Bitte beachten (wurde in meinen damaligen VO klar kommuniziert): Der gesamte Analysis-Zyklus ist stark an die (2005-2008 verfügbaren Ausgaben der) Bücher von Forster angelehnt, mit Zusätzen aus Heuser und Rudin im dritten Semester.

Vorlesungsskriptum zu EINFÜHRUNG IN DIE ANALYSIS

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CONTENTS

§0. REVIEW OF REAL AND COMPLEX NUMBERS

In this section we briefly recall some of the basic definitions and properties concerning the sets of real and complex numbers (cf. [SS07]). This material is part of the prerequisites and is presented here without proofs. It merely should remind the reader of facts used in due course without further mentioning and serves to fix some notation.

0.1. Field axioms ⟨*Körperaxiome*⟩ for $\mathbb{R}$:

Axioms of addition

(A1) Law of associativity ⟨*Assoziativgesetz*⟩: for all $x, y, z \in \mathbb{R}$

$$(x + y) + z = x + (y + z).$$

(A2) Law of commutativity ⟨*Kommutativgesetz*⟩: for all $x, y \in \mathbb{R}$

$$x + y = y + x.$$

(A3) Existence of zero: there exists a number $0 \in \mathbb{R}$ such that

$$x + 0 = x \qquad \forall x \in \mathbb{R}.$$

(A4) Existence of the additive inverse: for each $x \in \mathbb{R}$ there exists a number $-x \in \mathbb{R}$ such that

$$x + (-x) = 0.$$

Axioms of multiplication

(M1) Law of associativity: for all $x, y, z \in \mathbb{R}$

$$(xy)z = x(yz).$$

(M2) Law of commutativity: for all $x, y \in \mathbb{R}$

$$xy = yx.$$

(M3) Existence of a unit: there exists a number $1 \in \mathbb{R}$ such that

$$x \cdot 1 = x \qquad \forall x \in \mathbb{R}.$$

(M4) Existence of the multiplicative inverse: for each $x \in \mathbb{R}$ with $x \neq 0$ there exists a number $x^{-1} \in \mathbb{R}$ such that

$$xx^{-1} = 1.$$

Distributive law ⟨*Distributivgesetz*⟩

(D) For all $x, y, z \in \mathbb{R}$

$$x(y + z) = xy + xz.$$

Recall some of the immediate consequences of the above axioms: the uniqueness of the zero element $0 \in \mathbb{R}$, the unit $1 \in \mathbb{R}$, as well as of the additive and multiplicative inverses $-x$ and x^{-1} for any nonzero element $x \in \mathbb{R}$. Furthermore, there are no zero divisors ⟨*Nullteiler*⟩ in $\mathbb{R}$, i.e., for all $x, y \in \mathbb{R} \setminus \{0\}$ we have $xy \neq 0$. Integer powers x^n $(n \in \mathbb{N})$ of a real number x are always well-defined — as the n-fold product $x \cdots x$; finite sums and products are well-defined and obey extended versions of the commutative and distributive law, thus leading to standard notation involving the symbols $\sum$ and $\prod$ as, for example, in

$$\sum_{j=1}^{n} a_j \qquad \text{and} \qquad \prod_{j=1}^{n} a_j \qquad a_1, \ldots, a_n \in \mathbb{R}.$$

0.2. The complex number field $\mathbb{C}$: The field axioms for $\mathbb{C}$ are of course precisely the same with $\mathbb{R}$ replaced by $\mathbb{C}$ everywhere in the statements. (All axioms listed above are just instances of the abstract field axioms for the set $\mathbb{R}$.) Alternatively, we may identify $\mathbb{C}$ as a set with $\mathbb{R} \times \mathbb{R}$ equipped with operations of addition and multiplication for ordered pairs $(x_j, y_j) \in \mathbb{R}^2$ $(j = 1, 2)$ as follows:

$$\begin{aligned} (x_1, y_1) + (x_2, y_2) &:= (x_1 + x_2, y_1 + y_2) \\ (x_1, y_1) \cdot (x_2, y_2) &:= (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1). \end{aligned}$$

Then we can prove that the field axioms are satisfied with $(0, 0)$ as the zero element and $(1, 0)$ as the unit. Recall that the element $i := (0, 1)$ has the property that $i^2 = (-1, 0)$ and that every complex number z can be written as

$$z = x + iy,$$

where x and y are uniquely determined real numbers. We call x the *real part* ⟨*Realteil*⟩ and y the *imaginary part* ⟨*Imaginärteil*⟩ of z, denoted by $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$.

As an additional structure on $\mathbb{C}$ there is *complex conjugation* ⟨*komplexe Konjugation*⟩ $^{-}: \mathbb{C} \to \mathbb{C}$, defined by $z = x + iy \mapsto \bar{z} := x - iy$. As a map complex conjugation is a field automorphism ⟨*Körperautomorphismus*⟩ (i.e., isomorphism onto itself) with the additional property of involution ⟨*Involution*⟩, that is $\bar{\bar{z}} = z$ holds for all $z \in \mathbb{C}$.

Note that by embedding $\mathbb{R}$ as a subfield into $\mathbb{C}$, $x \mapsto x + i0$, we may identify $0 \in \mathbb{R}$ with $0 \in \mathbb{C}$ as well as $1 \in \mathbb{R}$ with $1 \in \mathbb{C}$.

0.3. $\mathbb{R}$ as an ordered field: There exists a relation $>$ on $\mathbb{R}$ with the following properties:

(O1) Trichotomy ⟨*Trichotomie*⟩: for each $x \in \mathbb{R}$ precisely one of the following holds

$$x > 0 \quad or \quad x = 0 \quad or \quad -x > 0.$$

Elements x satisfying $x > 0$ are called *positive* ⟨*positiv*⟩. If $-x > 0$ we also write $x < 0$ and call x *negative* ⟨*negativ*⟩. If x non-positive we thus have $x = 0$ or $x < 0$, which we summarize by writing $x \leq 0$. Similarly, we use the notation $x \geq 0$ for a non-negative element $x \in \mathbb{R}$.

(O2) For all $x, y \in \mathbb{R}$: If $x > 0$ and $y > 0$ then $x + y > 0$.

(O3) For all $x, y \in \mathbb{R}$: If $x > 0$ and $y > 0$ then $xy > 0$.

Definition: For $x, y \in \mathbb{R}$ we henceforth write

$$\begin{aligned}
x > y &\qquad \text{if } x - y > 0, \\
x < y &\qquad \text{if } y > x, \\
x \geq y &\qquad \text{if } x > y \text{ or } x = y, \\
x \leq y &\qquad \text{if } y - x \geq 0.
\end{aligned}$$

From (O1) we obtain that for $x, y \in \mathbb{R}$ precisely one of the relations $x < y$, $x = y$, or $x > y$ holds. (Thus the *maximum* and the *minimum* of two real numbers is well-defined.) We list a few more simple consequences for elements $x, y, a \in \mathbb{R}$:

transitivity ⟨*Transitivität*⟩: $x < y$ and $y < z$ implies $x < z$
if $x < y$ then $x + a < y + a$
if $x < y$ and $a > 0$ then $xa < ya$
$x < y \Longleftrightarrow -x > -y$
if $x \neq 0$ then $x^2 > 0$
$y > x > 0 \Longleftrightarrow x^{-1} > y^{-1} > 0$.

Recall that an *interval* ⟨*Intervall*⟩ in $\mathbb{R}$ is a set which is of any of the following types: let $a, b \in \mathbb{R}$ with $a < b$ and define
the open bounded interval $]a, b[:= \{x \in \mathbb{R} : a < x < b\}$,
the open half-bounded intervals $]-\infty, b[:= \{x \in \mathbb{R} : x < b\}$, $]a, \infty[:= \{x \in \mathbb{R} : a < x\}$,
the half-open intervals $]a, b] := \{x \in \mathbb{R} : a < x \leq b\}$, $[a, b[:= \{x \in \mathbb{R} : a \leq x < b\}$,
the closed half-bounded intervals $]-\infty, b] := \{x \in \mathbb{R} : x \leq b\}$, $[a, \infty[:= \{x \in \mathbb{R} : a \leq x\}$,
the closed bounded interval $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$,
and $]-\infty, \infty[= \mathbb{R}$.

0.4. Absolute value in $\mathbb{R}$: For a real number x we define its *absolute value* ⟨*Absolutbetrag oder Betrag*⟩ by

$$|x| := \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

The basic properties of the map $|\ |\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto |x|$ are

(i) for all $x \in \mathbb{R}$ we have $|x| \geq 0$ and $|x| = 0 \iff x = 0$.

(ii) for all $x, y \in \mathbb{R}$: $|xy| = |x| \cdot |y|$

(iii) Triangle inequality ⟨*Dreiecksungleichung*⟩: $|x + y| \leq |x| + |y|$ for all $x, y \in \mathbb{R}$.

Simple consequences are $|-x| = |x|$, $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ whenever $y \neq 0$, the so-called reverse triangle inequality

$$|x - y| \geq |x| - |y|$$

(or equivalently, $|x+y| \geq |x| - |y|$), and the following formulae for maximum and minimum

$$\max(x, y) = \frac{x + y + |x - y|}{2} \qquad \min(x, y) = \frac{x + y - |x - y|}{2}.$$

0.5. Countable and uncountable sets in $\mathbb{R}$: Recall that a set X is *countable* ⟨*abzählbar*⟩ if there exists a bijective map $F\colon X \to \mathbb{N}$. For example, $\mathbb{N}$, $\mathbb{N} \times \mathbb{N}$, and $\mathbb{Q}$ are countable sets, but $\mathbb{R}$ is not. Thus $\mathbb{R}$ is an uncountable set. In fact, any interval $]a, b[$ with $a < b$ is uncountable. (cf. [SS07, 4.4].)

0.6. Order completeness ⟨*Ordnungsvollständigkeit*⟩ and uniqueness of $\mathbb{R}$:

Axiom of order completeness: Any non-empty subset of $\mathbb{R}$ which is bounded above (resp. below) possesses a supremum (resp. infimum).

THEOREM (DEDEKIND): Up to isomorphism (of ordered fields) $\mathbb{R}$ is the unique order complete ordered field with the rational numbers $\mathbb{Q}$ as an ordered subfield[1].

0.7. Some consequences of the order completeness of $\mathbb{R}$: Based on the order completeness one can prove the following important properties of the set of real numbers (cf. [SS07]):

THE ARCHIMEDIAN PROPERTY: If[2] $a > 0$ and $y \in \mathbb{R}$ then there exists $n \in \mathbb{N}$ such that $na > y$.

DENSITY OF $\mathbb{Q}$ AND $\mathbb{R} \setminus \mathbb{Q}$ IN $\mathbb{R}$: If $x, y \in \mathbb{R}$ with $x < y$, then there exists a rational number $q \in \mathbb{Q}$ such that $x < q < y$ and an irrational number $s \in \mathbb{R} \setminus \mathbb{Q}$ such that $x < s < y$.

EXISTENCE OF ROOTS: If $a \in \mathbb{R}$ and $a > 0$, then for all $n \in \mathbb{N}$ there exists a unique $x \in \mathbb{R}$ such that $x^n = a$.

[1] Richard Dedekind (1831–1916) [ˈrɪçaʁt ˈdeːdəkɪnt], German mathematician

[2] Archimedes (287–212 B.C.) (ὁ Ἀρχιμήδης), one of the greatest ancient Greek mathematicians, physicists, engineers ...

CHAPTER I

SEQUENCES, SERIES, AND SUBSETS OF $\mathbb{R}$

§1. $\mathbb{N}$ AS A SUBSET OF $\mathbb{R}$ AND SOME CONSEQUENCES OF THE ARCHIMEDIAN PROPERTY

1.1. $\mathbb{N}$ as a subset of $\mathbb{R}$:

Here we briefly discuss how the natural numbers can be characterized as a subset of the real number field by means of the following notion.

Definition: A subset $X \subseteq \mathbb{R}$ is said to be *inductive* if

(i) $0 \in X$

(ii) $x \in X \Longrightarrow x + 1 \in X$.

Theorem: There exists a smallest inductive subset $\mathcal{N} \subseteq \mathbb{R}$. The set $\mathcal{N}$ together with the *successor map* $S\colon \mathcal{N} \to \mathbb{R}$, $n \mapsto n + 1$ satisfies *Peano's axioms*[1], that is

(PA1) $0 \in \mathcal{N}$

(PA2) $\forall n \in \mathcal{N}$: $S(n) \in \mathcal{N}$, i.e. $S(\mathcal{N}) \subseteq \mathcal{N}$

(PA3) $\nexists n \in \mathcal{N}$: $S(n) = 0$, i.e. $0 \notin S(\mathcal{N})$

(PA4) S is injective, i.e. $\forall n, m \in \mathcal{N}$: $S(n) = S(m) \Longrightarrow n = m$

(PA5) *Induction axiom:* if $\mathcal{M} \subseteq \mathcal{N}$ is inductive (as a subset of $\mathbb{R}$) then $\mathcal{M} = \mathcal{N}$.

Proof. Define $\mathcal{N}$ to be the intersection of all inductive subsets of $\mathbb{R}$. By construction $\mathcal{N}$ is contained in every inductive subset of $\mathbb{R}$, hence it is the smallest such set.

Furthermore, $\mathcal{N}$ is an inductive subset of $\mathbb{R}$, since both defining properties of inductive sets are preserved under intersection. Thus (PA1) and (PA2) are immediate.

(PA3): The set $\mathcal{P} := \{x \in \mathcal{N} : x \geq 0\}$ is inductive, since $0 \in \mathcal{P}$ and $x \geq 0$ implies $x + 1 \geq 0$. Therefore $\mathcal{P}$ must contain the smallest inductive set $\mathcal{N}$ (in fact, this yields $\mathcal{P} = \mathcal{N}$). In other words, all elements of $\mathcal{N}$ are non-negative. If $n \in \mathcal{N}$ with $S(n) = 0$ then $n + 1 = 0$. Hence $n = -1 < 0$ and n belongs to $\mathcal{N}$ — a contradiction $\lightning$.

(PA4) follows from the field axioms (in $\mathbb{R}$), since $n + 1 = m + 1$ implies $n = m$.

[1] Giuseppe Peano (1858–1932) [dʒu'seppe pe'aːno], Italian mathematician

(PA5): $\mathcal{M}$ is an inductive set and must therefore contain the smallest inductive set $\mathcal{N}$, which implies $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{M}$. Hence $\mathcal{M} = \mathcal{N}$. □

By the uniqueness of the set theoretic construction of the natural numbers $\mathbb{N}$ the set $\mathcal{N}$ is isomorphic to $\mathbb{N}$ (cf. [SS07]). Thus we obtain a model of $\mathbb{N}$ inside $\mathbb{R}$ in the sense of the following statement.

Corollary: We may identify $\mathbb{N}$ with $\mathcal{N} \subseteq \mathbb{R}$ and can henceforth consider $\mathbb{N}$ as a subset of $\mathbb{R}$.

Remark: In contrast with the totally ordered field $\mathbb{R}$ its subset $\mathbb{N}$ (as an ordered set with the order inherited from that on $\mathbb{R}$) is *well-ordered*, i.e., any non-empty subset of $\mathbb{N}$ possesses a minimum. To see this, let $\emptyset \neq A \subseteq \mathbb{N}$. If A is a finite subset, the minimum clearly exists (and can be found after finitely many comparisons of the elements in A). If A is not finite, choose $a \in A$ arbitrary and define $B := \{x \in A : x \leq a\}$, $C := A \setminus B$. Then $A = B \cup C$, every element in C is greater than any element in B, and B is finite, thus has a minimum. By construction, the minimum of B is the minimum of A as well.

The following results are simple, but important, consequences of the Archimedian property.

1.2. THEOREM:

(i) For all $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$ there exists $n \in \mathbb{N}$, $n \geq 1$, such that $\dfrac{1}{n} < \varepsilon$.

(ii) Let $r \in \mathbb{R}$ with $r \geq 0$. If for all $n \in \mathbb{N}$, $n \geq 1$, the inequality $r < \dfrac{1}{n}$ holds, then $r = 0$.

Proof. (i): Application of the Archimedian property (with $a = \varepsilon$ and $y = 1$) gives that $\exists n \in \mathbb{N}$ such that $n\varepsilon > 1$ (note that this implies $n > 0$). Hence we obtain $\varepsilon > 1/n$.

(ii): Let $r \geq 0$. If $r > 0$ then (i) implies the existence of $m \in \mathbb{N}$, $m \geq 1$, such that $0 < 1/m < r$ — a contradiction ↯ . Thus only $r = 0$ is possible. □

1.3. LEMMA: Let $x \in \mathbb{R}$, $x \geq -1$. Then we have

$$\forall n \in \mathbb{N}: \quad (1+x)^n \geq 1 + nx \qquad \textit{Bernoulli's inequality}^2.$$

[2] The Bernoulli family [ˈbɛrnʊli] was a family of Swiss mathematicians in the 17th and 18th century, who originally came from Holland.

Proof. By induction on n.

If $n = 0$ we clearly have $(1 + x)^0 = 1 \geq 1 + 0 \cdot x$.

Suppose the statement holds for n. Since $1 + x \geq 0$ we conclude

$$\begin{aligned}(1+x)^{n+1} = (1+x)^n(1+x) \geq (1+nx)(1+x) &= 1 + nx + x + nx^2 \\ &= 1 + (n+1)x + \underbrace{nx^2}_{\geq 0} \geq 1 + (n+1)x.\end{aligned}$$

□

1.4. PROPOSITION: Let $b \in \mathbb{R}$.

(i) If $b > 1$ then $\forall K \in \mathbb{R}\ \exists n \in \mathbb{N}$: $b^n > K$.

(ii) If $0 < b < 1$ then $\forall \varepsilon > 0\ \exists n \in \mathbb{N}$: $b^n < \varepsilon$.

Proof. (i): Let $x := b - 1$, then $x > 0$. For all $m \in \mathbb{N}$ Bernoulli's inequality gives

$$b^m = (1+x)^m \geq 1 + mx.$$

Let $K \in \mathbb{R}$. By the Archimedian property there is some $n \in \mathbb{N}$ such that $nx > K - 1$. Therefore using $m = n$ in the above inequality we obtain

$$b^n \geq 1 + nx > 1 + K - 1 = K.$$

(ii): Put $b_1 := 1/b$, then $b_1 > 1$. Let $\varepsilon > 0$. Then (i) can be applied to b_1 with $K := 1/\varepsilon$, i.e. there is some $n \in \mathbb{N}$ such that $b_1^n > K$. Thus we have

$$b^n = \frac{1}{b_1^n} < \frac{1}{K} = \varepsilon.$$

□

1.5. Geometric sums: Let $n \in \mathbb{N}$. We define the function $s_n \colon \mathbb{R} \to \mathbb{R}$ by

$$s_n(x) := \sum_{k=0}^{n} x^k = 1 + x + x^2 + \ldots + x^n \qquad \forall x \in \mathbb{R}.$$

If $x = 1$ we obtain

$$s_n(1) = \sum_{k=0}^{n} 1 = n + 1.$$

If $x \neq 1$ we take the differences on both sides of the following equations

$$\begin{aligned} s_n(x) &= 1 + x + x^2 + \ldots + x^{n-1} + x^n \\ x\, s_n(x) &= x + x^2 + x^3 + \ldots + x^n + x^{n+1} \end{aligned}$$

to obtain

$$\underbrace{s_n(x) - x s_n(x)}_{(1-x)s_n(x)} = 1 - x^{n+1},$$

which in turn yields the following formula

$$\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x} \qquad (n \in \mathbb{N}, x \in \mathbb{R} \setminus \{1\}). \tag{1.1}$$

Observation: We may rewrite (1.1) as

$$s_n(x) = \sum_{k=0}^{n} x^k = \frac{1}{1-x} - \frac{x^{n+1}}{1-x},$$

where the first term of the difference on the right-hand side is independent of n.

If $|x| < 1$ the absolute value of the second term of the above difference has numerator ⟨*Zähler*⟩ $|x|^{n+1}$, which becomes arbitrarily small when n gets large. To be more precise, for every $\varepsilon_1 > 0$ Proposition 1.4.(ii) guarantees that there exists an $N \in \mathbb{N}$ such that $|x|^N < \varepsilon_1$. In fact, the latter inequality then holds for all $n \in \mathbb{N}$ with $n \geq N$ as well: $|x|^n < \varepsilon_1$. Therefore we obtain that

$$\left| \frac{x^{n+1}}{1-x} \right| = \frac{|x|^{n+1}}{1-x} < \frac{\varepsilon_1}{1-x} \qquad \forall n \geq N.$$

Let $\varepsilon > 0$ be arbitrary and put $\varepsilon_1 := \varepsilon(1-x)$. Then the above inequality implies that

$$\left| s_n(x) - \frac{1}{1-x} \right| < \varepsilon \qquad \forall n \in \mathbb{N}, n \geq N.$$

Thus we see that for arbitrary fixed x with $|x| < 1$ the sum $s_n(x)$ is approximately equal to $1/(1-x)$ as n gets large, in the sense that the error can be made smaller than any given positive "tolerance" as soon as n is larger than an appropriately chosen "number of steps in the computation".

§2. SEQUENCES AND LIMITS

2.1. DEFINITION: A *sequence* ⟨*Folge*⟩ of real numbers is a map $a\colon \mathbb{N} \to \mathbb{R}$. Thus for every $n \in \mathbb{N}$ a number $a(n) \in \mathbb{R}$ is given.

We usually write sequences in indexed form, i.e. we set $a_n := a(n)$ and denote the sequence by $(a_n)_{n\in\mathbb{N}}$ or $(a_0, a_1, a_2, \ldots)$ or simply (a_n). Occasionally we will encounter index sets other than $\mathbb{N}$, as for example in the sequence $(a_k, a_{k+1}, \ldots)$ starting with index k, also denoted by $(a_n)_{n\geq k}$; sometimes k will be allowed to be a negative integer as well. (In fact, any countable set can serve as index set as long as it is totally ordered.)

2.2. EXAMPLES:

1) Let $c \in \mathbb{R}$ and $a_n = c$ for all $n \in \mathbb{N}$. This gives the constant sequence $(c, c, \ldots) = (c)_{n\in\mathbb{N}}$.

2) Let $a_n = \dfrac{1}{n}$ for all $n \in \mathbb{N}$ with $n \geq 1$. Then we get $\left(\dfrac{1}{n}\right) = \left(1, \dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{4}, \ldots\right)$.

3) $a_n = (-1)^n$ $(n \in \mathbb{N})$ defines $(a_n) = (1, -1, 1, -1, \ldots)$.

4) $a_n = \dfrac{n}{n+1}$ gives the sequence $\left(\dfrac{n}{n+1}\right) = \left(0, \dfrac{1}{2}, \dfrac{2}{3}, \ldots\right)$.

5) $a_n = \dfrac{n}{2^n}$ describes the sequence $\left(\dfrac{n}{2^n}\right) = \left(0, \dfrac{1}{2}, \dfrac{2}{4}, \dfrac{3}{8}, \ldots\right)$.

6) The *Fibonacci*[1] *numbers:* Let $f_0 = 0$ and $f_1 = 1$. If $n \geq 2$ define f_n inductively by

$$f_n = f_{n-1} + f_{n-2}.$$

Thus we obtain $(f_n) = (0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots)$.

7) Let $x \in \mathbb{R}$ and put $a_n = x^n$. Then we obtain the sequence $(a_n) = (1, x, x^2, x^3, \ldots)$.

8) Let $x \in \mathbb{R}$ and define $s_n = \sum_{k=0}^{n} x^k$ $(= \sum_{k=0}^{n} a_k$ with a_k as in 7)). This gives the sequence of geometric sums $(s_n) = (1, 1 + x, 1 + x + x^2, \ldots)$.

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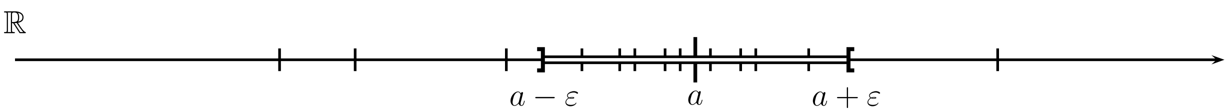
[1] Fibonacci [fiboˈnattʃi], actually Leonardo Pisano (1170–1250 [?]) [leoˈnardo piˈsaːno], an Italian mathematician, invented this series to solve a problem according to the breeding of rabbits.

With the following notion the subject of mathematical analysis really gets started.

2.3. DEFINITION: Let (a_n) be a sequence and $a \in \mathbb{R}$. The sequence (a_n) is said to be *convergent* ⟨*konvergent*⟩ to a if the following holds:

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ such that } \quad |a_n - a| < \varepsilon \quad \forall n \geq N.$$



In this case a is called the *limit*[2] ⟨*Grenzwert oder Limes*⟩ of the sequence (a_n). In symbols we describe this fact by $a = \lim\limits_{n\to\infty} a_n$, briefly $a = \lim a_n$, or $a_n \to a$ as $n \to \infty$, also $a_n \to a$ $(n \to \infty)$.

Equivalently, a is the limit of (a_n) if

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ such that } \quad a_n \in \,]a - \varepsilon, a + \varepsilon[\quad \forall n \geq N.$$

We then also say that the members a_n of the sequence eventually or finally lie in the interval $]a - \varepsilon, a + \varepsilon[$ or the property $|a - a_n| < \varepsilon$ holds for almost all $n \in \mathbb{N}$, i.e. for all but finitely many n (namely at most those with $n < N$) the statement is true.

If a sequence (a_n) is not convergent (to any $a \in \mathbb{R}$) it is said to be *divergent* ⟨*divergent*⟩.

A sequence converging to 0, i.e. $a_n \to 0$ as $n \to \infty$, is called a *zero sequence* or *null sequence* ⟨*Nullfolge*⟩.

Note: In the conditions stated above $\varepsilon > 0$ is "given arbitrarily" and our task in showing convergence is to find $N \in \mathbb{N}$ (which in general will depend on ε!) such that for all $n \geq N$ the sequence element a_n belongs to the *ε-neighborhood* ⟨*ε-Umgebung*⟩ $U_\varepsilon(a) :=]a - \varepsilon, a + \varepsilon[$ of $a \in \mathbb{R}$.

Furthermore, the property of convergence as well as the value of the limit remains unchanged upon alteration or dropping of finitely many members of a sequence.

2.4. Examples:

1) The constant sequence $(a_n) = (c)$ is convergent to c: for given $\varepsilon > 0$ choose $N = 0$, then $|a_n - c| = 0 < \varepsilon$ for all $n \geq N$.

[2] The term "limit" (from the Latin limes, literally "border") was first used by Isaac Newton.

2) $\left(\frac{1}{n}\right)$ is a null sequence: let $\varepsilon > 0$; by the Archimedian property there exists $N \in \mathbb{N}$ such that $N > 1/\varepsilon$; then we have for $n \geq N$ that $\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$.

3) $((-1)^n)$ is divergent, which we prove by contradiction ⟨*indirekt*⟩: suppose there is $a \in \mathbb{R}$ with $(-1)^n \to a$ $(n \to \infty)$; let $\varepsilon := 1/2$ and choose $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|a_n - a| < \varepsilon = 1/2$. Note that $|a_{n+1} - a_n| = |(-1)^{n+1} - (-1)^n| = |(-1)^n(-1-1)| = 2$. Hence we obtain

$$2 = |a_{n+1} - a_n| = |(a_{n+1} - a) + (a - a_n)| \underset{\text{[triangle inequ.]}}{\leq} |a_{n+1} - a| + |a - a_n| < \frac{1}{2} + \frac{1}{2} = 1$$

— a contradiction ↯ .

4) $\lim\limits_{n\to\infty} \frac{n}{n+1} = 1$, i.e. $\left(\frac{n}{n+1}\right)$ is convergent to 1: let $\varepsilon > 0$; choose $N \in \mathbb{N}$ such that $N > 1/\varepsilon$, then we have for all $n \geq N$

$$\left|\frac{n}{n+1} - 1\right| = \left|\frac{n-(n+1)}{n+1}\right| = \frac{1}{n+1} \leq \frac{1}{N} < \varepsilon.$$

5) $\lim\limits_{n\to\infty} \frac{n}{2^n} = 0$: as a lemma we state that

$$\forall n \geq 4\colon\ n^2 \leq 2^n \qquad \text{[proof by induction, exercise!].}$$

Using the above result we have $\frac{n}{2^n} \leq \frac{1}{n}$ if $n \geq 4$; let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $N \geq \max(4, 2/\varepsilon)$, then we have for $n \geq N$

$$\left|\frac{n}{2^n} - 0\right| = \frac{n}{2^n} \leq \frac{1}{n} \leq \frac{1}{N} \leq \frac{\varepsilon}{2} < \varepsilon.$$

2.5. Definition: A sequence (a_n) is said to be *bounded (from) above* ⟨*nach oben beschränkt*⟩ (resp. *bounded (from) below* ⟨*nach unten beschränkt*⟩), if there exists $K \in \mathbb{R}$ such that for all $n \in \mathbb{N}$ we have $a_n \leq K$ (resp. $a_n \geq K$). The sequence is ⟨*bounded*⟩ ⟨*beschränkt*⟩ if it is bounded above and below.

Thus we have the following equivalence:

$$(2.1) \qquad (a_n) \text{ is bounded} \quad \Longleftrightarrow \quad \exists K > 0\ \forall n \in \mathbb{N} : |a_n| \leq K.$$

Note that the constant K is independent of n.

2.6. Proposition: A convergent sequence is bounded.

Proof. Let $a = \lim a_n$ and choose $N \in N$ such that $|a_n - a| < 1$ holds when $n \geq N$. Then for all $n \geq N$

$$|a_n| = |a_n - a + a| \leq |a_n - a| + |a| < 1 + |a|.$$

Put $K := \max(|a_0|, |a_1|, \ldots, |a_{N-1}|, |a| + 1)$, then $|a_n| \leq K$ follows for all $n \in \mathbb{N}$. □

2.7. Remark: The converse statement of the above proposition is wrong. For example, the sequence $((-1)^n)$ is bounded (since $|a_n| = 1$ for all n) but not convergent [cf. Example 2.4.3)].

2.8. Examples:

1) The sequence (f_n) of Fibonacci numbers [cf. Example 2.2.6)] is divergent.

Proof. We assert that $\forall n \geq 5$: $f_n \geq n$.

Indeed, we have $f_5 = 5$ and then for $n \geq 6$ inductively

$$f_{n+1} = f_n + f_{n-1} \underset{\text{[ind. hyp.]}}{\geq} n + (n-1) \geq n + (2-1) = n + 1.$$

The above assertion now implies that the sequence (f_n) is unbounded [Archimedian property of $\mathbb{R}$] and hence cannot be convergent [by the negation of Proposition 2.6]. □

2) Let $x \in \mathbb{R}$ and consider the sequence $(x^n)_{n \in \mathbb{N}}$. The convergence properties depend on the value of x:

Case $|x| > 1$: Proposition 1.4.(i) implies that for every $K \in \mathbb{R}$ there is an $N \in \mathbb{N}$ such that $\forall n \geq N$ we have $|x|^n > K$. Therefore (x^n) is not bounded, hence cannot be convergent.

Case $|x| = 1$: If $x = -1$ then $(x^n) = ((-1)^n)$ is not convergent [cf. Example 2.4.3)]; if $x = 1$ then $(x^n) = (1)_{n \in \mathbb{N}}$ is a constant sequence and thus convergent to 1.

Case $|x| < 1$: If $x = 0$ then $(x^n)_{n \geq 1} = (0)_{n \geq 1}$ has limit 0.
Finally, if $0 < |x| < 1$ we can show that (x^n) also converges to 0: Let $\varepsilon > 0$. By Proposition 1.4.(ii) there is an $N \in \mathbb{N}$ such that $|x|^n < \varepsilon$ holds for all $n \geq N$. In other words,

$$|x^n - 0| = |x^n| = |x|^n < \varepsilon \qquad \forall n \geq N,$$

which proves that $\lim\limits_{n \to \infty} x^n = 0$.

2.9. Proposition (Uniqueness of the limit): If a sequence (a_n) converges to $a \in \mathbb{R}$ and to $b \in \mathbb{R}$, then $a = b$.

(In particular, the notation $\lim a_n = a$ is justified by this statement.)

Proof. (By contradiction.) Suppose $a \neq b$ and put $\varepsilon := \dfrac{|a-b|}{3}$. Then $\varepsilon > 0$ and we observe the following:

$\lim a_n = a \implies \exists N_1$ such that $\forall n \geq N_1$: $|a_n - a| < \varepsilon$

as well as

$\lim a_n = b \implies \exists N_2$ such that $\forall n \geq N_2$: $|a_n - b| < \varepsilon$.

Thus for $n \geq N := \max(N_1, N_2)$ both of the above inequalities are valid and yield

$$|a-b| = |a - a_n + a_n - b| \leq |a - a_n| + |a_n - b| < 2\varepsilon = \frac{2}{3}|a-b|.$$

By assumption, we have $|a-b| > 0$ and therefore obtain $1 < 2/3$ — a contradiction ↯ . $\square$

2.10. Basic operations with convergent sequences:

(i) **Sum and product:** If (a_n), (b_n) are convergent, then the sequences $(a_n + b_n)$ and $(a_n \cdot b_n)$ are also convergent and

$$\lim_{n\to\infty}(a_n + b_n) = \left(\lim_{n\to\infty} a_n\right) + \left(\lim_{n\to\infty} b_n\right) \tag{2.2}$$

$$\lim_{n\to\infty}(a_n b_n) = \left(\lim_{n\to\infty} a_n\right) \cdot \left(\lim_{n\to\infty} b_n\right). \tag{2.3}$$

Proof. Let $a := \lim a_n$ and $b := \lim b_n$.

Sum: We have to show that $a_n + b_n \to a + b \ \ (n \to \infty)$.
Let $\varepsilon > 0$. Then $\varepsilon/2 > 0$ as well, thus we have

$\lim a_n = a \implies \exists N_1$ such that $\forall n \geq N_1$: $|a_n - a| < \varepsilon/2$
and
$\lim b_n = b \implies \exists N_2$ such that $\forall n \geq N_2$: $|b_n - b| < \varepsilon/2$.

Therefore we obtain for $n \geq N := \max(N_1, N_2)$

$$|(a_n + b_n) - (a+b)| = |(a_n - a) + (b_n - b)| \leq |(a_n - a)| + |(b_n - b)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Product: We have to show that $a_n b_n \to ab \ \ (n \to \infty)$.
Since (a_n) is convergent it is bounded, thus $\exists K_1 > 0$ such that $|a_n| \leq K_1$ for all $n \in \mathbb{N}$.
Define $K := \max(K_1, |b|)$, then $K > 0$.

Let $\varepsilon > 0$. Using $\dfrac{\varepsilon}{2K}$ in place of ε in the defining property of convergence for (a_n) and (b_n) we obtain
$\exists M_1 \in \mathbb{N}$ such that $|a_n - a| < \varepsilon/(2K)$ for all $n \geq M_1$ as well as
$\exists M_2 \in \mathbb{N}$ such that $|b_n - b| < \varepsilon/(2K)$ for all $n \geq M_2$.

