in disjoint pieces.
- We can define an exchange of pieces.

Pisot conjecture

- We suppose that the Rauzy fractal tiles the plane.
- This is equivalent to say that the limit word is a model set.
- This is a generalization of the Pisot conjecture for substitutions.
- It is true for Brun algorithm.
- In that case, we can define a torus translation.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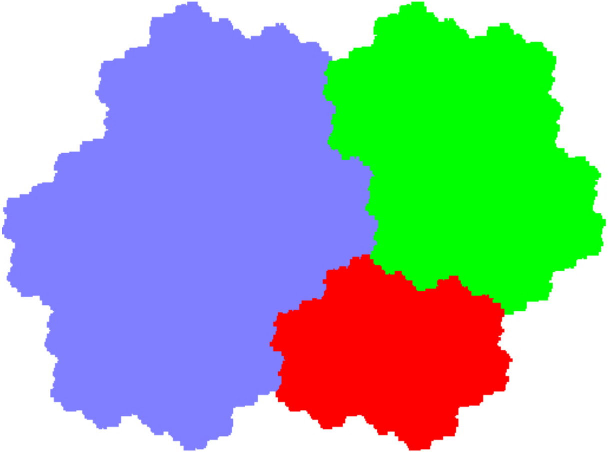
About the cover: abacabaabacababacabaabacabacaba ... (see page ???)

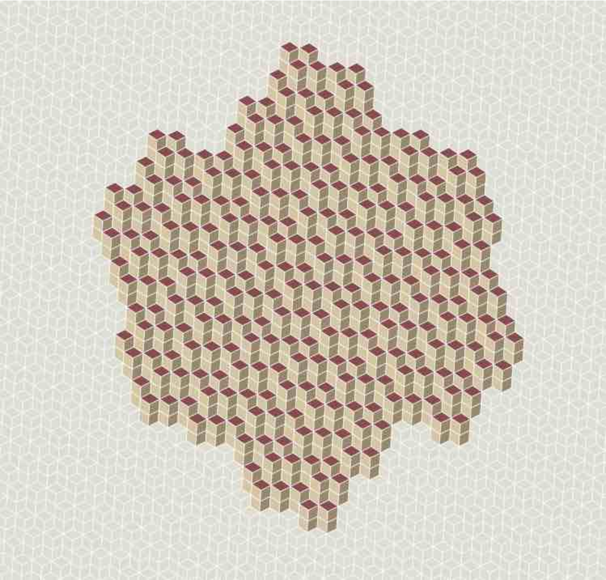
Some properties of the Rauzy fractal

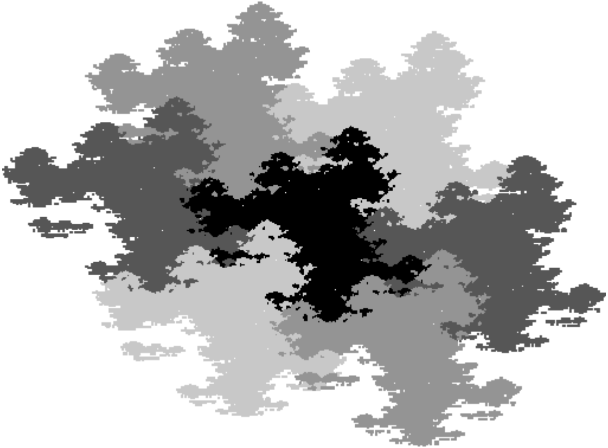
- The Rauzy fractal is compact, by the balance condition.
- Take $\mathbf{w}^* = (1, \dots, 1)$. H is the diagonal plane, and $H \cap \mathbb{Z}^d$ is a lattice Δ in H
- Let L be the *stepped line* associated with the limit point u.
- $\{L + \delta, \delta \in \Delta\}$ is a partition of $\mathbb{Z}^d$.
- Since $\mathbf{w}$ is totally irrational, $\pi(\mathbb{Z}^d)$ is dense in H.
- It follows immediately that $\{\mathcal{R}_{\mathbf{w}^*} + \delta\}$ is a locally finite covering of H.
- Hence $\mathcal{R}_{\mathbf{w}^*}$ contains a fundamental domain for Δ, and has nonempty interior.
- We would like $\mathcal{R}_{\mathbf{w}^*}$ to be a fundamental domain !

