On some aspects of local thermodynamical formalism

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Abstract

In 2007, Ye & Zhang introduced a version of local topological entropy. Since their entropy function is, as we show under mild conditions, constant for topologically transitive dynamical systems, we propose to adjust the notion in a way that does not neglect the initial transient part of an orbit. We investigate the properties of this "transient" version, which we call translocal entropy, and compute it in terms of Lyapunov exponents for various dynamical systems. We also investigate how this adjustment affects measure-theoretic local (Brin-Katok) entropy and local pressure functions, generalizing some partial variation principles of Ma & Wen.

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1 Introduction and motivation

Topological and measure-theoretic entropy can be interpreted as indications of complexity of a map $f : X \to X$, but they provide a global view, without specifying on which subsets of X the map is "more complicated" or "less complicated". This motivated the study of local versions of entropy and of pressure, as we will discuss presently.

Topological entropy is related to measure-theoretic entropy via the variational principle: If f is a continuous of a compact metric space (X, d) (an assumption we will make throughout the paper), then $h_{\text{top}}(f) = \sup_\mu h_\mu(f)$, where the supremum is taken over all f-invariant probability measures μ on X. Measures μ such that $h_{\text{top}}(f) = h_\mu(f)$ are called *measures of maximal entropy* or *maximal measures*. In thermodynamic formalism, one tries to maximize the sum of entropy and energy (represented by an integral over a potential function $\phi : X \to \mathbb{R}$), called *free energy* or *measure-theoretic pressure*. The *topological pressure* can then be defined by the variational principle as $P_{top}(f) = \sup_\mu \{h_\mu + \int_X \phi \, d\mu\}$, see e.g. [33, 37].

Local versions of topological entropy were used by Bowen [9] to get an upper estimate of the difference between $h_\mu(f)$ and the entropy of a partition with diameter less than ε and by

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Misiurewicz [29], who proved that vanishing of his local entropy implies existence of maximal measure. Different versions were used by Newhouse [30], and by and Buzzi & Ruette [13, 15], to address questions about the (unique) existence of a measure of maximal entropy. All these notions rely on *Bowen balls* (also called *dynamical balls*):

$$B_n(z;\varepsilon) = \{z' \in X : d(f^j(z), f^j(z') < \varepsilon \text{ for all } 0 \le j < n\}.$$

A local view on measure-theoretic entropy goes back to at least Brin & Katok [11]. In particular, Brin-Katok entropy $h_\mu(f,x)$ (see (6) in Section 3.1 for the definition), when integrated over the space yields the global measure-theoretic entropy:

$$\int h_\mu(f,x)\, d\mu(x) = h_\mu(f).$$

Variational principles have been derived for this local version as well, see e.g. [18].

1.1 The Ye & Zhang entropy function and translocal entropy.

In 2007, Ye & Zhang [39] introduced a new version of local entropy, which they called *entropy function* $h_{\text{top}} : X \to [0,\infty]$. Their definition, see (3) below, is the starting point of our paper, but one soon discovers that under mild conditions the entropy function is constant: $h_{\text{top}}(z) \equiv h_{\text{top}}(f)$ for very natural dynamical systems, see Theorem 2.1. It is not straightforward to find maps with non-constant entropy functions, see Section 2.1. The reason is that in the Ye & Zhang approach, only the limit behaviour of the orbit of z counts, not the "transient" behaviour at finite time scales. The entropy function does not see if entropy is created "immediately" in small neighbourhoods of z, or only at very distant time scales.

For this reason, we propose an adjustment to Ye & Zhang's definition, which we call the *translocal entropy function* and denote as h_ω, that takes into account the time scale needed for Bowen balls to reach unit size. As a result, $h_\omega(z)$ depends non-trivially on $z \in X$, and Lyapunov exponents at z play a central role. This interaction is most transparent in smooth one-dimensional maps, see Theorem 2.2, where each point has at most one Lyapunov exponent (because there is only one direction in which derivatives can be taken). As representatives of higher-dimensional maps, we study toral automorphisms $F_A : \mathbb{T}^d \to \mathbb{T}^d$ based on a $d \times d$ matrix A. Theorem 2.3 shows that its translocal entropy function is determined by the Lyapunov exponents, i.e., the logarithms of the eigenvalues of A.

In the second half of the paper, we turn to the local approach to pressure and study the translocal version of pressure. Ma & Wen in [27] obtained a partial variational principle for topological entropy. They noticed (see Theorem 3.2) the relations between topological entropy of a map and its upper and lower measure-theoretic entropies. Using a Carathéodory-like construction, elaborated by Pesin [32], we define a dimensional type of topological pressure $P_Z(f,\phi)$ for a continuous map $f : X \to X$ with potential ϕ, restricted to subsets $Z \subset X$. Fixing a Borel probability measure μ on X we generalize the Ma & Wen result by means of upper and lower local measure-theoretic pressures, see Theorem 3.3. In Theorem 3.4 we show the same result for translocal version of local measure-theoretic pressures.

1.2 Structure of the paper

Our paper is organized as follows: In Section 2, we introduce the entropy function by Huang & Ye, and study when it is constant and when not. In Section 2.2, we give our adaptation of the entropy function, and give its basic and more advanced properties in the remaining subsections of Section 2.