Therefore we have for all $n \geq M := \max(M_1, M_2)$

$$\begin{aligned}|a_n b_n - ab| = |a_n b_n - a_n b + a_n b - ab| &= |a_n(b_n - b) + (a_n - a)b| \\ &\leq |a_n||b_n - b| + |a_n - a||b| < K\frac{\varepsilon}{2K} + \frac{\varepsilon}{2K}K = \varepsilon.\end{aligned}$$

□

(ii) **Linearity of the limit:** As a corollary to (i) we obtain: If $\lambda, \mu \in \mathbb{R}$ and (a_n), (b_n) are convergent, then the linear combination $\lambda \cdot (a_n)_{n\in\mathbb{N}} + \mu \cdot (b_n)_{n\in\mathbb{N}} := (\lambda a_n + \mu b_n)_{n\in\mathbb{N}}$ converges as well and

$$\lim_{n\to\infty} (\lambda a_n + \mu b_n) = \lambda \lim_{n\to\infty} a_n + \mu \lim_{n\to\infty} b_n.$$

Proof. Let $a := \lim a_n$ and $b := \lim b_n$. The constant sequence (λ), resp. (μ), converges to λ, resp. μ, hence by (i) $\lambda a_n \to \lambda a$ and $\mu b_n \to \mu b$; furthermore, again by (i), we have the sum rule $(\lambda a_n) + (\mu b_n) \to \lambda a + \mu b$. □

(iii) **Quotient:** If (a_n), (b_n) are convergent and $b := \lim b_n$ is nonzero, then there exists $n_0 \in \mathbb{N}$ such that $b_n \neq 0$ for all $n \geq n_0$ and the sequence $\left(\dfrac{a_n}{b_n}\right)_{n\geq n_0}$ converges with limit

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim a_n}{\lim b_n}. \tag{2.4}$$

Proof. Let $a := \lim a_n$.

We first show that $\dfrac{1}{b_n} \to \dfrac{1}{b}$ $(n \to \infty)$.

Put $\varepsilon' := |b|/2$ and note that $\varepsilon' > 0$. By convergence of (b_n) to b, there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have

$$\frac{|b|}{2} = \varepsilon' > |b_n - b| \underset{\text{[reverse tri. inequ.]}}{\geq} |b| - |b_n|.$$

The outermost inequality implies $|b_n| > |b|/2$ and thus $b_n \neq 0$ for all $n \geq n_0$.

Let $\varepsilon > 0$, then, again by the basic convergence estimate of (b_n) applied to $\varepsilon'' := |b|^2\varepsilon/2 > 0$, we obtain the existence of $N_1 \in \mathbb{N}$ such that

$$|b_n - b| < \varepsilon'' = \frac{|b|^2\varepsilon}{2} \qquad \forall n \geq N_1.$$

We conclude that for $n \geq N := \max(n_0, N_1)$

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{b - b_n}{b_n b}\right| = \frac{1}{|b_n|}\frac{1}{|b|}|b - b_n| \leq \frac{2}{|b|}\frac{1}{|b|}\frac{|b|^2\varepsilon}{2} = \varepsilon.$$

Finally, the product rule (i) gives $\dfrac{a_n}{b_n} = a_n \cdot \dfrac{1}{b_n} \to a \cdot \dfrac{1}{b} = \dfrac{a}{b}$. $\square$

(iv) Example: Let $a_n = \dfrac{3n^2 + 13n}{n^2 + 2}$ $(n \in \mathbb{N})$.

We rewrite a_n (upon division of the numerator and the denominator by the highest order term with respect to n) in the form

$$a_n = \frac{3 + \frac{13}{n}}{1 + \frac{2}{n^2}}$$

and observe that repeated use of (i) and (ii) yields $3 + \dfrac{13}{n} = 3 + 13 \cdot \dfrac{1}{n} \to 3 + 13 \cdot 0 = 3$ and
$1 + \dfrac{2}{n^2} = 1 + 2 \cdot \dfrac{1}{n} \cdot \dfrac{1}{n} \to 1 + 2 \cdot 0 \cdot 0 = 1$, hence by the quotient rule (iii)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n^2 + 13n}{n^2 + 2} = \lim_{n\to\infty} \frac{3 + \frac{13}{n}}{1 + \frac{2}{n^2}} = \frac{\lim 3 + \frac{13}{n}}{\lim 1 + \frac{2}{n^2}} = \frac{3}{1} = 3.$$

(v) If (a_n), (b_n) are convergent and $a_n \leq b_n$ for almost all n (i.e. for all $n \geq n_0$), then

$$\lim a_n \leq \lim b_n.$$

Proof. Put $c_n := b_n - a_n$, then $c_n \geq 0$ for almost all n and $c := \lim c_n = \lim b_n - \lim a_n$ by (ii). Thus it suffices to show that $c \geq 0$, since this implies $\lim a_n \leq \lim b_n$.

Suppose the contrary, that is $c < 0$. Let $\varepsilon := -c$, then $\varepsilon > 0$ and there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$\varepsilon > |c_n - c| = |c_n - (-\varepsilon)| = |c_n + \varepsilon| \underset{[c_n \geq 0, \varepsilon > 0]}{=} c_n + \varepsilon.$$

But this implies that for all $n \geq N$ we would have $c_n < 0$ — a contradiction ↯ . $\square$

(vi) **The Sandwich Lemma:** If (a_n), (b_n), (c_n) are sequences such that $a_n \leq b_n \leq c_n$ holds for almost all n and $a_n \to a$, $c_n \to a$ $(n \to \infty)$, then also (b_n) is convergent and $\lim b_n = a$.

Proof. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that for all $n \geq N$ the inequalities $|a_n - a| < \varepsilon$ and $|c_n - a| < \varepsilon$ hold. Then

$$a - \varepsilon < a_n \leq b_n \leq c_n < a + \varepsilon \qquad \forall n \geq N$$

and upon subtracting a we obtain

$$-\varepsilon < b_n - a < \varepsilon \qquad \forall n \geq N.$$

In other words, the inequality $|b_n - a| < \varepsilon$ holds for all $n \geq N$, thus (b_n) converges to a. □

(vii) Example: Consider the sequence (b_n) given by

$$b_n := \sum_{k=n+1}^{2n} \frac{1}{k^2} = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \ldots + \frac{1}{(2n)^2}.$$

Note that $n + 1 \leq k \leq 2n$ implies $\dfrac{1}{k^2} < \dfrac{1}{n^2}$ and therefore

$$0 < b_n < \underbrace{\frac{1}{n^2} + \frac{1}{n^2} + \ldots + \frac{1}{n^2}}_{n \text{ terms}} \leq n \cdot \frac{1}{n^2} = \frac{1}{n} \to 0 \quad (n \to \infty).$$

Hence we may apply the Sandwich lemma with $a_n = 0$, $c_n = 1/n$ and conclude that (b_n) is convergent with limit 0.

Remark: Note that the above example shows that strict inequalities $a_n < b_n$ for all n do (in general) not imply a strict inequality for the respective limits, i.e. $a_n < b_n \not\Rightarrow \lim a_n < \lim b_n$, but of course $\lim a_n \leq \lim b_n$.

2.11. Series: Many sequences (s_m) in applications occur through summation over the first m members of a given sequence (a_n) of real numbers.

Definition: The sequence $(s_m)_{m \in \mathbb{N}}$ of *partial sums* ⟨Partialsummen⟩ is defined by

$$s_m := \sum_{k=0}^{m} a_k = a_0 + a_1 + \ldots + a_m \qquad (m \in \mathbb{N})$$

and is called (infinite) *series* ⟨*(unendliche) Reihe*⟩, usually denoted by $\sum_{k=0}^{\infty} a_k$. If (s_m) is convergent, then $\lim s_m$ is called the *sum* of the series and we write

$$\sum_{k=0}^{\infty} a_k = \lim_{m \to \infty} s_m = \lim_{m \to \infty} \sum_{k=0}^{m} a_k.$$

REMARK: (i) We will also consider series with summation starting at $n_0 \in \mathbb{N}$, that is $\sum\limits_{k=n_0}^{\infty} a_k$ with corresponding partial sums $(s_m)_{m \geq n_0}$.

(ii) Every sequence can be interpreted as a series: Let (c_n) be an arbitrary sequence and define $a_k := c_k - c_{k-1}$ $(k \geq 1)$ and $a_0 := c_0$. Then we obtain

$$c_n = \sum_{k=0}^{n} a_k.$$

EXAMPLES: 1) Let $a_k = \dfrac{1}{k(k+1)}$ $(k \geq 1)$, then the corresponding series is $\sum\limits_{k=1}^{\infty} \dfrac{1}{k(k+1)}$.

Observe that $a_k = \dfrac{k}{k+1} - \dfrac{k-1}{k}$, hence

$$s_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \left(\frac{k}{k+1} - \frac{k-1}{k} \right) =$$
$$\left(\frac{1}{1+1} - \frac{1-1}{1} \right) + \left(\frac{2}{2+1} - \frac{2-1}{2} \right) + \ldots + \left(\frac{n-1}{n} - \frac{n-2}{n-1} \right) + \left(\frac{n}{n+1} - \frac{n-1}{n} \right) = \frac{n}{n+1}$$

is convergent to 1, that is $\sum\limits_{k=1}^{\infty} \dfrac{1}{k(k+1)} = \lim\limits_{n \to \infty} s_n = 1$.

2) *The geometric series:* Let $x \in \mathbb{R}$ and consider $\sum\limits_{k=0}^{\infty} x^k$.

Recall that we have already determined the values of the partial sums (s_m) in 1.5, where we have obtained

$$s_m = m + 1 \quad \text{if } x = 1, \qquad s_m = \frac{1 - x^{m+1}}{1 - x} \quad \text{if } x \neq 1.$$

<u>Case $x = 1$:</u> $s_m = m + 1$ is unbounded, hence (s_m) is divergent.

<u>Case $x = -1$:</u> $s_{2n} = \frac{1-(-1)}{2} = 1$ and $s_{2n+1} = \frac{1-(1)}{2} = 0$, hence (s_m) is not convergent (argue as in Example 2.4.3)).

<u>Case $|x| > 1$:</u> $s_m = \frac{1}{1-x} - \frac{x^{m+1}}{1-x}$; as observed in Example 2.8.2), the sequence (x^{m+1}) is unbounded and thus (s_m) as well. Therefore the series is not convergent.

<u>Case $|x| < 1$:</u> $s_m = \frac{1}{1-x} - \frac{x^{m+1}}{1-x}$, where $x^{m+1} \to 0$ [cf. Example 2.8.2)]. Applying the rules for the computation of limits in 2.10 yields $\lim s_m = \frac{1}{1-x}$, that is

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \qquad (|x| < 1).$$

3) As special cases of 2) consider $x = \pm\frac{1}{2}$ in the geometric sum, then we obtain

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \frac{1}{1-\frac{1}{2}} = 2$$
$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}.$$

PROPOSITION: If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are convergent series and $\lambda, \mu \in \mathbb{R}$, then $\sum_{n=0}^{\infty}(\lambda a_n + \mu b_n)$ is convergent and

$$\sum_{n=0}^{\infty}(\lambda a_n + \mu b_n) = \lambda \sum_{n=0}^{\infty} a_n + \mu \sum_{n=0}^{\infty} b_n.$$

Proof. Apply the limit rules 2.10 to the partial sums. $\square$

2.12. Infinite or improper limits:

DEFINITION: A sequence (a_n) is said to have *(improper or) infinite limit* $+\infty$ $\langle$*ist uneigentlich konvergent oder bestimmt divergent gegen* $+\infty\rangle$, if for all $K \in \mathbb{R}$ there is an $N \in \mathbb{N}$ such that $a_n > K$ for all $n \geq N$. We then denote this fact by $\lim_{n\to\infty} a_n = +\infty$.

A sequence (a_n) is said to have *(improper or) infinite limit* $-\infty$, if $(-a_n)$ has infinite limit $+\infty$.

Note that a sequence with infinite limit is necessarily unbounded, namely not bounded above if the improper limit is $+\infty$ and not bounded below if the improper limit is $-\infty$.

EXAMPLES: 1) $\lim_{n\to\infty} n = +\infty$ 2) $\lim_{n\to\infty}(-n^2) = -\infty$

3) Let $a_n = (-1)^n n$ $(n \in \mathbb{N})$, then (a_n) is unbounded hence not convergent. But (a_n) is not improperly convergent either, since $a_{2n} \to +\infty$ and $a_{2n+1} \to -\infty$.

4) Let $a_n = n$ for even n, and $a_n = 0$ for odd n. Then (a_n) is unbounded, divergent, and also not improperly convergent.

5) Some rules for operations with improper limits: if (a_n) is convergent to $a \in \mathbb{R}$ and (b_n), (c_n) have improper limit $+\infty$, then

$$\begin{aligned}
\lim(a_n + b_n) &= \lim(b_n + a_n) = +\infty \\
\lim(c_n + b_n) &= \lim(b_n + c_n) = +\infty \\
\lim(a_n - b_n) &= \lim(-b_n + a_n) = -\infty \\
\text{if } a > 0 \text{ then} \qquad \lim(a_n b_n) &= \lim(b_n a_n) = +\infty \\
\lim(c_n b_n) &= \lim(b_n c_n) = +\infty.
\end{aligned}$$

WARNING: There are no general limit relations for differences of sequences with infinite limit $+\infty$ or products of a zero sequence with an improperly convergent sequence. For example,

$\lim n = +\infty$, $\lim n^2 = +\infty$ and $\lim(n - n) = 0$, but $\lim(n - n^2) = -\infty$
and
$\lim \frac{1}{n} = 0$, $\lim \frac{1}{n^2} = 0$ and $\lim \left(n \cdot \frac{1}{n}\right) = 1$, but $\lim \left(n \cdot \frac{1}{n^2}\right) = 0$.

PROPOSITION: (i) Let (a_n) have infinite limit $+\infty$ or $-\infty$, then there exists $n_0 \in \mathbb{N}$ such that $\left(\dfrac{1}{a_n}\right)_{n \geq n_0}$ is well-defined and $\lim \dfrac{1}{a_n} = 0$.

(ii) If (a_n) is a zero sequence with $a_n > 0$ (resp. $a_n < 0$) for all $n \in \mathbb{N}$, then $\lim \dfrac{1}{a_n} = +\infty$ (resp. $\lim \dfrac{1}{a_n} = -\infty$)

Proof. (i) It suffices to consider the case $\lim a_n = +\infty$. If we put $K = 0$, then by definition there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $a_n > K = 0$. Thus $\left(\dfrac{1}{a_n}\right)_{n \geq n_0}$ is well defined. Note that in addition $\frac{1}{a_n} > 0$ $(n \geq n_0)$.

Let $\varepsilon > 0$. Putting $K = 1/\varepsilon$ we obtain some $N \in \mathbb{N}$ such that $a_n > K = 1/\varepsilon$ whenever $n \geq N$. Therefore we have for $n \geq \max(n_0, N)$ the inequality $0 < \frac{1}{a_n} < \varepsilon$, which shows convergence of $(\frac{1}{a_n})$ to 0.

(ii) is left as an exercise. $\square$

EXAMPLE: $\lim \frac{n}{2^n} = 0$ [cf. Example 2.4.5)] and $\frac{n}{2^n} > 0$, hence (ii) of the above Proposition implies that $\left(\dfrac{2^n}{n}\right)$ has infinite limit $+\infty$.

REMARK: If (a_n), (b_n) satisfy $a_n \leq b_n$ for almost all n and $\lim a_n = +\infty$, then $\lim b_n = +\infty$ follows directly.

§3. COMPLETENESS OF $\mathbb{R}$ AND CONVERGENCE PRINCIPLES

In this section we investigate important consequences of order completeness of $\mathbb{R}$ [cf. Section 0] for sequences of real numbers. We recall the statement of the

3.1. Axiom of order completeness: A non-empty subset of $\mathbb{R}$ which is bounded above (resp. below) possesses a supremum (resp. infimum).

We define the key notions for sequences which allow us to deduce strong methods for convergence tests of sequences.

3.2. Definition: Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.

(i) If $(n_k)_{k\in\mathbb{N}}$ is a sequence of natural numbers (i.e. $n_k \in \mathbb{N}$ for all k) satisfying $n_0 < n_1 < n_2 < \ldots$ (i.e. $n_k < n_{k+1}$ for all k) then the sequence $(a_{n_k})_{k\in\mathbb{N}} = (a_{n_0}, a_{n_1}, a_{n_2}, \ldots)$ is called a *subsequence* ⟨Teilfolge⟩ of the sequence (a_n).

(ii) A real number a is called *cluster point* ⟨Häufungswert⟩ of the sequence (a_n), if there exists a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ converging to a.

3.3. Proposition: A real number a is a cluster point of the real sequence (a_n) if and only if every ε-neighborhood of a contains infinitely many members of (a_n), that is

$$\forall \varepsilon > 0\, \forall N \in \mathbb{N}\; \exists m \geq N : \quad a_m \in U_\varepsilon(a) =]a-\varepsilon, a+\varepsilon[\tag{3.1}$$

(recall that $a_m \in U_\varepsilon(a)$ is equivalent to $|a_m - a| < \varepsilon$).

Proof. • We show the 'only if'-part (i.e. *necessity*): a is cluster point $\Rightarrow$ (3.1).

Let $(a_{n_k})_{k\in\mathbb{N}}$ be a subsequence with $\lim_{k\to\infty} a_{n_k} = a$ and $\varepsilon > 0$ arbitrary. We can choose $k_0 \in \mathbb{N}$ such that $a_{n_k} \in U_\varepsilon(a)$ holds for all $k \geq k_0$.

Let $N \in \mathbb{N}$, then there exists $k_1 \geq k_0$ such that $n_{k_1} \geq N$ [since $\ldots < n_{k-1} < n_k < n_{k+1} < \ldots$ by definition of a subsequence]. Therefore, if we put $m := n_{k_1}$, then $a_m = a_{n_{k_1}} \in U_\varepsilon(a)$ by construction.

• We show the 'if'-part (i.e. *sufficiency*): (3.1) $\Rightarrow$ a is cluster point.

Putting $\varepsilon := 1$ and $N := 0$ we obtain an $n_0 \in \mathbb{N}$ sucht that $a_{n_0} \in U_1(a)$. Then construct a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ with the property

$$\forall k \in \mathbb{N}, k \geq 1,\ \exists n_k \in \mathbb{N} : \quad n_k > n_{k-1} \text{ and } a_{n_k} \in U_{\frac{1}{k+1}}(a)$$

inductively: For the basic case $k = 1$, (3.1) with $\varepsilon := 1/2$ and $N := n_0 + 1$ gives some $n_1 \geq N > n_0$ such that $a_{n_1} \in U_{\frac{1}{2}}(a)$. If $a_{n_1}, \ldots, a_{n_k}$ have been defined, then applying (3.1) with $\varepsilon := 1/(k+2)$ and $N := n_k + 1$ we obtain $n_{k+1} \geq N > n_k$ such that $a_{n_{k+1}} \in U_{\frac{1}{k+2}}(a)$.

We assert that $(a_{n_k})_{k\in\mathbb{N}}$ converges to a: Let $\varepsilon > 0$. Choose $k_0 \in \mathbb{N}$ such that $\frac{1}{k_0+1} < \varepsilon$ [cf. 1.2.(i)], then we have for all $k \geq k_0$ that

$$|a_{n_k} - a| < \frac{1}{k_0 + 1} < \varepsilon.$$

$\square$

3.4. Examples: 1) If $a_n = (-1)^n$ then (a_n) has 1 and -1 as cluster points, since the subsequences $(a_0, a_2, a_4, \ldots)$ (with even index only) and $(a_1, a_3, a_5, \ldots)$ (with odd index only) converge to these values:

$$\lim_{k\to\infty} a_{2k} = \lim 1 = 1 \qquad \lim_{k\to\infty} a_{2k+1} = \lim(-1) = -1.$$

2) $\left((-1)^n + \dfrac{1}{n}\right)_{n\geq 1}$ also has cluster points 1 and -1, since

$$\lim_{k\to\infty} a_{2k} = \lim\left(1 + \frac{1}{2k}\right) = 1+0 = 1 \qquad \lim_{k\to\infty} a_{2k+1} = \lim\left(-1 + \frac{1}{2k+1}\right) = -1+0 = -1.$$

3) (n) has no cluster points, since every subsequence is unbounded, hence divergent.

4) Let $a_n = n$, if n is even, and $a_n = \frac{1}{n}$, if n is odd. The sequence (a_n) is unbounded, since for example $a_{2k} \to \infty$ $(k \to \infty)$. But (a_n) has 0 as cluster point, because the subsequence $(a_{2k+1})_{k\in\mathbb{N}} = \left(\frac{1}{2k+1}\right)$ converges to 0.

3.5. Remark: If (a_n) is a sequence with limit a, then a is the only cluster point of (a_n) [cf. the Exercises].

3.6. Theorem (of Bolzano-Weierstraß[1]): Every bounded sequence of real numbers has a cluster point (that is, possesses at least one cluster point).

Proof. Let (a_n) be a bounded sequence, then there exists $K \in \mathbb{R}$ such that $|a_n| \leq K$ holds for all $n \in \mathbb{N}$. Consider the subset

$$A := \{x \in \mathbb{R} : a_n > x \text{ holds for at most finitely many } n\} \subseteq \mathbb{R}.$$

Since no a_n can be larger than K, we have $K \in A$, thus A is nonempty.

A is bounded from below: If $x < -K$, then $x \notin A$ [since $a_n \geq -K$ for all n]; hence $-K-1$ is a lower bound for A.

By order completeness A has an infimum. Let $a := \inf A$.

Claim: a is a cluster point of (a_n).

Let $\varepsilon > 0$.

• Since $a + \varepsilon > a$ the number $a + \varepsilon$ is not a lower bound of A [property of the infimum a!]. Hence there exists $y \in A$ such $a \leq y < a + \varepsilon$.

By definition of A we have $a_n \leq y < a + \varepsilon$ for almost all n. Therefore we obtain

$$(\star) \qquad \exists n_0 \in \mathbb{N}\ \forall n \geq n_0 : a_n < a + \varepsilon.$$

• Since a is a lower bound of A we have that $a - \varepsilon \notin A$. Hence $a_n > a - \varepsilon$ holds for infinitely many n, that is

$$(\star\star) \qquad \forall N \in \mathbb{N}\ \exists m \geq N : a - \varepsilon < a_m.$$

Combining $(\star)$ and $(\star\star)$ gives (3.1), therefore a is a cluster point of (a_n). □

3.7. Remark: Note that in the above proof we have found the greatest cluster point of a given bounded sequence (a_n). Indeed, if $b > a$ — using the notation as in the proof — we may choose $c \in A$ with $a < c < b$. If $\varepsilon := b - c > 0$ then the ε-neighborhood $U_\varepsilon(b)$ contains only finitely many members a_n. Thus b is not a cluster point of (a_n). Similarly, we can prove that a smallest cluster point exists.

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[1] Bernhard Bolzano (1781–1848) [ˈbeɐnhaʁt bɔlˈtsaːno], German mathematician
Karl Weierstraß (1815–1897) [kaʁl ˈvaɪəɐʃtras], German mathematician, was the one who introduced the letter ε to mathematical analysis.

3.8. Definition: (i) Let (a_n) be a bounded sequence. The greatest cluster point of the sequence is called the *limit superior* ⟨*Limes superior*⟩ of (a_n) and is denoted by $\limsup a_n$ or $\overline{\lim} a_n$. The smallest cluster point of (a_n) is called the *limit inferior* ⟨*Limes inferior*⟩ of (a_n) and is denoted by $\liminf a_n$ or $\underline{\lim} a_n$.

(ii) If (a_n) is not bounded above (resp. below), then we set $\limsup a_n = +\infty$ (resp. $\liminf a_n = -\infty$).

3.9. Examples: 1) Let $a_n = (-1)^n \left(1 + \frac{1}{n}\right)$, then (a_n) has cluster points 1 and -1, thus

$$\limsup (-1)^n \left(1 + \frac{1}{n}\right) = 1, \qquad \liminf (-1)^n \left(1 + \frac{1}{n}\right) = -1.$$

2) (n) has no (real) cluster points and $\limsup n = +\infty$.

In deciding wether a certain sequence is convergent or not the defining property of convergence requires to already be in the possession of a good guess for the value of the prospective limit. Unless the sequence is simple enough to be analyzed directly by means of the basic rules for computation of limits [cf. 2.10] (or is seen to be unbounded), it might be difficult or even hopeless to guess the limit with complicated terms or when the sequence members are not defined by an explicit formula or procedure.

To deal with such situations we strive for the development of methods that allows to decide the question of convergence without having to know a candidate for the limit in advance. In some cases, this then also leads to a successful determination of the limit a posteriori. There are also situations where a reasonable candidate for the limit is easily guessed or the only possible value can be determined (under the assumption that the sequence converges), but direct convergence proofs are inaccessible. The principles of Cauchy[2] and that of monotone bounded sequences [cf. 3.10 and 3.12] are among the most powerful alternative methods to prove convergence of real sequences.

3.10. Cauchy sequences:

Definition: A real sequence (a_n) is a *Cauchy sequence* if

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \quad \forall n \geq N \; \forall m \geq N : \quad |a_n - a_m| < \varepsilon. \tag{3.2}$$

Theorem: Let (a_n) be a real sequence. The following are equivalent:

(i) (a_n) is convergent

(ii) (a_n) is a Cauchy sequence.

[2]Augustin Louis Cauchy (1789–1857) [ogys'tɛ̃ lwi ko'ʃi], French mathematician

REMARK: Note that condition (ii) can be checked without knowing the limit. Moreover, if (ii) fails to hold we may conclude that the sequence is divergent.

Proof. (i) $\Rightarrow$ (ii): Let $a := \lim a_n$ and $\varepsilon > 0$. Choose N such that $|a_n - a| < \varepsilon/2$ holds $\forall n \geq N$. Then we have for all $n, m \geq N$

$$|a_n - a_m| = |(a_n - a) + (a - a_m)| \leq |a_n - a| + |a - a_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(ii) $\Rightarrow$ (i): Let (a_n) be a Cauchy sequence.

Step 1: (a_n) is bounded.

We put $\varepsilon = 1$ in (3.2) and obtain that there is some N such that $n, m \geq N$ implies $|a_n - a_m| < \varepsilon = 1$. In particular, if $m = N$ this means that $|a_n| - |a_N| \leq |a_n - a_N| < 1$, hence $|a_n| \leq 1 + |a_N|$, holds for all $n \geq N$. Since clearly $|a_n| \leq 1 + \max(|a_0|, \ldots, |a_N|) =: K$ for $n = 1, \ldots, N$, we therefore have $|a_n| \leq K$ for all $n \in \mathbb{N}$.

Step 2: By the Theorem of Bolzano-Weierstraß [cf. 3.6] (a_n) has a cluster point $a \in \mathbb{R}$.

Step 3: $a_n \to a$ $(n \to \infty)$

Let $\varepsilon > 0$. Choose N such that $|a_n - a_m| < \varepsilon/2$ for all $n, m \geq N$. Since a is a cluster point there exists $k \geq N$ such that $|a_k - a| < \varepsilon/2$. Combining these facts we obtain that for $n \geq N$

$$|a_n - a| = |(a_n - a_k) + (a_k - a)| \leq |a_n - a_k| + |a_k - a| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

3.11. Example: Let (a_k) be a sequence and suppose there is some $\theta \in]0, 1[$ such that $|a_k| \leq \theta$ for all k. Consider the partial sums $s_n := \sum_{k=0}^{n} a_k{}^k$. If $m < n$ then

$$\begin{aligned} |s_n - s_m| \leq \sum_{k=m+1}^{n} |a_k|^k \leq \sum_{k=m+1}^{n} \theta^k = \sum_{k=0}^{n} \theta^k - \sum_{k=0}^{m} \theta^k = \frac{1-\theta^{n+1}}{1-\theta} - \frac{1-\theta^{m+1}}{1-\theta} \\ = \frac{\theta^{m+1} - \theta^{n+1}}{1-\theta} = \theta^{m+1}\frac{1-\theta^{n-m}}{1-\theta} \leq \theta^{m+1}\frac{1}{1-\theta}. \end{aligned}$$

Let $\varepsilon > 0$. Since $0 \leq \theta < 1$ we can choose N such that $0 \leq \theta^{m+1} < \varepsilon(1-\theta)$ holds for all $m \geq N$. Therefore we have for all $N \leq m < n$ that

$$|s_n - s_m| \leq \theta^{m+1}\frac{1}{1-\theta} < \varepsilon.$$

If $n = m \geq N$ we have $s_n - s_m = 0$ and if $N \leq n \leq m$ the roles of n and m can be interchanged in the above estimates. In summary, we obtain that (s_n) is a Cauchy sequence.

We conclude that the series $\sum_{k=0}^{\infty} a_k{}^k$ is convergent, although we have no clue what its sum might be — and in the general case there is no way to determine the limit.

3.12. Monotone sequences:

Definition: A real sequence (a_n) is *(strictly) increasing* ⟨*(streng) monoton wachsend*⟩ if for all $n \in \mathbb{N}$: $a_n \leq a_{n+1}$ (resp. $a_n < a_{n+1}$). A real sequence (a_n) is *(strictly) decreasing* ⟨*(streng) monoton fallend*⟩ if for all $n \in \mathbb{N}$: $a_n \geq a_{n+1}$ (resp. $a_n > a_{n+1}$). If $n_0 \in \mathbb{N}$ such that $(a_n)_{n \geq n_0}$ is (strictly) increasing (or decreasing) we shall occasionally express this by saying that (a_n) has the corresponding property for $n \geq n_0$.

Remark: A decreasing sequence (a_n) which is bounded below is bounded. Indeed, if C is a lower bound we have for all $n \in \mathbb{N}$

$$C \leq a_n \leq a_{n-1} \leq \cdots \leq a_0.$$

Similarly for increasing sequences which are bounded above.

Example: 1) The Fibonacci sequence (f_n) [cf. Example 2.2.6)] is increasing and strictly increasing for $n \geq 2$. We have $f_0 = 0 < f_1 = 1 = f_2$ and $f_n > 0$ when $n \geq 1$, therefore

$$f_{n+1} = f_n + f_{n-1} > f_n + 0 = f_n \qquad \forall n \geq 2.$$

2) Let $x_0 > 0$ and define the sequence (x_n) recursively by

$$x_{n+1} = \frac{1}{2}(x_n + \frac{3}{x_n}) \qquad (n \in \mathbb{N}).$$

Note that by induction $x_n > 0$ for all n, hence the recursion defines a sequence (x_n).

Claim 1: $\forall n \geq 1$: $x_n^2 \geq 3$

This follows from

$$\begin{aligned} x_{n+1}{}^2 - 3 = \frac{1}{4}(x_n + \frac{3}{x_n})^2 - 3 = \frac{1}{4}(x_n{}^2 + 6 + \frac{9}{x_n{}^2}) - 3 \\ = \frac{1}{4}(x_n{}^2 - 6 + \frac{9}{x_n{}^2}) = \frac{1}{4}(x_n - \frac{3}{x_n})^2 \geq 0. \end{aligned}$$

Claim 2: (x_n) is decreasing for $n \geq 1$

We have for $n \geq 1$

$$x_n - x_{n+1} = x_n - \frac{x_n}{2} - \frac{3}{2x_n} = \frac{1}{2x_n}(x_n^2 - 3),$$

which is nonnegative by claim 1, hence $x_n \geq x_{n+1}$.

The theorem which we prove below will guarantee that the sequence (x_n) is convergent, since it is bounded below (by $\sqrt{3}$) and decreasing for $n \geq 1$. What is the value of the limit $x := \lim x_n$?

In this case we can make use of the recursion relation. Note first that $x \geq \sqrt{3} > 0$ [since $x_n \geq \sqrt{3}$] and then take limits for $n \to \infty$ on both sides of the recursion relation

$$\begin{array}{ccc} x_{n+1} & = & \frac{1}{2}(x_n + \frac{3}{x_n}) \\ \downarrow & & \downarrow \\ x & = & \frac{1}{2}(x + \frac{3}{x}). \end{array}$$

Therefore $x^2 = 3$, that is $x = \sqrt{3}$.

THEOREM: If (a_n) is increasing (for $n \geq n_0$) and bounded above, then (a_n) is convergent. A corresponding statement holds for decreasing sequences that are bounded below.

Proof. Let $A := \{a_n : n \in \mathbb{N}\}$. As noted in a remark above (a_n) is bounded, thus the set A is bounded. Put $a := \sup A$. We show that (a_n) converges to a:

Let $\varepsilon > 0$. Since a is the supremum of A we can find $m \in \mathbb{N}$ such that $a - \varepsilon < a_m \leq a$. By monotonicity we obtain that $a - \varepsilon < a_m \leq a_n \leq a$ holds for every $n \geq m$. Hence $|a - a_n| < \varepsilon$ for every $n \geq m$ which proves that $a_n \to a$.

□

As an application of the preceding theorem we present a useful alternative to describe the limit superior (resp. limit inferior) of an arbitrary bounded sequence as the limit of a specifically constructed decreasing (resp. increasing) sequence.

3.13. Proposition: Let (a_n) be a bounded real sequence, then

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} (\sup\{a_k : k \geq n\})$$

and

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} (\inf\{a_k : k \geq n\}).$$

A corresponding statement holds for unbounded sequences with improper values $+\infty$ or $-\infty$ assigned to sup, inf, lim sup, or lim inf in the appropriate way.

Proof. (For lim sup, the case of lim inf being very similar.)
Let $M_n := \{a_k : k \geq n\}$ and $x_n := \sup M_n$. Since $M_{n+1} \subseteq M_n$ the sequence (x_n) is decreasing.

Furthermore, if $C > 0$ is a bound for (a_n), that is $|a_n| \leq C$ for all n, then it also is one for (x_n), hence (x_n) is bounded too.

By Theorem 3.12 (x_n) is convergent. Let $x := \lim x_n$.

Assertion: x is a cluster point of (a_n).

Let $\varepsilon > 0$.
• We may choose $N \in \mathbb{N}$ such that for all $n \geq N$: $x \leq x_n < x + \varepsilon$. By construction $a_n \leq x_n$, hence

$$\forall n \geq N : \quad a_n < x + \varepsilon.$$

• Since x_n is the supremum of the set $M_n = \{a_k : k \geq n\}$ (and $x_n - \varepsilon < x_n$), we have the following:

$$\forall n \in \mathbb{N}\ \exists m \geq n : a_m > x_n - \varepsilon \geq x - \varepsilon.$$

Combining the two inequalities above we find that for all n there is some $m \geq n$ such that $a_m \in U_\varepsilon(x)$. [Proposition 3.3] then implies that x is a cluster point of (a_n).

Finally, if (a_{n_k}) is an arbitrary subsequence of (a_n), then clearly $a_{n_k} \leq x_{n_k}$ holds for all k. Therefore, if (a_{n_k}) converges we must have

$$\lim_{k\to\infty} a_{n_k} \leq \lim_{k\to\infty} x_{n_k} = x.$$

Thus x is the largest cluster point, i.e. $x = \limsup a_n$. $\square$

3.14. Adherent points and accumulation points of subsets of $\mathbb{R}$:

DEFINITION: Let $A \subseteq \mathbb{R}$ and $a \in \mathbb{R}$.

(i) a is an *adherent point* ⟨*Berührpunkt*⟩ of A if $\forall \varepsilon > 0$ we have that $U_\varepsilon(a) \cap A \neq \emptyset$.
(Every ε-neighborhood of a contains at least one point of A.)

(ii) a is an *accumulation point* ⟨*Häufungspunkt*⟩ of A if

$$\forall \varepsilon > 0 : \quad U_\varepsilon(a) \cap A \quad \text{contains infinitely many points.}$$

BASIC PROPERTIES: Clearly, every accumulation point of A and every element of A is an adherent point of A.