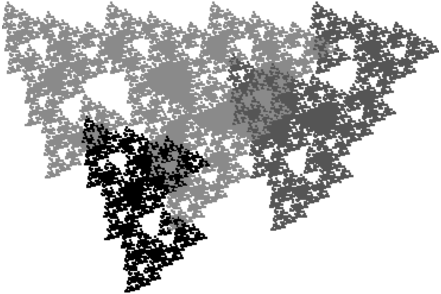
One simple consequence

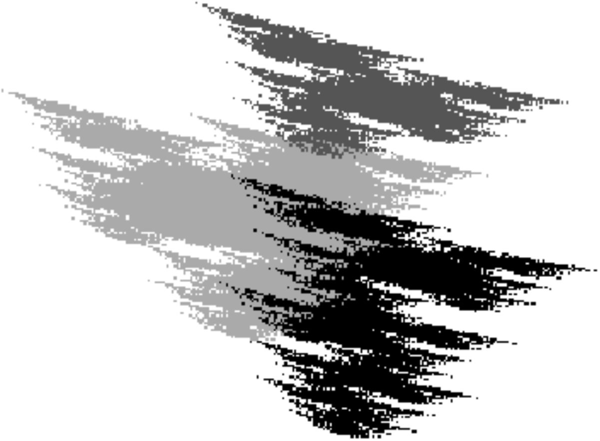
- The symbolic dynamical system is a finite extension of a toral translation.
- Indeed, on H/Δ, all $\pi(\mathbf{e}_a)$ project, by construction, to the same vector.
- but the difference between two consecutive vertices of the stepped line is some $\mathbf{e}_a$.
- Hence the symbolic dynamical system has a large equicontinuous factor.

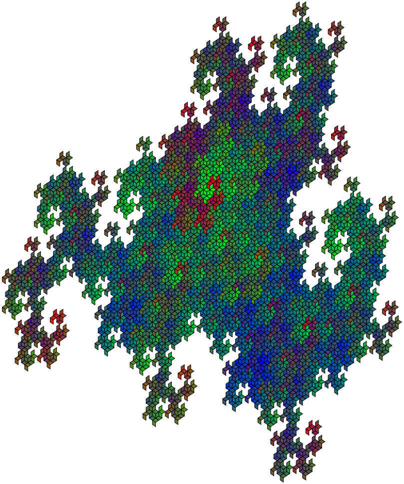












Dragon de Rauzy

The basic equation

- Let $\mathcal{V}^{(n)} := \{\mathbf{l}(P) : P \text{ is a prefix of } u^{(n)}\}$.
- Let $\mathcal{V}^{(n)}(a) = \{\mathbf{l}(P) : Pa \text{ is a prefix of } u^{(n)}\}$.
- We have a combinatorial decomposition :
- $$\mathcal{V}^{(n)}(a) = \coprod_{(P,a,S):\, \sigma_{[n,k)}(b)=PaS} \mathbf{l}(P) + M_{[n,k)}\mathcal{V}^{(k)}(b)$$
- This decomposition immediately translates by projection to a (nonstationary) IFS for the Rauzy fractal.