Section 3 is devoted to local measure-theoretic approach to pressure. Section 3.1 deals with measure-theoretic versions of local entropy, starting with the Brin-Katok entropy. We discuss (local) topological pressure in Section 3.2 and give a variational principle in Section 3.3. The variational principle of our translocal adaptation is covered in Section 3.4. We finish the paper in Section 4 with a discussion and comparison of various notions of local entropy in the literature.

2 Local topological entropy in sense of Ye & Zhang

Let $f : X \to X$ be a continuous map on a compact metric space (X, d). For closed subset $K \subset X$ we define the topological entropy $h_{\text{top}}(f, K)$ restricted to K as the limit of exponential growth rates of (n, ε)-separated subsets of K as $\varepsilon \to 0$, that is

$$h_{\text{top}}(f, K) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log S(n, \varepsilon, K), \tag{1}$$

where $S(n, \varepsilon, K)$ is the maximal cardinality of (n, ε)-separated subsets of K. A point $z \in X$ is an *entropy point* if there are arbitrarily small closed neighbourhoods K such that $h_{\text{top}}(f, K) > 0$ and a *full entropy point* if in addition $h_{\text{top}}(f, K) = h_{\text{top}}(f)$, see [34, Definition 5.1] and [39, Section 2]. Hence, only maps with positive entropy can have full entropy points.

It seems natural to define the *local entropy* of f at a point z as

$$\lim_{\delta \to 0} \inf\{h_{\text{top}}(f, K) : K \text{ is a closed } \delta\text{-neighbourhood of } z\}. \tag{2}$$

However, in [34] this step is not taken, and in fact, if f is locally eventually onto, for instance, then there is N_K such that $f^{N_K}(K) = X$. Notice that $(n - N_k, \varepsilon)$-separated points $y_1, y_2 \in f^{N_k}(K)$ determine points $x_1, x_2 \in K$, with $y_1 = f^{N_k}(x_1)$ and $y_2 = f^{N_k}(x_2)$, such that x_1, x_2 are (n, ε)-separated. Therefore $S(n, \varepsilon, K) \geq S(n - N_k, \varepsilon, f^{N_k}(K))$. Thus the above quantity in (2) is constant $h_{\text{top}}(f)$.

Instead, in [39, Definition 4.1] the local entropy function is defined as

$$h_{\text{top}}(x) = \lim_{\varepsilon \to 0} h_{\text{top}}(x, \varepsilon), \tag{3}$$

where

$$h_{\text{top}}(x, \varepsilon) = \inf\{\limsup_{n \to \infty} \frac{1}{n} \log S(n, \varepsilon, K) : K \text{ is a closed } \delta\text{-neighbourhood of } x\}.$$

This differs from (2) by the order in which the limits $\varepsilon \to 0$ and $\delta \to 0$ are taken. In general, $h_{\text{top}}(x) = h_{\text{top}}(f(x))$, so the local entropy is constant along forward orbits, see [39, Proposition 4.4(1)]. In fact, we have

Lemma 2.1 *The entropy function takes its minimum at transitive points, i.e., points with a dense orbit.*

Proof. Suppose $x \in X$ has a dense orbit. Let $y \in Y$ with closed neighbourhood K_y be arbitrary. Then there is $n \geq 0$ such that $f^n(x) \in K_y$ and K_x such that $f^n(K_y) \subset K_x$. It follows that $h_{\text{top}}(f, K_x) \leq h_{\text{top}}(f, K_y)$, and $h_{\text{top}}(x) \leq h_{\text{top}}(y)$ follows. □

2.1 When is the entropy function non-constant?

The next theorem show the limitations of the local entropy function: under mild assumptions it is constant. A version of this theorem, for much more specific dynamical systems, can be found in [8].

A map $f : X \to X$ is *topologically exact* or *locally eventually onto* if for every open $U \subset X$, there is $N \in \mathbb{N}$ such that $f^N(U) = X$. Topological exactness gives a simple sufficient condition for the entropy function to be constant.

Lemma 2.2 *If $f : X \to X$ is topologically exact, then $h_{top}(z) \equiv h_{top}(f)$ for any $z \in X$.*

Proof. Given a closed neighbourhood $K \ni z$, take N such that $f^N(K) \supset f^N(\mathring{K}) = X$. Then $\frac{1}{n} \log S(n, \varepsilon, K) \geq \frac{1}{n} \log S(n - N, \varepsilon, X) = \frac{n-N}{n} \frac{1}{n-N} \log S(n - N, \varepsilon, X)$. So after taking limits $n \to \infty$ and $\varepsilon \to 0$, we find $h_{\text{top}}(z) \geq h_{\text{top}}(f)$. □

Theorem 2.1 *Let $f : X \to X$ be a continuous topological transitive map defined on a compact metric space (X, d). If $h_{top}(f) = 0$ or if each measure of maximal entropy is fully supported, then $h_{top}(z) \equiv h_{top}(f)$ for every $z \in X$.*

Proof. Since $h_{\text{top}}(f) \geq h_{\text{top}}(z)$ for every $z \in X$, only the case that $h_{\text{top}}(f) > 0$ needs a proof.