(a) a is an adherent point of $A \iff$ there exists a sequence (a_n) in A: $a_n \to a$ $(n \to \infty)$
$[(a_n)$ in A means $a_n \in A$ for all $n]$

$\boxed{\Rightarrow}$ For $n \in \mathbb{N}$, $n \geq 1$, choose $a_n \in U_{1/n}(a) \cap A$. Then (a_n) is a sequence in A with $a_n \to a$.

$\boxed{\Leftarrow}$ By assumption there exists (a_n) with $a_n \in A$ for all n and satisfying for all $\varepsilon > 0$ that there is some N such that $a_n \in U_\varepsilon(a)$ for all $n \geq N$. In particular, $U_\varepsilon(a) \cap A$ is not empty.

(b) a is an accumulation point of $A \iff a$ is an adherent point of $A \setminus \{a\}$

Proof as an exercise.

EXAMPLES: 1) 0 is an accumulation point of $A = \{\frac{1}{n} : n \in \mathbb{N}, n \geq 1\}$ (but 0 does not belong to A).

2) Let $A = \{1\}$ and $a_n = 1$ for all n. Then 1 is an adherent point of A and a cluster point of (a_n), but it is not an accumulation point of A.
Thus we learn that, in general, a cluster points of a sequence (a_n) need not be an accumulation point of the set $\{a_n : n \in \mathbb{N}\}$.

3) Consider $\mathbb{Q} \subseteq \mathbb{R}$. Then every point $x \in \mathbb{R}$ is an accumulation point of $\mathbb{Q}$. This follows from the density of $\mathbb{Q}$ in $\mathbb{R}$ [cf. 0.7].

4) Let $a < x_0 < b$ and $A =]a, x_0[\cup]x_0, b[$. Every $x \in [a, b]$ is an accumulation point of A.

REMARK: Let A be a bounded subset of $\mathbb{R}$.

(i) By the Theorem of Bolzano-Weierstraß, every sequence (a_n) in A has a cluster point, which is then also an adherent point of A.

(ii) The supremum (resp. infimum) of A is an adherent point of A. Moreover, $\sup A$ (resp. $\inf A$) is the limit of an increasing (resp. decreasing) sequence in A: Let $\alpha := \sup A$, choose $\alpha - 2 < a_0 \leq \alpha$, and construct (a_n) inductively with the property

$$\forall n \in \mathbb{N}, n \geq 1 : \quad a_n \in A, \quad a_n \geq a_{n-1} \quad \text{and } \alpha - \frac{1}{n} \leq a_n \leq \alpha.$$

3.15. Principle of nested intervals ⟨*Intervallschachtelungsprinzip*⟩:

Let $a \le b$ and I be the closed bounded interval $[a,b]$, then

$$\operatorname{diam}(I) := b - a \tag{3.3}$$

is called the *diameter*[3] or *length* of I. Note that

$$\forall x, y \in I : \quad |x - y| \le \operatorname{diam}(I).$$

Theorem: Let I_n ($n \in \mathbb{N}$) be a sequence of closed bounded intervals with the following properties:

(i) $I_0 \supseteq I_1 \supseteq \ldots \supseteq I_n \supseteq I_{n+1} \ldots$

(ii) $\lim_{n\to\infty} \operatorname{diam}(I_n) = 0$.

Then there is a unique point $x \in \mathbb{R}$ which belongs to every I_n ($n \in \mathbb{N}$), i.e.

$$\bigcap_{n\in\mathbb{N}} I_n = \{x\}.$$

Proof. Let $I_n = [a_n, b_n]$ ($n \in \mathbb{N}$), where $a_n \le b_n$. By property (i) we have for all $n, N \in \mathbb{N}$

$$a_0 \le a_1 \le \ldots \le a_n \le a_{n+1} \le b_{N+1} \le b_N \le \ldots \le b_1 \le b_0.$$

Thus (a_n) is a Cauchy sequences, since by property (ii) for any $n, m \ge N$ the right-hand side of

$$0 \le |a_m - a_n| = a_{\max(m,n)} - a_{\min(m,n)} \le b_N - a_N$$

can be made arbitrarily small when N is sufficiently large.

Hence there exists $x \in \mathbb{R}$ such that $x = \lim a_n$.

For all $n \ge k$ we have the inequalities $a_k \le a_n \le b_n \le b_k$.

Sending $n \to \infty$ yields $a_k \le x \le b_k$ for every k, thus $x \in \bigcap_{n\in\mathbb{N}} I_n$.

If z_1 and z_2 are arbitrary points in $\bigcap_{n\in\mathbb{N}} I_n$ then (ii) implies

$$0 \le |z_2 - z_1| \le \operatorname{diam}(I_k) \to 0 \quad (k \to \infty).$$

Thus $\bigcap_{n\in\mathbb{N}} I_n$ contains only a single point which then has to be x.

To summarize, $\bigcap_{n\in\mathbb{N}} I_n = \{x\}$, where x is uniquely determined as the limit of (a_n). □

[3] The word "diameter" was originally derived from the Greek ἡ διάμετρος (literally "measurements") and is now common in most languages.

3.16. Completeness revisited: The current section has been devoted to the derivation of consequences of the Axiom of order completeness 3.1. Along the way, we proved in succession the Theorem of Bolzano-Weierstraß [cf. 3.6], the Cauchy convergence principle [cf. 3.10], and (via the principle of monotone bounded sequences) also the principle of nested intervals [cf. 3.15].

It is an intriguing fact, that in turn the statement of order completeness becomes a provable theorem when any of the other three results is taken as an axiom instead. For example, below we give a proof that the principle of nested intervals implies order completeness.

We summarize this situation in a diagram:

$$\begin{array}{ccc}
\boxed{\text{order completeness}} & \Leftarrow & \boxed{\text{principle of nested intervals}} \\
\Downarrow & & \Uparrow \\
\boxed{\text{Bolzano-Weierstraß theorem}} & \Rightarrow & \boxed{\text{Cauchy principle}}
\end{array}$$

Theorem: The principle of nested intervals implies order completeness.

Proof. Let A be a bounded and nonempty subset of $\mathbb{R}$. We will construct a nested sequence of intervals $I_n = [a_n, b_n]$ $(n \in \mathbb{N})$ with the following properties

(i) every b_n is an upper bound for A

(ii) no a_n is an upper bound for A.

Let b_0 be an upper bound for A [A is bounded] and choose $\alpha \in A$ arbitrary [A is nonempty!]. Then $\alpha \le b_0$ and $a_0 := \alpha - 1 \le b_0$ cannot be an upper bound for A.

We proceed by induction: supppose that $[a_0, b_0] \supseteq \ldots \supseteq [a_n, b_n]$ have been constructed satisfying (i) and (ii). Let $m := (b_n - a_n)/2$ be the midpoint of I_n and define

$$[a_{n+1}, b_{n+1}] := \begin{cases} [a_n, m] & \text{if } m \text{ is an upper bound of } A \\ [m, b_n] & \text{otherwise.} \end{cases}$$

By the principle of nested intervals we have $\bigcap\limits_{n\in\mathbb{N}} I_n = \{s\}$ for some $s \in \mathbb{R}$.

Claim 1: s is an upper bound for A

Assume the contrary, then there exists $x \in A$ such that $x > s$. Since $a_n \le s \le b_n$ for all n and $b_n - a_n \to 0$ $(n \to \infty)$ there exists N such that

$$b_N - s \le b_N - a_N < x - s.$$

Therefore $b_N < x$, which is a contradiction to property (i) above.

Claim 2: s is the least upper bound of A

Supppose that $s' < s$ is also an upper bound of A. Then there is $n \in \mathbb{N}$ such that diam $(I_n) < s - s'$. Since $s \in I_n$ we have

$$s - a_n \leq \text{diam } (I_n) < s - s',$$

which implies that $a_n > s'$ and thus a_n is an upper bound of A — a contradiction ↯ to (ii) above. □

§4. CONVERGENCE OF SERIES

Let $(a_k)_{k\in\mathbb{N}}$ be a sequence of real numbers. Recall that the series $\sum_{k=0}^{\infty} a_k$ is defined to be convergent if and only if the corresponding sequence $(s_m)_{m\in\mathbb{N}}$ of partial sums $s_m = \sum_{k=0}^{m} a_k$ is convergent, with $\lim s_m$ being called the sum of the series.

4.1. Proposition (Cauchy principle for series): The series $\sum\limits_{k=0}^{\infty} a_k$ is convergent if and only if

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n \geq m \geq N: \quad \left| \sum_{k=m}^{n} a_k \right| < \varepsilon. \tag{4.1}$$

Proof. The sequence of partial sums (s_m) converges if and only if it is a Cauchy sequence [cf. Theorem 3.10]. The latter is equivalent to the property that

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}: \quad |s_n - s_{m-1}| < \varepsilon \quad \forall n, m-1 \geq N,$$

where we may assume in addition that $n \geq m-1$, which yields $s_n - s_{m-1} = \sum_{k=m}^{n} a_k$. $\square$

4.2. Remark: Note that condition (4.1) is not affected by changing finitely many a_k's in the series. Thus the convergence behavior of a series does not depend on alteration of a finite number of terms. (However, the value of the sum might change, of course.)

4.3. Corollary: If $\sum_{k=0}^{\infty} a_k$ is convergent, then (a_n) is a null sequence.

Proof. Let $\varepsilon > 0$ and put $m = n \geq N$ in condition (4.1). Thus $|a_n| < \varepsilon$ holds for all $n \geq N$, hence $a_n \to 0$ $(n \to \infty)$. $\square$

4.4. Proposition: Let $\sum a_n$ be a series of nonnegative numbers, i.e. $a_n \geq 0$ for all $n \in \mathbb{N}$, then

$$\sum_{n=0}^{\infty} a_n \text{ is convergent} \qquad \Longleftrightarrow \qquad \text{the sequence of partial sums } (s_m) \text{ is bounded.}$$

Proof. $\boxed{\Rightarrow}$ (s_m) is convergent, hence bounded.
$\boxed{\Leftarrow}$ $s_{m+1} = s_m + a_{m+1} \geq s_m$ shows that (s_m) is increasing. Since, by assumption, (s_m) is also bounded [3.12, Theorem] implies convergence. □

4.5. Examples: 1) $\sum_{n=0}^{\infty} (-1)^n$ is divergent, since $(-1)^n \not\to 0$.

2) The *harmonic series* $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, since the partial sums are unbounded:

$$s_{2^k} = \sum_{n=1}^{2^k} \frac{1}{n} = 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{\geq 2\frac{1}{4} = \frac{1}{2}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{\geq 4\frac{1}{8} = \frac{1}{2}} + \ldots + \underbrace{\left(\frac{1}{2^{k-1}+1} + \ldots + \frac{1}{2^k}\right)}_{\geq 2^{k-1}\frac{1}{2^k} = \frac{1}{2}}$$

$$\geq 1 + \underbrace{\frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{2}}_{k \text{ terms}} \geq 1 + \frac{k}{2}.$$

Note that in this example $a_n := \frac{1}{n} \to 0$ $(n \to \infty)$ but $\sum a_n$ is divergent. Thus Corollary 4.3 gives a necessary condition for convergence which is not sufficient ($a_n \to 0$ does not imply that $\sum a_n$ converges)!

3) If $s \in \mathbb{N}$ with $s \geq 2$, then $\sum_{n=1}^{\infty} \frac{1}{n^s}$ is convergent.

Since all terms are nonnegative it suffices to prove boundedness of the partial sums: Let $m \in \mathbb{N}$ and choose $l \in \mathbb{N}$ such that $m \leq 2^{l+1} - 1$, then we obtain

$$s_m = \sum_{n=1}^{m} \frac{1}{n^s} \leq \sum_{n=1}^{2^{l+1}-1} \frac{1}{n^s} = 1 + \underbrace{\left(\frac{1}{2^s} + \frac{1}{3^s}\right)}_{\leq 2\frac{1}{2^s}} + \ldots + \underbrace{\left(\sum_{2^l}^{2^{l+1}-1} \frac{1}{n^s}\right)}_{\leq 2^l \frac{1}{2^{ls}}}$$

$$\leq \sum_{j=0}^{l} 2^j \frac{1}{2^{js}} = \sum_{j=0}^{l} \left(\frac{1}{2^{s-1}}\right)^j \leq \sum_{j=0}^{\infty} \left(\frac{1}{2^{s-1}}\right)^j = \frac{1}{1 - \frac{1}{2^{s-1}}},$$

which gives an upper bound independent of m. (The same proof would work for any $s \in]1, \infty[$.)

4.6. Theorem (Leibniz's[1] criterion for alternating series): Let $a_n \geq 0$ for all $n \in \mathbb{N}$ and consider the series with alternating signs

$$\sum_{n=0}^{\infty}(-1)^n a_n = a_0 - a_1 + a_2 - a_3 + a_4 - \dots$$

If in addition (a_n) satisfies

(i) $a_n \geq a_{n+1}$ for all n, i.e. (a_n) is decreasing,

and

(ii) $a_n \to 0$ $(n \to \infty)$,

then the series $\sum(-1)^n a_n$ is convergent.

Furthermore, for the partial sums $s_m = \sum_{n=0}^{m}(-1)^n a_n$ and the sum of the series $s = \lim_{m\to\infty} s_m = \sum_{n=0}^{\infty} a_n$ we have the error estimate

$$|s - s_m| \leq a_{m+1} \qquad \forall m \in \mathbb{N}. \tag{4.2}$$

(In other words, the error of each partial sum is not larger than the first neglected term.)

Proof. Observe that for all k

$$\begin{aligned} s_{2k+1} &= s_{2k} - a_{2k+1} \leq s_{2k} \\ s_{2k+2} &= s_{2k} - \underbrace{(a_{2k+1} - a_{2k+2})}_{\geq 0 \text{ by (i)}} \leq s_{2k} \\ s_{2k+3} &= s_{2k+1} + \underbrace{(a_{2k+2} - a_{2k+3})}_{\geq 0 \text{ by (i)}} \geq s_{2k+1}, \end{aligned}$$

which we summarize by

$$s_1 \leq s_3 \leq s_5 \leq \dots \leq s_{2k+1} \leq s_{2k} \leq \dots \leq s_4 \leq s_2 \leq s_0.$$

In other words, we have a sequence of nested intervals $I_k := [s_{2k+1}, s_{2k}]$ $(k \in \mathbb{N})$ with diam $(I_k) = s_{2k} - s_{2k+1} = a_{2k+1} \to 0$ as $k \to \infty$ [by property (ii)]. Therefore [Theorem 3.15] implies that $\bigcap_{k\in\mathbb{N}} I_k = \{s\}$.

Since $s_{2k+1} \leq s \leq s_{2k}$ for all k we obtain for every $m \in \mathbb{N}$ that

$$|s - s_m| \leq |s_{m+1} - s_m| = a_{m+1},$$

which gives (4.2) and also proves convergence of (s_n): Let $\varepsilon > 0$ and choose N such that $0 \leq a_n < \varepsilon$ for all $n \geq N$. Then for all $n \geq N$

$$|s_n - s| \leq a_{n+1} < \varepsilon.$$

□

[1] Gottfried Wilhelm Leibnitz (1646–1716) [ˈɡɔtfriːt ˈvɪlhɛlm ˈlaɪbnɪts], German philosopher and scientist

4.7. Example: The alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent, since $\left(\frac{1}{n}\right)$ is decreasing and converges to 0.

4.8. Remark and warning: The value of the sum — and even the convergence behavior (!) — of a convergent series may depend on the order of summation. To be more precise, if $\sum_{n=0}^{\infty} a_n$ is a convergent series and $\tau\colon \mathbb{N} \to \mathbb{N}$ is bijective, then the *rearrangement* ⟨*Umordnung*⟩ $\sum_{n=0}^{\infty} a_{\tau(n)}$ need not have the same sum or need not converge at all.

For example, consider again the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots,$$

which is convergent as we have shown above. If we rearrange the terms in the series in the following way

$$\begin{aligned}
&\underbrace{1 - \frac{1}{2}}_{1/2} - \frac{1}{4} + \underbrace{\frac{1}{3} - \frac{1}{6}}_{1/6} - \frac{1}{8} + \underbrace{\frac{1}{5} - \frac{1}{10}}_{1/10} - \frac{1}{12} + \underbrace{\frac{1}{7} - \frac{1}{14}}_{1/14} - \frac{1}{16} + \ldots \\
&\qquad = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \ldots = \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots\right),
\end{aligned}$$

then we obtain half of the original sum.

We can even find a divergent rearrangement of the (originally convergent) alternating harmonic series, where the negative terms occur with more and more delay as we progress: Let $n \geq 2$ and observe that

$$\begin{aligned}
&\underbrace{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}}_{>0} + \underbrace{\left(\frac{1}{5} + \frac{1}{7}\right)}_{>2/8=1/4} - \frac{1}{6} + \underbrace{\left(\frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15}\right)}_{>4/16=1/4} - \frac{1}{8} + \ldots \\
&\qquad + \underbrace{\left(\frac{1}{2^n+1} + \frac{1}{2^n+3} + \ldots + \frac{1}{2^{n+1}-1}\right)}_{>2^{n-1}/2^{n+1}=1/4} - \frac{1}{2n+2} \\
&\qquad > \frac{n-1}{4} - \left(\frac{1}{6} + \frac{1}{8} + \ldots + \frac{1}{2n+2}\right) \geq \frac{n-1}{4} - \frac{n-1}{6} = \frac{n-1}{12},
\end{aligned}$$

thus the partial sums corresponding to such a rearrangement must be unbounded.

The following definition introduces the appropriate convergence notion for series which is stable under rearrangements.

4.9. Definition: A series $\sum a_n$ is *absolutely convergent* ⟨absolut konvergent⟩ if $\sum |a_n|$ is convergent.

4.10. Remark: (i) Since $|a_n| \geq 0$ absolute convergence is thus equivalent to the boundedness of $s_m = \sum_{n=0}^{m} |a_n|$.

(ii) Convergence does not imply absolute convergence, as can be seen from the example of the alternating harmonic series, where $a_n = \frac{(-1)^{n-1}}{n}$ $(n \geq 1)$: in this case $\sum |a_n| = \sum \frac{1}{n}$ is the harmonic series, which is divergent.

Absolute convergence is a stronger condition than convergence.

4.11. Proposition: Every absolutely convergent series is convergent.

Proof. Suppose $\sum |a_n|$ is convergent. Let $\varepsilon > 0$. By the Cauchy principle for series we can find $N \in \mathbb{N}$ such that

$$\forall n \geq m \geq N : \quad \sum_{k=m}^{n} |a_k| < \varepsilon.$$

Applying the triangle inequality (for finitely many terms) we obtain

$$\forall n \geq m \geq N : \quad \left| \sum_{k=m}^{n} a_k \right| \leq \sum_{k=m}^{n} |a_k| < \varepsilon,$$

which in turn by the Cauchy principle for series yields convergence of $\sum a_n$. □

4.12. Rearrangement theorem for absolutely convergent series: Let $\sum a_n$ be an absolutely convergent series. Then every rearrangement $\sum a_{\tau(n)}$, where $\tau \colon \mathbb{N} \to \mathbb{N}$ is a bijection, is absolutely convergent and has the same limit.

Proof. Let $s := \sum_{n=0}^{\infty} a_n$.

Claim 1: $\sum_{n=0}^{\infty} a_{\tau(n)}$ is convergent with sum s

Let $\varepsilon > 0$ and choose N such that for all $l \geq N$ we have $\sum_{k=N}^{l} |a_k| < \dfrac{\varepsilon}{2}$ [Cauchy principle for the convergent series $\sum |a_k|$]. Sending $l \to \infty$ yields $\sum_{k=N}^{\infty} |a_k| \leq \dfrac{\varepsilon}{2}$ and therefore for all

$m \geq N$ also that

$$\left|\sum_{k=0}^{m} a_k - \sum_{k=0}^{N-1} a_k\right| = \left|\sum_{k=N}^{m} a_k\right| \leq \sum_{k=N}^{m} |a_k| \leq \sum_{k=N}^{\infty} |a_k| \leq \frac{\varepsilon}{2}.$$

Upon taking limits as $m \to \infty$ we find

$$\left|s - \sum_{k=0}^{N-1} a_k\right| \leq \frac{\varepsilon}{2}.$$

Choose $M \in \mathbb{N}$ with $M \geq N$ and large enough to ensure $\{\tau(0), \tau(1), \ldots, \tau(M)\} \supset \{0, 1, \ldots, N-1\}$. [Since τ is a bijection of $\mathbb{N}$ such an M exists.] Then we have for all $m \geq M$

$$\left|\sum_{k=0}^{m} a_{\tau(k)} - s\right| \leq \left|\sum_{k=0}^{m} a_{\tau(k)} - \sum_{k=0}^{N-1} a_k\right| + \left|\sum_{k=0}^{N-1} a_k - s\right| \leq \sum_{k=N}^{\infty} |a_k| + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

thus $\sum_{n=0}^{\infty} a_{\tau(n)}$ converges to s.

Claim 2: $\sum |a_{\tau(n)}|$ is convergent

This follows by application of Claim 1 to the series $\sum b_n$ with $b_n := |a_n|$ (and its corresponding sum $s' := \sum b_n$). □

4.13. Remark: One can prove that absolute convergence of a series is, in fact, equivalent to the property that all its rearrangements converge to the same limit. [cf. [BF00]]

Now that we have established the importance of absolute convergence we come to the question of how to determine wether a given series is absolutely convergent.

4.14. Proposition (Basic comparison test): (i) Let $\sum c_n$ be convergent with nonnegative terms $c_n \geq 0$. If $|a_n| \leq c_n$ holds for almost all n, then $\sum a_n$ is absolutely convergent. ⟨$\sum c_n$ *ist eine konvergente Majorante für* $\sum a_n$.⟩

(ii) Let $\sum d_n$ be divergent and have nonnegative terms $d_n \geq 0$. If $a_n \geq d_n$ holds for almost all n, then $\sum a_n$ is divergent. ⟨$\sum d_n$ *ist eine divergente Minorante für* $\sum a_n$.⟩

Proof. (i) WLOG (= without loss of generality) we may assume that $|a_n| \leq c_n$ holds for all n. $\sum |a_k|$ is a series with nonegative terms and for all $m \in \mathbb{N}$

$$0 \leq \sum_{k=0}^{m} |a_k| \leq \sum_{k=0}^{\infty} c_n$$

proves that the partial sums are bounded. Thus the convergence of $\sum |a_n|$ follows from [Proposition 4.4].

(ii) If $\sum a_n$ were convergent, then by (i) $\sum d_n$ would converge — a contradiction $\lightning$. $\square$

4.15. Examples: 1) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent, since

$$\forall n \geq 1 : \frac{1}{\sqrt{n}} \geq \frac{1}{n} \qquad \text{and } \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges.}$$

2) Let (a_n) be a sequence with $|a_n| \leq 1$ for all n and let $q \in]0,1[$. Since $|a_n q^n| \leq q^n$ and the geometric sum $\sum q^n$ is convergent, we deduce that

$$\sum_{n=0}^{\infty} a_n q^n \quad \text{is absolutely convergent.}$$

4.16. Proposition (Root test ⟨*Wurzeltest*⟩**):** The series $\sum a_n$ is

(a) absolutely convergent, if there exist $\theta \in \mathbb{R}$ with $0 \leq \theta < 1$ and $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0 : \quad |a_n|^{1/n} \leq \theta$$

(b) divergent, if

$$|a_n|^{1/n} \geq 1 \quad \text{for infnitely many } n.$$

Proof. (a) Since $|a_n| \leq \theta^n$ for almost all n and the geometric series $\sum \theta^n$ is convergent, the basic comparison test implies convergence of $\sum |a_n|$.

(b) Since it follows that $|a_n| \geq 1$ for infinitely many n, the sequence (a_n) does not tend to 0, thus $\sum a_n$ is divergent. $\square$

4.17. Remark: (i) There are variants of the above statement, for example [cf. [BF00]]: Let $\alpha := \limsup |a_n|^{1/n}$, then $\sum a_n$ is

(a) absolutely convergent, if $\alpha < 1$

(b) divergent, if $\alpha > 1$.

(ii) In practice, we often have that $\beta := \lim |a_n|^{1/n}$ exists. Then $\beta < 1$ implies case (a) of the root test, hence absolute convergence, and $\beta > 1$ implies condition (b), hence divergence. (Note that also $\alpha = \beta$ in such situations.)

(iii) Note however, that achieving $\theta = 1$ in condition (a) of the root test (or $\alpha = 1$ or $\beta = 1$) is not conclusive! For example,

$$\sum \frac{1}{n^2} \quad \text{is convergent and} \quad \left(\frac{1}{n^2}\right)^{1/n} < 1,$$
$$\sum \frac{1}{n} \quad \text{is divergent and} \quad \left(\frac{1}{n}\right)^{1/n} < 1.$$

4.18. Proposition (Ratio test ⟨*Quotiententest*⟩**):** Let $a_n \neq 0$ for almost all n. The series $\sum a_n$ is

(a) absolutely convergent, if there exist $\theta \in \mathbb{R}$ with $0 \leq \theta < 1$ and $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0: \quad \left|\frac{a_{n+1}}{a_n}\right| \leq \theta$$

(b) divergent, if there exists $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0: \quad \left|\frac{a_{n+1}}{a_n}\right| \geq 1.$$

Proof. (a) We obtain for all $n \geq n_0$

$$|a_{n+1}| \leq \theta |a_n| \leq \ldots \leq \theta^{n-n_0} |a_{n_0}|,$$

where $\sum \theta^{n-n_0} |a_{n_0}| = |a_{n_0}| \theta^{-n_0} \sum \theta^n$ is convergent. Thus the comparison theorem implies convergence of $\sum |a_n|$.

(b) Let $n_1 \geq n_0$ such that $a_{n_1} \neq 0$ then the stated condition implies that $|a_n| \geq |a_{n_1}| > 0$ for all $n \geq n_1$. Thus (a_n) cannot be a null sequence and therefore $\sum a_n$ diverges. □

4.19. Remark: (i) Again, there exist variants of the ratio test [e.g., cf. [BF00]]: Let $\alpha := \limsup \left|\frac{a_{n+1}}{a_n}\right|$ and $\gamma := \liminf \left|\frac{a_{n+1}}{a_n}\right|$, then $\sum a_n$ is

(a) absolutely convergent, if $\alpha < 1$

(b) divergent, if $\gamma > 1$.

(ii) Suppose that $\beta := \lim \frac{a_{n+1}}{a_n}$ exists. Then $\beta < 1$ implies case (a) of the ratio test, hence absolute convergence, and $\beta > 1$ implies condition (b), hence divergence. (Note that consequently $\alpha = \beta = \gamma$.)

(iii) As with the root test, having $\theta = 1$ in condition (a) of the ratio test (or $\alpha = 1$ or $\gamma = 1$ or $\beta = 1$) is not conclusive! The same examples illustrate this:

$$\sum \frac{1}{n^2} \quad \text{is convergent and} \quad \frac{a_{n+1}}{a_n} = \frac{n^2}{(n+1)^2} < 1,$$
$$\sum \frac{1}{n} \quad \text{is divergent and} \quad \frac{a_{n+1}}{a_n} = \frac{n}{n+1} < 1.$$

(iv) One can prove that condition (a) of the ratio test implies that condition (a) of the root test holds [cf. [BF00, Section 5.3]]. In other words, whenever the ratio test concludes positively the root test is applicable as well. On the other hand, there are absolutely convergent series for which the root test is successful, whereas the ratio test is inconclusive. (We give an example below.)

4.20. Examples: 1) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ is absolutely convergent, since

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^2\, 2^n}{2^{n+1}\, n^2} = \frac{1}{2}\left(1 + \frac{1}{n}\right)^2 \to \frac{1}{2} \quad (n \to \infty).$$

2) Consider

$$a_n = \begin{cases} 2^{-n} & \text{if } n \text{ is even} \\ 3^{-n} & \text{if } n \text{ is odd.} \end{cases}$$

Then $\sum a_n$ is absolutely convergent as can be seen by the root test:

$$|a_n|^{1/n} \le \max\left(\frac{1}{2}, \frac{1}{3}\right) \le \frac{1}{2} \qquad \forall n$$

showing that we may choose $\theta = 1/2$. In this case the quotient test is not conclusive, since

$$\frac{a_{2k+1}}{a_{2k}} = \frac{1}{3}\left(\frac{2}{3}\right)^{2k} \to 0, \qquad \frac{a_{2k+2}}{a_{2k+1}} = \frac{1}{2}\left(\frac{3}{2}\right)^{2k+1} \to \infty \qquad (k \to \infty).$$

4.21. Decimal and b-adic expansion of real numbers: In everyday life we are used to seeing rational numbers (e.g. purchase prices of certain products) expressed as decimal numbers like 17,8041 (or 17.8041). The corresponding number $x \in \mathbb{Q}$ is in this case determined by the expansion with basis 10 and decimal digits from $\{0, 1, 2, \ldots, 9\}$ as

$$x = 1 \cdot 10^1 + 7 \cdot 10^0 + 8 \cdot 10^{-1} + 0 \cdot 10^{-2} + 4 \cdot 10^{-3} + 1 \cdot 10^{-4}$$
$$= \frac{10^5 + 7 \cdot 10^4 + 8 \cdot 10^3 + 4 \cdot 10 + 1}{10^4} = \frac{178041}{10000}.$$

There are two immediate ideas to generalize such representations: The above expansion has finitely many terms. Can this be extended to infinite sums of the same type and be considered then as a representation of the limit? A very similar kind of expansion can be defined with an integer basis other than 10.

Definition: Let $b \in \mathbb{N}$, $b \geq 2$, $N \in \mathbb{Z}$ and $a_n \in \{0, 1, 2, \ldots, b-1\}$ ($n \in \mathbb{Z}$, $n \geq N$). The series

$$\pm \sum_{n=N}^{\infty} a_n b^{-n}$$

is called a *b-adic expansion* ⟨b-adische Entwicklung⟩ with *digits* ⟨Ziffern⟩ a_n ($n \geq -N$).

Three questions arise:

1. Is every b-adic expansion convergent?
2. Can every $x \in \mathbb{R}$ be represented as the limit of such an expansion?
3. Are the digits of an expansion uniquely determined by x?

The answer to the third one is negative, as can be seen from the following simple example with decimal expansions:

$$0,999\ldots = \sum_{n=1}^{\infty} 9 \cdot 10^{-n} = 9 \cdot 10^{-1} \sum_{n=0}^{\infty} \left(10^{-1}\right)^n = \frac{9}{10} \cdot \frac{1}{1 - 1/10} = \frac{9}{10} \cdot \frac{10}{10-1} = 1 = 1,000\ldots$$

But questions 1 and 2 can be shown to have positive answers.

Theorem: Let $b \in \mathbb{N}$, $b \geq 2$.

(i) Every b-adic expansion is convergent (in $\mathbb{R}$).

(ii) Every real number x is the sum of a b-adic expansion (with a sequence of digits that can be constructed recursively).

Proof. (i) Since $|a_n b^{-n}| \leq (b-1) b^{-n}$ for all n and $(b-1) \sum b^{-n}$ is convergent, this follows from the basic comparison theorem.

(ii) It suffices to show this for the case $x \geq 0$.

By the Archimedian property there is $m \in \mathbb{N}$ such that $x < b^{m+1}$. Let m_0 be the minimum of all $m \in \mathbb{N}$ such that $0 \leq x < b^{m+1}$. Put $N := -m_0$. We shall construct a sequence $(a_n)_{n \geq N}$ of integers $a_n \in \{0, 1, \ldots, b-1\}$ inductively such that

$$(\star) \qquad \forall n \geq N \quad \exists \xi_n \text{ with } 0 \leq \xi_n < b^{-n} : \quad x = \sum_{k=N}^{n} a_k b^{-k} + \xi_n.$$

Since by $(\star)$ $\lim \xi_n = 0$, hence $x = \sum_{k=N}^{\infty} a_k b^{-k}$, the assertion of the theorem will then follow.

Before we proceed, recall that the *floor function* $\lfloor \ \rfloor : \mathbb{R} \to \mathbb{R}$ is defined by $\lfloor y \rfloor := \max\{l \in \mathbb{Z} : l \leq y\}$, and that $0 \leq y - \lfloor y \rfloor < 1$ holds.

<u>Induction base, $n = N$:</u> The definition of N implies $0 \leq x \cdot b^N < b$. We define $a_N \in \{0, 1, \ldots, b-1\}$ by

$$a_N := \lfloor x b^N \rfloor \quad \text{ and } \quad \xi_N := (x b^N - a_N) \cdot b^{-N}$$

Then we clearly have

$$x = a_N b^{-N} + \xi_N \quad \text{ and } \quad 0 \leq \xi_N < b^{-N}.$$

<u>Induction step, $n \to n+1$:</u> Property $(\star)$ yields $0 \leq \xi_n \cdot b^{n+1} < b$. If we define $a_{n+1} \in \{0, 1, \ldots, b-1\}$ by

$$a_{n+1} := \lfloor \xi_n b^{n+1} \rfloor \quad \text{ and } \quad \xi_{n+1} := (\xi_n b^{n+1} - a_{n+1}) \cdot b^{-n-1},$$

then $\xi_n = a_{n+1} b^{-n-1} + \xi_{n+1}$ and therefore

$$x = \sum_{k=N}^{n} a_k b^{-k} + a_{n+1} b^{-n-1} + \xi_{n+1},$$

where $0 \leq \xi_{n+1} < b^{-n-1}$. Thus $(\star)$ holds with n replaced by $n+1$. □

COROLLARY: Every real number is the limit of a sequence of rational numbers. ($\mathbb{Q}$ is dense in $\mathbb{R}$. Compare with the variant of this statement in [0.7].)

Proof. Let $x \in \mathbb{R}$ have decimal expansion $x = \sum_{n=N}^{\infty} a_n 10^{-n}$. For every $m \geq N$ the partial sum $s_m := \sum_{n=N}^{m} a_n 10^{-n}$ is a rational number and $x = \lim s_m$. □

REMARK: Let $x = \pm \sum_{n=N}^{\infty} a_n b^{-n}$ be a b-adic expansion. One can prove (cf. [AE02, II.7]) that x is a rational number if and only if the sequence of digits is periodic from a certain index $m \geq N$ onward, that is, with some $p \in \mathbb{N} \setminus \{0\}$ we have $a_{n+p} = a_n$ for all $n \geq m$ (including the case that almost all a_n vanish, which decsribes a finite sum).

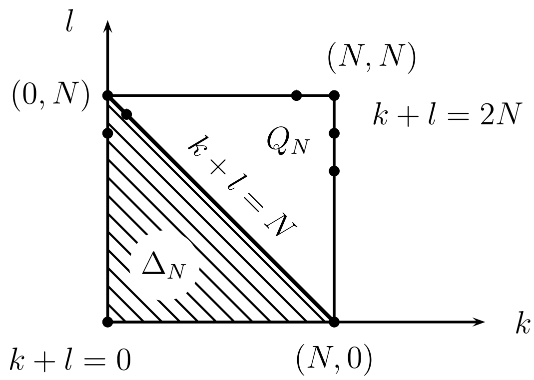
4.22. The Cauchy product of absolutely convergent series:

We begin with a simple observation about the product of finite sums.