Coincidence conditions

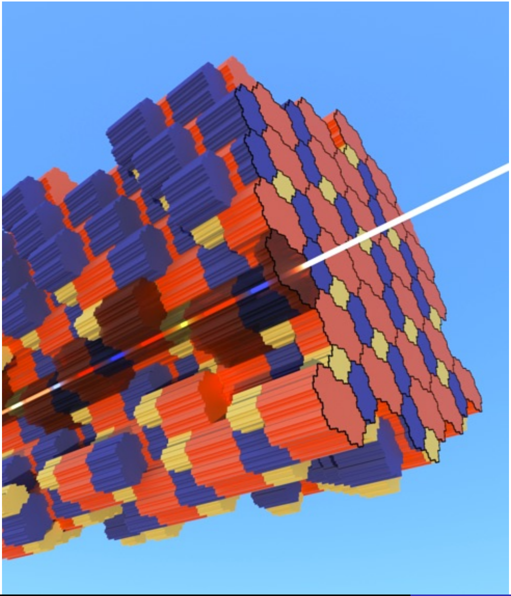
- $\boldsymbol{\sigma}$ has a *coincidence* for $k \in \mathbb{Z}$ if there exist $\ell > k$ and $b \in \mathcal{A}$ such that, for each letter $a \in \mathcal{A}$, $\sigma_{[k,\ell)}(a) = p_a b s_a$ with either same prefix abelianization $\mathbf{l}(p_a)$ or same suffix abelianization $\mathbf{l}(s_a)$ for all $a \in \mathcal{A}$.
- $\boldsymbol{\sigma}$ satisfies the *strong coincidence condition* if it has a coincidence for every $k \in \mathbb{Z}$.
- In that case, all the components of the Rauzy fractal are disjoint.
- The symbolic dynamical system is conjugate to an exchange of pieces.
- The components of the stepped line, seen as a model set, project to disjoint parts of the window.

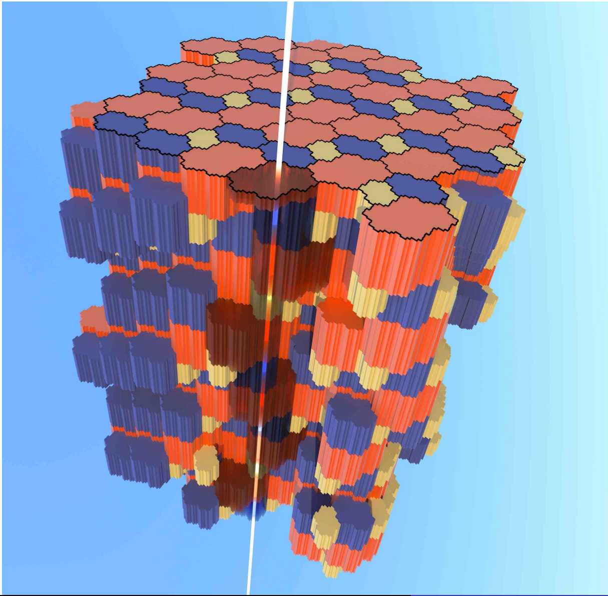
The Pisot conjecture

- Is it really a model set ?
- Does the Rauzy fractal tile the diagonal plane ?
- Hard problem in general, even in the periodic case : Pisot conjecture.
- We can prove it in special cases,
- For example for the Brun substitutions (delicate combinatorics).

The dual space

- The eigenvector is defined by the future.
- The projection plane can be defined arbitrarily for a given future.
- But an arbitrary choice will not work for all paths :
- We should chose a left eigenvector.
- In that case, this defines a dual configuration, on which we can project the basis vectors.
- This allows us to define a product set, with basis the Rauzy fractal, and height the projection of the basis vectors.
- One can prove that in this way we define a fundamental domain of $\mathbb{Z}^d \backslash \mathbb{R}^d$.
- If the Pisot conjecture is satisfied.





Some examples

- Are there non trivial examples ?
- A first case is that of a periodic sequence, powers of a Pisot matrix ;
- in that case, we have terminating algorithms to decide everything,
- but we do not know if the answer is always positive.
- Are there nonstationary examples ?

Some examples : Brun and AR algorithms

- Oseledets theorem tells us that, for a "good" cocycle, Lyapunov exponents are almost everywhere defined (and constant)
- We can compute θ_2 in some cases.
- Avila-Delecroix proved that, for Brun and AR algorithms, $\theta_2 < 0$
- by showing that we can modify the matrices along the eigenvector to obtain stochastic matrices with the same non-dominant eigenvectors : this proves that all other coefficient must be negative.
- These algorithms provide a set of full measure of sequences of substitutions satisfying all the required properties.
- In this case, we have completely realised the goal given at the beginning.