Assume by contradiction that there is $z \in X$ is such that $h_{\text{top}}(z) < h_{\text{top}}(f)$. Take compact neighbourhoods K', K of z such that K' is compactly contained in K, such that

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log S(n, \varepsilon, K) < b := \frac{h_{\text{top}}(z) + h_{\text{top}}(f)}{2}.$$

The set $Y = \overline{X \setminus \bigcup_{n \geq 0} f^{-n}(K')}$ is compact and forward invariant. Since measures of maximal entropy are fully supported, $h' := h_{\text{top}}(f|_Y) < h_{\text{top}}(f)$. By the variational principle, we can take a sequence of f-invariant probability measures μ_n such that $h_{\mu_n}(f) \to h_{\text{top}}(f)$. Take n so large that $h_{\mu_n}(f) > h'$. This means in particular that $\text{supp}(\mu_n) \setminus Y \neq \emptyset$ and by invariance of $\text{supp}(\mu_n)$, also $\text{supp}(\mu_n) \cap K' \neq \emptyset$. The Birkhoff Ergodic Theorem implies that $\text{orb}_f(x)$ is dense in $\text{supp}(\mu_n)$ for μ_n-a.e. x. By the Ye & Zhang result, see [39, Proposition 4.4], we can pick x such that additionally $h_{\text{top}}(x) \geq h'$. But there is $k \in \mathbb{N}$ such that $f^k(x) \in K'$. But then there is a compact neighbourhood $K'' \ni x$ such that $f^k(K'') \subset K$, which implies that $h_{\text{top}}(x) \leq h_{\text{top}}(z) < h'$. This contradiction proves the theorem. □

The assumption that the measures of maximal entropy are fully supported is not entirely automatic, although it is conjectured that **on manifolds** the measure of maximal entropy (if existent) is automatically fully supported if the map is topologically transitive, see [40, 14] for results supporting this conjecture. In the context of subshifts (X, σ) of positive entropy (so X is a Cantor set, not a manifold), Kwietniak et al. [25] presented a counter-example with all of its measures of maximal entropy supported on a proper subsets of X. Loosely inspired by their construction, we have the following topologically transitive (and in fact coded) subshift with a non-constant entropy function.

Example 2.1 *We construct a coded subshift of $\{0, 1, 2\}^{\mathbb{Z}}$ on which the local entropy function is not constant. Let $\{w_k\}_{k \in \mathbb{N}}$ be a denumeration of all finite words in $\{0, 1\}^+$ such that $|w_k| \leq |w_{k+1}|$ for all $k \in \mathbb{N}$. Let $\mathcal{C} = \{C_k\}_{k \in \mathbb{N}}$ the collection of code words, where*

$$C_k = 20^{(10+k)!} w_k 0^{(10+k)!} 2.$$

A coded shift *is the shift of which the language consists of subwords of free concatenations of code words, see [26] and [31]. The structure of the code words in $\mathcal{C}$ implies that* 22 *is a* synchronizing word *(i.e., if* $u22$ *and* $22v$ *are both allowed words in* $X_{\mathcal{C}}$*, then so is* $u22v$*), but more importantly,* $h_{top}(X_{\mathcal{C}}, \sigma) \geq \log 2$*, because the full shift on* $\{0, 1\}$ *is a subshift of* $X_{\mathcal{C}}$*.*

Now consider the cylinder set $[2.2]$ *and let* $z \in [2.2]$ *be such that* $\mathrm{orb}_\sigma(z)$ *is dense in* $X_{\mathcal{C}}$ *and* $z \in K \subset [2.2]$ *any closed neighbourhood. From the theory of coded shifts, see [12, Section 3.3], we can compute the exponential growth rate of the number of centered words* x *of length* $2n+1$ *and with* $x_{-1} = x_0 = 2$*, namely, this is the unique positive solution* h *of* $\sum_k e^{-h|C_k|} = 1$*, and it is clear that* $h < \log 2$*. It follows that* $h_{top}(z) \leq h < h_{top}(X_{\mathcal{C}}, \sigma)$*. In this case, work of Pavlov [31, Theorem 1.1] implies that its measures of maximal entropy are supported on* $\overline{\mathrm{orb}_\sigma(z)} \setminus \mathrm{orb}_\sigma(z)$*, i.e.,* $\{0, 1\}^{\mathbb{Z}}$*. On the other hand, for every closed neighbourhood* K *of* z*, there is* n *such that* $\sigma^n(K) \supset [2.2]$*, and therefore* $h_{top}(z) = \log h < h_{top}(X_{\mathcal{C}}, \sigma) = \log 2$*.*

Without the assumption of transitivity, maps with non-constant entropy are not hard to find. A simple example of a map where $h_{\text{top}}(z)$ is not constant, and in fact, $\sup_z h_{\text{top}}(z) = \infty$, is given in Figure 1; see [23] for Hölder continuous maps of this type. Maps of this type (i.e., with infinitely many transitive components, each with its own value for local entropy, and infinite entropy altogether, are C^0-generic on any manifold of dimension ≥ 2, see [17]. For such examples, the values that local entropy assume forms a countable dense subset of $[0, \infty)$, and with some more work (but giving up on genericity) one can find smooth examples with these properties.