Let $A_N := \sum_{n=0}^{N} a_n$ and $B_N := \sum_{n=0}^{N} b_n$, then

$$A_N \cdot B_N = \left(\sum_{k=0}^{N} a_k\right) \cdot \left(\sum_{l=0}^{N} b_l\right) = \sum_{0 \leq k \leq N} \sum_{0 \leq l \leq N} a_k b_l.$$

For the investigation of convergence (as $N \to \infty$) of such a double sum it is helpful to introduce some notation and to illustrate rearrangements of the terms in a 2-dimensional picture:



Here, $Q_N := \{(k,l) \in \mathbb{N}^2 : k \leq N, l \leq N\}$ and $\Delta_N := \{(k,l) \in Q_N : k + l \leq N\}$. Then the above double sum can be rewritten as

$$\begin{aligned} A_N \cdot B_N = \sum_{(k,l) \in Q_N} a_k b_l &= \sum_{(k,l) \in \Delta_N} a_k b_l + \sum_{(k,l) \in Q_N \setminus \Delta_N} a_k b_l \\ &= \sum_{n=0}^{N} \sum_{\substack{(k,l) \in \Delta_N \\ k+l=n}} a_k b_l + \sum_{(k,l) \in Q_N \setminus \Delta_N} a_k b_l = \sum_{n=0}^{N} \sum_{k=0}^{n} a_k b_{n-k} + \sum_{(k,l) \in Q_N \setminus \Delta_N} a_k b_l. \end{aligned}$$

Proposition: Let $\sum a_n$ and $\sum b_n$ be absolutely convergent series and define

$$c_n := \sum_{k=0}^{n} a_k b_{n-k} \qquad (n \in \mathbb{N}).$$

Then $\sum c_n$ is absolutely convergent and

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{k=0}^{\infty} a_k\right) \cdot \left(\sum_{l=0}^{\infty} b_l\right).$$

Proof. We use the notation introduced above. Let $S_N := \sum_{n=0}^{N} c_n$, then

$$A_N \cdot B_N - S_N = \sum_{(k,l)\in Q_N \setminus \Delta_N} a_k b_l.$$

Claim 1: $\lim\limits_{N\to\infty} S_N = \left(\sum\limits_{k=0}^{\infty} a_k\right) \cdot \left(\sum\limits_{l=0}^{\infty} b_l\right)$

Put $A_N^* := \sum\limits_{n=0}^{N} |a_n|$ and $B_N^* := \sum\limits_{n=0}^{N} |b_n|$, then $A_N^* B_N^* = \sum\limits_{(k,l)\in Q_N} |a_k||b_l|$. Using $Q_{\lfloor N/2 \rfloor} \subseteq \Delta_N \subseteq Q_N$ we obtain

$$(\star) \qquad |A_N B_N - S_N| \leq \sum_{(k,l)\in Q_N \setminus \Delta_N} |a_k||b_l| \leq \sum_{(k,l)\in Q_N \setminus Q_{\lfloor N/2 \rfloor}} |a_k||b_l|$$
$$= A_N^* B_N^* - A_{\lfloor N/2 \rfloor}^* B_{\lfloor N/2 \rfloor}^*.$$

By hypothesis, $(A_N^* B_N^*)_{N\in\mathbb{N}}$ is convergent, hence a Cauchy sequence, which implies that $\lim\limits_{N\to\infty} A_N^* B_N^* - A_{\lfloor N/2 \rfloor}^* B_{\lfloor N/2 \rfloor}^* = 0$. Since $\lim\limits_{N\to\infty} A_N B_N = \left(\sum\limits_{k=0}^{\infty} a_k\right) \cdot \left(\sum\limits_{l=0}^{\infty} b_l\right)$, the assertion now follows from $(\star)$.

Claim 2: $\sum c_n$ is absolutely convergent

Let $S_N^* := \sum\limits_{n=0}^{N} \sum\limits_{k=0}^{n} |a_k||b_{n-k}|$, then an application of Claim 1 shows that $(S_N^*)_{N\in\mathbb{N}}$ is convergent. Since $\sum |c_n| \leq S_N^*$ the assertion follows. $\square$

4.23. The exponential function: For every $x \in \mathbb{R}$ the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is absolutely convergent, as can be verified by the ratio test: The terms $a_n := x^n/n!$ are all nonzero if $x \neq 0$, and in this case

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{n+1} n!}{|x|^n (n+1)!} = \frac{|x|}{n+1} \to 0 \quad (n \to \infty);$$

in the case $x = 0$ absolute convergence is clear.

We may thus define a real-valued function on $\mathbb{R}$ by assigning the value of the sum $\sum \frac{x^n}{n!}$ to each x.

Definition: The *exponential function* ⟨Exponentialfunktion⟩ $\exp\colon \mathbb{R} \to \mathbb{R}$ is defined by

$$\exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (x \in \mathbb{R}).$$

The value $e := \exp(1)$ is called *Euler's number* ⟨Eulersche Zahl⟩[2].

Many important properties of the exponential function are a direct consequence of the so-called functional equation of the exponential function, which we prove in the following theorem. In fact, the exponential function can be characterized as the unique function satisfying this particular equation on $\mathbb{R}$ and being bounded on some closed finite interval [cf.[BF00, Section 7.5]].

Theorem (Functional equation for the exponential function): For all $x, y \in \mathbb{R}$

$$\exp(x+y) = \exp(x) \cdot \exp(y). \tag{4.3}$$

Proof. Both series $\sum \frac{x^n}{n!}$ and $\sum \frac{y^n}{n!}$ are absolutely convergent, hence the Cauchy product [4.22] gives $\exp(x)\exp(y) = \sum c_n$, where

$$c_n = \sum_{k=0}^{n} \frac{x^k}{k!} \frac{y^{n-k}}{(n-k)!} = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \underset{\text{[Binom. thm.]}}{=} \frac{(x+y)^n}{n!}.$$

Thus we obtain

$$\exp(x)\exp(y) = \sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = \exp(x+y).$$

□

[2] Leonhard Euler (1707–1783) [ˈleɔnhaʁt ˈɔɪlɐ], Swiss mathematician and physicist, introduced many of the modern mathematic's symbols (e.g. $e, \pi, i, \sum, f(x)$ etc.).

Corollary: (i) For all $x \in \mathbb{R}$: $\exp(x) > 0$ and $\exp(-x) = \dfrac{1}{\exp(x)}$.
(ii) For all $n \in \mathbb{Z}$: $\exp(n) = e^n$.

Proof. (i) The functional equation (4.3) implies

$$1 = \exp(0) = \exp(x - x) = \exp(x)\exp(-x),$$

which proves the second equation in the assertion. If $x \geq 0$ then

$$\exp(x) = 1 + x + \frac{x^2}{2} + \ldots \geq 1 > 0,$$

since all neglected terms are nonnegative. If $x < 0$ then $\exp(x) = 1/\exp(-x) > 0$ by what we have just proved.

(ii) Since $\exp(-n) = 1/\exp(n)$ it suffices to show this for $n \in \mathbb{N}$. First, $\exp(0) = 1 = e^0$ and then inductively

$$\exp(n+1) = \exp(n)\exp(1) = e^n \cdot e^1 = e^{n+1}.$$

□

Remark: The above Theorem and Corollary (i) show that exp is a group homomorphism of the additive group $(\mathbb{R}, +)$ into the multiplicative group $(]0, \infty[, \cdot)$.

Finally, we provide a (crude, but useful!) bound on the error of partial sum approximations of the exponential function. (The error bound shall be improved on later as we learn more about functions of a real variable and Taylor series.)

Proposition: Let $N \in \mathbb{N}$. For all $x \in \mathbb{R}$ we have

$$\exp(x) = \sum_{n=0}^{N} \frac{x^n}{n!} + R_{N+1}(x), \tag{4.4}$$

where the remainder term $R_{N+1}(x)$ satisfies the following estimate for all $x \in \mathbb{R}$ with $|x| \leq 1 + \frac{N}{2}$

$$|R_{N+1}(x)| \leq 2\, \frac{|x|^{N+1}}{(N+1)!}. \tag{4.5}$$

Proof. The remainder term is

$$R_{N+1}(x) := \exp(x) - \sum_{n=0}^{N} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^{N} \frac{x^n}{n!} = \sum_{n=N+1}^{\infty} \frac{x^n}{n!},$$

which is absolutely convergent. Therefore we have for all $|x| \le 1 + N/2$ the following chain of inequalities

$$|R_{N+1}(x)| \le \sum_{n=N+1}^{\infty} \frac{|x|^n}{n!} = \frac{|x|^{N+1}}{(N+1)!}\Big(1 + \frac{|x|}{N+2} + \underbrace{\frac{|x|^2}{(N+2)(N+3)}}_{\le \frac{|x|^2}{(N+2)^2}} + \dots\Big)$$

$$\le \frac{|x|^{N+1}}{(N+1)!} \sum_{k=0}^{\infty} \left(\frac{|x|}{N+2}\right)^k \underset{[|x| \le (N+2)/2]}{\le} \frac{|x|^{N+1}}{(N+1)!} \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 2 \cdot \frac{|x|^{N+1}}{(N+1)!}.$$

$\square$

EXAMPLE: Since $e = \exp(1) = \sum_{n=0}^{N} \frac{1}{n!} + R_{N+1}(1)$ and $1 \le 1 + N/2$ for all $N \in \mathbb{N}$, we obtain for $N = 2$

$$e = 1 + 1 + \frac{1}{2} + R_3(1), \qquad \text{where } 0 < R_3(1) \le 2\,\frac{1}{3!} = \frac{2}{6} = \frac{1}{3}.$$

Therefore

$$2 < \frac{5}{2} < e \le \frac{5}{2} + \frac{1}{3} = \frac{17}{6} < 3.$$

As a matter of fact, the exponential series is numerically efficient. For example, one obtains a value for e which is accurate up to 100 digits by summation of 73 terms; to give just the first few digits

$$e \approx 2{,}71828\dots$$

CHAPTER II

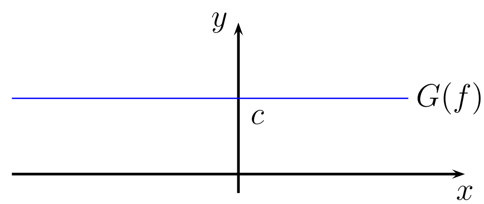
CONTINUOUS FUNCTIONS OF A REAL VARIABLE

§5. CONTINUITY

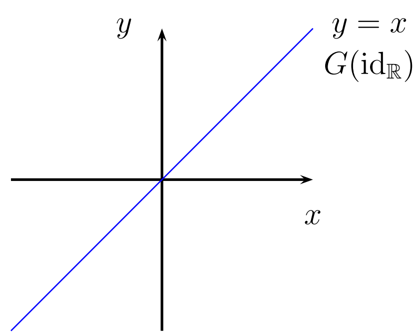
In this section we study real (valued) functions[1] on subsets of $\mathbb{R}$, i.e., maps $f\colon D \to \mathbb{R}$, where $D \subseteq \mathbb{R}$. Recall that the *graph* ⟨*Graph*⟩ of f is defined as the following subset of $\mathbb{R}^2$:

$$G(f) := \{(x, f(x)) \in \mathbb{R}^2 : x \in D\}.$$

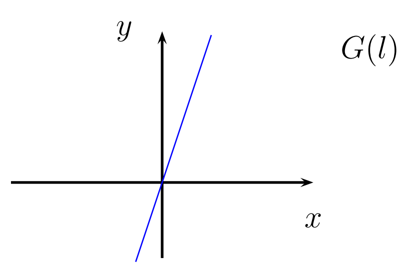
5.1. Examples: 1) Let $c \in \mathbb{R}$ arbitrary, then $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) := c$ for all $x \in \mathbb{R}$ defines a *constant function*.



2) The *identity map* ⟨*identische Abbildung*⟩ on $\mathbb{R}$ is given by $\mathrm{id}_{\mathbb{R}}\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto x$.

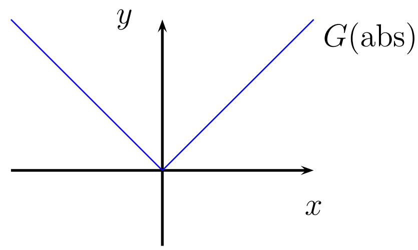


Slightly more general are linear functions $l\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto a \cdot x$, where $a \in \mathbb{R}$ gives the slope of the graph:

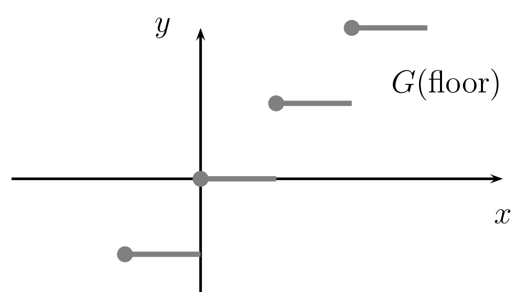


[1] The mathematical term, "function" (from the Latin functio, meaning performance, execution) was first used by Leibniz in 1694 to describe curves.

3) The *absolute value (function)* ⟨*Betragsfunktion oder Absolutbetrag*⟩ is defined by abs: $\mathbb{R} \to \mathbb{R}$, $x \mapsto |x|$.

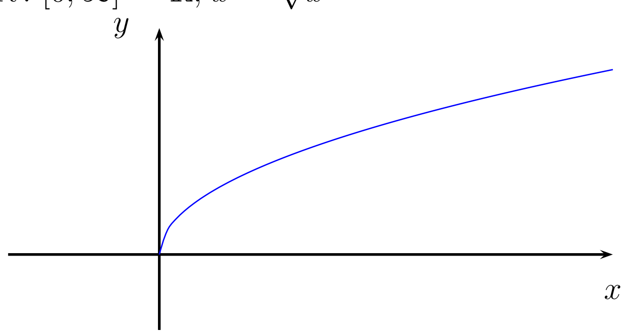


4) floor: $\mathbb{R} \to \mathbb{R}$, $x \mapsto \lfloor x \rfloor$, where (as on page 39) $\lfloor x \rfloor = \max\{n \in \mathbb{Z} \colon n \leq x\}$.



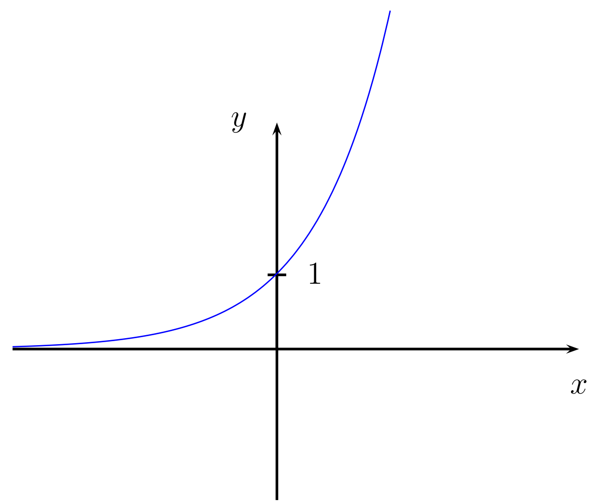
The floor function is sometimes called *Gauß bracket*[2] ⟨*Gaußklammer*⟩ and the values are also denoted by $[x]$ $(x \in \mathbb{R})$.

5) The square root sqrt: $[0, \infty] \to \mathbb{R}$, $x \mapsto \sqrt{x}$



[2]Carl Friedrich Gauß (1777–1855) [kaʁl ˈfriːtrɪç gaʊs], one of the most outstanding German mathematicians

6) The exponential function $\exp\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto \exp(x)$ as defined in 4.23.

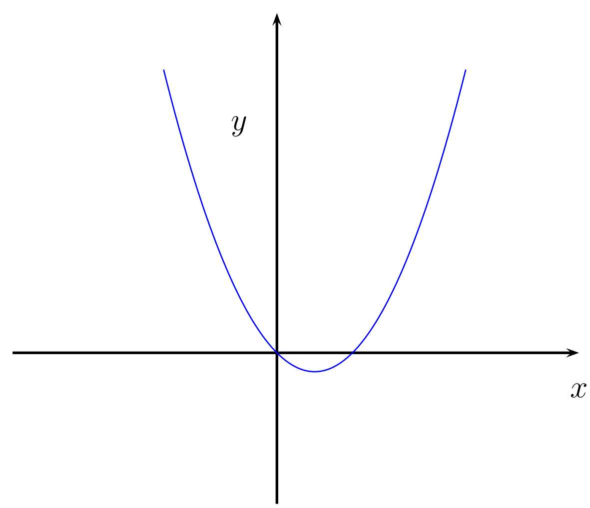


7) *Polynomial functions* ⟨*Polynomfunktionen*⟩: Let $m \in \mathbb{N}$ and $a_0, a_1, \ldots, a_m \in \mathbb{R}$. We define

$$p\colon \mathbb{R} \to \mathbb{R} \quad \text{by} \quad p(x) := a_m x^m + a_{m-1}x^{m-1} + \ldots a_1 x + a_0 \qquad \forall x \in \mathbb{R}.$$

The constants $a_0, \ldots a_m$ are called the *coefficients* ⟨*Koeffizienten*⟩ of the polynomial function. If $a_m \neq 0$ then p is said to be of *degree* m ⟨*vom Grad* m⟩.

For example, when $m = 2$ and $a_0 = 0$, $a_1 = -1$, $a_2 = 1$ we obtain $p(x) = x^2 - x$



8) *Rational functions* ⟨*rationale Funktionen*⟩: Let p and q be polynomial functions, that is

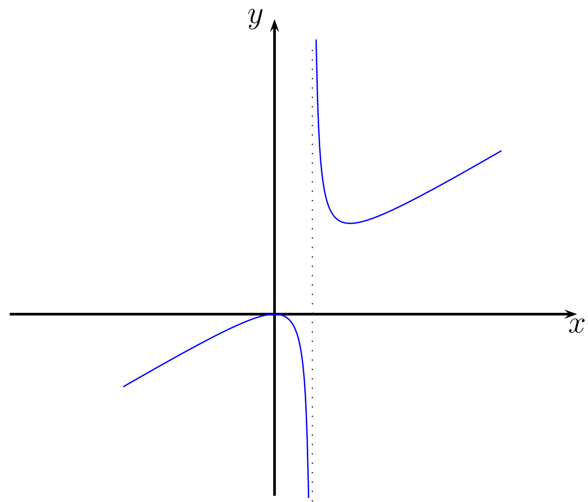
$$p(x) = a_m x^m + \ldots + a_1 x + a_0 \quad \text{and} \quad q(x) = b_n x^n + \ldots + b_1 x + b_0$$

with given coefficients $a_0, \ldots, a_m, b_0, \ldots b_n \in \mathbb{R}$. Then a rational function is the quotient function with domain $D := \{x \in \mathbb{R} : q(x) \neq 0\}$, defined by

$$r\colon D \to \mathbb{R}, \quad x \mapsto \frac{p(x)}{q(x)}.$$

Note that polynomial functions are just rational functions with denominator $q \equiv 1$.

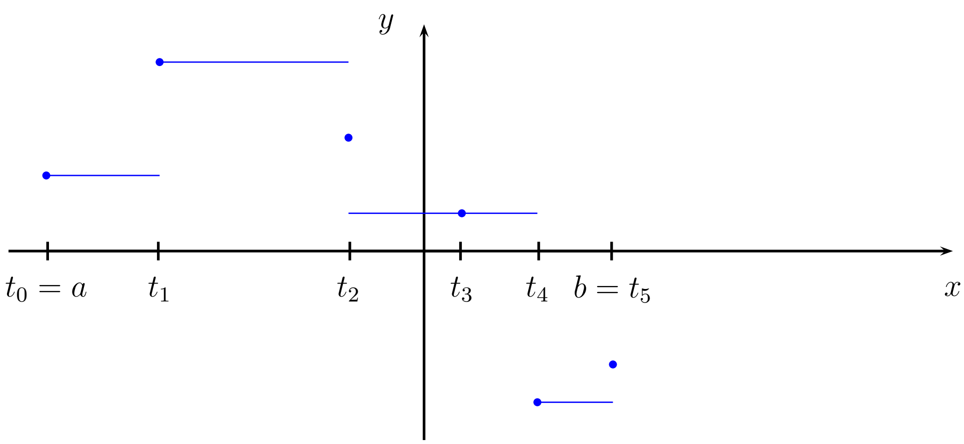
For example, here is the graph of the rational function $r\colon \mathbb{R} \setminus \{1\} \to \mathbb{R}$, $r(x) = x^2/(x-1)$



9) *Simple functions (or step functions)* ⟨Treppenfunktionen⟩: Let $a, b \in \mathbb{R}$ with $a < b$. A function $\varphi\colon [a,b] \to \mathbb{R}$ is called a simple function (or step function), if there is a finite partition $a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$ of the interval $[a,b]$ and coefficients $c_1, \ldots c_n \in \mathbb{R}$ such that

$$\varphi(x) = c_k \qquad \text{when } x \in]t_{k-1}, t_k[\quad (1 \leq k \leq n).$$

Therefore φ is constant on each open subinterval $]t_{k-1}, t_k[$ $(1 \leq k \leq n)$ but the finitely many values $\varphi(t_k)$ $(0 \leq k \leq n)$ are arbitrary.



Note that the restriction floor $|_{[a,b]}$ of the floor function provides an example of a simple function.

10) The *characteristic function of* $\mathbb{Q}$ ⟨charakteristische Funktion von $\mathbb{Q}$⟩ or *Dirichlet function*[3] is given by

$$\mathbb{1}_{\mathbb{Q}}\colon \mathbb{R} \to \mathbb{R}, \quad \mathbb{1}_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

In this case the graph is

$$G(\mathbb{1}_{\mathbb{Q}}) = \{(q,1) : q \in \mathbb{Q}\} \cup \{(s,0) : s \in \mathbb{R} \setminus \mathbb{Q}\},$$

which would be somewhat hard to depict ...

5.2. Review of basic operations with functions:

Let $f, g\colon D \to \mathbb{R}$ be functions on $D \subseteq \mathbb{R}$ and $\lambda \in \mathbb{R}$.

- Then the functions

  $$f + g\colon D \to \mathbb{R}, \qquad \lambda f\colon D \to \mathbb{R}, \qquad f \cdot g\colon D \to \mathbb{R}$$

  are defined in terms of the corresponding pointwise operations (with real numbers) for all $x \in D$ by

  $$\begin{aligned} (f+g)(x) &:= f(x) + g(x), \\ (\lambda f)(x) &:= \lambda \cdot f(x), \\ (f \cdot g)(x) &:= f(x) \cdot g(x). \end{aligned}$$

  Remark: It is easy to check that the set $\mathcal{F}(D) := \{f\colon D \to \mathbb{R}\}$ of all real valued functions on the set D together with the addition and scalar multiplication as defined by the first two lines above forms a vector space over $\mathbb{R}$.

- Let $D' := \{x \in D : g(x) \neq 0\}$. The quotient function is defined by

  $$\frac{f}{g}\colon D' \to \mathbb{R}, \quad x \to \frac{f(x)}{g(x)}.$$

- Let $E \subseteq \mathbb{R}$ such that $f(D) \subseteq E$ and $h\colon E \to \mathbb{R}$. Recall that the composition of f and h is given by

  $$h \circ f\colon D \to \mathbb{R}, \quad (h \circ f)(x) := h(f(x)) \qquad \forall x \in D.$$

[3] Johann Peter Gustav Lejeune Dirichlet (1805–1859) [ˈjoːhan ˈpeːtɐ ˈɡʊstaf ləˈʒœn diʀiˈkle], German mathematician with Belgish origins (the French words Lejeune Dirichlet literally mean "the young chap from Richelet")

Examples: 1) If $q\colon \mathbb{R} \to \mathbb{R}$, $q(x) = x^2$, then $q = \mathrm{id} \cdot \mathrm{id}$.

2) More generally, if p is a polynomial function, given by

$$p(x) = a_m x^m + \ldots a_1 x + a_0,$$

then

$$p = a_m \cdot \underbrace{(\mathrm{id} \cdot \mathrm{id} \cdots \mathrm{id})}_{m \text{ factors}} + \ldots a_1 \cdot \mathrm{id} + a_0 \cdot \mathbb{1},$$

where $\mathbb{1}$ denotes the constant function $\mathbb{1}(x) = 1$ for all $x \in \mathbb{R}$.

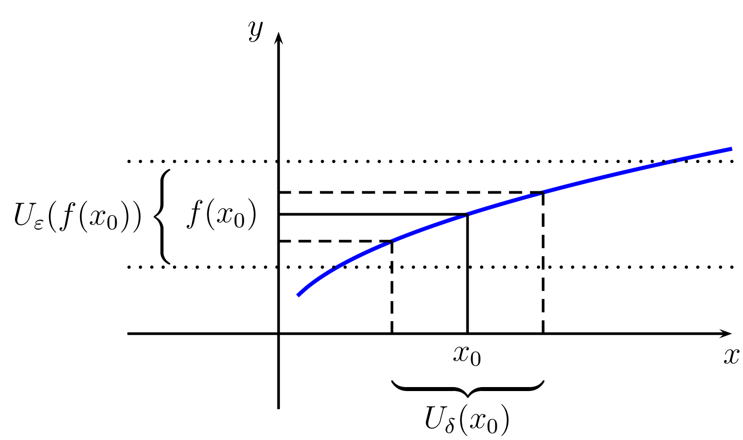
3) With q as in example 1) we have $\mathrm{abs} = \mathrm{sqrt} \circ q$, since for all $x \in \mathbb{R}$

$$\big(\mathrm{sqrt} \circ q\big)(x) = \sqrt{x^2} = |x| = \mathrm{abs}(x).$$

5.3. Continuity ⟨*Stetigkeit*⟩:

The notion of continuity of a function is a precise way to express an intuitive requirement, which is often implicitly made in model applications: Namely, that small perturbations of a function argument should not result in extreme changes of the function values.

How to specify such a property for a given function f near a point x_0 of its domain? It might seem practically desirable to first prescribe the acceptable tolerance around the value $f(x_0)$ and then to look for a safety interval around the argument x_0 on which function values near $f(x_0)$ within tolerance are guaranteed. If the tolerance is given in terms of an interval $]f(x_0) - \varepsilon, f(x_0) + \varepsilon[$ with $\varepsilon > 0$ and the safety interval is sought in the form $]x_0 - \delta, x_0 + \delta[$ with $\delta > 0$ we obtain the following picture:



By requiring that for every tolerance $\varepsilon > 0$ — chosen arbitrarily small — an appropriate safety guard $\delta > 0$ can (in principle) be found we arrive at the notion of continuity.

DEFINITION: Let $x_0 \in D \subseteq \mathbb{R}$ and $f\colon D \to \mathbb{R}$. The function f is *continuous* ⟨*stetig*⟩ at x_0 if

$$\text{(5.1)} \qquad \forall \varepsilon > 0 \, \exists \delta > 0 : \quad \forall x \in D : \quad |x - x_0| < \delta \quad \Longrightarrow \quad |f(x) - f(x_0)| < \varepsilon.$$

Equivalently, upon recalling that $]x_0 - \delta, x_0 + \delta[\, = U_\delta(x_0)$ and $]f(x_0) - \varepsilon, f(x_0) + \varepsilon[\, = U_\varepsilon(f(x_0))$, we can define the continuity of f at x_0 in terms of neighborhoods:

$$\forall \varepsilon > 0 \, \exists \delta > 0 : \quad f\big(U_\delta(x_0) \cap D\big) \subseteq U_\varepsilon\big(f(x_0)\big).$$

The function f is said to be *continuous (on D)* if it is continuous at each point in D. If f is not continuous at a point $b \in D$ then f is said to be *discontinuous* ⟨*unstetig*⟩ at b.

EXAMPLES: 1) Clearly, a constant function f is continuous (at every point x_0 in its domain), since $f(x) - f(x_0) = 0$ and therefore (5.1) is satisfied for all $\varepsilon > 0$ and $\delta > 0$ arbitrary.

2) Every linear function $f\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto ax$, is continuous (at every $x_0 \in \mathbb{R}$): If $a = 0$ this is clear from Example 1), hence consider $a \neq 0$. Let $\varepsilon > 0$. From the preparatory observation $|f(x) - f(x_0)| = |a||x - x_0|$ we learn that we can simply choose $\delta := \varepsilon/|a|$ to achieve (5.1): Indeed, if $|x - x_0| < \delta = \varepsilon/|a|$ then

$$|f(x) - f(x_0)| = |a| \, |x - x_0| < |a| \, \delta = \varepsilon.$$

3) The exponential function $\exp\colon \mathbb{R} \to \mathbb{R}$ is continuous: Let $x_0 \in \mathbb{R}$ and $\varepsilon > 0$. By the properties of the exponential function we have

$$|\exp(x) - \exp(x_0)| = \exp(x_0) \, |\exp(x - x_0) - 1|,$$

where $\exp(x_0) > 0$. From (4.5) we obtain for $|x - x_0| \leq 1$ that

$$|\exp(x - x_0) - 1| \leq 2|x - x_0|.$$

Thus, putting $\delta := \min(1, \frac{\varepsilon}{4\exp(x_0)})$ and combining the above inequalities we obtain for all x with $|x - x_0| < \delta$ the required estimate

$$|\exp(x) - \exp(x_0)| \leq 2\exp(x_0) \, |x - x_0| < 2\exp(x_0) \, \delta < \varepsilon.$$

4) $\mathrm{abs}\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto |x|$, is continuous: Let $x_0 \in \mathbb{R}$ and $\varepsilon > 0$. Put $\delta := \varepsilon$ then we have for all $x \in U_\delta(x_0)$

$$|\mathrm{abs}(x) - \mathrm{abs}(x_0)| = ||x| - |x_0|| \leq |x - x_0| < \delta = \varepsilon.$$

5) The Dirichlet function $\mathbb{1}_{\mathbb{Q}}$ [Example 5.1, 10)] is discontinuous at every point in $\mathbb{R}$: Let $x_0 \in \mathbb{R}$ and put $\varepsilon = 1/2$.

If $x_0 \in \mathbb{R} \setminus \mathbb{Q}$ then $\mathbb{1}_{\mathbb{Q}}(x_0) = 0$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, for every $\delta > 0$ we might choose the interval $U_\delta(x_0) =]x_0 - \delta, x_0 + \delta[$ will always contain some (in fact, many) rational number(s) r [cf. 0.7 or the Corollary in 4.21]. In other words, we can find r with $|r - x_0| < \delta$ but

$$|\mathbb{1}_{\mathbb{Q}}(r) - \mathbb{1}_{\mathbb{Q}}(x_0)| = |1 - 0| = 1 \geq \frac{1}{2} = \varepsilon.$$

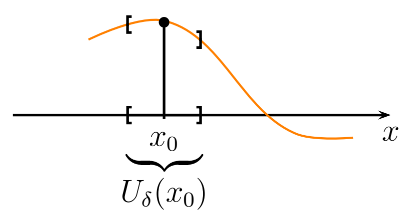
If $x_0 \in \mathbb{Q}$ then $\mathbb{1}_{\mathbb{Q}}(x_0) = 1$. Recall that also $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$ [cf. 0.7]. Hence for every $\delta > 0$ we can find $s \in U_\delta(x_0) \cap (\mathbb{R} \setminus \mathbb{Q})$, which implies

$$|\mathbb{1}_{\mathbb{Q}}(s) - \mathbb{1}_{\mathbb{Q}}(x_0)| = |0 - 1| = 1 \geq \frac{1}{2} = \varepsilon$$

while $|s - x_0| < \delta$.

Knowing that a specific value of a continuous function has positive distance to a certain real number c already guarantees that the function values will stay away from c in a whole neighborhood. In the following statement we formulate this for the special case with $c = 0$. This can easily be adapted to the case $c \neq 0$ by a simple translation of the function graph.

5.4. Lemma: Let $f\colon D \to \mathbb{R}$ be continuous at $x_0 \in D$ and assume that $f(x_0) \neq 0$. Then there is $\delta > 0$ such that for all $x \in U_\delta(x_0) \cap D$ we have $f(x) \neq 0$.



Proof. Put $\varepsilon := |f(x_0)|/2$. Then clearly $\varepsilon > 0$ and by continuity there exists some $\delta > 0$ such that for all $x \in D$ with $|x - x_0| < \delta$ we have $|f(x) - f(x_0)| < \varepsilon = |f(x_0)|/2$. Therefore $x \in U_\delta(x_0) \cap D$ implies

$$|f(x)| = |f(x_0) + f(x) - f(x_0)| \geq |f(x_0)| - |f(x) - f(x_0)| > |f(x_0)| - \varepsilon = \frac{|f(x_0)|}{2} > 0.$$

□

5.5. Continuity test by sequences:

THEOREM: Let $a \in D \subseteq \mathbb{R}$ and $f\colon D \to \mathbb{R}$. The following are equivalent:

(i) f is continuous at a.

(ii) For every sequence (x_n) with $x_n \in D$ we have: if $\lim x_n = a$ then $\lim f(x_n) = f(a)$,

i.e., $\boxed{f(\lim x_n) = \lim f(x_n)}$ as an abbreviated slogan.

Proof. (i) $\Rightarrow$ (ii): Let $x_n \in D$ ($n \in \mathbb{N}$) with $\lim x_n = a$ and let $\varepsilon > 0$. Choose $\delta > 0$ such that the continuity condition (5.1) is satisfied. There exists $n_0 \in \mathbb{N}$ such that $|x_n - a| < \delta$ holds for all $n \geq n_0$. Thus (5.1) implies

$$|f(x_n) - f(a)| < \varepsilon \quad \forall n \geq n_0,$$

which proves that $\lim f(x_n) = f(a)$.

(ii) $\Rightarrow$ (i): (proof by contradiction) Assume that (ii) holds but (5.1) is false. That is,

$$\exists \varepsilon > 0\ \forall \delta > 0 : \exists x \in U_\delta(a) \cap D : f(x) \notin U_\varepsilon(f(a)).$$

In particular, with this same $\varepsilon > 0$, we can choose the δ-values to be $1/n$ ($n \in \mathbb{N}$, $n \geq 1$) successively and obtain:

$$\forall n \in \mathbb{N}, n \geq 1 : \exists x_n \in D : |x_n - a| < \frac{1}{n}, \text{ but } |f(x_n) - f(a)| \geq \varepsilon.$$

Therefore $\lim x_n = a$ whereas $f(x_n) \not\to f(a)$ $(n \to \infty)$ — a contradiction $\lightning$. $\square$

EXAMPLE: The function floor: $\mathbb{R} \to \mathbb{R}$, $x \mapsto \lfloor x \rfloor$ is continuous in $\mathbb{R} \setminus \mathbb{Z}$ and discontinuous in all points $a \in \mathbb{Z}$.

If $a \in \mathbb{Z}$ then $\lfloor a \rfloor = a$ and the sequence $x_n := a - \frac{1}{n}$ $(n \geq 1)$ has $\lim x_n = a$ but $\lim \lfloor x_n \rfloor = \lim(a - 1) = a - 1 \neq \lfloor a \rfloor$.

If $a \in \mathbb{R} \setminus \mathbb{Z}$ then $\lfloor a \rfloor < a < \lfloor a \rfloor + 1$. Hence for every sequence (x_n) with $\lim x_n = a$ there exists some n_0 such that $\lfloor a \rfloor < x_n < \lfloor a \rfloor + 1$ when $n \geq n_0$. Therefore $\lfloor x_n \rfloor = \lfloor a \rfloor$ for all $n \geq n_0$, in particular $\lim \lfloor x_n \rfloor = \lfloor a \rfloor$.

5.6. Basic operations and continuity: The following results show that we do not leave the class of continuous functions when applying the basic operations summarized in 5.2 to continuous functions. In other words, we can generate many "new" continuous functions from a set of given continuous functions simply by pointwise addition, scalar multiplication, multiplication, division (when the denominator does not vanish), and composition (where the images and domains match appropriately).