A dual viewpoint

- take a basis $(\mathbf{u}_1, \ldots, \mathbf{u}_{d-1})$ of the projection plane, with $\mathbf{w} = \mathbf{u}_d$.
- Consider $\mathbb{Z}^d$ as a lattice in this base.
- We can associate to it a matrix B.
- We consider the group G which preserve u_d and the measure on the horizontal plane (projection plane).
- The natural extension can be seen as a section of a flow acting on the homogeneous space
- $SL(d, \mathbb{Z}) \backslash SL(d, R) / G$

The Weyl chamber flow

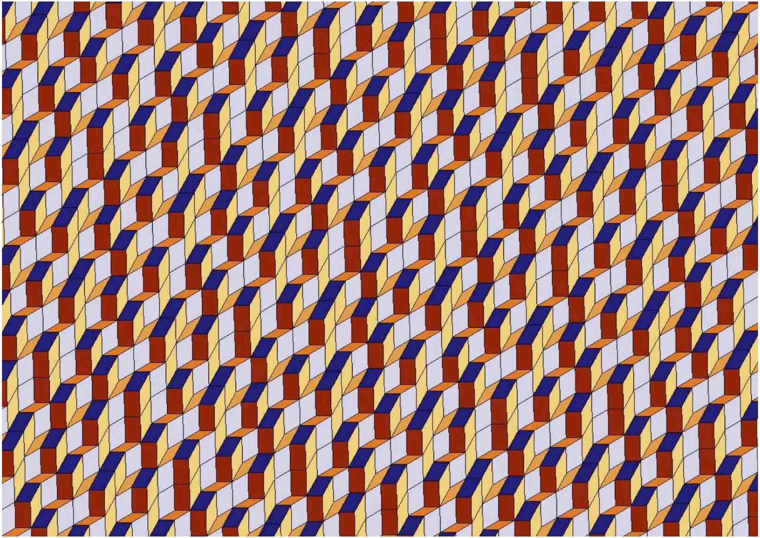
- This is a variant of the Weyl chamber flow, given by the action of the diagonal group on the set of lattices $SL(d,\mathbb{Z})\backslash SL(d,R)$.
- A small extension allows us to give complete symbolic models for this flow.