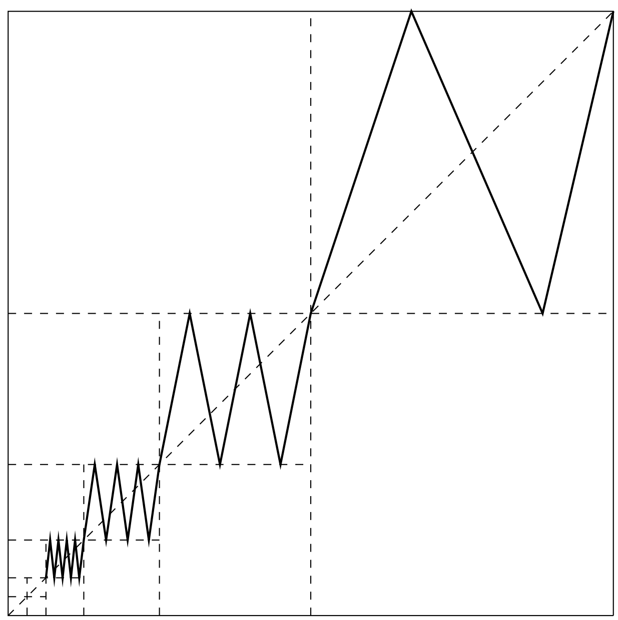

Figure 1: An infinite entropy map where $h_{\text{top}}(x) = \log(2n+1)$ for $x \in (2^{-n}, 2^{1-n}]$ and $h_{\text{top}}(0) = \infty$.

2.2 Translocal entropy

Definition 2.1 *For $\omega \geq 0$ and $z \in X$, the* upper translocal entropy *at $z \in X$ is*

$$\overline{h}_\omega(z) = \max\{\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log S(n,\varepsilon,\overline{B(z,e^{-\omega n})})\ ,\ 0\}.$$

The lower translocal entropy *$\underline{h}_\omega(z)$ at $z \in X$ is the same with* lim inf *instead of* lim sup*. If the two are the same, then $h_\omega(z) := \overline{h}_\omega(z) = \underline{h}_\omega(z)$ is the* translocal entropy *at z.*

Clearly $\overline{h}_\omega(z)$ is decreasing in ω. Compared to the quantity (2), $h_{\text{top}}(x)$ is obtained by swapping the limit $\lim_{\varepsilon\to 0}$ and $\inf_K$, and $\underline{h}_\omega(z)$ etc. comes from a simultaneous limit in between the two. As a result:

Lemma 2.3 *Let $f : X \to X$ be a continuos map on a compact metric space. For all $x \in X$ and $\omega \geq 0$, we have*

$$\begin{aligned} 0 &= \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\inf\left\{\frac{1}{n}\log S(n,\varepsilon,K) : K \text{ is a closed } \delta\text{-neighbourhood of } x\right\} \\ &\leq \underline{h}_\omega(x) \ \leq\ \overline{h}_\omega(x) \qquad (4) \\ &\leq \inf\{h_{top}(f,K) : K \text{ is a closed } \delta\text{-neighbourhood of } x\} \leq\ h_{top}(x). \end{aligned}$$

Proof. If K is sufficiently small compared to n, then $S(n,\varepsilon,K) = 1$, so the first equality in (4) is obvious. Since $S(n,\varepsilon,K) \geq 1$, we have $\frac{1}{n}\log S(n,\varepsilon,K) \geq 0$ for all $\varepsilon > 0$, $n \geq 1$ and closed neighbourhoods K of x. Also, $\frac{1}{n}\log S(n,\varepsilon,K)$ is decreasing in n and in ε, but increasing in K. Write $S(\varepsilon,K) = \lim_{n\to\infty} S(n,\varepsilon,K)$; due to the subadditivity of $\log S(n,\varepsilon,K)$, the limit exists and is bounded by $h_{\text{top}}(f)$. Therefore, for every closed neighbourhood K_0 of x,

$$0 \leq \lim_{\operatorname{diam} K\to 0} S(\varepsilon,K) \leq S(\varepsilon,K_0) \leq h_{\text{top}}(f).$$

Taking the limit $\varepsilon \to 0$ on both sides gives

$$0 \leq \lim_{\varepsilon\to 0}\lim_{\operatorname{diam} K\to 0} S(\varepsilon,K) \leq \lim_{\varepsilon\to 0} S(\varepsilon,K_0) \leq h_{\text{top}}(f).$$

Finally take the limit $\operatorname{diam} K_0 \to 0$, and the lemma follows. □

The next example explores the role of Lyapunov exponents in the translocal entropy function. For one-dimensional maps, the Lyapunov exponent is defined as

$$\lambda(x) = \lim_{n\to\infty}\frac{1}{n}\log|Df^n(x)|,$$

provided the derivatives and the limit exist.