PROPOSITION: (i) Let $a \in D \subseteq \mathbb{R}$ and $\lambda \in \mathbb{R}$. If $f, g\colon D \to \mathbb{R}$ are continuous at a then also

$$f + g\colon D \to \mathbb{R}, \qquad \lambda f\colon D \to \mathbb{R}, \qquad f \cdot g\colon D \to \mathbb{R}$$

are continuous at a. Furthermore, if $a \in D' := \{x \in D : g(x) \neq 0\}$ then

$$\frac{f}{g}\colon D' \to \mathbb{R}$$

is continuous at a.

(ii) Let $D \subseteq \mathbb{R}$, $E \subseteq \mathbb{R}$ and $f\colon D \to \mathbb{R}$, $g\colon E \to \mathbb{R}$ such that $f(D) \subseteq E$. If f is continuous at $a \in D$ and g is continuous at $b := f(a) \in E$ then the composition $g \circ f\colon D \to \mathbb{R}$ is continuous at a.

Proof. (i) Let (x_n) be a sequence in D, respectively D', such that $x_n \to a$. Then by the corresponding properties of basic operations with convergent sequences in 2.10 we obtain that

$$(f+g)(x_n) = f(x_n) + g(x_n) \to f(a) + g(a) = (f+g)(a) \quad (n \to \infty)$$

and similarly for the other types of operations. Thus Theorem 5.5 proves continuity at a.

(ii) Let (x_n) be a sequence in D such that $x_n \to a$. Since f is continuous at a we have $y_n := f(x_n) \to f(a) = b$. Continuity of g at b implies $g(y_n) \to g(b)$. Therefore

$$\lim_{n\to\infty} (g \circ f)(x_n) = \lim g(f(x_n)) = \lim g(y_n) = g(b) = g(f(a)) = (g \circ f)(a)$$

and again by Theorem 5.5 the continuity at a follows. □

Corollary: Polynomial functions and rational functions are continuous (on their respective domains).

Proof. By 5.3, Examples 1) and 2), constant functions and the identity map $\mathrm{id}\colon \mathbb{R} \to \mathbb{R}$ are continuous. In 5.2, Example 2), we noted that polynomial functions are just finite linear combinations of products of id by itself plus a constant function, thus the above Proposition (i) shows continuity.

Rational functions are quotients of polynomial functions, defined where the denominator does not vanish, and are therefore also continuous by the second part of (i) in the above Proposition. □

Example: 1) $p(x) := -x^2$ defines a continuous function on $\mathbb{R}$ and exp is continuous on $\mathbb{R}$. Hence the function $\exp \circ p\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto \exp(-x^2)$ is continuous $\mathbb{R} \to \mathbb{R}$.

2) The *hyperbolic sine and cosine* ⟨hyperbolischer Sinus und Cosinus⟩ are defined by

$$\sinh(x) := \frac{\exp(x) - \exp(-x)}{2} \quad \text{and} \quad \cosh(x) := \frac{\exp(x) + \exp(-x)}{2} \qquad (x \in \mathbb{R}),$$

hence are continuous functions on $\mathbb{R}$.

5.7. Limit of a function: Recall that $a \in \mathbb{R}$ is an adherent point of $D \subseteq \mathbb{R}$ if and only if there exists a sequence (x_n) in D (i.e., $x_n \in D$ for all n) such that $x_n \to a$ $(n \to \infty)$. If a is an element of D then the latter condition is clearly satisfied by the constant sequence $x_n = a$ for all n. In general, an adherent point of D need not be a member of the set D.

DEFINITION: Let $f\colon D \to \mathbb{R}$ and a an adherent point of D. The function f has *limit* $c \in \mathbb{R}$ *as x tends to a*, if every sequence (x_n) in D such that $x_n \to a$ $(n \to \infty)$ satisfies $\lim_{n\to\infty} f(x_n) = c$. A short-hand notation for this fact is

$$\lim_{x\to a} f(x) = c \qquad \text{or} \qquad f(x) \to c \quad (x \to a).$$

We also define $c \in \mathbb{R}$ to be *the limit of f at a from the right* ⟨rechtsseitiger Grenzwert⟩

$$\lim_{x\searrow a} f(x) = c, \quad \text{or also } \lim_{x\to a+} f(x) = c$$

if a is an adherent point of $D \cap \,]a,\infty[$ and for all sequences (x_n) with $x_n \in D$ and $x_n > a$ such that $x_n \to a$ we have $\lim f(x_n) = c$.
The notion of *limit from the left* ⟨linksseitiger Grenzwert⟩ $\lim_{x\nearrow a} f(x)$, also denoted by $\lim_{x\to a-} f(x)$, is defined analogously using $]-\infty, a[\, \cap D$ and $x_n < a$ instead.

Finally, we define *limits of f at infinity* as follows:

$$\lim_{x\to\infty} f(x) = c$$

means that D is unbounded from above and for every sequence (x_n) with $x_n \in D$ and $x_n \to \infty$ we have $\lim f(x_n) = c$.
We define $\lim_{x\to-\infty} f(x)$ similarly when D is unbounded from below using $x_n \to -\infty$.

Of course, we will often find it convenient to also use the above notions with improper limits $c = \pm\infty$. The required adaptations of the definition should be routine and are left to the reader.

EXAMPLES: 1) For the rational function $f\colon \mathbb{R}\setminus\{1\} \to \mathbb{R}$, $f(x) = (x^2-1)/(x-1)$, we have

$$\lim_{x\to 1} f(x) = 2.$$

Indeed, if $x_n \to 1$ with $x_n \neq 1$ then

$$f(x_n) = \frac{(x_n-1)(x_n+1)}{x_n-1} = x_n + 1 \to 2 \quad (n\to\infty).$$

2) $\lim_{x\searrow 1} \lfloor x \rfloor = 1$, since $\lfloor x_n \rfloor = 1$ when $1 < x_n < 2$. On the other hand, $\lim_{x\nearrow 1} \lfloor x \rfloor = 0$ as $\lfloor x_n \rfloor = 0$ when $0 \le x_n < 1$.
We conclude that $\lim_{x\to 1} \lfloor x \rfloor$ does not exist, because otherwise the limits from the left and from the right would have to be equal.

3) Let $m \in \mathbb{N}$, $m \ge 1$, and $p\colon \mathbb{R} \to \mathbb{R}$ be a polynomial function of the form

$$p(x) = x^m + a_{m-1}x^{m-1} + \ldots + a_0.$$

Then we have $\lim\limits_{x\to\infty} p(x) = \infty$ and $\lim\limits_{x\to\infty} \dfrac{1}{p(x)} = 0.$

To see this, we first note that for all $x > 0$ we have the estimate

$$p(x) = x^m \left(1 + \frac{a_{m-1}}{x} + \ldots + \frac{a_0}{x^m}\right) \geq x^m \left(1 - \frac{|a_{m-1}|}{|x|} - \ldots - \frac{|a_0|}{|x^m|}\right).$$

Let $x \geq M := 2m \cdot \max(1, |a_{m-1}|, \ldots, |a_0|)$, then the above inequality implies

$$p(x) \geq x^m \left(1 - m \cdot \frac{1}{2m}\right) = \frac{x^m}{2} \qquad \text{(in particular, } p(x) \geq \frac{1}{2}\text{)}.$$

Let $x_n \to \infty$ and choose $n_0 \in \mathbb{N}$ such that $x_n \geq M$ for all $n \geq n_0$. Then we obtain for $n \geq n_0$

$$p(x_n) \geq \frac{x_n^m}{2} \to \infty \qquad (n \to \infty),$$

therefore $\lim p(x_n) = \infty$, which proves the first assertion above. The second assertion follows immediately from the first, if we note that $1/p(x)$ is defined for $x \geq M$ (since $p(x) \geq 1/2$ then, as noted above).

Remark: (i) Note that if $a \in D$ and $\lim\limits_{x\to a} f(x)$ exists then the limit has to be $f(a)$ (since $x_n = a$ is a special sequence in D converging to a).

(ii) An ε-δ-version for a function $f \colon D \to \mathbb{R}$ to have limit $c \in \mathbb{R}$ reads as follows:

$$\forall \varepsilon > 0\, \exists \delta > 0 : \ x \in D, |x - a| < \delta \quad \Longrightarrow \quad |f(x) - c| < \varepsilon.$$

Warning: The notion of 'limit of a function' is not used in exactly the same way as we do here throughout the literature. Some texts (e.g. [Heu88]) require the admissible sequences (x_n) in the definition to be in $D \setminus \{a\}$, so that the special choice $x_n = a$ is excluded even in the case where a belongs to D. If a is an adherent point of D and does not belong to D, both notions give the same result concerning existence and value of the function limit. However, if $a \in D$ the conclusions may differ, as can be seen from the following example: Let $D := \mathbb{R} \setminus \{0\}$ and define $f \colon D \to \mathbb{R}$ by $f(x) := 1$ if $x \neq 0$, and $f(0) := 0$. Then in the sense of our definition f does not have a limit at 0, whereas we obtain for all sequences (x_n) with $x_n \neq 0$ and $x_n \to 0$ that $\lim f(x_n) = 1$ (note that this value differs from $f(0)$), which would give existence of the limit of f at 0 in the alternative definition.

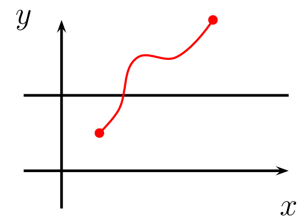
Since the notion of 'limit of a function' is essentially a short-hand notation to describe the way how a function translates converging sequences into sequences of corresponding function values, we can rephrase the sequence test of continuity 5.5 in these terms.

Proposition: A function $f \colon D \to \mathbb{R}$ is continuous at a point $a \in D$ if and only if

$$\lim_{x\to a} f(x) = f(a).$$

Proof. This is immediate from Theorem 5.5 and the remark made above. □

5.8. The intermediate value property ⟨*Zwischenwertsatz*⟩:



THEOREM: Suppose $f\colon [a,b] \to \mathbb{R}$ is continuous and $c \in \mathbb{R}$ lies between $f(a)$ and $f(b)$, that is $f(a) \le c \le f(b)$ or $f(b) \le c \le f(a)$. Then there exists $x_0 \in [a,b]$ such that $f(x_0) = c$.

In other words, a continuous function on $[a,b]$ attains every value between $f(a)$ and $f(b)$ at least once — there are no gaps in $f([a,b])$.

An important **special case** of the Theorem is the following: If $f\colon [a,b] \to \mathbb{R}$ is continuous and $f(a) < 0$ and $f(b) > 0$ (resp. $f(a) > 0$ and $f(b) < 0$), then f has a *zero* ⟨*Nullstelle*⟩ in $[a,b]$, i.e., $\exists\, x_0 \in [a,b]\colon\ f(x_0) = 0$.

EXAMPLE OF AN APPLICATION: Let $p\colon \mathbb{R} \to \mathbb{R}$ be a polynomial function of odd degree $m = 2n+1$ (with $n \in \mathbb{N}$), say,

$$p(x) = b_{2n+1}x^{2n+1} + b_{2n}x^{2n} + \ldots + b_0 \qquad (x \in \mathbb{R}),$$

where $b_{2n+1} \neq 0$. Then p has at least one <u>real</u> zero.

To show this, we first write

$$p(x) = b_{2n+1} \cdot \big(x^{2n+1} + \frac{b_{2n}}{b_{2n+1}}x^{2n} + \ldots + \frac{b_0}{b_{2n+1}}\big) = b_{2n+1} \cdot q(x),$$

where the polynomial function q is of the form $q(x) = x^{2n+1} + a_{2n}x^{2n} + \ldots + a_0$ (with $a_j = b_j/b_{2n+1}$ for $j = 0, \ldots, 2n$).

By 5.7, Example 3), we have $\lim\limits_{x\to\infty} q(x) = \infty$, hence there exists $x_+ > 0$ such that $q(x_+) > 0$. Similarly, upon observing that

$$q(-x) = -x^{2n+1} + a_{2n}x^{2n} - \ldots + a_0 = -(x^{2n+1} - a_{2n}x^{2n} + \ldots - a_0)$$

we obtain that $\lim\limits_{x\to-\infty} q(x) = -\infty$, hence there exists $x_- < 0$ such that $q(x_-) < 0$.

Since $q\,|_{[x_-,x_+]}\colon [x_-,x_+] \to \mathbb{R}$ is continuous and $q(x_-) < 0 < q(x_+)$, the above Theorem implies that there exists $x_0 \in [x_-,x_+]$ such that $q(x_0) = 0$ (in fact, $x_- < x_0 < x_+$, because the values of q at $x_\pm$ are known to be nonzero). Therefore also $p(x_0) = b_{2n+1}\,q(x_0) = 0$.

Proof of the Theorem.

WLOG (:= without loss of generality) ⟨*OBdA (:= ohne Beschränkung der Allgemeinheit)*⟩ we may assume that $f(a) < c < f(b)$. [otherwise we just have to consider $-f$ instead]

If $c \neq 0$ we can reduce the statement to that of the special case of a zero by putting $f_1(x) := f(x) - c$. Then $f_1(a) = f(a) - c < 0$ and $f_1(b) = f(b) - c > 0$ and the assertion of the Theorem is equivalent to the existence of a zero $x_0 \in [a,b]$ of the function f_1.

So, again WLOG we may assume that $c = 0$ and $f(a) < 0 < f(b)$. We have to show that there is some $x_0 \in [a, b]$ such that $f(x_0) = 0$.

We will find x_0 by constructing a sequence of nested intervals in the fashion of a so-called *bisection method*. To be more precise, we claim that we can define $[a_n, b_n] \subseteq [a, b]$ for all $n \in \mathbb{N}$ with the following properties:

1. $\forall n \in \mathbb{N}$, $n \geq 1$: $[a_n, b_n] \subseteq [a_{n-1}, b_{n-1}]$
2. $b_n - a_n = \dfrac{b-a}{2^n}$
3. $f(a_n) < 0$ and $f(b_n) \geq 0$.

Put $a_0 := a$ and $b_0 := b$, then properties 2 and 3 are satisfied. We proceed inductively and assume that $[a_0, b_0]$, ..., $[a_n, b_n]$ have been defined satisfying properties 1-3. Let $m := (a_n + b_n)/2$ (this is the midpoint of $[a_n, b_n]$) and distinguish two cases:

If $f(m) \geq 0$ put $a_{n+1} := a_n$ and $b_{n+1} := m$

If $f(m) < 0$ put $a_{n+1} := m$ and $b_{n+1} := b_n$.

Then properties 1–3 are valid for $[a_{n+1}, b_{n+1}]$ as well. By the principle of nested intervals we obtain that (a_n) and (b_n) converge to the same limit $x_0 \in [a, b]$, that is $\lim a_n = \lim b_n = x_0$. Since f is continuous we have that

$$f(x_0) = \lim f(a_n) = \lim f(b_n).$$

By property 3 we obtain in addition

$$f(x_0) = \lim f(a_n) \leq 0 \leq \lim f(b_n) = f(x_0),$$

which proves that $f(x_0) = 0$. □

Corollary: Let $I \subseteq \mathbb{R}$ be a nonempty interval and $f\colon I \to \mathbb{R}$ be continuous. Then $f(I) \subseteq \mathbb{R}$ is an interval as well.

Proof. Let $A := \inf f(I)$ and $B := \sup f(I)$, where we allow for the improper values $A = -\infty$ (unbounded below) and $B = \infty$ (unbounded above). If $A = B$ then $f(I)$ contains just a single point, in which case the statement is true. So we henceforth assume that $A < B$.

We assert that $]A, B[\subseteq f(I)$: Let $y \in]A, B[$, then there exist $r, s \in I$ such that $f(r) < y < f(s)$. By the above Theorem we have some $x_0 \in I$ such $f(x_0) = y$. Therefore $y \in f(I)$.

To summarize, $]A, B[\subseteq f(I) \subseteq [A, B]$, hence $f(I)$ equals one of the intervals $]A, B[$ or $]A, B]$ or $[A, B[$ or $[A, B]$. □

5.9. Continuous functions on bounded closed intervals:

DEFINITION: A function $f\colon D \to \mathbb{R}$ is called *bounded* ⟨*beschränkt*⟩ if the image set $f(D) \subseteq \mathbb{R}$ is bounded, i.e.,

$$\exists M > 0 \ \ \forall x \in D: \ |f(x)| \leq M.$$

THEOREM: Let $f\colon [a,b] \to \mathbb{R}$ be continuous. Then f is bounded and attains maximum and minimum values, i.e., there exist $x_1, x_2 \in [a,b]$ such that

$$\begin{aligned} f(x_1) &= \min f([a,b]) = \min \{f(x) : x \in [a,b]\} \qquad && (= \inf f([a,b])) \\ f(x_2) &= \max f([a,b]) = \max \{f(x) : x \in [a,b]\} \qquad && (= \sup f([a,b])). \end{aligned}$$

REMARK: (i) In the hypothesis of this theorem it is essential that the interval $[a,b]$, where f is defined and continuous, is <u>bounded</u> (i.e., $-\infty < a \leq b < \infty$) and <u>closed</u> (i.e., the boundary points a and b belong to the interval). Otherwise the statement is not true in general as can be seen from the following examples: Consider the continuous functions

$$f_1\colon]0,1] \to \mathbb{R}, x \mapsto \frac{1}{x}, \qquad f_2\colon]0,1[\to \mathbb{R}, x \mapsto x, \qquad f_3\colon [0,\infty[\to \mathbb{R}, x \mapsto x.$$

Then f_1 and f_3 are unbounded and do not attain a maximum, f_2 does neither attain a maximum nor a minimum.

(ii) As is shown by the simple example of a constant function, the locations x_1 and x_2 of a minimum or a maximum need not be unique.

Proof of the Theorem. It suffices to give the proof for boundedness from above and concerning the maximum, the case of minimum and boundedness from below can be reduced to the latter by switching to $-f$.

Let $A := \sup f([a,b]) \in \mathbb{R} \cup \{\infty\}$, then there exists a sequence (a_n) in $[a,b]$ such that $f(a_n) \to A$ $(n \to \infty)$.

Since $[a,b]$ is a bounded subset of $\mathbb{R}$ the Theorem of Bolzano-Weierstraß implies that there is a convergent subsequence $(a_{n_k})_{k\in\mathbb{N}}$. Let $x_2 := \lim\limits_{k\to\infty} a_{n_k} \in [a,b]$.

Since f is continuous we obtain that

$$\mathbb{R} \ni f(x_2) = \lim_{k\to\infty} f(a_{n_k}) = A = \sup f([a,b]).$$

In particular, $f([a,b])$ is bounded above and the supremum is a maximum, which is attained by f at $x_2 \in [a,b]$. $\square$

5.10. Uniform continuity: If we are to check continuity of a real valued function f at a certain point x_0 in its domain D, then for given $\varepsilon > 0$ we have to find $\delta > 0$ such that the condition $|f(x) - f(x_0)| < \varepsilon$ is met whenever $x \in D$ satisfies $|x - x_0| < \delta$. We observe that, in general, δ will dependend on $\varepsilon > 0$ as well as on the point x_0. Consider the following example, where the range of possible values for δ is strictly shrinking as ε gets smaller or x_0 varies: Let $D =]0,1]$ and $f\colon D \to \mathbb{R}$ with $f(x) = 1/x$, which is continuous in every point $x_0 \in D$.

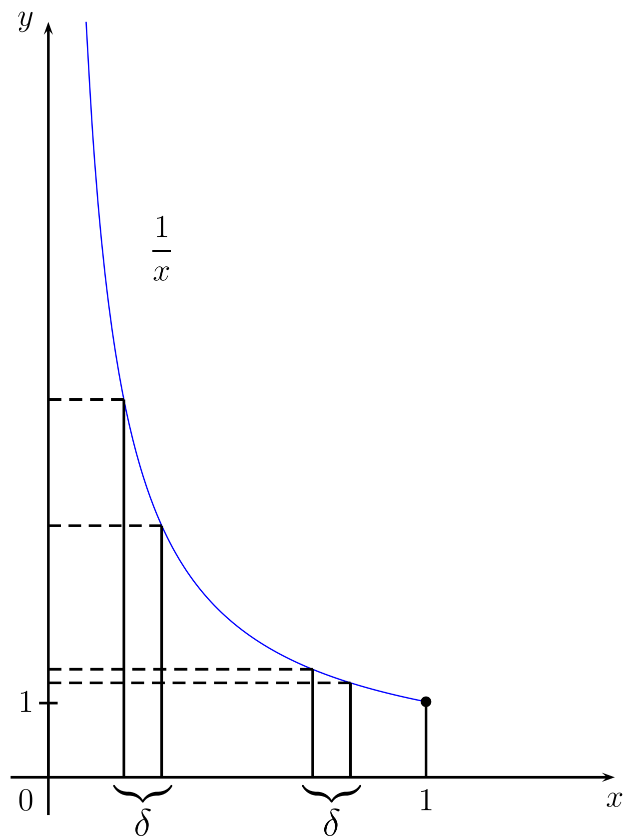
Fix some $x_0 \in D$ and $\varepsilon > 0$ arbitrarily and let us test the allowed tolerance in varying the argument in $0 < x \leq x_0$ while maintaining $|f(x) - f(x_0)| < \varepsilon$. For every $0 < \delta < x_0$ let $x_\delta := x_0 - \delta$. Then

$$|f(x_\delta) - f(x_0)| = \frac{1}{x_\delta} - \frac{1}{x_0} = \frac{x_0 - x_\delta}{x_0\, x_\delta} = \frac{\delta}{x_0\,(x_0 - \delta)}.$$

Thus requiring $|f(x) - f(x_0)| < \varepsilon$ for all $x \in]0,1]$ with $|x - x_0| < \delta$ implies $\varepsilon > \frac{\delta}{x_0^2 - x_0\delta}$. Equivalently, $\varepsilon x_0^2 - \varepsilon x_0 \delta > \delta$ and hence

$$\delta < \frac{\varepsilon x_0^2}{1 + \varepsilon\, x_0} < \varepsilon\, x_0^2.$$

This shows that the smaller $x_0 > 0$ is the smaller we have to choose $\delta > 0$ (even at fixed value of $\varepsilon > 0$).



We thus obtain a stronger form of continuity notion, if we require that for each $\varepsilon > 0$ a suitable $\delta > 0$ can be found which guarantees the typical ε-δ-estimate to hold for all pairs of points in the domain of relative distance less than δ.

DEFINITION: Let $D \subseteq \mathbb{R}$. A function $f\colon D \to \mathbb{R}$ is *uniformly continuous* ⟨*gleichmäßig stetig*⟩ (in D), if the following holds:

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x, x' \in D: \ |x - x'| < \delta \Longrightarrow |f(x) - f(x')| < \varepsilon.$$

REMARK: It is immediate from the definition that every uniformly continuous function is continuous (at every point in the domain). The converse is not true as is illustrated by the example above with $D =]0,1]$, $f(x) = 1/x$: If $x_n = 1/n$ and $x'_n = 1/(2n)$ ($n \in \mathbb{N}$, $n \geq 1$) then

$$|x_n - x'_n| = \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n}$$

is arbitrarily small when n is sufficiently large, but

$$|f(x_n) - f(x'_n)| = 2n - n = n$$

is unbounded, hence will never stay below a given ε-tolerance.

However, as the following theorem will show, there is no distinction between continuity and uniform continuity if D is a bounded closed interval.

THEOREM: If $f\colon [a,b] \to \mathbb{R}$ is continuous then f is uniformly continuous (on $[a,b]$).

Proof. (by contradiction) If f is not uniformly continuous then

$$\exists \varepsilon > 0\ \forall n \in \mathbb{N}, n > 0\ \exists x_n, x'_n \in [a,b]:\ |x_n - x'_n| < \frac{1}{n} \text{ and } |f(x_n) - f(x'_n)| \geq \varepsilon.$$

The sequence (x_n) is bounded, thus by the Theorem of Bolzano-Weierstraß possesses a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$. Let $x_0 := \lim x_{n_k} \in [a,b]$.

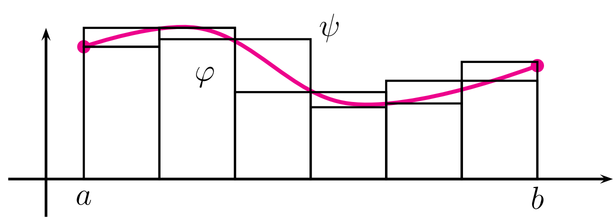
Since $|x_{n_k} - x'_{n_k}| < 1/n_{n_k} \to 0$ $(k \to \infty)$ we have that $\lim x'_{n_k} = \lim x_{n_k} = x_0$. Then the continuity of f at x_0 yields

$$0 < \varepsilon \leq |f(x_{n_k}) - f(x'_{n_k})| \to 0 \qquad (k \to \infty),$$

— a contradiction ↯ . ∎

5.11. Approximation by step (or simple) functions:

As an application of the above Theorem 5.10 we show that "the area under the graph of a continuous function" can be approximated by sums over areas of small vertical rectangles. This will be used later in the chapter on integration theory.



The rectangles can be represented as graphs of step or simple functions and the approximation result is stated in terms of these as a uniform approximation from above and from below.

Proposition: Let $f\colon [a,b] \to \mathbb{R}$ be continuous. For every $\varepsilon > 0$ there exist simple functions $\varphi, \psi\colon [a,b] \to \mathbb{R}$ with the following properties valid for all $x \in [a,b]$:

(a) $\varphi(x) \leq f(x) \leq \psi(x)$

(b) $|\psi(x) - \varphi(x)| = \psi(x) - \varphi(x) \leq \varepsilon$.

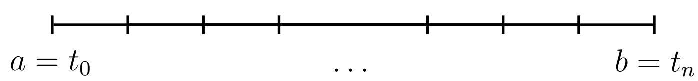
Proof. By Theorem 5.10 f is uniformly continuous on $[a,b]$. Therefore we can find $\delta > 0$ such that

$$\forall x, x' \in [a,b] : \ |x - x'| < \delta \Longrightarrow |f(x) - f(x')| < \varepsilon.$$

Choose $n \in \mathbb{N}$ large enough to ensure $(b-a)/n < \delta$ and define partition points

$$t_k := a + k \cdot \frac{b-a}{n} \qquad (k = 0, \ldots, n).$$

In this way we obtain an equidistant partition of $[a,b]$



by $t_0 = a < t_1 < \ldots < t_n = b$ with

$$t_k - t_{k-1} = \frac{b-a}{n} < \delta.$$

As heights of the approximating rectangles we choose the maximum or minimum values of f on the corresponding subintervals $[t_{k-1}, t_k]$ $(k = 1, \ldots, n)$ of the partition, that is we define

$$c_k := \max\{f(x) : t_{k-1} \leq x \leq t_k\}, \quad c_k' := \min\{f(x) : t_{k-1} \leq x \leq t_k\}.$$

By Theorem 5.9 there exist $\xi_k, \xi_k' \in [t_{k-1}, t_k]$ such that $f(\xi_k) = c_k$ and $f(\xi_k') = c_k'$ $(k = 1, \ldots, n)$. Since $|\xi_k - \xi_k'| < \delta$ we have $|c_k - c_k'| < \varepsilon$ from the uniform continuity property noted in the beginning.

Finally, we define the simple functions $\varphi, \psi\colon [a,b] \to \mathbb{R}$ as follows:

Let $\varphi(a) := f(a)$ and $\psi(a) := f(a)$,

for $t_{k-1} < x \leq t_k$ we set $\psi(x) := c_k$ and $\varphi(x) := c_k'$ $(k = 1, \ldots, n)$.

Then the conditions (a) and (b) follow by construction of φ and ψ. $\square$

5.12. Continuous inverse function: Let $A, B \subseteq \mathbb{R}$. Assume that the function $f\colon A \to B$ is bijective, then the inverse function $f^{-1}\colon B \to A$ exists. If we know that f is continuous, does this imply that f^{-1} is also continuous? In general, the answer is 'no' (see the exercises for an example).

It turns out that there is a positive answer to the above question under the two additional hypotheses of strict monotonicity on f and that A is an interval.

Recall that f is strictly increasing (resp. decreasing) if $x_1 < x_2$ implies $f(x_1) < f(x_2)$ (resp. $f(x_1) > f(x_2)$) and that a strictly monotone function necessarily is injective.

THEOREM: Let $I \subseteq \mathbb{R}$ be an interval and $f\colon I \to \mathbb{R}$ be continuous and strictly increasing (resp. decreasing). Then f maps the interval I bijectively onto the interval $J := f(I)$ and the corresponding inverse function $J \to I$ is also continuous and strictly increasing (resp. decreasing).

(Strictly speaking, we deal here with the inverse of the map $\tilde{f}\colon I \to J$, $x \mapsto f(x)$; but we will follow the common abuse of language and denote $\tilde{f}$ again by f and its inverse by $f^{-1}\colon J \to I$.)

Proof. We present the proof for the case that f is strictly increasing, the case where f is strictly decreasing is reduced to this by considering $-f$ instead.

Corollary 5.8 implies that $J = f(I)$ is an interval. Since f is strictly increasing it is injective, hence f is bijective as a map $I \to J$. Let $f^{-1}\colon J \to I$ denote the inverse of this map.

Note that for $x_1, x_2 \in I$ the inequality $f(x_1) < f(x_2)$ in turn implies $x_1 < x_2$ (since then $x_1 = x_2$ is impossible with different function values and $x_1 > x_2$ contradicts the fact that f increases), therefore we have

$$\forall x_1, x_2 \in I: \quad x_1 < x_2 \Longleftrightarrow f(x_1) < f(x_2),$$

which shows that f^{-1} is strictly increasing as well.

It remains to prove that f^{-1} is continuous at every point $b \in J$.

Case 1, if $b \in J$ is not a boundary point of J: Let $a := f^{-1}(b) \in I$. Then a is not a boundary point of I (for otherwise by monotonicity b would have to be boundary point of J). Choose $\varepsilon > 0$ so small that both $a - \varepsilon$ and $a + \varepsilon$ belong to I. Since f is strictly increasing we have

$$f(a - \varepsilon) < f(a) = b < f(a + \varepsilon).$$

Thus we can find $\delta > 0$ such that

$$f(a - \varepsilon) < b - \delta < b + \delta < f(a + \varepsilon),$$

which simply means that

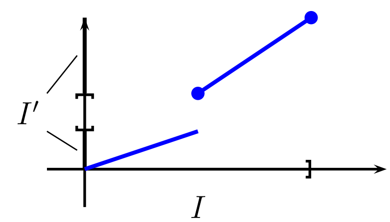
$$f^{-1}(U_\delta(b)) \subseteq U_\varepsilon(f^{-1}(b))$$

and therefore proves the continuity of f^{-1} at b.

Case 2, if $b \in J$ is the left boundary point of J: Then $a := f^{-1}(b)$ has to be the left boundary point of I (since f is strictly increasing). We can copy the proof of case 1 with the only changes that we use $U_\delta \cap J$ and $U_\varepsilon(f^{-1}(b)) \cap I$ as neighborhoods and the chain of inequalities reads $f(a) = b < b + \delta < f(a + \varepsilon)$.

Case 3, if $b \in J$ is the right boundary point of J: Similarly to case 2. □

REMARK: The second part of the above proof shows, in fact, the following result: If $I \subseteq \mathbb{R}$ is an interval and $f\colon I \to \mathbb{R}$ is strictly increasing (not necessarily continuous!), then $f^{-1}\colon f(I) \to I$ is continuous. But $f(I)$ need not be an interval, if f is discontinuous:



ROOT FUNCTIONS: As an application of the above Theorem we consider for any $k \in \mathbb{N}$, $k \geq 1$, the functions[4]

$$f_{2k}\colon [0,\infty[\to [0,\infty[,\ x \mapsto x^{2k}, \qquad \text{and} \qquad f_{2k+1}\colon \mathbb{R} \to \mathbb{R},\ x \mapsto x^{2k+1}.$$

All these functions are continuous, strictly increasing, and bijective, therefore the corresponding inverse functions

$$f_{2k}^{-1}\colon [0,\infty[\to [0,\infty[\qquad \text{and} \qquad f_{2k+1}^{-1}\colon \mathbb{R} \to \mathbb{R}$$

are continuous and strictly increasing. We use the the following notation for their function values (for x in the appropriate domain)

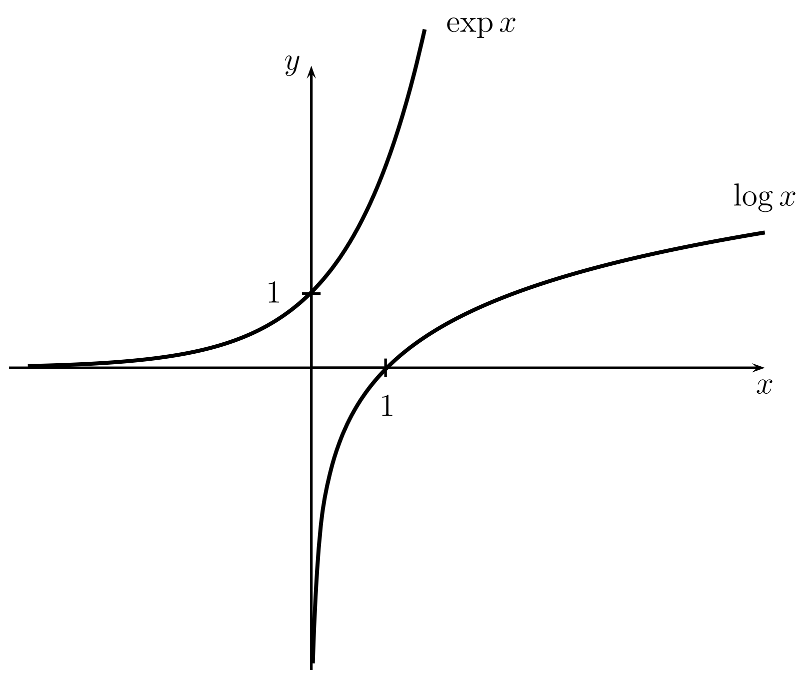
$$n \geq 2: \quad \sqrt[n]{x} = x^{\frac{1}{n}} := f_n^{-1}(x).$$

[4]Altough the origin of the radical symbol $\sqrt{\ }$ is rather unclear, many believe that it is an abbreviation of the Latin word radix (root). The symbol was first used in Germany in the 16th century without the winkulum (i.e. the term $\sqrt{a+b}$ was originally denoted by $\surd(a+b)$)

§6. ELEMENTARY TRANSCENDENTAL FUNCTIONS

6.1. Proposition: The exponential function $\exp\colon \mathbb{R} \to \mathbb{R}$ is continuous, strictly increasing, and $\exp(\mathbb{R}) = \,]0,\infty[$. Its inverse function $\log\colon]0,\infty[\to \mathbb{R}$ is continuous, strictly increasing and is called the *natural logarithm* ⟨natürlicher Logarithmus⟩.[1] Furthermore, the following functional equation holds for all $x, y \in \,]0,\infty[$:

$$\log(x \cdot y) = \log(x) + \log(y). \tag{6.1}$$



[1]Logarithms have been introduced by the Scottish mathematician John Napier in 1614, the term logarithm being derived from the Greek words λόγος ("proportion") and ἀριθμός ("number").
Nowadays, there are different notations used for logarithmic functions: While mathematicians often write $\log(x)$ for the natural logarithm and $\log_b(x)$ for the base-b logarithm, in many calculus textbooks a notations such as $\ln(x)$ can be found for the natural logarithm, $\lg(x)$ for the base-10 logarithm etc. In these lecture notes we will always use $\log(x)$ to denote the natural logarithm and $\log_b(x)$ for the base-b logarithm.