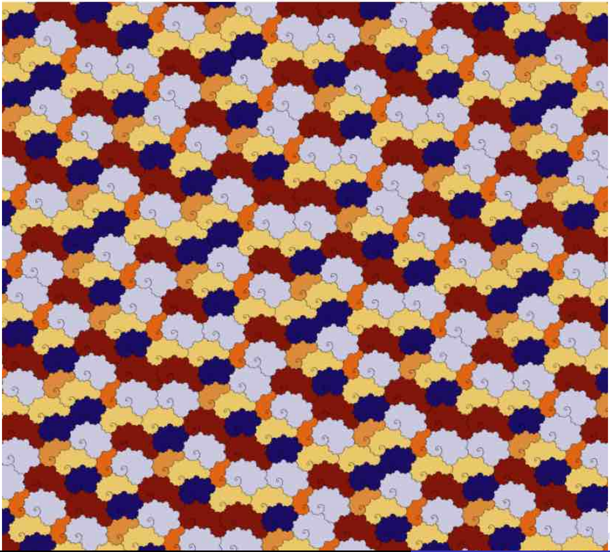
Some consequences

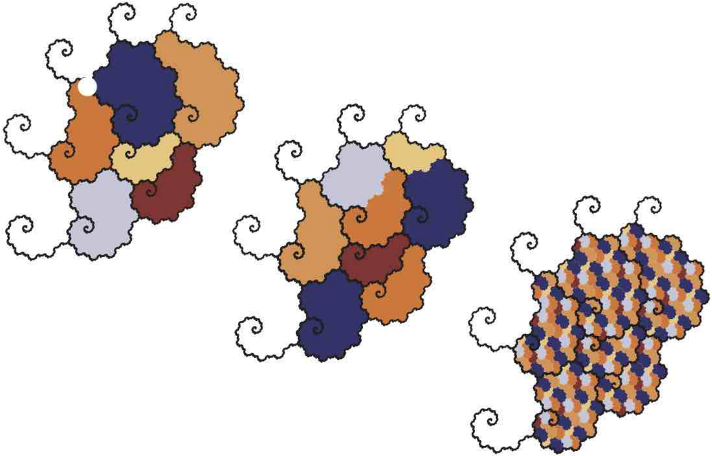
- A large number of information can be retrieved from this flow.
- In particular, periodic orbits
- correspond to toral automorphisms and their Markov partitions.

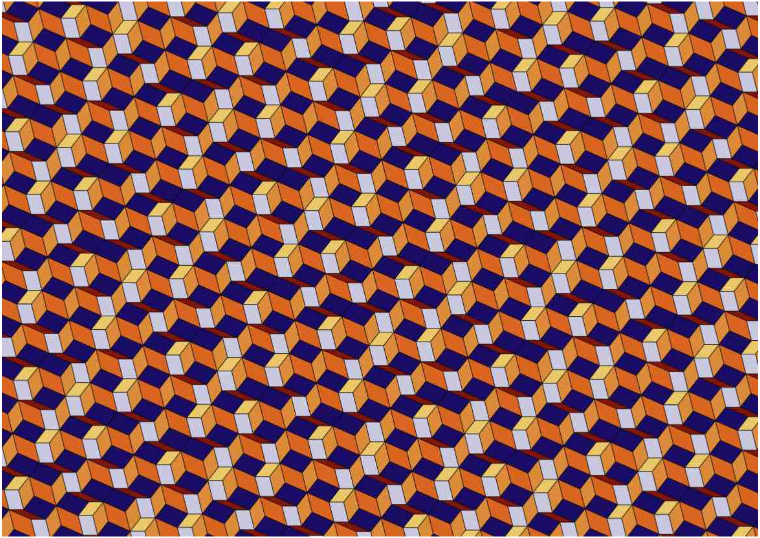
After the Pisot condition ?

- Can we study some non-Pisot cases ?
- This has been done in special cases.
- We can consider hyperbolic automorphisms of the torus,
- and define generalizations of the Rauzy fractal.
- replacing the stepped line by a stepped surface.
- We have to consider free group automorphisms with special properties,
- and extend the notion of substitution to higher dimension.
- We are far from being able to study the nonstationary case...









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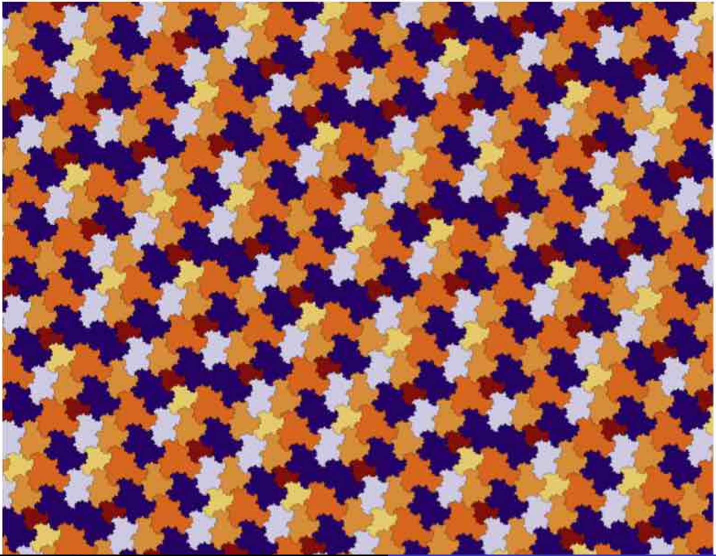