Example 2.2 *(i) Let $f : \mathbb{S}^1 \to \mathbb{S}^1$, $x \mapsto 3x \mod 1$. For any $x \in \mathbb{S}^1$ and $K = [x-\delta, x+\delta]$, we have $f^{n_0}(K) = \mathbb{S}^1$ for $n_0 \geq -\frac{\log 2\delta}{\log 3}$, and $S(n,\varepsilon,K) \geq 3^{n-n_0}/\varepsilon$. Therefore $h_{top}(f,K) = \log 3 = \lambda(x)$ for every $x \in \mathbb{S}^1$, and so every point is a full entropy point. Regarding $h_\omega(z)$, if $K_n(z) = \overline{B(z,e^{-\omega n})}$, then after $k = \omega n/\log 3$ iterates, $f^k(K_n(z)) = \mathbb{S}^1$. Therefore*

$$\begin{aligned} \frac{1}{n}\log S(n,\varepsilon,K_n(z)) &\sim \frac{1}{n}\log S(n-k,\varepsilon,[0,1]) \\ &= (1-\frac{\omega}{\log 3})\frac{1}{n-k}\log S(n-k,\varepsilon,[0,1]) \to (1-\frac{\omega}{\log 3})h_{top}(f). \end{aligned}$$

This is independent of z because the Lyapunov exponent is the same for every z.

(ii) However, if $g : \mathbb{S}^1 \to \mathbb{S}^1$ is given by

$$g(x) = \begin{cases} 2x & \text{if } 0 \le x < \frac{1}{2}, \\ 4x - 2 & \text{if } \frac{1}{2} \le x < \frac{3}{4}, \\ 4x - 3 & \text{if } \frac{3}{4} \le x < 1, \end{cases}$$

then $h_{top}(g, K) = \log 3$, independent of the Lyapunov exponent which is non-constant for this maps. Again every point is a full entropy point. But this time, the Lyapunov exponent $\lambda(z)$ varies with the point, and it takes $k \sim \frac{\omega}{\lambda(z)}$ iterates for $K_n(z)$ to reach large scale. If the limit $\lambda(z)$ exists, then $h_\omega(z) = (1 - \frac{\omega}{\lambda(z)})h_{top}(f)$.

(iii) The same method shows that $h_\omega(z) = 0$ for $z = 0$ and the Pomeau-Manneville map

$$g_{PM}(x) = \begin{cases} \frac{x}{1-x} & \text{if } 0 \le x < \frac{1}{2}, \\ 2x - 1 & \text{if } \frac{1}{2} \le x < 1. \end{cases}$$

For this map, $g_{PM}(\frac{1}{n}) = \frac{1}{n+1}$, so if $K_n(0) = [0, e^{-\omega n}]$, then it take $e^{\omega n} \gg n$ iterates to reach unit size. The above computation gives $h_\omega(0) = 0$ for all $\omega > 0$.

(iv) On the other hand, $h_\omega(z) = \log 2$ for $z = 0$ and every $\omega \ge 0$ for the map

$$g_{\sqrt{}}(x) = \begin{cases} \sqrt{2x} & \text{if } 0 \le x < \frac{1}{2}, \\ 2x - 1 & \text{if } \frac{1}{2} \le x < 1. \end{cases}$$

This time, $g_{\sqrt{}}^k(e^{-\omega n}) \approx 2^k e^{-\omega n/2^k}$, so it takes no more that $k \approx \log n$ iterates to reach large scale. Hence, $h_\omega(0) = h_{top}(g_{\sqrt{}}) = \log 2$ for all $\omega \ge 0$.

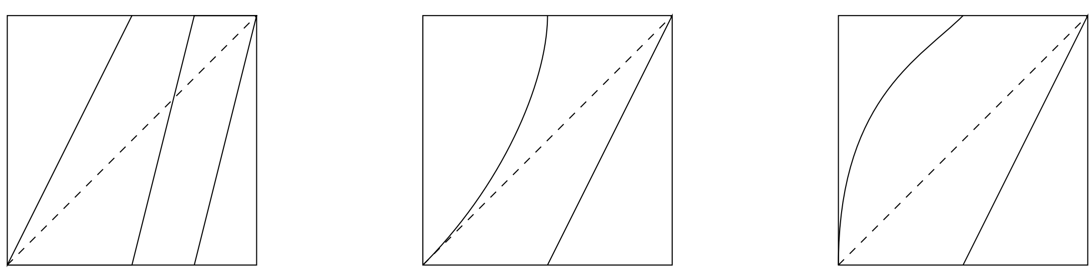

Figure 2: The maps g, g_{PM} and $g_{\sqrt{}}$.

2.3 Properties of translocal entropy

As the translocal entropy $h_\omega(z) = 0$ if f is constant on a neighbourhood of z, we will assume in the remainder of this paper that $f : X \to X$ is a local homeomorphism.

Lemma 2.4 *The translocal entropy for iterates of the map f satisfies $h_\omega(f^R, z) = Rh_\omega(f, z)$ for integers $R \ge 0$ and also for $h_\omega(f^{-R}, z) = Rh_\omega(f, z)$ if f is invertible.*

Proof. The proof goes as for the analogous statement on topological entropy. □

Definition 2.2 *We say that u has a* super-exponential approach *to v if*

$$\limsup_{k\to\infty} -\frac{1}{k}\log d(f^k(u),v) = \infty$$

.

Clearly u has a *super-exponential approach* to every $v \in \text{orb}_f(u)$, but the definition refers to other situations too. The following example shows this, but also that *super-exponential approach* is not a symmetric or transitive notion.