Proof. The continuity of exp has already been established in the previous section.

Step 1: We show that exp is strictly increasing.

For every $\xi > 0$ we have

$$\exp(\xi) = 1 + \xi + \sum_{k=2}^{\infty} \frac{\xi^k}{k!} > 1 + \xi > 1.$$

Let $x_1 < x_2$ then $\xi := x_2 - x_1 > 0$ and

$$\exp(x_2) = \exp(x_1 + \xi) = \exp(x_1) \cdot \exp(\xi) > \exp(x_1).$$

Step 2: We show that $\exp(\mathbb{R}) =]0, \infty[$.

Since $\exp(x) > 0$ for all $x \in \mathbb{R}$ [4.23] we have $\exp(\mathbb{R}) \subseteq]0, \infty[$. To show the reverse inclusion relation, it suffices to show that

$$\lim_{n\to\infty} \exp(n) = \infty \quad \text{and} \quad \lim_{n\to\infty} \exp(-n) = 0,$$

since then by the intermediate value theorem all values in $]0, \infty[$ are indeed attained.

For $n \in \mathbb{N}$ we had shown $\exp(n) = e^n$. Since $e > 2$ we therefore have $e^n \to \infty$ $(n \to \infty)$, which implies that

$$\exp(-n) = \frac{1}{\exp(n)} = \frac{1}{e^n} \to 0 \quad (n \to \infty).$$

We may thus define the function $f\colon \mathbb{R} \to]0, \infty[$, $f(x) := \exp(x)$, which is again continuous and strictly increasing. Due to Theorem 5.12 the inverse function $\log := f^{-1}\colon\ ]0, \infty[\to \mathbb{R}$ is also continuous and strictly increasing.

Step 3: We prove the functional equation (6.1).

Let $x, y \in]0, \infty[$ and put $\xi := \log(x)$, $\eta := \log(y)$. Then $\exp(\xi + \eta) = \exp(\xi) \cdot \exp(\eta) = x \cdot y$ and therefore

$$\log(x \cdot y) = \xi + \eta = \log(x) + \log(y).$$

□

Remark: As a simple consequence we obtain

$$\log(x^k) = k \log(x) \qquad \text{for all } x > 0 \text{ and } k \in \mathbb{N}.$$

6.2. Real powers and general exponentials: We can use the exponential function and the logarithm to give a simple definition of expressions of the form r^s when $r > 0$ and $s \in \mathbb{R}$. Observe that if s is a natural number then $r^s = \exp(\log(r^s)) = \exp(s \log(r))$.

Definition: (i) Let $r > 0$ and $s \in \mathbb{R}$ then

$$r^s := \exp(s \log(r)) \in \,]0, \infty[.$$

As an immediate consequence of this definition we thus obtain the formula

$$\log(r^s) = s \log(r) \qquad (r > 0, s \in \mathbb{R}).$$

(ii) For any $\alpha \in \mathbb{R}$ we define general power or root functions $w_\alpha \colon]0, \infty[\to \mathbb{R}$ by

$$x \mapsto x^\alpha = \exp(\alpha \log(x)).$$

(iii) The Exponential function with base $a \in]0, \infty[$ is given by

$$\exp_a \colon \mathbb{R} \to \mathbb{R}, \quad \exp_a(x) := a^x = \exp(x \log(a)).$$

Note that $\exp(x) = \exp_e(x) = e^x$ for all $x \in \mathbb{R}$.

We list basic properties of the exponential function with base $a > 0$, which are immediate consequences of those for the exponential function and the logarithm.

Proposition: $\exp_a$ is continuous on $\mathbb{R}$ and we have the following:

(i) If $a > 1$ then $\exp_a$ is strictly increasing, if $0 < a < 1$ then $\exp_a$ is strictly decreasing.

(ii) The functional equation: $a^{x+y} = a^x \cdot a^y$ for all $x, y \in \mathbb{R}$.

(iii) Let $a > 0$. For all $m \in \mathbb{Z}$: $\exp_a(m) = a^m = a \cdot a \cdots a$ (m factors).
(In other words, the notation a^m is consistent with the algebraically defined integer powers.)

(iv) Let $a > 0$. If $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, $q \geq 1$, then $a^{\frac{p}{q}} = \sqrt[q]{a^p} = (a^p)^{\frac{1}{q}}$.
(Consistency with the root functions as defined in 5.12.)

(v) Let $a > 0$. For all $x, y \in \mathbb{R}$: $(a^x)^y = a^{xy} = (a^y)^x$.

(vi) For all $a > 0$, $b > 0$, and $x \in \mathbb{R}$: $a^x \cdot b^x = (a \cdot b)^x$.

(vii) Let $a > 0$. For all $x \in \mathbb{R}$: $(\frac{1}{a})^x = a^{-x}$.

Proof. Immediate from the definition. $\square$

6.3. A collection of useful limits:

1) For all $k \in \mathbb{N}$: $\lim\limits_{x \to \infty} \dfrac{e^x}{x^k} = \infty$.

We may assume that $x > 0$, which yields $e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} > \dfrac{x^{k+1}}{(k+1)!}$. Therefore

$$\frac{e^x}{x^k} > \frac{x}{(k+1)!} \to \infty \qquad \text{as } x \to \infty.$$

2) For all $k \in \mathbb{N}$: $\lim\limits_{x\to\infty} \dfrac{x^k}{e^x} = 0$. (Follows directly from 1).)

3) For all $k \in \mathbb{N}$: $\lim\limits_{x\searrow 0} x^k e^{1/x} = \infty$.

Writing $y = 1/x$ gives $\lim\limits_{x\searrow 0} x^k e^{1/x} = \lim\limits_{y\to\infty} \dfrac{e^y}{y^k} = \infty$ (by 1)).

4) $\lim\limits_{x\to\infty} \log(x) = \infty$ and $\lim\limits_{x\searrow 0} \log(x) = -\infty$.

Both assertions follows from Proposition 6.1, which implies that $\log\colon]0,\infty[\to \mathbb{R}$ is strictly increasing and bijective.

5) For all $\alpha > 0$: $\lim\limits_{x\searrow 0} x^\alpha = 0$ and $\lim\limits_{x\searrow 0} x^{-\alpha} = \infty$.

The second assertion follows from the first. To prove the first we write $x = e^{-y/\alpha}$ (equivalently, $y = -\alpha\log(x)$) and compute

$$\lim_{x\searrow 0} x^\alpha = \lim_{y\to\infty} e^{-y} = 0.$$

6) For all $\alpha > 0$: $\lim\limits_{x\to\infty} \dfrac{\log(x)}{x^\alpha} = 0$.

We may assume that $x > 0$ and write $x^\alpha = e^y$ (equivalently, $y = \alpha\log(x)$) to obtain

$$\lim_{x\to\infty} \frac{\log(x)}{x^\alpha} = \frac{1}{\alpha}\cdot \lim_{y\to\infty} \frac{y}{e^y} = 0.$$

7) For all $\alpha > 0$: $\lim\limits_{x\searrow 0} x^\alpha \log(x) = 0$.

Upon writing $x = 1/y$ (so that $y \to \infty$) we have $x^\alpha \log(x) = -\log(y)/y^\alpha$, then use 6).

8) $\lim\limits_{\substack{x\to 0\\ x\neq 0}} \dfrac{e^x - 1}{x} = 1$

The remainder term estimate (4.4), (4.5) for the exponential sum gives for all x with $|x| \le 3/2$ that

$$|e^x - (1+x)| \le 2\,\frac{|x|^2}{2!} = |x|^2.$$

In other words, if $0 < |x| \le 3/2$ then

$$|\frac{e^x - 1}{x} - 1| = \frac{|e^x - 1 - x|}{|x|} \le |x| \to 0 \qquad \text{as } x \to 0.$$

6.4. The Landau-symbols[2] — comparison of asymptotic growth:

DEFINITION: (i) Let $a \in \mathbb{R}$ and $f, g\colon]a, \infty[\to \mathbb{R}$. We write

$$f(x) = o(g(x)) \quad (x \to \infty),$$

to mean that $\forall \varepsilon\ \exists R > a$: $|f(x)| \leq \varepsilon\, |g(x)|$ holds for all $x \geq R$.
("$f(x)$ is a little-oh of $g(x)$ as x tends to infinity".)

We write

$$f(x) = O(g(x)) \quad (x \to \infty),$$

to mean that $\exists K > 0\ \exists R > a$: $|f(x)| \leq K\, |g(x)|$ holds for all $x \geq R$.
("$f(x)$ is a big-oh of $g(x)$ as x tends to infinity".)

(ii) Let $D \subseteq \mathbb{R}$ and x_0 be an adherent point of D and $f, g\colon D \to \mathbb{R}$. We write

$$f(x) = o(g(x)) \quad (x \to x_0, x \in D),$$

to mean that $\forall \varepsilon > 0\ \exists \delta > 0$: $|f(x)| \leq \varepsilon\, |g(x)|$ holds $\forall x \in U_\delta(x_0) \cap D$.

We write

$$f(x) = O(g(x)) \quad (x \to x_0, x \in D),$$

to mean that $\exists K > 0\ \exists \delta > 0$: $|f(x)| \leq K\, |g(x)|$ holds $\forall x \in U_\delta(x_0) \cap D$.

REMARK: (i) If, for example, $g(x) \neq 0$ for all x near x_0 and $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)} = 0$, then $f(x) = o(g(x))$ $(x \to x_0)$.

(ii) We will occasionally make use of a notation like

$$f(x) = h(x) + O(g(x))$$

to mean that $f(x) - h(x) = O(g(x))$.

EXAMPLES: 1) If $\alpha > 0$ then $\log(x) = o(x^\alpha)$ $(x \to \infty)$ [cf. 6.3.6)].

2) $e^x = 1 + x + O(x^2)$ $(x \to 0)$, since $|e^x - 1 - x| \leq |x|^2$ when $|x| \leq 3/2$ [as seen in 6.3.8)].

3) $f(x) = f(x_0) + o(1)$ $(x \to x_0)$ $\iff$ $\lim\limits_{x \to x_0} f(x) = f(x_0)$ $\iff$ f is continuous at x_0.

4) If p is a polynomial function of degree m, then $p(x) = O(x^m)$ $(x \to \infty)$.

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[2] Edmund Georg Hermann Landau (1877–1938) [ˈɛdmʊnt ˈgeˑɔɐk ˈhɛɐman ˈlandaʊ], German mathematician

6.5. A digression into basic analysis on $\mathbb{C}$:

Let $z = x + iy \in \mathbb{C}$, so that $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$ and $\overline{z} = x + iy \in \mathbb{C}$. Then the product $z\overline{z} = (x+iy)(x-iy) = x^2+y^2$ always gives a non-negative real number and we may define the *absolute value* of z by

$$|z| := \sqrt{z\overline{z}} = \sqrt{x^2+y^2} = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}.$$

Note that, since real numbers x are embedded as complex numbers[3] of the form $x + i0$, the absolute value of x as a real number is the same as its absolute value as a complex number.

LEMMA: The absolute value as a map $|.|\colon \mathbb{C} \to \mathbb{R}$ has the following properties, valid for all $z, z_1, z_2 \in \mathbb{C}$:

(i) $|z| \geq 0$ and $|z| = 0 \Leftrightarrow z = 0$

(ii) $|\overline{z}| = |z|$

(iii) $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

(iv) $|\mathrm{Re}(z)| \leq |z|$ and $|\mathrm{Im}(z)| \leq |z|$

(v) $|z_1 + z_2| \leq |z_1| + |z_2|$ (triangle inequality).

Proof. Let $z = x + iy$, $z_k = x_k + iy_k$ $(k = 1, 2)$.

(i): $|z| \geq 0$ and $|0| = 0$ is immediate. If $|z| = 0$, then $0 \leq x^2 \leq x^2 + y^2 = 0$ as well as $0 \leq y^2 \leq x^2 + y^2 = 0$, hence $x = 0$ and $y = 0$.

(ii): Clear from the definition.

(iii): $|z_1 z_2|^2 = (z_1 z_2)(\overline{z_1 z_2}) = (z_1\overline{z_1})(z_2\overline{z_2}) = |z_1|^2|z_2|^2$.

(iv): $|x|^2 = x^2 \leq x^2 + y^2$ and $|y|^2 = y^2 \leq x^2 + y^2$.

(v): $|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1} + \overline{z_2}) = |z_1|^2 + 2\,\mathrm{Re}(z_1\overline{z_2}) + |z_2|^2 \leq$ [by (iv)]
$|z_1|^2 + 2|z_1||z_2| + |z_2|^2 = (|z_1| + |z_2|)^2$. □

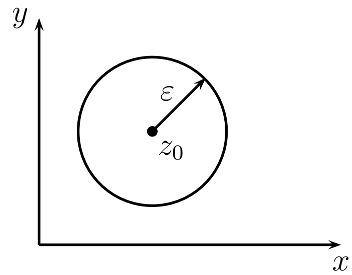
(a) Convergence in $\mathbb{C}$:

DEFINITION: (i) Let $z_0 \in \mathbb{C}$ and $\varepsilon > 0$, then the *ε-neighborhood* $U_\varepsilon(z_0)$ of z_0 is defined by

$$U_\varepsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| \leq \varepsilon\}.$$

In a planar representation of the complex numbers, $U_\varepsilon(z_0)$ is an open disk with radius ε around z_0:

[3] Square roots of negative numbers were “invented” by the Italian mathematicians Gerolamo Cardano and Raffaele Bombelli. In modern mathematics, complex numbers are generally denoted by $z = a + bi$ or sometimes $z = a + bj$.



(ii) A *sequence of complex numbers* is a map $c\colon \mathbb{N} \to \mathbb{C}$. We use the notation $(c_n)_{n\in\mathbb{N}}$ with $c_n := c(n)$.

(iii) The complex sequence (c_n) *converges to* $z_0 \in \mathbb{C}$, denoted by $c_n \to z_0$ $(n \to \infty)$ or $\lim\limits_{n\to\infty} c_n = z_0$, if

$$\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n \geq n: \quad |c_n - z_0| < \varepsilon.$$

Equivalently, we may require that

$$\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n \geq n: \quad c_n \in U_\varepsilon(z_0).$$

PROPOSITION: Let (c_n) be a complex sequence. Then the following are equivalent:

(i) (c_n) is convergent (in $\mathbb{C}$).

(ii) Both sequences $(\operatorname{Re} c_n)$ and $(\operatorname{Im} c_n)$ converge (in $\mathbb{R}$).

In this case we have $\lim c_n = \lim \operatorname{Re} c_n + i \lim \operatorname{Im} c_n$.

Proof. Let $a_n := \operatorname{Re} c_n$, $b_n := \operatorname{Im} c_n$ $(n \in \mathbb{N})$.

(i) $\Rightarrow$ (ii): Let $c := \lim c_n$, $a := \operatorname{Re} c$ and $b := \operatorname{Im} c$.

If $\varepsilon > 0$ is given arbitrarily, we can find $n_0 \in \mathbb{N}$ such that $|c_n - c| < \varepsilon$ holds for all $n \geq n_0$. Therefore we have for all $n \geq n_0$

$$|a_n - a| = |\operatorname{Re}(c_n - c)| \leq |c_n - c| < \varepsilon \quad \text{as well as} \quad |b_n - b| = |\operatorname{Re}(c_n - c)| \leq |c_n - c| < \varepsilon,$$

which proves that $a_n \to a$ and $b_n \to b$.

(ii) $\Rightarrow$ (i): Let $\varepsilon > 0$. Put $a := \lim a_n$, $b := \lim b_n$, and $c := a + ib$. Choose $n_0 \in \mathbb{N}$ such that $|a_n - a| < \varepsilon/2$ and $|b_n - b| < \varepsilon/2$ holds for all $n \geq n_0$. Then we have for $n \geq n_0$

$$|(a_n + ib_n) - (a + ib)| = |(a_n - a) + i(b_n - b)| \leq |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

thus $c_n \to c$ $(n \to \infty)$.

$\square$

COROLLARY: If (c_n) is a convergent complex sequence, then $\lim \overline{c_n} = \overline{\lim c_n}$.

Proof. $\overline{\lim c_n} = \lim \operatorname{Re} c_n - i \lim \operatorname{Im} c_n = \lim \operatorname{Re} \overline{c_n} + i \lim \operatorname{Im} \overline{c_n} = \lim \overline{c_n}$. □

Precisely as in the case of real sequences one proves the following rules for basic operations with convergent sequences:

If (c_n), (d_n) are convergent complex sequences and $\lambda \in \mathbb{C}$, then

$$\begin{aligned} \lim(c_n + d_n) &= \lim c_n + \lim d_n \\ \lim(\lambda c_n) &= \lambda \lim c_n \\ \lim(c_n d_n) &= (\lim c_n)(\lim d_n) \\ \lim \frac{c_n}{d_n} &= \frac{\lim c_n}{\lim d_n} \quad \text{(if } d_n \neq 0 \text{ for almost all } n\text{).} \end{aligned}$$

THEOREM (COMPLETENESS OF $\mathbb{C}$): A sequence (c_n) of complex numbers converges if and only if it is a *Cauchy sequence*, i.e.,

(6.2) $$\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n, m \geq n_0 : \quad |c_n - c_m| < \varepsilon.$$

Proof. (6.2) $\Leftrightarrow$ both $(\operatorname{Re} c_n)$ and $(\operatorname{Im} c_n)$ are Cauchy sequences in $\mathbb{R}$ $\Leftrightarrow$ both $(\operatorname{Re} c_n)$ and $(\operatorname{Im} c_n)$ are convergent in $\mathbb{R}$ $\underset{\text{[Prop.]}}{\Leftrightarrow}$ (c_n) is convergent in $\mathbb{C}$. □

(b) Complex series:

DEFINITION: Let (c_n) be a sequence of complex numbers. The series $\sum_{k=0}^{\infty} c_k$ is *convergent*, if the corresponding sequence (s_n) of partial sums $s_n := \sum_{k=0}^{n} c_k$ is convergent (in $\mathbb{C}$). The series $\sum_{k=0}^{\infty} c_k$ is *absolutely convergent*, if the (real) series $\sum_{k=0}^{\infty} |c_k|$ converges (in $\mathbb{R}$).

PROPOSITION: (i) Basic comparison test: Let (a_n) be a sequence with $a_n \geq 0$ (thus, real!) and such that $\sum a_n$ is convergent. If (c_n) is a complex sequence with the property

$$\exists n_0 \in \mathbb{N}\ \forall n \geq n_0 : \quad |c_n| \leq a_n,$$

then the series $\sum_{n=0}^{\infty} c_n$ is absolutely convergent.

(ii) The root test and the ratio test both are valid for complex sequences literally as stated in Section 4. In particular, if a complex sequence (c_n) with $c_n \neq 0$ (for almost all n) satisfies

$$\exists \theta \in [0, 1[: \quad \left| \frac{c_{n+1}}{c_n} \right| \leq \theta,$$

then the series $\sum_{n=0}^{\infty} c_n$ is absolutely convergent.

(iii) The Proposition concerning the Cauchy product for absolutely convergent series holds literally as stated in Section 4.

Proof. Can be literally copied from those of the corresponding statements about real series. $\square$

(c) Continuity of functions of a complex variable:

DEFINITION: Let $D \subseteq \mathbb{C}$, $z_0 \in D$. A function $f\colon D \to \mathbb{C}$ is *continuous* at z_0, if

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall z \in D: \quad |z - z_0| < \delta \Longrightarrow |f(z) - f(z_0)| < \varepsilon,$$

or equivalently

$$\forall \varepsilon > 0\ \exists \delta > 0: \quad f(U_\delta(z_0) \cap D) \subseteq U_\varepsilon(f(z_0)).$$

f is said to be continuous on D, if f is continuous at all points in D.

REMARK: As in the case of functions on $\mathbb{R}$, continuity can be tested by sequences (the proof is also a literal translation of that in the real case): $f\colon D \to \mathbb{C}$ is continuous at $w \in D$ if and only if or all sequences (z_n) with $z_n \in D$ and $z_n \to w$ $(n \to \infty)$ we have that $\lim f(z_n) = f(w)$. We also express the latter fact by $\lim\limits_{z \to w} f(z) = f(w)$.

6.6. The complex exponential function:

THEOREM: (i) For all $z \in \mathbb{C}$ the series $\sum\limits_{k=0}^{\infty} \frac{z^k}{k!}$ is absolutely convergent. We thus define the complex exponential function $\exp\colon \mathbb{C} \to \mathbb{C}$ by

$$\exp(z) = e^z := \sum_{k=0}^{\infty} \frac{z^k}{k!} \qquad (z \in \mathbb{C}).$$

When restricted to $\mathbb{R}$ it coincides with the exponential function defined in Section 4. We will continue to use the same notation for both functions.

(ii) For all $N \in \mathbb{N}$

$$\exp(z) = \sum_{k=0}^{N} \frac{z^k}{k!} + R_{N+1}(z),$$

where

$$|R_{N+1}(z)| \leq 2\frac{|z|^{N+1}}{(N+1)!} \qquad (z \in \mathbb{C}, |z| \leq 1 + \frac{N}{2}).$$

(iii) We have the functional equation

$$\forall z_1, z_2 \in \mathbb{C}: \quad \exp(z_1 + z_2) = \exp(z_1) \cdot \exp(z_2).$$

(iv) For all $z \in \mathbb{C}$: $\exp(\overline{z}) = \overline{\exp(z)}$.

(v) For all $z \in \mathbb{C}$: $\exp(z) \neq 0$.

(vi) $\displaystyle\lim_{z\neq 0, z\to 0} \frac{e^z - 1}{z} = 1$.

(vii) $\exp\colon \mathbb{C} \to \mathbb{C}$ is continuous (at all points of $\mathbb{C}$).

Proof. (i): If $z = 0$ the assertion is trivial. If $z \neq 0$ we apply the ratio test with $c_n = z^k/k!$. For all $n \geq 2|z|$ we have

$$\left|\frac{c_{n+1}}{c_n}\right| = \left|\frac{z^{n+1}n!}{z^n(n+1)!}\right| = \frac{|z|}{n+1} \leq \frac{1}{2} < 1,$$

which proves absolute convergence.

If we temporarily use the notation $\exp_{\mathbb{R}}$ for the (real) exponential function defined in Section 4, then for $x \in \mathbb{R}$ we have $\exp(x + i0) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \exp_{\mathbb{R}}(x)$.

(ii), (iii): Literally as in the corresponding proofs in Section 4.

(iv): Let $s_n(z) := \sum_{k=0}^{n} z^k/k!$ and use 6.5(a): $\exp(\overline{z}) = \lim s_n(\overline{z}) = \overline{\lim s_n(z)} = \overline{\exp(z)}$.

(v): The functional equation gives $\exp(z)\exp(-z) = \exp(z - z) = \exp(0) = 1$, hence $\exp(z) \neq 0$.

(vi): By (ii) we have $|e^z - 1 - z| \leq 2\frac{|z|^2}{2} = |z|^2$ for all $|z| \leq 3/2$, hence

$$\left|\frac{e^z - 1}{z} - 1\right| \leq |z| \to 0 \quad (z \to 0).$$

(vii): By (vi) we have $e^z - 1 = O(z)$ $(z \to 0)$. Therefore $\lim_{z\to 0} \exp(z) = 1 = \exp(0)$, which shows continuity of exp at 0.

Let $w \in \mathbb{C}$ arbitrary and assume that (z_n) is a sequence in $\mathbb{C}$ such that $z_n \to w$ $(n \to \infty)$. Then $z_n - w \to 0$, thus

$$1 = \exp(0) = \lim_{n\to\infty} \exp(z_n - w) = \lim_{n\to\infty} \exp(z_n)\exp(-w),$$

which implies that $\lim \exp(z_n) = \exp(w)$, hence the continuity of exp at w. $\square$

6.7. Trigonometric functions ⟨*trigonometrische Funktionen oder Winkelfunktionen*⟩[4]:

Definition: We define the *cosine (function)* ⟨*Cosinus (Funktion)*⟩ by

$$\cos\colon \mathbb{R} \to \mathbb{R}, \quad \cos(x) := \mathrm{Re}(\exp(ix)) = \mathrm{Re}(e^{ix}),$$

and the *sine (function)* ⟨*Sinus (Funktion)*⟩ by

$$\sin\colon \mathbb{R} \to \mathbb{R}, \quad \sin(x) := \mathrm{Im}(\exp(ix)) = \mathrm{Im}(e^{ix}).$$

Basic properties: (i) Since $e^{ix} = \mathrm{Re}(e^{ix}) + i\,\mathrm{Im}(e^{ix})$ we obtain *Euler's formula*

$$\forall x \in \mathbb{R}: \qquad e^{ix} = \cos(x) + i\sin(x). \tag{6.3}$$

Furthermore, cos and sin are continuous $\mathbb{R} \to \mathbb{R}$, since $\exp(ix_n) \to \exp(ia)$ if and only if $\mathrm{Re}(\exp(ix_n)) \to \mathrm{Re}(\exp(ia))$ and $\mathrm{Im}(\exp(ix_n)) \to \mathrm{Im}(\exp(ia))$.

(ii) Geometric interpretation: Since any real t gives $|e^{it}|^2 = e^{it} \cdot \overline{(e^{it})} = e^{it}e^{-it} = e^0 = 1$, we obtain

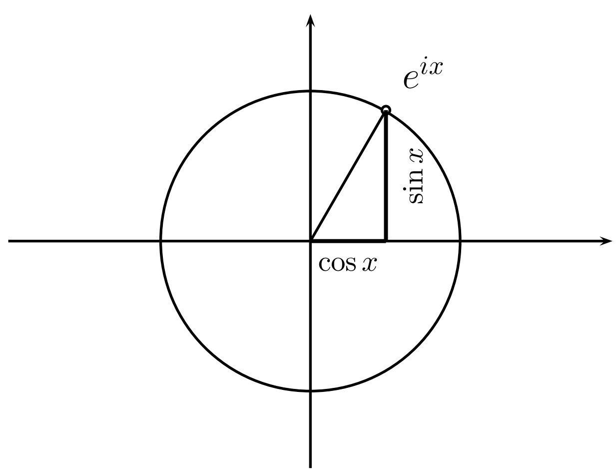
$$|e^{it}| = 1 \qquad \forall t \in \mathbb{R}.$$

Therefore every number of the form e^{it} lies on the unit circle

$$S^1 := \{z \in \mathbb{C} : |z| = 1\} \cong \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$$

and $(\cos(t), \sin(t))$ represents the (Cartesian) coordinates in the plane. In particular, we have the relation

$$\cos^2(x) + \sin^2(x) = 1 \qquad \forall x \in \mathbb{R}. \tag{6.4}$$



[4]These so-called trigonometric functions have a very long history: They were first used by the Babylonians in around 1900 BC and later in the Hellenistic world, in medieval India, in the Islamic Persia and in the medieval Europe. The terms sine and cosine (from the Latin sinus, i.e. "arch ") were introduced by the German mathematician Georg von Peuerbach.

Remark: Note that we avoided any reference to notions like arc length ⟨*Bogenlänge*⟩ or angle ⟨*Winkel*⟩ in defining the trigonometric functions for reasons of a deductive presentation. Arc length will be introduced rigorously, and in more generality, later during the course (based on the notion of integrals along curves), but it certainly is useful to have the intuitive meaning at hand already as suggested by the above geometric interpretation.

(iii) Recall that for any complex number w the real and imaginary part can be obtained from $\mathrm{Re}(w) = (w + \overline{w})/2$ and $\mathrm{Im}(w) = (w - \overline{w})/2i$. Therefore we have

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}, \qquad \sin(x) = \frac{e^{ix} - e^{-ix}}{2i},$$

which in turn implies

$$\cos(-x) = \cos(x) \quad \text{and} \quad \sin(-x) = -\sin(x),$$

telling that cos is an even ⟨*gerade*⟩ function (the graph is symmetric with respect to the vertical axis) and sin is an odd ⟨*ungerade*⟩ function (the graph is reflected by lines through the origin $(0,0)$).

(iv) The fundamental relations for the addition of arguments (“angles”) ⟨*Additionstheoreme*⟩ are the following: For all $x, y \in \mathbb{R}$

$$\begin{aligned} \cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \\ \sin(x+y) &= \cos(x)\sin(y) + \sin(x)\cos(y) \end{aligned}$$

and

$$\begin{aligned} \cos(x) - \cos(y) &= -2\sin\frac{x+y}{2}\sin\frac{x-y}{2} \\ \sin(x) - \sin(y) &= 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}. \end{aligned}$$

Proof. The first two equations are obtained by taking real and imaginary parts in the relation

$$e^{i(x+y)} = e^{ix} \cdot e^{iy}.$$

The third (resp. fourth) equation follows from the first (resp. second) equation upon setting $u = (x+y)/2$ and $v = (x-y)/2$ ($\Leftrightarrow x = u + v$, $y = u - v$):

$$\begin{aligned} \cos(x) - \cos(y) &= \cos(u+v) - \cos(u-v) \\ &= \cos(u)\cos(v) - \sin(u)\sin(v) - \big(\cos(u)\cos(v) + \sin(u)\sin(v)\big) \\ &= -2\sin(u)\sin(v) = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}, \end{aligned}$$

and similarly for the last equation. $\square$

(v) The natural integer powers of i show a simple repetitive pattern: Since $i^2 = -1$, $i^3 = i^2 i = -i$, $i^4 = i^3 i = -i^2 = 1$, we have for $n \in \mathbb{N}$ that

$$i^n = \begin{cases} 1 & \text{if } n = 4m \text{ for some } m \in \mathbb{N} \quad (\Leftrightarrow n \equiv 0 \mod 4) \\ i & \text{if } n = 4m+1 \text{ for some } m \in \mathbb{N} \quad (\Leftrightarrow n \equiv 1 \mod 4) \\ -1 & \text{if } n = 4m+2 \text{ for some } m \in \mathbb{N} \quad (\Leftrightarrow n \equiv 2 \mod 4) \\ -i & \text{if } n = 4m+3 \text{ for some } m \in \mathbb{N} \quad (\Leftrightarrow n \equiv 3 \mod 4). \end{cases}$$

Therefore we obtain for all $x \in \mathbb{R}$

$$\begin{aligned} \cos(x) + i\sin(x) = e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty} \frac{i^n x^n}{n!} \\ = \underbrace{\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}}_{\mathrm{Re}(e^{ix})} + i \cdot \underbrace{\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}}_{\mathrm{Im}(e^{ix})}, \end{aligned}$$

which proves the following series expansions for cosine and sine:

$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}, \qquad \sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}.$$

(vi) Suppose we are to use the above series expansions to approximate cosine and sine for small x by simply dropping all terms that contain x to quadratic or higher order. If justified, this would give the following simple heuristic relations when $|x|$ is small:

$$\cos(x) \approx 1 \quad \text{and} \quad \sin(x) \approx x.$$

As a matter of fact, we have the limit equations

$$\lim_{x \neq 0, x \to 0} \frac{\cos(x) - 1}{x} = 0, \qquad \lim_{x \neq 0, x \to 0} \frac{\sin(x)}{x} = 1.$$

Proof. By Theorem 6.6(vi) we have that

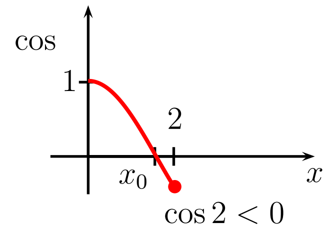
$$1 + i \cdot 0 = \lim_{x \to 0} \frac{e^{ix} - 1}{ix} = \lim_{x \to 0} \mathrm{Re}(\frac{e^{ix} - 1}{ix}) + i \cdot \lim_{x \to 0} \mathrm{Im}(\frac{e^{ix} - 1}{ix}).$$

Therefore we have for $x \in \mathbb{R}$ as $x \to 0$

$$\begin{aligned} \frac{\cos(x) - 1}{x} &= -\mathrm{Im}(\frac{\cos(x) - 1 + i\sin(x)}{ix}) = -\mathrm{Im}(\frac{e^{ix} - 1}{ix}) \to 0 \\ \frac{\sin(x)}{x} &= \mathrm{Re}(\frac{\cos(x) - 1 + i\sin(x)}{ix}) = \mathrm{Re}(\frac{e^{ix} - 1}{ix}) \to 1. \end{aligned}$$

$\square$

6.8. Definition of π: We will show that cos is strictly decreasing on the interval $[0,2]$ and possesses a unique zero x_0 in that interval. We will define π as the value of $2x_0$. We postpone the precise identification of π with half the (length of the) circumference of the unit circle until integration theory allows us to provide a simple calculation.[5]



LEMMA: (i) $\cos(0) = 1$ and $\cos(2) \leq -1/3$.

(ii) If $0 < x \leq 2$ then $\sin(x) > 0$.

(iii) cos is strictly decreasing on $[0,2]$.

Proof. (i): We clearly have $\cos(0) = \mathrm{Re}(e^{i0}) = 1$. The series expansion for the cosine function gives the alternating sum

$$\cos(2) = 1 - \frac{2^2}{2!} + \sum_{k=2}^{\infty}(-1)^k \frac{2^{2k}}{(2k)!} = -1 + r,$$

where r represents the error when approximating $\cos(2)$ by the partial sum $s_1 = -1$. Thus the error estimate (4.2) from the Leibniz criterion tells that $|r|$ is bounded by the absolute value of the first neglected term. Therefore we have

$$\cos(2) \leq -1 + |r| \leq -1 + \frac{2^4}{4!} = -1 + \frac{16}{24} = -1 + \frac{2}{3} = -\frac{1}{3}.$$

(ii): Let $0 < x \leq 2$. We have the alternating sum for the sine function

$$\sin(x) = x + \sum_{k=1}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!} = x + r(x),$$

where $r(x)$ now denotes the error when approximating $\sin(x)$ by the partial sum $s_1(x) = x$. We apply again the estimate (4.2), which now reads

$$|r(x)| \leq \frac{x^3}{3!} = x \cdot \frac{x^2}{6} \leq x \cdot \frac{4}{6} = \frac{2x}{3},$$

and therefore

$$\sin(x) \geq x - |r(x)| \geq x - \frac{2x}{3} = \frac{x}{3} > 0.$$

(iii): Let $0 \leq x_1 < x_2 \leq 2$, then we have $0 < (x_1+x_2)/2 \leq 2$ as well as $0 < (x_2-x_1)/2 \leq 2$. By 6.7(iv) and property (ii) proved above we obtain

$$\cos(x_2) - \cos(x_1) = -2 \cdot \underbrace{\sin\frac{x_2+x_1}{2}}_{>0} \cdot \underbrace{\sin\frac{x_2-x_1}{2}}_{>0} < 0,$$

hence $\cos(x_2) < \cos(x_1)$. □

[5] This constant was first named "π" by the Welsh scientist William Jones in 1706 because it is the first letter of the Greek words περιφερεία ("periphery") and περίμετρος ("circumference"). This notation was later adopted by Euler.

Proposition: There exists a unique $x_0 \in [0,2]$ such that $\cos(x_0) = 0$.