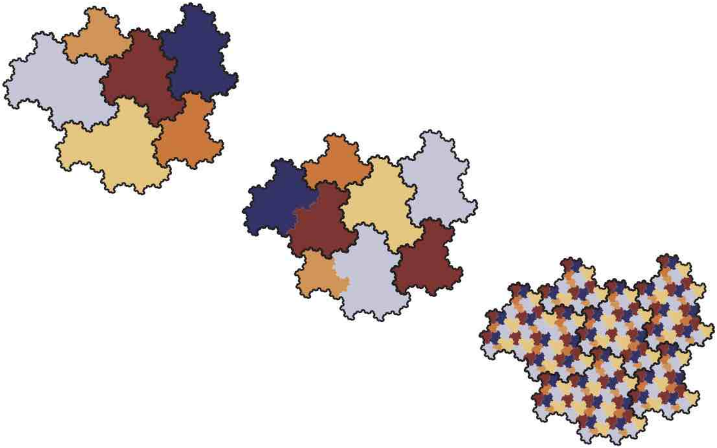

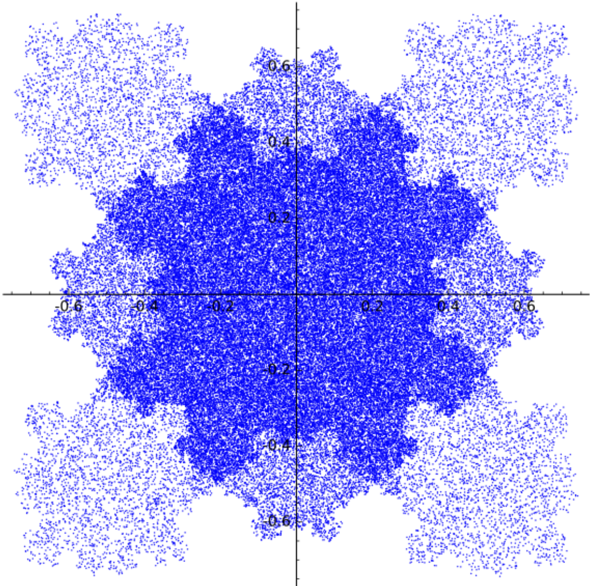
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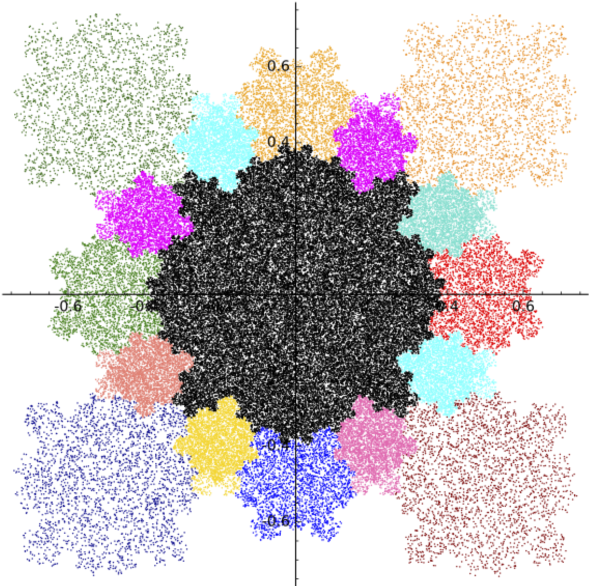
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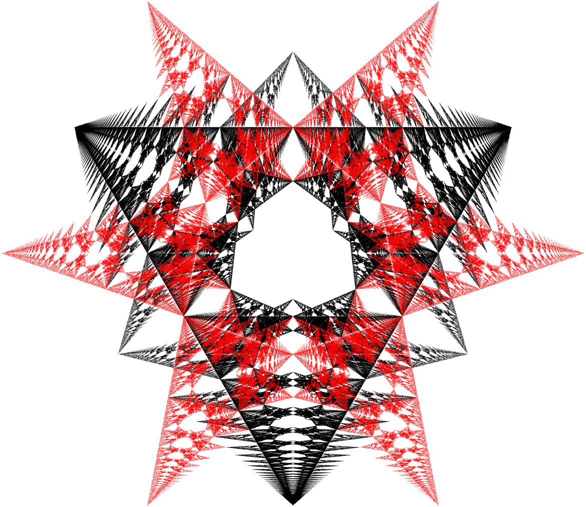