Example 2.3 *Let $X = \{0,1\}^{\mathbb{Z}}$ and $\sigma : X \to X$ be the left shift. Let $w = 0^\infty.0^\infty \in X$ the sequence that is constant 0. Let*

$$v = 1^\infty.101^{3!-2!}0^{4!-3!}1^{5!-4!}0^{6!-5!}1^{7!-6!}0^{8!-7!}\dots$$

Note that the number of iterates $2+\sum_{j=1}^{n-1} j! - (j-1)! = (n-1!)+1$ before the n-th block of 0s is reach is negligible compared to the length $n! - (n-1)! = (n-1)(n-1)!$ of that block, so v exponentially approaches w, but as w is a fixed point, w does not approach v. Finally, let

$$u = 1^\infty.1^{2!}v_1\dots v_{3!-2!}1^{4!-3!}v_1\dots v_{5!-4!}1^{6!-5!}v_1\dots v_{7!-6!}1^{8!-7!}\dots$$

Then u has super-exponential approach to v, but not to w.

Proposition 2.1 *Let $f : X \to X$ be a Lipschitz continuous map on a metric space. If u has a* super-exponential approach *to v, then $\underline{h}_\omega(u) \le \overline{h}_\omega(v)$.*

Proof. If $\omega = 0$, then the translocal entopy function coincides with the topological entropy, and there is nothing to prove. So assume that $\omega > 0$.

Let L be a global Lipschitz constant of f; for simplicity, we assume that $\log 2L > 1 + 2\omega$. Choose $\varepsilon > 0$ and $R > \log L$ arbitrary and pick $k \in \mathbb{N}$ such that $d(f^k(u), v) \le e^{-Rk} < \varepsilon/4$. Then for $n = \lceil Rk/\omega \rceil$ we have

$$f^k(\overline{B(u, e^{-\omega(n+k)})}) \subset B(f^k(u), e^{-\omega(n+k)+k\log L}) \subset B(v, 2e^{-\omega(n+k)+k\log L}) \subset \overline{B(v, e^{-\omega(n-k')})}$$

for $k' := k(\frac{\log 2L}{\omega} - 1) > k$.

We can assume that $\operatorname{diam} f^j(\overline{B(w, e^{-\omega(n+k)})}) < \varepsilon$ for $j \le k$, so in the first k iterates, no points in $\overline{B(w, e^{-\omega(n+k)})}$ are ε-separated. Therefore

$$\begin{aligned}
\frac{1}{n+k}\log S(n+k,\varepsilon,\overline{B(u,e^{-\omega(n+k)})}) &\le \frac{1}{n+k}\log S(n,\varepsilon,\overline{B(f^k(u),e^{-\omega(n+k)+k\log L})}) \\
&\le \frac{1}{n+k}\log\left(S(n,\varepsilon,\overline{B(v,e^{-\omega(n-k')})})\right) \\
&\le \frac{1}{n+k}\log\left(S(n-k',\varepsilon,\overline{B(v,e^{-\omega(n-k')})})\cdot e^{(k'+k)h_{\text{top}}(f)}\right) \\
&\le \frac{1}{n-k'}\log S(n-k',\varepsilon,\overline{B(v,e^{-\omega(n-k')})}) + \frac{h_{\text{top}}(f)\log 2L}{R},
\end{aligned}$$

where in the last line we used $\frac{k'+k}{n+k} < \frac{\log 2L}{R}$. Since $h_{\text{top}}(f) \le \log L$ for Lipschitz maps and ε and R are arbitrarily, $\underline{h}_\omega(u) \le \overline{h}_\omega(v)$, as claimed. □

Example 2.4 *This example shows that no other inequality holds in Proposition 2.1. Let v and w be the one-sided versions of the sequences v and w in Example 2.3, and interpret them as the itineraries of two points $p_v \in [0,1]$ and $p_w = 0$ for the map g in Example 2.2(ii), with respect to the partition $I_0 = [0, \frac{1}{2})$, $I_1 = [\frac{1}{2}, \frac{3}{4})$ and $I_2 = [\frac{3}{4}, 1]$. Then v approaches w super-exponentially, but $\overline{h}_\omega(p_u) = (1 - \frac{\omega}{\log 4}) \log 3 > (1 - \frac{\omega}{\log 2}) \log 3 = h_\omega(p_w)$ for every $\omega \in (0, \log 2]$.*

2.4 Translocal entropy for one-dimensional maps

The next theorem is exemplary for the translocal entropy of one-dimensional maps. It holds by and large also for smooth interval maps with critical points, but dealing with the technicalities of distortion control is not the purpose of this paper, so we only state it for expanding circle maps.

Theorem 2.2 *Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a C^2 expanding circle map and $\omega \geq 0$, then for every $z \in \mathbb{S}^1$*

$$\overline{h}_\omega(z) = \big(1 - \omega/\overline{\lambda}(z)\big)\, h_{top}(f),$$

where $\overline{\lambda}(z)$ is the upper Lyapunov exponent of z, and

$$\underline{h}_\omega(z) = (1 - \omega/\underline{\lambda}(z))\, h_{top}(f),$$

where $\underline{\lambda}(z)$ is the lower Lyapunov exponent of z.

Proof. If $\omega = 0$, then $B(z, e^{-\omega n})$ is independent of n, and the topological transitivity gives $\underline{h}_0(z) = \overline{h}_0(z) = h_{\text{top}}(f)$ for all $z \in \mathbb{S}^1$. So let us fix $\omega > 0$.