Proof. By the above lemma, cos is strictly decreasing on $[0,2]$, hence $\cos|_{[0,2]}$ is injective. Furthermore, the same lemma gives that $\cos(0) > 0$ and $\cos(2) < 0$. Since cos is continuous, the intermediate value theorem implies the existence of a zero $x_0 \in [0,2]$. This zero must be unique, since cos is injective on that interval. $\square$

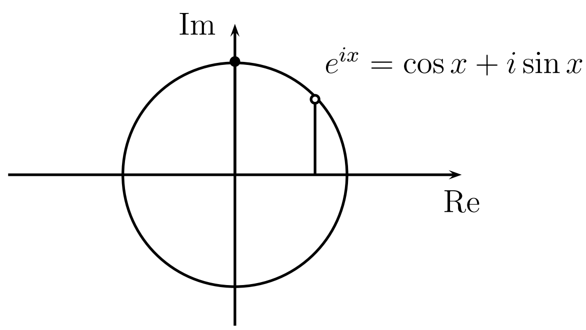
Definition: Let x_0 denote the unique zero of cos in the interval $[0,2]$ (according to the above proposition). Then the real number π is defined by $\pi := 2x_0$.

The properties of cos and sin established above can now be reformulated in more familiar terms: For example, we obtain that

$$\cos(x) > 0 \quad \text{for } 0 \leq x < \frac{\pi}{2}, \quad \cos(\frac{\pi}{2}) = 0, \quad \cos(x) < 0 \quad \text{for } \frac{\pi}{2} < x \leq 2.$$

Since $\sin^2(\frac{\pi}{2}) = 1 - \cos^2(\frac{\pi}{2}) = 1$ and $\sin(\frac{\pi}{2}) > 0$ (by the above lemma), we have

$$\sin(\frac{\pi}{2}) = 1 \quad \text{and} \quad e^{i\frac{\pi}{2}} = \cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2}) = i.$$



Taking integer powers for all $k \in \mathbb{Z}$ we obtain $e^{ik\frac{\pi}{2}} = \left(e^{i\frac{\pi}{2}}\right)^k = i^k$. In particular,

$$e^{i\,0} = 1 = \cos(0) + i\sin(0), \quad e^{i\frac{\pi}{2}} = i = \cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2}), \quad e^{i\pi} = -1 = \cos(\pi) + i\sin(\pi),$$
$$e^{i\frac{3\pi}{2}} = -i = \cos(\frac{3\pi}{2}) + i\sin(\frac{3\pi}{2}), \quad e^{i2\pi} = 1 = \cos(2\pi) + i\sin(2\pi),$$

which is summarized in the table

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin(x)$	0	1	0	-1	0
$\cos(x)$	1	0	-1	0	1

6.9. Further properties of the trigonometric functions: For all $x \in \mathbb{R}$ we have the following properties:

(a) cos and sin are *periodic* ⟨*periodisch*⟩ with *period* ⟨*Periode*⟩ of 2π, i.e.,

$$\cos(x + 2\pi) = \cos(x), \qquad \sin(x + 2\pi) = \sin(x).$$

This follows from 6.7(iv) and the fact that $\cos(2\pi) = 1$, $\sin(2\pi) = 0$:

$$\cos(x + 2\pi) = \cos(x)\cos(2\pi) - \sin(x)\sin(2\pi) = \cos(x).$$

(b) Since $\cos(x + \pi) = \cos(x)\cos(\pi) - \sin(x)\sin(\pi) = -\cos(x)$, and a similar calculation for sin, we have

$$\cos(x + \pi) = -\cos(x), \qquad \sin(x + \pi) = -\sin(x).$$

(c) By $\sin(\frac{\pi}{2} - x) = \sin(\frac{\pi}{2})\cos(-x) + \cos(\frac{\pi}{2})\sin(-x) = \cos(x)$, and a similar calculation for cos, we obtain

$$\sin(\frac{\pi}{2} - x) = \cos(x), \qquad \cos(\frac{\pi}{2} - x) = \sin(x).$$

(d) $\sin(x) = 0 \iff x \in \pi\mathbb{Z} := \{k\,\pi : k \in \mathbb{Z}\}$

Proof. By 2π-periodicity it suffices to show the assertion for $x \in [0, 2\pi[$.

Let $0 < x < \pi$ arbitrary, then $\frac{\pi}{2} - x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ and therefore $\sin(x) = \cos(\frac{\pi}{2} - x) > 0$.

Furthermore, note that $]\pi, 2\pi[= \{r + \pi : 0 < r < \pi\}$ and $\sin(x + \pi) = -\sin(x) < 0$.

Thus, 0 and π are the only zeros of sin in the interval $[0, 2\pi[$, which proves the assertion. □

(e) $\cos(x) = 0 \iff x \in \{\frac{\pi}{2}\} + \pi\mathbb{Z} := \{\frac{\pi}{2} + k\,\pi : k \in \mathbb{Z}\}$

Proof. Use $\cos(x) = -\sin(x - \frac{\pi}{2})$ and apply (d). □

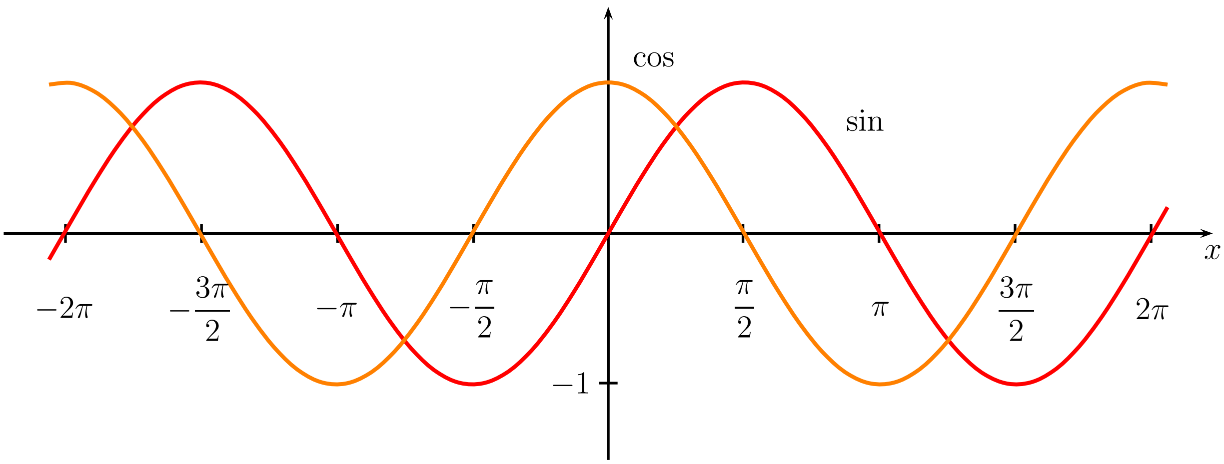
(f) $e^{ix} = 1 \iff x \in 2\pi\mathbb{Z} := \{2k\,\pi : k \in \mathbb{Z}\}$

Proof. We have $e^{ix} - 1 = e^{i\frac{x}{2}} \cdot \left(e^{i\frac{x}{2}} - e^{-i\frac{x}{2}}\right) = 2i\, e^{i\frac{x}{2}}\, \sin\frac{x}{2}$ and $2i\, e^{i\frac{x}{2}} \neq 0$. Therefore

$$e^{ix} = 1 \iff \sin(\frac{x}{2}) = 0 \iff \frac{x}{2} \in \pi\mathbb{Z} \iff x \in 2\pi\mathbb{Z}.$$

□

Using the above list of basic properties of cos and sin we can get a good qualitative picture of their graphs. Note in particular the following features: a shift of the graph of cos by $\frac{\pi}{2}$ along the horizontal axes gives the graph of sin; cos is even and strictly decreasing on $[0, \pi]$ (thus increasing on $[-\pi, 0]$), sin is odd and strictly increasing on $[-\frac{\pi}{2}, \frac{\pi}{2}]$; besides the zeros we can read off locations of maximum and minimum values, where each functions changes monotonicity type from increasing to decreasing or vice versa.



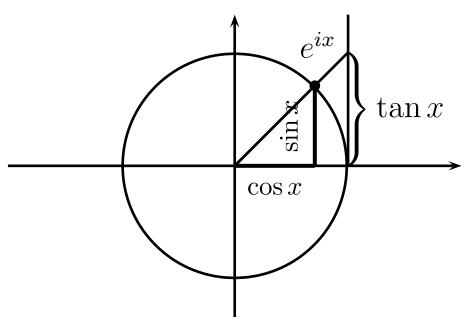
Definition: (i) The *tangent (function)*[6] ⟨*Tangens (-Funktion)*⟩ $\tan\colon \mathbb{R}\setminus(\frac{\pi}{2}+\pi\mathbb{Z})\to\mathbb{R}$ is given by

$$\tan(x) := \frac{\sin(x)}{\cos(x)}.$$

(ii) The *cotangent (function)* ⟨*Cotangens (-Funktion)*⟩ $\cot\colon \mathbb{R}\setminus\pi\mathbb{Z}\to\mathbb{R}$ is given by

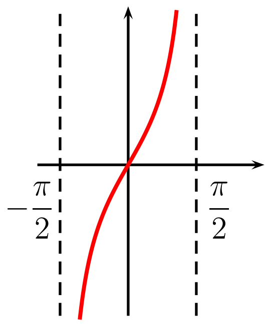
$$\cot(x) := \frac{\cos(x)}{\sin(x)}.$$

A geometric interpretation of $\tan(x)$, when $-\frac{\pi}{2} < x < \frac{\pi}{2}$, is easy by comparing the right-angled triangles in the following illustration:



[6]The term tangent was first used by the Danish mathematician Thomas Fincke in 1583.

Here is the part of the graph of tan above the interval $]-\frac{\pi}{2},\frac{\pi}{2}[$, whose basic qualitative features can be derived from the properties of cos and sin:

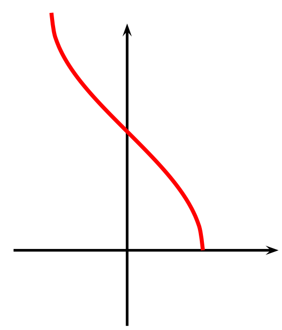


Note that $\tan(x+\pi) = \frac{\sin(x+\pi)}{\cos(x+\pi)} = \frac{-\sin(x)}{-\cos(x)} = \tan(x)$, so that the complete graph of tan can be obtained from shifts of the basic part on $]-\frac{\pi}{2},\frac{\pi}{2}[$ by integer multiples of π.

6.10. Inverse trigonometric functions ⟨*Arcusfunktionen*⟩:

Arc cosine: We assert that cos is strictly decreasing on $[0,\pi]$ and $\cos([0,\pi]) = [-1,1]$.

Indeed, that cos is strictly decreasing on $[0,\frac{\pi}{2}]$ follows from the Lemma in 6.8; since $\cos(\pi - x) = -\cos(x)$ the same follows for the interval $[\frac{\pi}{2},\pi]$; by continuity and injectivity, $\cos([0,\pi]) = [\cos(\pi),\cos(0)] = [-1,1]$.



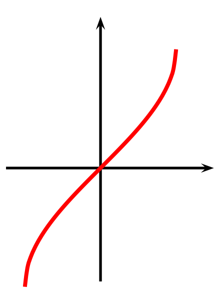
Thus cos is continuous, strictly decreasing, and bijective as a map $[0,\pi] \to [-1,1]$, hence possesses a strictly decreasing continuous inverse function

$$\arccos\colon [-1,1] \to [0,\pi],$$

called the *arc cosine (function)* ⟨*Arcus Cosinus*⟩.

We have for all $x \in [0,\pi]$ that $\arccos(\cos(x)) = x$ and $\cos(\arccos(y)) = y$ for all $y \in [-1,1]$.

Of course we could have constructed similar inverses on any interval of strict monotonicity for cos. Unless stated otherwise we will usually refer to the one constructed above as arccos.



Arc sine: Using $\sin(x) = \cos(\frac{\pi}{2} - x)$ and (i) we deduce the following: sin is strictly increasing on $[-\frac{\pi}{2},\frac{\pi}{2}]$ and $\sin([-\frac{\pi}{2},\frac{\pi}{2}]) = [-1,1]$. The corresponding inverse function

$$\arcsin\colon [-1,1] \to [-\frac{\pi}{2},\frac{\pi}{2}],$$

called the *arc sine (function)* ⟨*Arcus Sinus*⟩, is continuous and strictly increasing.

Arc tangent: We claim that tan is strictly increasing on $]-\frac{\pi}{2},\frac{\pi}{2}[$ and $\tan(]-\frac{\pi}{2},\frac{\pi}{2}[)=\mathbb{R}$.

Proof. Since

$$\tan(-x)=\frac{\sin(-x)}{\cos(-x)}=\frac{-\sin(x)}{\cos(x)}=-\tan(x),$$

it suffices to consider the subinterval $[0,\frac{\pi}{2}[$. If $0\leq x<x'<\frac{\pi}{2}$ then $\sin(x)<\sin(x')$ and $\cos(x)>\cos(x')>0$, hence

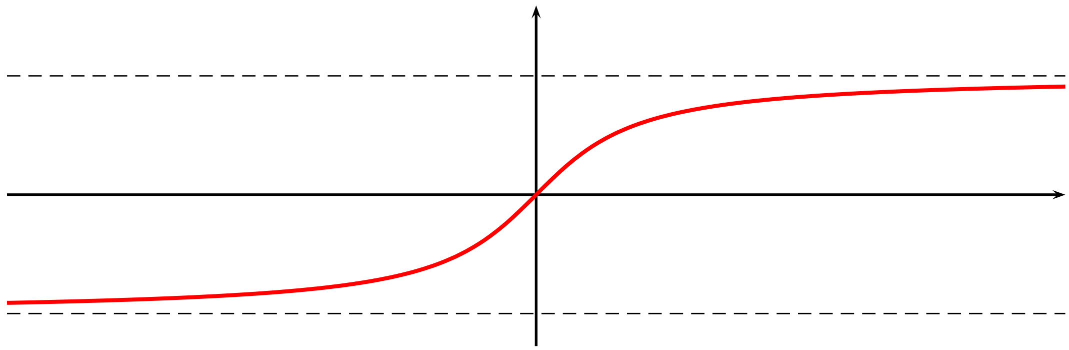
$$\tan(x)=\frac{\sin(x)}{\cos(x)}<\frac{\sin(x')}{\cos(x')}=\tan(x').$$

Note that $\frac{\cos(x)}{\sin(x)}>0$ for all $x\in]0,\frac{\pi}{2}[$ and that

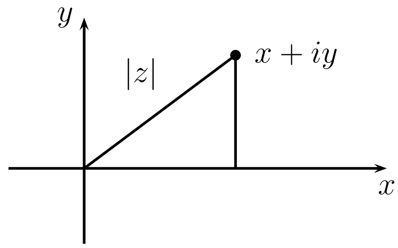
$$\lim_{x\nearrow\frac{\pi}{2}}\frac{\cos(x)}{\sin(x)}=\frac{\cos(\frac{\pi}{2})}{\sin(\frac{\pi}{2})}=0.$$

Therefore we obtain that $\tan(x)\to\infty$ as $x\nearrow\frac{\pi}{2}$ and by the intermediate value theorem (tan is continuous!) that $\tan([0,\frac{\pi}{2}[)=[0,\infty[$. By symmetry of tan we obtain that $\tan(]-\frac{\pi}{2},\frac{\pi}{2}[)=]-\infty,\infty[$. $\square$

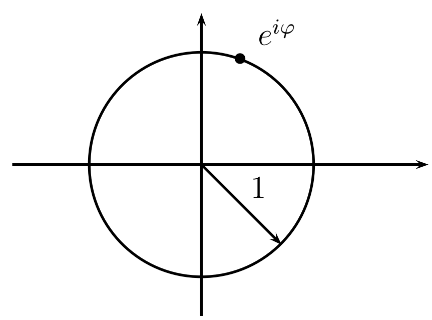
We conclude that the restriction of tan to $]-\frac{\pi}{2},\frac{\pi}{2}[$ has a continuous and strictly increasing inverse function $\arctan\colon\mathbb{R}\to]-\frac{\pi}{2},\frac{\pi}{2}[$, called *arc tangent (function)* ⟨*Arcus Tangens*⟩.



6.11. Polar coordinates[7] for complex numbers: If $z = x + iy \in \mathbb{C}$ we may interpret the absolute value $|z| = \sqrt{x^2 + y^2}$ as the distance of z to the origin in the plane:



How do we obtain information on the direction towards z as seen from the origin with respect to the positive real axis (the x-axis)?



Recall that for any $\varphi \in \mathbb{R}$ we have $|e^{i\varphi}| = 1$ and $e^{i\varphi} = \cos(\varphi) + i\,\sin(\varphi)$.

Furthermore, the points where the unit circle S^1 intersects the Cartesian axes are given by $e^{i0} = 1$, $e^{i\frac{\pi}{2}} = i$, $e^{i\pi} = -1$, $e^{i\frac{3\pi}{2}} = -i$ and we have 2π-periodicity

$$e^{i(\varphi+2\pi)} = e^{i\varphi}.$$

Let $z \neq 0$ and set $w := \frac{z}{|z|}$. Then w lies on the unit circle and can be written in the form

$$w = \xi + i\,\eta, \quad \text{where } \xi, \eta \in \mathbb{R} \text{ are such that} \quad 1 = |w|^2 = \xi^2 + \eta^2.$$

Therefore $\xi \in [-1, 1]$ and $\alpha := \arccos(\xi) \in [0, \pi]$ and we have

$$\sin^2(\alpha) = 1 - \cos^2(\alpha) = 1 - \xi^2 = \eta^2,$$

hence $\sin(\alpha) = \eta$ or $\sin(\alpha) = -\eta$. If we define

$$\varphi := \begin{cases} \alpha & \text{if } \sin(\alpha) = \eta \\ -\alpha & \text{if } \sin(\alpha) = -\eta, \end{cases}$$

then we obtain $w = \cos(\varphi) + i\,\sin(\varphi) = e^{i\varphi}$, which in turn yields the *polar representation* ⟨*Polardarstellung*⟩ of z in the form

$$z = |z| \cdot e^{i\varphi}.$$

In this representation φ, the so-called the *argument* ⟨*Argument*⟩ of z, $\varphi = \arg(z)$, is unique up to an addition of integer multiples of 2π.

[7]The history of polar coordinates is about as long as the one of trigonometry. The term polar coordinates was introduced by 18th-century Italian mathematicians.

CHAPTER III

DIFFERENTIATION

§7. DIFFERENTIABILITY AND DERIVATIVE

7.1. Definition: Consider $V \subseteq \mathbb{R}$, $f\colon V \to \mathbb{R}$, and let $x \in V$ be an accumulation point of V. Then f is said to be *differentiable at x* ⟨*differenzierbar in x*⟩ if the limit

$$f'(x) := \lim_{\substack{\xi \to x \\ \xi \in V \setminus \{x\}}} \frac{f(\xi) - f(x)}{\xi - x} \tag{7.1}$$

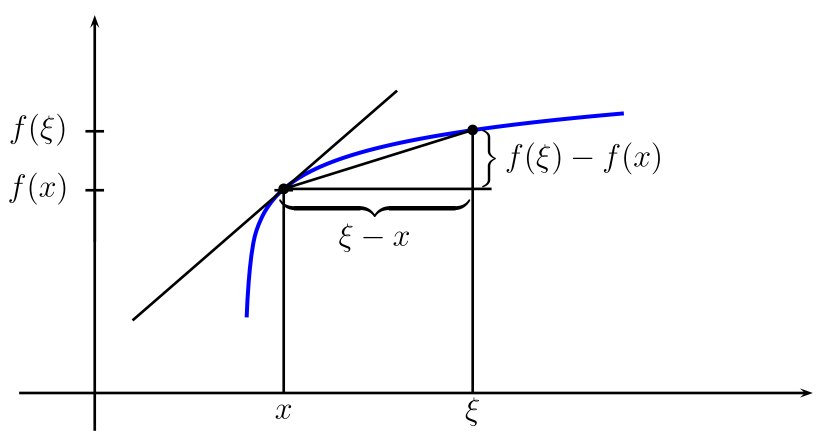
exists. The value $f'(x)$ is the *derivative of f at x* ⟨*Ableitung von f an der Stelle x*⟩. The function f is called differentiable in V, if f is differentiable at all points $x \in V$.

7.2. Remark: (i) An alternative expression to (7.1) is

$$f'(x) := \lim_{\substack{h \to 0 \\ x+h \in V, h \neq 0}} \frac{f(x+h) - f(x)}{h} \tag{7.2}$$

(We will often supress the additional conditions on h in the specification of the limit.)

(ii) **Geometric interpretation**: Let V be an interval and consider the graph of f of a differentiable function $f\colon V \to \mathbb{R}$.



The *difference quotient* ⟨Differenzenquotient⟩

$$\frac{f(\xi) - f(x)}{\xi - x}$$

gives the slope of the straight line, the so-called secant ⟨*Sekante*⟩, through the points $(x, f(x))$ and $(\xi, f(\xi))$. As ξ is getting closer and closer to x, the slopes of the corresponding secant lines approach the limit $f'(x)$, which therefore can be thought of being the slope of the tangent ⟨*Tangente*⟩ to the graph of f at the point $(x, f(x))$.[1]

(iii) A common notation for $f'(x)$ is the *differential quotient* ⟨Differenzialquotient⟩ $\dfrac{df(x)}{dx}$, which however is not a quotient but merely a reminder of the fact that $f'(x)$ is the limit of the difference quotients $\dfrac{\Delta f(x)}{\Delta x} = \dfrac{f(\xi) - f(x)}{\xi - x}$.

(iv) If f is differentiable in V then the derivative of f defines a function $f' \colon V \to \mathbb{R}$, $x \mapsto f'(x)$.

7.3. Examples: 1) Let $c \in \mathbb{R}$. The constant function $f \colon \mathbb{R} \to \mathbb{R}$, $f(x) = c$, is differentiable in $\mathbb{R}$ and has derivative

$$f'(x) = \lim_{\substack{h \to 0 \\ h \neq 0}} \frac{f(x+h) - f(x)}{h} = \lim_{\substack{h \to 0 \\ h \neq 0}} \frac{c - c}{h} = 0.$$

2) Let $c \in \mathbb{R}$ and $n \in \mathbb{N}$ with $n \geq 1$. The function $f \colon \mathbb{R} \to \mathbb{R}$, $x \mapsto cx^n$, is differentiable in $\mathbb{R}$ and has derivative

$$\begin{aligned} f'(x) &= \lim_{0 \neq h \to 0} \frac{c(x+h)^n - cx^n}{h} = c \lim_{0 \neq h \to 0} \frac{(x+h)^n - x^n}{h} \\ &= c \lim_{0 \neq h \to 0} \frac{x^n + \sum_{k=1}^n \binom{n}{k} h^k x^{n-k} - x^n}{h} = c \lim_{0 \neq h \to 0} \frac{h \cdot \sum_{k=1}^n \binom{n}{k} h^{k-1} x^{n-k}}{h} \\ &= c \lim_{0 \neq h \to 0} \left(\binom{n}{1} x^{n-1} + \sum_{k=2}^n \binom{n}{k} h^{k-1} x^{n-k} \right) = c\, n\, x^{n-1}. \end{aligned}$$

In particular, we have $(cx)' = cx^0 = c$ and $(x^2)' = 2x$.

3) $f \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $x \mapsto \frac{1}{x}$, is differentiable at every $x \neq 0$ and

$$f'(x) = \lim_{h \to 0} \frac{1}{h} \left(\frac{1}{x+h} - \frac{1}{x} \right) = \lim_{h \to 0} \frac{1}{h} \cdot \frac{-h}{x(x+h)} = \lim_{h \to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}.$$

[1] This solution of the so-called “tangent problem” was developed independently by Isaac Newton and Gottfried Wilhelm Leibnitz in the 17th century, thereby inventing modern calculus.

4) The exponential function is differentiable and

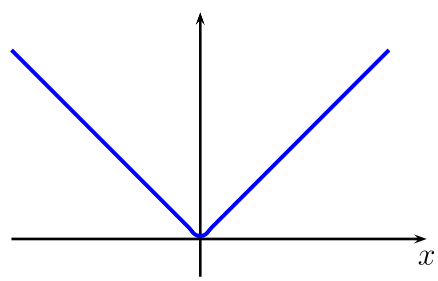
$$\exp'(x) = \lim_{h\to 0} \frac{\exp(x+h)-\exp(x)}{h} = \exp(x) \lim_{h\to 0} \frac{\exp(h)-1}{h} \underset{[6.6(vi)]}{=} \exp(x)\cdot 1 = \exp(x).$$

5) $\sin\colon \mathbb{R} \to \mathbb{R}$ is differentiable and

$$\begin{aligned} \sin'(x) &= \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} \underset{[6.7(iv)]}{=} \lim_{h\to 0} \frac{2\cos(x+\frac{h}{2})\sin(\frac{h}{2})}{h} \\ &= \lim_{h\to 0} \underbrace{\cos(x+\frac{h}{2})}_{[\cos\ \text{continuous}]} \underbrace{\frac{\sin(h/2)}{h/2}}_{[\to 1 \text{ by } 6.7(\text{vi})]} = \lim_{h\to 0}\cos(x+\frac{h}{2})\cdot \lim_{h\to 0}\frac{\sin(h/2)}{h/2} = \cos(x). \end{aligned}$$

Similarly, one can show that cos is differentiable and $\cos'(x) = -\sin(x)$ (left as an exercise).

6) $\mathrm{abs}\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto |x|$, is differentiable for all $x \neq 0$. This follows easily from the fact that $\mathrm{abs}\,|_{]0,\infty[} = \mathrm{id}$ and $\mathrm{abs}\,|_{]-\infty,0[} = -\,\mathrm{id}$.



But $\mathrm{abs}'(0)$ does not exist: Let $h_n := (-1)^n/n$, then $\lim h_n = 0$ and the difference quotients do not converge

$$\frac{\mathrm{abs}(0+h_n)-\mathrm{abs}(0)}{h_n} = \frac{1/n}{(-1)^n/n} = (-1)^n.$$

Thus abs is a continuous function on $\mathbb{R}$ but not differentiable at the point 0.

7.4. One-sided derivatives: Let x be an accumulation point of $V \subseteq \mathbb{R}$ such that $x \in V$. $f\colon V \to \mathbb{R}$ is *differentiable from the right* ⟨rechtsseitig differenzierbar⟩ at $x \in V$, if

$$f'_+(x) := \lim_{\substack{\xi \searrow x \\ \xi \in V\setminus\{x\}}} \frac{f(\xi)-f(x)}{\xi - x}$$

exists. f is *differentiable from the left* ⟨linksseitig differenzierbar⟩ at $x \in V$, if

$$f'_-(x) := \lim_{\substack{\xi \nearrow x \\ \xi \in V\setminus\{x\}}} \frac{f(\xi)-f(x)}{\xi - x}$$

exists.

For example, $\mathrm{abs}'_-(0) = -1$ and $\mathrm{abs}'_+(0) = 1$. (Compare with the graph!)

7.5. Proposition: If $f\colon V \to \mathbb{R}$ is differentiable at $x \in V$, then f is continuous at x.

Proof. Let $V \ni \xi \neq x$, then we have

$$f(\xi) - f(x) = \frac{f(\xi) - f(x)}{\xi - x} \cdot (\xi - x) \to f'(x) \cdot 0 = 0 \quad (\xi \to x).$$

□

7.6. Proposition: Let $f, g\colon V \to \mathbb{R}$ be differentiable in $x \in V$.

(i) If $\lambda, \mu \in \mathbb{R}$ then $\lambda f + \mu g$ is differentiable at x and

$$(\lambda f + \mu g)'(x) = \lambda f'(x) + \mu g'(x).$$

(ii) *Leibniz or product rule:* $f \cdot g$ is differentiable at x and

$$(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x).$$

(iii) *Quotient rule:* If $g(\xi) \neq 0$ for all $\xi \in V$ then $\dfrac{f}{g}\colon V \to \mathbb{R}$ is differentiable at x and

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Proof. (i) Follows from the basic rules for limits.

(ii) Let $h \neq 0$ such that $x + h \in V$. Then we have

$$\begin{aligned} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} &= \frac{1}{h}\Big(f(x+h)\big(g(x+h) - g(x)\big) + \big(f(x+h) - f(x)\big)g(x)\Big) \\ &= f(x+h) \cdot \frac{g(x+h) - g(x)}{h} + \frac{f(x+h) - f(x)}{h} \cdot g(x) \\ &\underset{(h \to 0)}{\longrightarrow} \quad f(x) \cdot g'(x) + f'(x) \cdot g(x) \end{aligned}$$

since f is continuous at x by Proposition 7.5.

(iii) Consider first the case that $f(\xi) = 1$ for all $\xi \in V$. Then we have with $h \neq 0$ such that $x + h \in V$

$$\frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h} = \frac{g(x) - g(x+h)}{hg(x)g(x+h)} \longrightarrow -\frac{g'(x)}{g(x)^2} \quad (h \to 0).$$

Thus $(1/g)'(x) = -g'(x)/g(x)^2$ and the general case now follows from the product rule:

$$\left(\frac{f}{g}\right)'(x) = \left(f \cdot \frac{1}{g}\right)'(x) = f'(x)\frac{1}{g(x)} - f(x)\frac{g'(x)}{g(x)^2} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

□

7.7. Examples: 1) Let $n \in \mathbb{N}$, $n \geq 1$ and $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $x \mapsto 1/x^n$:

$$f'(x) = \left(\frac{1}{x^n}\right)' = \frac{-nx^{n-1}}{x^{2n}} = -\frac{n}{x^{n+1}},$$

in other words, $(x^{-n})' = -n\, x^{-n-1}$.

2) $\tan\colon \mathbb{R} \setminus (\frac{\pi}{2} + \pi\mathbb{Z}) \to \mathbb{R}$ is differentiable and

$$(\tan x)' = \left(\frac{\sin(x)}{\cos(x)}\right)' = \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)}$$
$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = 1 + \tan^2(x).$$

7.8. The derivative as linear approximation: Let $V \subseteq \mathbb{R}$ and $x_0 \in V$ be an accumulation point of V. Suppose that $f\colon V \to \mathbb{R}$ is differentiable at x_0. We have seen that in a geometric interpretation the tangent to the graph of f at the point $(x_0, f(x_0)$ has slope $f'(x_0)$. Thus we define the *tangent* ⟨*Tangente*⟩ to the graph of f at $(x_0, f(x_0))$ as the straight line given by the following linear equation for $(x, y) \in \mathbb{R}^2$

$$y = f(x_0) + f'(x_0)(x - x_0). \tag{7.3}$$

Intuitively, if f is sufficiently "well-behaved" then the tangent, considered as the affine linear function $x \mapsto f(x_0) + f'(x_0)(x - x_0)$, should give a reasonable approximation of the function values when x is close to x_0, i.e., $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$ as $x \to x_0$. The following statement makes this more precise.

Theorem: The following statements are equivalent:

(i) f is differentiable at x_0.

(ii) There exist a constant $c \in \mathbb{R}$ and a function $r\colon V \to \mathbb{R}$ with $\lim\limits_{\substack{x \to x_0 \\ x \neq x_0}} \dfrac{r(x)}{x - x_0} = 0$ such that

$$f(x) = f(x_0) + c \cdot (x - x_0) + r(x) \qquad \forall x \in V. \tag{7.4}$$

In this case $c = f'(x_0)$.

Note that, in particular, $r(x) = o(|x - x_0|)$ $(x \to x_0)$.

Proof. (i) $\Rightarrow$ (ii): Put $r(x) := f(x) - f(x_0) - f'(x_0)(x - x_0)$ and let $x \in V$ with $x \neq x_0$. Then

$$\frac{r(x)}{x - x_0} = \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \longrightarrow f'(x_0) - f'(x_0) = 0 \qquad (x \to x_0).$$

<u>(ii) $\Rightarrow$ (i):</u> Let $x \in V$ with $x \neq x_0$. Then (7.4) implies

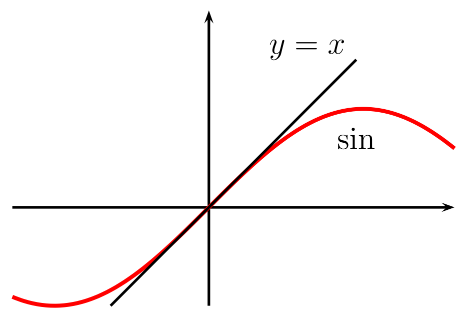
$$\frac{f(x) - f(x_0)}{x - x_0} = c + \frac{r(x)}{x - x_0} \longrightarrow c + 0 \qquad (x \to x_0).$$

Thus f is differentiable at x_0 and $f'(x_0) = c$. $\square$

EXAMPLE: Consider $\sin\colon \mathbb{R} \to \mathbb{R}$ with $\sin'(0) = \cos(0) = 1$. We obtain the relation

$$\begin{aligned}\sin(x) &= \sin(0) + \sin'(0)(x - 0) + r(x)\\ &= 0 + 1 \cdot x + r(x) = x + r(x),\end{aligned}$$

where $r(x) = \sin(x) - x = o(x)$ $(x \to 0)$.



7.9. The chain rule:

THEOREM: Let $V, W \subseteq \mathbb{R}$ and $f\colon V \to \mathbb{R}$, $g\colon W \to \mathbb{R}$ such that $f(V) \subseteq W$. If f is differentiable at $x \in V$ and g is differentiable at $y := f(x) \in W$, then the composition $g \circ f\colon V \to \mathbb{R}$ is differentiable at x and

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x).$$

Idea of the proof: We will need a little extra technical finesse to make use of the following heuristics: for $\xi \in V$ with $\xi \neq x$

$$\frac{g(f(\xi)) - g(f(x))}{\xi - x} = \frac{g(f(\xi)) - g(f(x))}{f(\xi) - f(x)} \cdot \frac{f(\xi) - f(x)}{\xi - x} \approx g'(f(x)) \cdot f'(x);$$

note that we cannot guarantee that $f(\xi) - f(x) \neq 0$.

Proof. Define the modified difference quotient $g^*\colon W \to \mathbb{R}$ of g at y by

$$g^*(\eta) := \begin{cases} \frac{g(\eta) - g(y)}{\eta - y} & \text{if } \eta \neq y \\ g'(y) & \text{if } \eta = y. \end{cases}$$

Since $\lim_{\eta \to y} g^*(\eta) = g'(y) = g^*(y)$ we deduce that g^* is continuous. Furthermore, by construction we have for all $\eta \in W$

$$g(\eta) - g(y) = g^*(\eta)\,(\eta - y).$$

Therefore we obtain for $\xi \in V$ with $\xi \neq x$

$$\frac{g(f(\xi)) - g(f(x))}{\xi - x} = \frac{g^*(f(\xi))\,(f(\xi) - f(x))}{\xi - x} = g^*(f(\xi)) \cdot \frac{f(\xi) - f(x)}{\xi - x} \underset{(\xi \to x)}{\longrightarrow} g^*(y) \cdot f'(x) = g'(f(x)) \cdot f'(x).$$

□

Application: Let $D, W \subseteq \mathbb{R}$ and $f\colon D \to W$ be a bijective map. Suppose that f is differentiable at $x \in D$ and that f^{-1} is differentiable at $y := f(x) \in W$ as well. Then the chain rule applied to $f^{-1} \circ f = \mathrm{id}_D$ yields

$$(f^{-1})'(f(x)) \cdot f'(x) = \mathrm{id}'(x) = 1.$$

In particular, this implies $f'(x) \neq 0$ and thus we may rewrite the above relation with $y = f(x)$ in the form

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}.$$

7.10. Differentiability and derivative of the inverse function: Let $I, J \subseteq \mathbb{R}$ be intervals and $f\colon I \to J$ be bijective, continuous, and strictly increasing with (therefore continuous!) inverse function $f^{-1}\colon J \to I$.