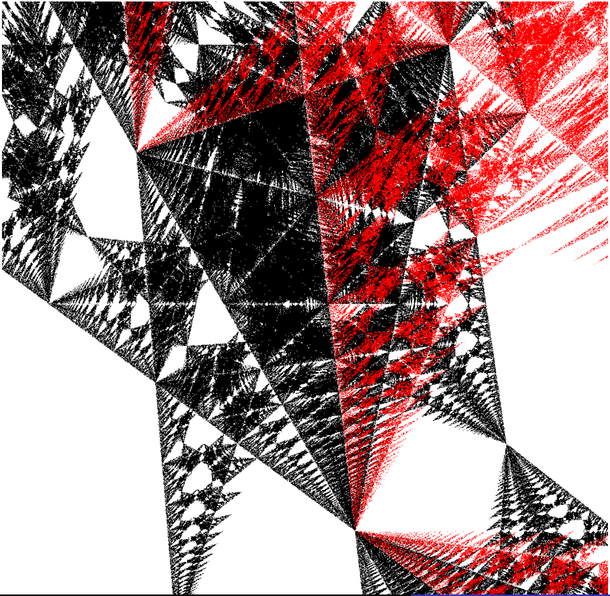

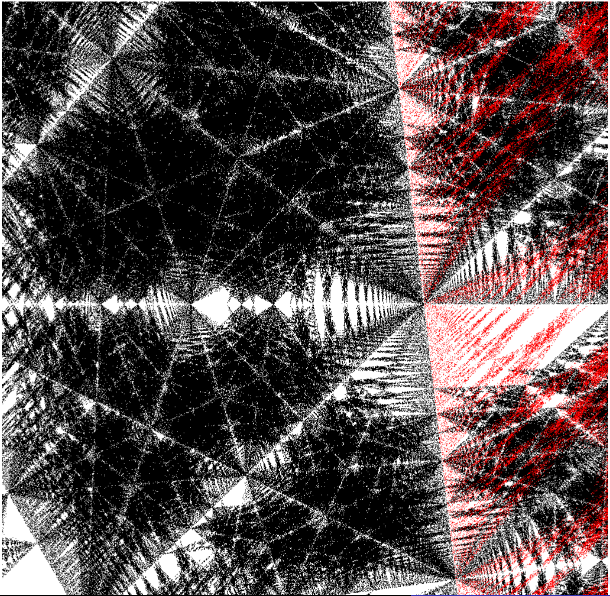

Une lettre de Gauss à Legendre

Soit M une quantité inconnue entre les limites 0 et 1, pour laquelle toutes les valeurs sont ou egalement probable ou plus ou moins selon une loi donnée : qu'on la suppose convertie en une fraction continue

$$M = \cfrac{1}{a' + \cfrac{1}{a'' + \text{etc}}}$$

Quelle est la probabilité, qu'en s'arretant dans le developpement à un terme fini, $a^{(n)}$, la fraction suivante

$$\cfrac{1}{a^{(n+1)} + \cfrac{1}{a^{(n+2)} + \text{etc}}}$$

soit entre les limites 0 et x ?

Une lettre de Gauss à Legendre

Pour les cas où n est plus grand, la valeur exacte de $P(n,x)$ semble intraitable. Cependant j'ai trouvé par des raisonnements tres simples que pour n infini on a

$$P(n,x) = \frac{\log(1+x)}{\log 2}$$

Mais les efforts que j'ai fait lors de mes recherches pour assigner

$$P(n,x) - \frac{\log(1+x)}{\log 2}$$

pour une valeur tres grande de n, mais pas infinie, ont été infructueux.