To explain the basic idea of the proof, we first assume that f is a C^2 expanding circle map, i.e., there are $c, c^+, A > 0$ such that for all $x \in \mathbb{S}^1$ we have expansion: $c^+ \geq |f'(x)| \geq c > 1$ and Adler's condition: $A(f,x) := |f''(x)|/|f'(x)|^2 \leq A$ for all $x \in \mathbb{S}^1$ (see [1, 10]). This means that Adler's conditions holds uniformly over all iterates. Indeed, due to a recursive formula $A(f \circ g, x) = (Af, g(x)) + \frac{1}{f' \circ g(x)} \cdot A(g,x)$, we get

$$\frac{|(f^n)''|}{|(f^n)'|^2} \leq \frac{Ac}{c-1} \qquad \text{for all } n \geq 0.$$

Fix $\eta := \frac{c-1}{2cA}$ and abbreviate $K_n(z) = \overline{B(z, e^{-\omega n})}$. Let

$$k_n(z) = \max\{k \geq 0 : \operatorname{diam}(f^k(K_n(z))) \leq \eta\}.$$

Take $\xi \in K_n(z)$ such that $(f^{k_n(z)})'(\xi)|K_n(z)| = |f^{k_n(x)}(K_n(z))| \in [\eta/c^+, \eta]$. Then

$$\left| \frac{1}{(f^{k_n(z)})'(\xi)} - \frac{1}{(f^{k_n(z)})'(z)} \right| = \left| \int_z^\xi \left(\frac{1}{(f^{k_n(z)})'(x)} \right)' dx \right| = \int_z^\xi A(f^{k_n(z)}, x)\, dx \leq \frac{Ac}{c-1} |\xi - z|.$$

This gives

$$\left| 1 - \left| \frac{(f^{k_n(z)})'(\xi)}{(f^{k_n(z)})'(z)} \right| \right| \leq \frac{Ac}{c-1} \left| (f^{k_n(z)})'(\xi) \right| \, |\xi - z| \leq \frac{Ac}{c-1} \eta = \frac{1}{2},$$

so that $\frac{1}{2} \leq \left| \frac{(f^{k_n(z)})'(\xi)}{(f^{k_n(z)})'(z)} \right| \leq \frac{3}{2}$. Recalling that $|K_n(z)| = 2e^{-\omega n}$, we get $\frac{\eta}{3c^+} e^{\omega n} \leq |(f^{k_n(z)})'(z)| \leq \eta e^{\omega n}$, so that

$$\omega + \frac{1}{n} \log \frac{\eta}{3c^+} \leq \frac{k_n(z)}{n} \frac{1}{k_n(z)} \log |(f^{k_n(z)})'(z)| \leq \omega + \frac{1}{n} \log \eta.$$

In the inferior/superior limit, we get $\liminf_n \frac{k_n(z)}{n} = \frac{\omega}{\overline{\lambda}(z)}$ and $\limsup_n \frac{k_n(z)}{n} = \frac{\omega}{\underline{\lambda}(z)}$, respectively.

Let N_η be such that $f^{N_\eta}(J) \supset \mathbb{S}^1$ for every interval J of length η. Then for the statement on the lim sup, we get

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n} \log S(n, \varepsilon, K_n(z)) &\geq \limsup_{n\to\infty} \frac{1}{n} \log S\left(n - k_n(z) - N_\eta, \varepsilon, f^{k_n(z)+N_\eta}(K_n(z))\right) \\
&= \limsup_{n\to\infty} \frac{n - k_n(z) - N_\eta}{n} \frac{\log S(n - k_n(z) - N_\eta, \varepsilon, \mathbb{S}^1)}{n - k_n(z) - N_\eta} \\
&= \left(1 - \frac{\omega}{\overline{\lambda}(z)}\right) \limsup_{n\to\infty} \frac{\log S(n - k_n(z) - N_\eta, \varepsilon, \mathbb{S}^1)}{n - k_n(z) - N_\eta} \\
&\to_{\varepsilon\to 0} \left(1 - \frac{\omega}{\overline{\lambda}(z)}\right) h_{\text{top}}(f).
\end{aligned}$$

For the other inequality, note that restrictions $f^j|_{\overline{B(z,e^{-\omega n})}}$, $1 \leq j \leq k_n(z)$, are homeomorphism, and therefore it can separate at most $1/\varepsilon$ points per iterate. Therefore

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n} \log S(n, \varepsilon, K_n(z)) &\leq \limsup_{n\to\infty} \frac{1}{n} \log \left(S\left(n - k_n(z), \varepsilon, f^{k_n(z)}(K_n(z))\right) \cdot \frac{k_n(z)}{\varepsilon} \right) \\
&\leq \limsup_{n\to\infty} \frac{n - k_n(z)}{n} \frac{\log S(n - k_n(z), \varepsilon, \mathbb{S}^1)}{n - k_n(z)} + \frac{1}{n} \log \frac{k_n(z)}{\varepsilon} \\
&= \left(1 - \frac{\omega}{\overline{\lambda}(z)}\right) \limsup_{n\to\infty} \frac{\log S(n - k_n(z), \varepsilon, \mathbb{S}^1)}{n - k_n(z)} \\
&\to_{\varepsilon\to 0} \left(1 - \frac{\omega}{\overline{\lambda}(z)}\right) h_{\text{top}}(f).
\end{aligned}$$