Theorem: Under the assumptions stated above: If f is differentiable at $x \in I$ with $f'(x) \neq 0$, then f^{-1} is differentiable at $y := f(x) \in J$ and

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}. \tag{7.5}$$

Proof. Let (η_n) be a sequence with $\eta_n \in J \setminus \{y\}$ (for all $n \in \mathbb{N}$) such that $\eta_n \to y$ ($n \to \infty$). If we put $\xi_n := f^{-1}(\eta_n)$ then $\xi_n \in I \setminus \{x\}$ and $\xi_n \to x$ as $n \to \infty$ (by continuity of f^{-1} [cf. 5.12]). Therefore

$$\frac{f^{-1}(\eta_n) - f^{-1}(y)}{\eta_n - y} = \frac{\xi_n - x}{f(\xi_n) - f(x)} \longrightarrow \frac{1}{f'(x)} \quad (n \to \infty),$$

which proves that f^{-1} is differentiable at y and $(f^{-1})'(y)$ equals $1/f'(f^{-1}(y))$. □

7.11. Examples: 1) Let $x > 0$ then

$$\log'(x) = \frac{1}{\exp'(\log(x))} = \frac{1}{\exp(\log(x))} = \frac{1}{x}.$$

A simple application of this result proves a famous limit relation: Since $\log'(1) = 1$ we have (using the continuity of exp)

$$1 = \lim_{n\to\infty} \frac{\log(1+\frac{1}{n}) - \log(1)}{\frac{1}{n}} = \lim n \cdot \log(1+\frac{1}{n}) = \lim \log(1+\frac{1}{n})^n$$
$$= \log \exp \lim \log(1+\frac{1}{n})^n = \log(\lim(1+\frac{1}{n})^n)$$

and upon exponentiation

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = \exp(1) = e. \tag{7.6}$$

2) If $x \in]-1,1[$ then

$$\arcsin'(x) = \frac{1}{\sin'(\arcsin(x))} = \frac{1}{\cos(\arcsin(x))} \underset{[\cos=\sqrt{1-\sin^2}]}{=} \frac{1}{\sqrt{1-x^2}}.$$

3) $\arctan'(x) = \dfrac{1}{\tan'(\arctan(x))} = \dfrac{1}{1+\tan^2(\arctan(x))} = \dfrac{1}{1+x^2}.$

4) If $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable and $a, b \in \mathbb{R}$ then $g(x) := f(ax+b)$ is differentiable and by the chain rule

$$g'(x) = f'(ax+b)\,(ax+b)' = a\,f'(ax+b).$$

5) For $\alpha \in \mathbb{R}$ the function $x \mapsto x^\alpha$, $]0,\infty[\to \mathbb{R}$ is differentiable and

$$(x^\alpha)' = (\exp(\alpha\,\log(x)))' = \exp'(\alpha\,\log(x))\,(\alpha\,\log(x))'$$
$$= \exp(\alpha\,\log(x))\,\alpha\,\log'(x) = x^\alpha\,\alpha\,\frac{1}{x} = \alpha\,x^{\alpha-1}.$$

7.12. Higher derivatives: Let $f\colon V \to \mathbb{R}$ be differentiable and $f'\colon V \to \mathbb{R}$ be its derivative. If f' is differentiable in $x \in V$, then

$$f''(x) := (f')'(x)$$

is called the *second derivative* ⟨*zweite Ableitung*⟩ of f at x. A classical notation for $f''(x)$ is also $\dfrac{d^2 f(x)}{dx^2}$.

Suppose that f' is differentiable in $U_\varepsilon(x) \cap V$. Then we can consider the function $f''\colon U_\varepsilon(x) \cap V \to \mathbb{R}$ and investigate its differentiability properties in turn. In this way, we can define the property of a function f to be k *times differentiable at* x ($k \in \mathbb{N}$, $k \geq 2$) inductively, if there exists $\varepsilon > 0$ such that $f\,|_{U_\varepsilon(x)\cap V}$ is $(k-1)$ times differentiable (in all points of $U_\varepsilon(x) \cap V$) and the $(k-1)$st derivative $f^{(k-1)}$ is differentiable at x. In this case we set

$$f^{(k)}(x) := (f^{(k-1)})'(x).$$

It is common to put $f^{(0)}(x) = f(x)$. A function $f\colon V \to \mathbb{R}$ is called *infinitely (often) differentiable* ⟨unendlich oft differenzierbar⟩ at x if it is k times differentiable for all $k \in \mathbb{N}$.[2]

EXAMPLE: sin is infinitely (often) differentiable on all of $\mathbb{R}$. We have $\sin' = \cos$, $\sin'' = -\sin$, $\sin^{(3)} = -\cos$, and $\sin^{(4)} = \sin$ etc.

Let $\omega \in \mathbb{R}$ be a constant (frequency). Then $u(x) := \sin(\omega\, x)$ is infinitely differentiable on $\mathbb{R}$. We have $u'(x) = \omega \cos(\omega\, x)$ and $u''(x) = -\omega^2 \sin(\omega\, x) = -\omega^2\, u(x)$. Therefore u satisfies the differential equation of the isochronous pendulum

$$u'' + \omega^2\, u = 0.$$

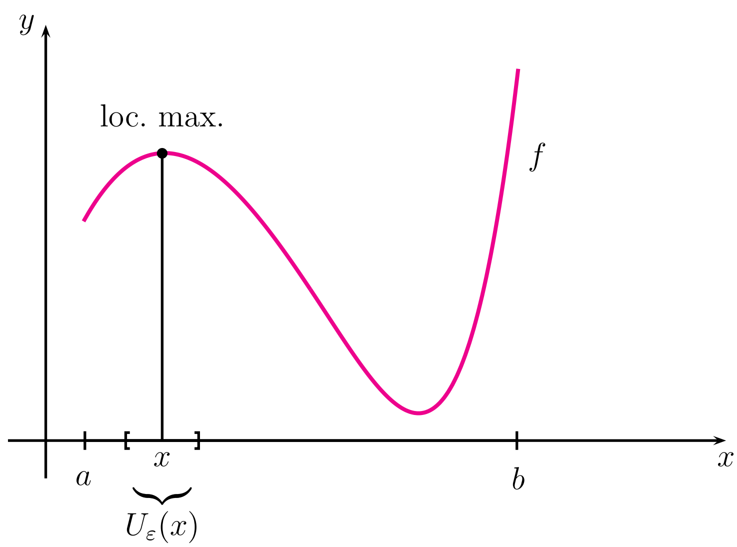
[2]Note that the case $k = 1$ is not included in the above definition and that mere differentiability of a function at a point does not imply differentiability in a whole ε-neighborhood. For example, consider the function $f\colon \mathbb{R} \to \mathbb{R}$, defined as $f(x) = x^2$ when $x \in \mathbb{Q}$ and $f(x) = 0$ otherwise. Then f is discontinuous at every $x \neq 0$ but differentiable in $x = 0$.

§8. BASIC PROPERTIES OF DIFFERENTIABLE FUNCTIONS

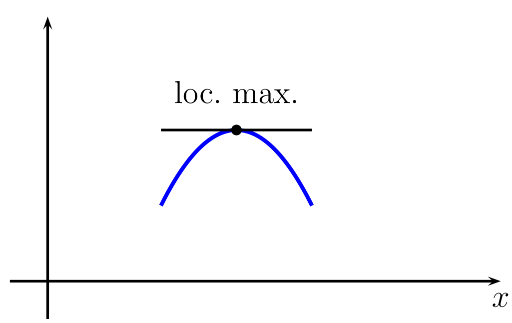
8.1. Definition: Let $f\colon\,]a,b[\to \mathbb{R}$. A point $x \in]a,b[$ is a *local maximum* ⟨*lokales Maximum*⟩ (resp. *local minimum* ⟨*lokales Minimum*⟩) of f, if

$$\exists \varepsilon > 0\ \forall \xi \in U_\varepsilon(x) : f(x) \geq f(\xi) \quad (\text{resp. } f(x) \leq f(\xi)).$$

If $f(x) > f(\xi)$ (resp. $f(x) < f(\xi)$) for all $\xi \in U_\varepsilon(x)$ with $\xi \neq x$, then x is a *strict local maximum* (resp. minimum) ⟨*striktes lokales Maximum (bzw. Minimum)*⟩. In either case x is called a *local extreme* ⟨*lokles Extremum*⟩ of f.



Geometrically, if f is differentiable we expect that the graph of f has a horizontal tangent in a local extreme, which means that the slope $f'(x)$ must be 0.



8.2. Proposition: Suppose that $f\colon]a,b[\to \mathbb{R}$ has a local extreme in x and is differentiable at x. Then $f'(x) = 0$.

Proof. Let x be a local maximum of f (the case of a local minimum is analogous). There is $\varepsilon > 0$ such that

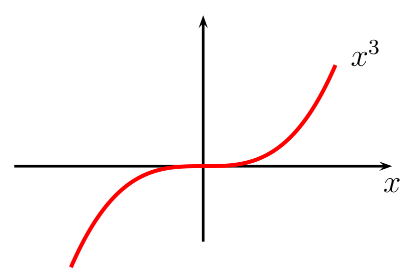
$$\forall \xi \in U_\varepsilon(x): \quad f(x) \geq f(\xi).$$

Since f is differentiable at x we deduce that

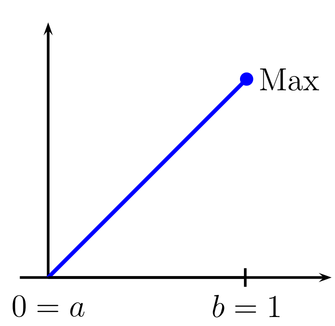
$$\lim_{\xi \searrow x} \underbrace{\frac{f(\xi) - f(x)}{\xi - x}}_{\leq 0} = f'(x) = \lim_{\xi \nearrow x} \underbrace{\frac{f(\xi) - f(x)}{\xi - x}}_{\geq 0}$$

and therefore $0 \leq f'(x) \leq 0$, hence $f'(x) = 0$. $\square$

8.3. Remark: (i) The above proposition gives a necessary condition, namely $f'(x) = 0$, for a local extreme at a point x where f is differentiable. This condition is, in general, *not sufficient* as the following simple example shows: Let $f\colon]-1,1[\to \mathbb{R}$, $x \mapsto x^3$; then $f'(0) = 0$, but $x = 0$ is neither a local maximum nor a local minimum.



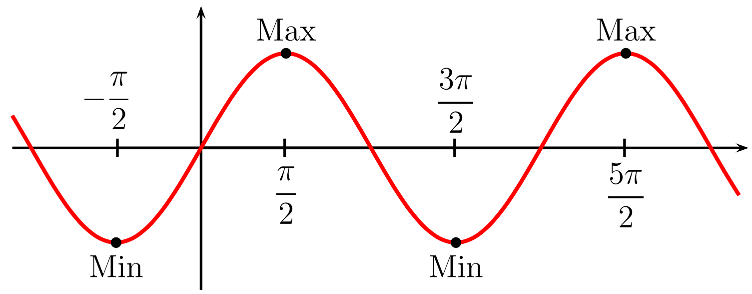
Thus a solution of the equation $f'(x) = 0$ merely provides a candidate for a local extremum.



(ii) Let $f\colon [a,b] \to \mathbb{R}$ be continuous. By the results in Section 5 we know that f possesses a (global) maximum and a minimum in the closed interval $[a,b]$. Such an extreme point can lie at the boundary, i.e., $x = a$ or $x = b$, in which case the derivative need not vanish. For example, the function $f\colon [0,1] \to \mathbb{R}$, $f(x) = x$, has a maximum in $x = 1$, but $f'(1) = 1 \neq 0$.

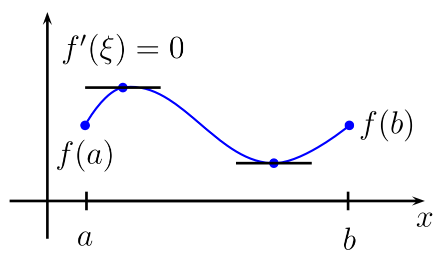
8.4. Example: Consider the function $\sin\colon \mathbb{R} \to \mathbb{R}$.

We already deduced from its strict monotonicity properties on the intervals $[k\frac{\pi}{2}, (k+1)\frac{\pi}{2}]$ that we have strict maxima at the points $\frac{\pi}{2} + 2k\pi$ and strict minima at the points $\frac{\pi}{2} + (2k+1)\pi$ $(k \in \mathbb{Z})$. And indeed we have $\sin'(\frac{\pi}{2} + k\pi) = \cos(\frac{\pi}{2} + k\pi) = 0$.



8.5. The mean value theorem of differential calculus: In the following statements we suppose that $f\colon [a,b] \to \mathbb{R}$ is continuous and differentiable in $]a,b[$.[1]

Proposition (Rolle's theorem): If $f(a) = f(b)$ then there exists $\xi \in]a,b[$ such that $f'(\xi) = 0$.[2]



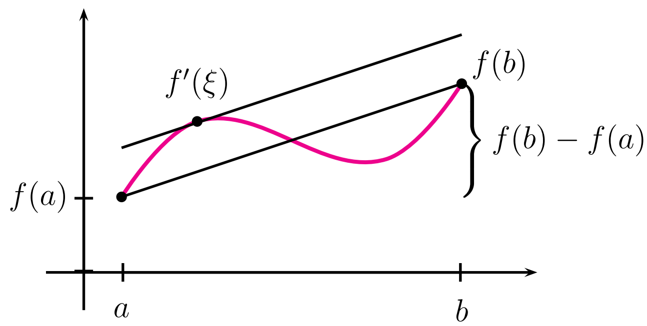
Proof. If f is a constant function the assertion is trivial, hence we may assume that f is not constant. Then there exists $x_0 \in]a,b[$ such that $f(x_0) > f(a) = f(b)$ or $f(x_0) < f(a) = f(b)$. Thus the maximum or minimum of f must be attained at some $\xi \in]a,b[$. Since ξ has to be a local extreme as well we obtain that $f'(\xi) = 0$. □

Mean value theorem: There exists $\xi \in]a,b[$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(\xi) \tag{8.1}$$

[1]In case you are wondering about the relation between the two conditions 'continuity on the closed interval $[a,b]$' and 'differentiability in the open interval $]a,b[$' or their implying 'differentiability on the closed interval $[a,b]$': First, recall that continuity at a point x does not imply differentiablity at x; second, consider the examples $f_1\colon [0,1] \to \mathbb{R}$, $f_1(x) = \sqrt{x}$ and $f_2\colon]0,1] \to \mathbb{R}$, $f_2(x) = 1/x$; then f_1 is continuous on $[0,1]$, differentiable in $]0,1[$, but not differentiable at 0 and f_2 is differentiable in $]0,1]$ but has no continuous extension to $[0,1]$.

[2]Michel Rolle (1652–1719) [miˈʃɛl ʀɔl], French mathematician



Proof. Define $g\colon [a,b] \to \mathbb{R}$ by $g(x) := f(x) - \dfrac{f(b)-f(a)}{b-a} \cdot (x-a)$.
Then g is continuous on $[a,b]$, differentiable in $]a,b[$, and $g(a) = f(a) = g(b)$. Therefore Rolle's theorem implies that there is some $\xi \in]a,b[$ such that

$$0 = g'(\xi) = f'(\xi) - \frac{f(b)-f(a)}{b-a},$$

which proves the assertion. □

COROLLARY: (i) If f' is bounded, i.e., there are $m, M \in \mathbb{R}$ such that $m \leq f'(\xi) \leq M$ holds for all $\xi \in]a,b[$, then we have for all $x_1, x_2 \in [a,b]$ with $x_1 \leq x_2$

$$m\,(x_2 - x_1) \leq f(x_2) - f(x_1) \leq M\,(x_2 - x_1).$$

In particular, f is *Lipschitz continuous* ⟨Lipschitz-stetig⟩ with *Lipschitz constant* $L := \max(|m|, |M|)$, that is

$$|f(x_2) - f(x_1)| \leq L\,|x_2 - x_1| \qquad \forall x_1, x_2 \in [a,b].$$

(ii) If $f'(x) = 0$ for all $x \in]a,b[$ then f is constant.

Proof. (i) Apply (8.1) to each interval $[x_1, x_2]$.
(ii) follows from (i) with $m = M = 0$. □

APPLICATION: Let $c \in \mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable. If f satisfies the differential equation

$$f'(x) = c \cdot f(x) \qquad \forall x \in \mathbb{R},$$

then f has to be of the form $f(x) = f(0)\exp(c\,x)$.

To prove this we put $g(x) := f(x)\exp(-cx)$ and observe that

$$g'(x) = -ce^{-cx}\,f(x) + e^{-cx}\,f'(x) = -ce^{-cx}\,f(x) + e^{-cx}\,c\,f(x) = 0.$$

Thus $g(x) = g(0) = f(0)$ for all x, which implies $f(x) = f(0)\exp(c\,x)$.

8.6. Proposition: Suppose $f\colon [a,b] \to \mathbb{R}$ is continuous and differentiable in $]a,b[$.

(i) If $f'(x) \geq 0$ (resp. $f'(x) > 0$) for all $x \in]a,b[$ then f is increasing (resp. strictly increasing) on $[a,b]$.

(ii) If f is increasing on $[a,b]$ then $f'(x) \geq 0$ for all $x \in\ [a,b]$.

Corresponding statements hold for decreasing functions.

Proof. (i) Proof by contradiction: If f is not increasing (resp. strictly increasing) there are $x_1, x_2 \in [a,b]$, $x_1 < x_2$ such that $f(x_1) > f(x_2)$ (resp. $f(x_1) \geq f(x_2)$). By the mean value theorem there is a $\xi \in]x_1, x_2[$ such that

$$f'(\xi) = \frac{f(x_2) - f(x_1)}{x_2 - x_1},$$

which implies that $f'(\xi) < 0$ (resp. $f'(\xi) \leq 0$) — a contradiction $\lightning$.

(ii) Since f is increasing we have for all $x, \xi \in]a,b[$ with $x \neq \xi$ that

$$\frac{f(\xi) - f(x)}{\xi - x} \geq 0.$$

Thus $f'(x) \geq 0$. $\square$

Remark: Note that strict monotonicity does not imply that the derivative is strcitly greater or smaller than 0: For example, $f\colon [-1,1] \to \mathbb{R}$, $f(x) = x^3$, is strictly increasing whereas $f'(0) = 0$.

8.7. Corollary (Sufficient condition for a local extreme): Let $f\colon]a,b[\to \mathbb{R}$ be differentiable. Suppose that $x \in]a,b[$ and that f is twice differentiable at x. If

$$f'(x) = 0 \quad \text{and} \quad f''(x) > 0 \quad (\text{resp. } f''(x) < 0)$$

then x is a strict local minimum (resp. maximum) of f.

Proof. Assume that $f'(x) = 0$ and $f''(x) > 0$ (the case $f''(x) < 0$ is analogous.)

We obtain that

$$0 < f''(x) = \lim_{x \neq \xi \to x} \frac{f'(\xi) - f'(x)}{\xi - x}.$$

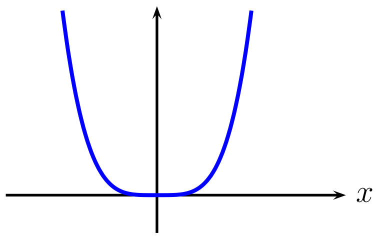
Hence there exists $\varepsilon > 0$ such that for all ξ with $0 < |\xi - x| < \varepsilon$

$$\frac{f'(\xi)}{\xi - x} = \frac{f'(\xi) - f'(x)}{\xi - x} > 0.$$

From this we deduce the following: If $\xi \in]x - \varepsilon, x[$ then $f'(\xi) < 0$ and if $\xi \in]x, x + \varepsilon[$ then $f'(\xi) > 0$. Therefore f is strictly decreasing in $[x - \varepsilon, x]$ and strictly increasing in $[x, x + \varepsilon]$, which implies that x is a strict local minimum for f. $\square$

8.8. Examples: 1) We already know that $\sin\colon \mathbb{R} \to \mathbb{R}$ has strict local maxima at the points $\frac{\pi}{2} + 2k\pi$ and strict local minima at $\frac{\pi}{2} + (2k+1)\pi$, where $k \in \mathbb{Z}$. We can now confirm this earlier observation by observing $\sin'(\frac{\pi}{2} + l\pi) = \cos(\frac{\pi}{2} + l\pi) = 0$ for all $l \in \mathbb{Z}$
and that $\sin''(\frac{\pi}{2} + 2k\pi) = -\sin(\frac{\pi}{2} + 2k\pi) = -1 < 0$,
$\sin''(\frac{\pi}{2} + (2k+1)\pi) = -\sin(\frac{\pi}{2} + (2k+1)\pi) = 1 > 0$.

2) The function $f\colon]-1,1[\to \mathbb{R}$, $f(x) = x^4$, has a strict (global) minimum at 0 since $f(x) = x^4 > 0$ for all $x \neq 0$ and $f(0) = 0$. But we also have $f''(0) = 0$.



This example illustrates that the condition $f''(x) > 0$ is *sufficient* but *not necessary* for a strict local extreme.

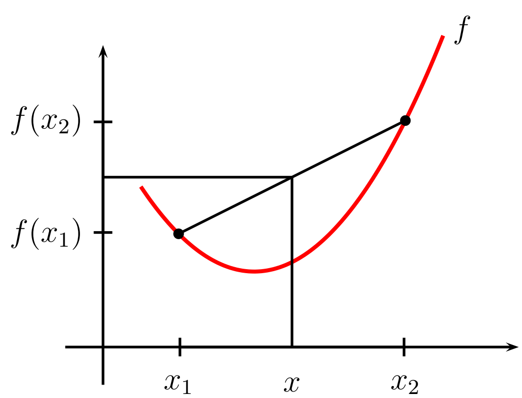
8.9. Convexity:

Definition: Let $I \subseteq \mathbb{R}$ be an interval and $f\colon I \to \mathbb{R}$. The function f is *convex* ⟨konvex⟩ if for all $x_1, x_2 \in I$ and for all $\lambda \in [0,1]$

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2). \tag{8.2}$$

f is said to be *concave* ⟨konkav⟩ if $-f$ is convex.

When the parameter λ runs through $[0,1]$ the corresponding point $x = \lambda x_1 + (1-\lambda)x_2 \in I$ runs through the closed interval with boundary points x_1 and x_2. The inequality (8.2) means that the points $(x, f(x))$ of the graph of f do not lie above the secant line through $(x_1, f(x_1)$ and $(x_2, f(x_2))$.



Remark: A subset B of a real vector space W is *convex* ⟨konvex⟩ if for all $u, v \in B$ the entire line segment $\{\lambda u + (1-\lambda)v : 0 \leq \lambda \leq 1\}$ connecting u and v belongs to B. In the definition given above the convexity of a function $f\colon I \to \mathbb{R}$ corresponds to convexity of the subset $B := \{(x,y) \in \mathbb{R}^2 : x \in I, y \geq f(x)\} \subseteq \mathbb{R}^2$ that lies above the graph of f.

PROPOSITION: Let $I \subseteq \mathbb{R}$ be an open interval and $f\colon I \to \mathbb{R}$ twice differentiable. Then

$$f''(x) \geq 0 \quad \forall x \in I \qquad \Longleftrightarrow \qquad f \text{ is convex.}$$

Proof. $\boxed{\Rightarrow}$ When checking (8.2) we may assume that $x_1 < x_2$ and $0 < \lambda < 1$. If $x := \lambda x_1 + (1-\lambda)x_2$ then $x_1 < x < x_2$.

By the Proposition in 8.5 f' is increasing and the mean value theorem provides us with $\xi_1 \in]x_1, x[$ and $\xi_2 \in]x, x_2[$ such that

$$(\star) \qquad \frac{f(x) - f(x_1)}{x - x_1} = f'(\xi_1) \leq f'(\xi_2) = \frac{f(x_2) - f(x)}{x_2 - x}.$$

Since $x - x_1 = (1-\lambda)(x_2 - x_1) > 0$ and $x_2 - x = \lambda(x_2 - x_1) > 0$ we obtain from $(\star)$ that

$$\frac{f(x) - f(x_1)}{1 - \lambda} \leq \frac{f(x_2) - f(x)}{\lambda},$$

which implies

$$f(x) \leq \lambda f(x_1) + (1-\lambda) f(x_2),$$

thus the convexity of f.

$\boxed{\Leftarrow}$ (Proof by contradition) Suppose there is an $x_0 \in I$ such that $f''(x_0) < 0$.

Let $\varphi(x) := f(x) - f'(x_0)(x - x_0)$ $(x \in I)$. Then $\varphi\colon I \to \mathbb{R}$ is twice differentiable and $\varphi'(x_0) = f'(x_0) - f'(x_0) = 0$. Since $\varphi''(x_0) = f''(x_0) < 0$ we deduce that φ possesses a strict local maximum at x_0.

Therefore we have $\varepsilon_0 > 0$ (sufficiently small) such that $\varphi(x) < \varphi(x_0)$ holds for all $x \in U_\varepsilon(x_0) \subseteq I$. In particular, we obtain for $0 < \varepsilon < \varepsilon_0$ that $\varphi(x_0 - \varepsilon) < \varphi(x_0)$ and $\varphi(x_0 + \varepsilon) < \varphi(x_0)$, which in turn yields

$$f(x_0) = \varphi(x_0) > \frac{1}{2}\big(\varphi(x_0 - \varepsilon) + \varphi(x_0 + \varepsilon)\big) = \frac{1}{2}\big(f(x_0 - \varepsilon) + f(x_0 + \varepsilon)\big).$$

If we put $\lambda := 1/2$, $x_1 := x_0 - \varepsilon$, and $x_2 := x_0 + \varepsilon$, the latter inequality means $f(\lambda x_1 + (1-\lambda)x_2) > \lambda f(x_1) + (1-\lambda) f(x_2)$ — a contradiction ↯ to the convexity of f. $\square$

EXAMPLES: 1) Consider the quadratic polynomial function $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$ are constants, $a \neq 0$. Since $f''(x) = 2a$ we have:

f is convex if and only if $a > 0$ 

f is concave if and only if $a < 0$ 

2) $\exp'' = (\exp')' = \exp' = \exp > 0$, hence exp is convex.

3) $\log'(x) = 1/x$ $(x > 0)$ and $\log''(x) = -1/x^2 < 0$, hence $\log\colon]0, \infty[\to \mathbb{R}$ is concave.

An interesting application of the concavity of log is the following: Let $p, q \in]1, \infty[$ such that $\frac{1}{p} + \frac{1}{q} = 1$. For all $x, y > 0$ we have that $x/p + y/q = x/p + (1 - 1/p)\, y$ is a convex combination and thus

$$\log(\frac{1}{p}x + \frac{1}{q}y) \geq \frac{1}{p}\log(x) + \frac{1}{q}\log(y) = \log(x^{1/p}) + \log(y^{1/q}) = \log(x^{1/p} \cdot y^{1/q}).$$

Upon applying the exponential function we obtain the following inequality

$$\frac{x}{p} + \frac{y}{q} \geq x^{1/p} \cdot y^{1/q} \qquad (x, y > 0; p, q > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1). \tag{8.3}$$

As a special case with $p = q = 2$ we have proved the inequality between the *arithmetic* and the *geometric mean* ⟨*Ungleichung zwischen arithmetischem und geometrischem Mittel*⟩

$$\frac{x + y}{2} \geq \sqrt{xy} \qquad (x, y \geq 0). \tag{8.4}$$

8.10. Generalized mean value theorem and de l'Hospital's rules:

LEMMA: Let $f, g\colon [a, b] \to \mathbb{R}$ continuous and differentiable in $]a, b[$. Then there exists $\xi \in]a, b[$ such that

$$\big(f(b) - f(a)\big)\, g'(\xi) = \big(g(b) - g(a)\big)\, f'(\xi).$$

(Generalized mean value theorem.)

Proof. Define $h\colon [a, b] \to \mathbb{R}$ by

$$h(x) := \big(f(b) - f(a)\big)\, g(x) - \big(g(b) - g(a)\big)\, f(x).$$

Then h is continuous on $[a, b]$ and differentiable in $]a, b[$. Furthermore, $h(a) = f(b)g(a) - g(b)f(a) = h(b)$ implies that there is some $\xi \in]a, b[$ such that $0 = h'(\xi) = (f(b) - f(a))\, g'(\xi) - (g(b) - g(a))\, f'(\xi)$, which proves the assertion. $\square$

PROPOSITION (RULES OF DE L'HOSPITAL[3]): Let $-\infty \leq a < b \leq \infty$ and $f, g\colon]a, b[\to \mathbb{R}$ be differentiable functions such that $g'(x) > 0$ for all $a < x < b$ (resp. $g'(x) < 0$ for all $a < x < b$). Suppose that the following limit exists

$$\eta := \lim_{x \nearrow b} \frac{f'(x)}{g'(x)}. \tag{8.5}$$

[3]Guillaume François Antoine, Marquis de l'Hospital (1661–1704) [gijoːm fʀɑ̃swa maʀˈki də lopiˈtal] was a French mathematician. He is the author of the first known textbook on differential calculus, l'Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes, published in 1696. It includes the lectures of his teacher, Johann Bernoulli, who was paid 300 Francs a year to tell de L'Hospital about his discoveries which, including the above rule, were then published under the Marquis' name.

(i) If in addition $\lim_{x\nearrow b} f(x) = \lim_{x\nearrow b} g(x) = 0$, then $g(x) \neq 0$ for all $a < x < b$ and

$$\lim_{x\nearrow b} \frac{f(x)}{g(x)} = \eta.$$

(ii) If in addition $\lim_{x\nearrow b} g(x) = \pm\infty$, then there is some $x_0 \in]a, b[$ such that $g(x) \neq 0$ for all $x_0 < x < b$ and

$$\lim_{x\nearrow b} \frac{f(x)}{g(x)} = \eta.$$

Analogous statements hold for the limits $x \searrow a$.

REMARK: The exact same statement is true if the limit in (8.5) is improper (cf. [Heu88, Abschnitt 50]).

Proof. (i) Since g' is positive (resp. negative) g is strictly increasing (resp. decreasing), hence g is injective. Therefore $\lim_{x\nearrow b} g(x) = 0$ implies that $g(x) \neq 0$ for all x (since $g(x_0) = 0$ with $x_0 < b$ would contradict the strict monotonicity).

Let $\varepsilon > 0$. By (8.5) we may choose $\beta \in]a, b[$ such that $f'(r)/g'(r) \in U_\varepsilon(\eta)$ for all $\beta < r < b$.

Let $x, y \in]\beta, b[$, $x \neq y$. By the above Lemma there is some ξ between x and y such that

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(\xi)}{g'(\xi)} \in U_\varepsilon(\eta).$$

In other words, $\left|\frac{f(x) - f(y)}{g(x) - g(y)} - \eta\right| < \varepsilon$. Since $g(x) \neq 0$ we may send $y \to b$ while keeping x fixed and obtain

$$\left|\frac{f(x)}{g(x)} - \eta\right| \leq \varepsilon,$$

which in turn shows that $f(x)/g(x) \to \eta$ as $x \to b$.

(ii) There is some $x_0 \in]a, b[$ such that $|g(x)| \geq 1$ for all $x \geq x_0$. In particular, $g(x) \neq 0$ for $x \geq x_0$.

Let $\varepsilon > 0$ and choose $\beta \in]x_0, b[$ such that

$$\forall r \geq \beta : \qquad \frac{f'(r)}{g'(r)} \in U_\varepsilon(\eta) \quad \text{and} \quad |g(r)| \geq \frac{1}{\varepsilon}.$$

Let (x_n) be a sequence in $]\beta, b[$ such that $\lim x_n = b$. For each $n \in \mathbb{N}$ the above Lemma provides some y_n between β and x_n such that

$$\frac{f(x_n) - f(\beta)}{g(x_n) - g(\beta)} = \frac{f'(y_n)}{g'(y_n)} \in U_\varepsilon(\eta).$$

Thus we have
$$\frac{f(x_n)}{g(x_n)} = \frac{g(x_n) - g(\beta)}{g(x_n)} \cdot \frac{f'(y_n)}{g'(y_n)} + \frac{f(\beta)}{g(x_n)}$$
and
$$\begin{aligned}
\left|\frac{f(x_n)}{g(x_n)} - \eta\right| &\leq \left|\frac{g(x_n) - g(\beta)}{g(x_n)} \cdot \frac{f'(y_n)}{g'(y_n)} - \eta\right| + \left|\frac{f(\beta)}{g(x_n)}\right| \\
&= \left|\frac{g(x_n) - g(\beta)}{g(x_n)}\left(\frac{f'(y_n)}{g'(y_n)} - \eta\right) + \left(\frac{g(x_n) - g(\beta)}{g(x_n)} - 1\right)\eta\right| + \left|\frac{f(\beta)}{g(x_n)}\right| \\
&\leq \left|\frac{g(x_n) - g(\beta)}{g(x_n)}\right|\left|\frac{f'(y_n)}{g'(y_n)} - \eta\right| + \left|\frac{g(x_n) - g(\beta)}{g(x_n)} - 1\right||\eta| + \left|\frac{f(\beta)}{g(x_n)}\right| \\
&= \left|1 - \frac{g(\beta)}{g(x_n)}\right|\left|\frac{f'(y_n)}{g'(y_n)} - \eta\right| + \left|-\frac{g(\beta)}{g(x_n)}\right||\eta| + \left|\frac{f(\beta)}{g(x_n)}\right| \\
&\leq (1 + \varepsilon\,|g(\beta)|)\,\varepsilon + \varepsilon\,|g(\beta)||\eta| + \varepsilon\,|f(\beta)| \\
&= \big(1 + \varepsilon\,|g(\beta)| + |g(\beta)||\eta| + |f(\beta)|\big) \cdot \varepsilon \leq K\,\varepsilon,
\end{aligned}$$
for some $K > 0$ if ε stays less than 1, say. Therefore $f(x_n)/g(x_n) \to \eta$ as $n \to \infty$. □

EXAMPLES: 1) We can give an alternative proof for 6.3,6): $\lim_{x\to\infty} \frac{\log(x)}{x^\alpha} = 0$ $(\alpha > 0)$.

With $f(x) := \log(x) \to \infty$, $g(x) := x^\alpha \to \infty$, $f'(x) = 1/x$, and $g'(x) = \alpha x^{\alpha-1}$ we obtain that
$$\frac{f'(x)}{g'(x)} = \frac{1}{\alpha x^\alpha} \to 0 =: \eta \qquad (x \to \infty),$$
hence Proposition (i) proves the above limit assertion.

2) $\lim_{x\searrow 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = \boxed{\ ?\ }$

Note that $\frac{1}{\sin(x)} - \frac{1}{x} = \frac{x - \sin x}{x \sin x}$, hence we attempt to apply the Proposition with $f(x) := x - \sin x \to 0$ and $g(x) := x \sin x \to 0$ as $x \to 0$.

We have

$f'(x) = 1 - \cos x \to 0$ and $g'(x) = \sin x + x \cos x \to 0$ as $x \to 0$

$f''(x) = \sin x \to 0$ and $g''(x) = \cos x + \cos x - x \sin x \to 2$ as $x \to 0$.

At this point we may observe that de l'Hospital's rule is applicable with f replaced by f' and g replaced by g'. Therefore we may summarize as follows
$$\lim_{x\searrow 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\searrow 0}\frac{f(x)}{g(x)} = \lim_{x\searrow 0}\frac{f'(x)}{g'(x)} = \lim_{x\searrow 0}\frac{f''(x)}{g''(x)} = \frac{0}{2} = 0.$$

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