Similarly, for the statement on the lim inf, we get

$$\begin{aligned}
\liminf_{n\to\infty} \frac{1}{n} \log S(n, \varepsilon, K_n(z)) &\geq \liminf_{n\to\infty} \frac{1}{n} \log S(n - k_n(z) - N_\eta, \varepsilon, f^{k_n(z)+N_\eta}(K_n(z))) \\
&= \liminf_{n\to\infty} \frac{n - k_n(z) - N_\eta}{n} \frac{\log S(n - k_n(z) - N_\eta, \varepsilon, \mathbb{S}^1)}{n - k_n(z) - N_\eta} \\
&= \left(1 - \frac{\omega}{\underline{\lambda}(z)}\right) \liminf_{n\to\infty} \frac{\log S(n - k_n(z) - N_\eta, \varepsilon, \mathbb{S}^1)}{n - k_n(z) - N_\eta} \\
&\to_{\varepsilon\to 0} \left(1 - \frac{\omega}{\underline{\lambda}(z)}\right) h_{\text{top}}(f).
\end{aligned}$$

The other inequality is obtained as follows:

$$\begin{aligned}
\liminf_{n\to\infty} \frac{1}{n} \log S(n, \varepsilon, K_n(z)) &\leq \liminf_{n\to\infty} \frac{1}{n} \log \left(S\left(n - k_n(z), \varepsilon, f^{k_n(z)}(K_n(z))\right) \cdot \frac{k_n(z)}{\varepsilon} \right) \\
&\leq \liminf_{n\to\infty} \frac{n - k_n(z)}{n} \frac{\log S(n - k_n(z), \varepsilon, \mathbb{S}^1)}{n - k_n(z)} + \frac{1}{n} \log \frac{k_n(z)}{\varepsilon} \\
&= \left(1 - \frac{\omega}{\underline{\lambda}(z)}\right) \liminf_{n\to\infty} \frac{\log S(n - k_n(z), \varepsilon, \mathbb{S}^1)}{n - k_n(z)} \\
&\to_{\varepsilon\to 0} \left(1 - \frac{\omega}{\underline{\lambda}(z)}\right) h_{\text{top}}(f).
\end{aligned}$$

This finishes the proof. $\square$

2.5 Translocal entropy for higher-dimensional maps

Lemma 2.5 *If $F = f \times g : X \to Y \to X \times Y$ is a Cartesian product and $z = (x, y)$, then $h_\omega(F, z) = h_\omega(f, x) + h_\omega(g, y)$.*

Proof. We can take Cartesian product of (n, ε)-separated sets as (n, ε)-separated sets for the Cartesian product $f \times g$, and there are no more efficient sets. Inserting this in the definition of translocal entropy proves the result. $\square$

Theorem 2.3 *For affine toral automorphisms $F_A : \mathbb{T}^d \to \mathbb{T}^d$, $h_\omega(z) = \#\{i : \log|\lambda_i| \geq \omega\}(\log|\lambda_i| - \omega)$, where λ_i, $i = 1, \dots, d$, are the eigenvalues of the $d \times d$-matrix A that produces F_A.*

Proof. Let $\mu_1 > \mu_2 > \dots > \mu_k$ be the set of absolute values of eigenvalues. Each μ_j represents (possibly several) eigenvalues, and the direct sums of their generalized eigenspaces is denoted as E_j; it is a hyperplane and we let $1 \leq d_j \leq \dim(E_j)$ be the size of the largest Jordan block in the Jordan representation of $F_A|_{E_j}$. If $K_{n,j}(z)$ is the intersection of $B(z, e^{-\omega n})$ and a local plane in the direction of E_j, then $F_A^k(K_{n,j}(z))$ reaches unit size after $k_n(z) = Cn^{1-d_j} e^{n \log \mu_j}$ iterates, for some uniform constant $C > 0$. Here we assume that $\log \mu_j \geq \omega$, because otherwise $F_A^k(K_{n,j}(z))$ does not reach unit size for $k \leq n$ iterates. Therefore, $S(n, \varepsilon, K_{n,j}(z)) \approx S(n - k_n(z), \varepsilon, E_j(z))$, where $E_j(z)$ is a unit size κ_j-dimensional ball centered at z in the direction of E_j. Taking the logarithm and the limit $n \to \infty$, we obtain $h_\omega(z) = \lim_{n\to\infty} \frac{n-k_n(z)}{n} \frac{1}{n-k_n(z)} \log S(n - k_n(z), \varepsilon, E_j(z)) = \kappa_j \max\{\log \mu_j - \omega, 0\}$. Locally, F_A acts as the Cartesian product of $F_A|_{E_1(z)} \times \dots \times F_A|_{E_k(z)}$. Therefore $h_\omega(z) = \sum_{\log \mu_j \geq \omega} (\log \mu_j - \omega) = \sum_{\log|\lambda_i| \geq \omega} (\log|\lambda_i| - \omega)$, as claimed. $\square$

In fact, if F is a toral automorphism (or Anosov map) that is not necessarily affine, but has bounded distortion, then the formula becomes $\overline{h}_\omega(z) = h_{\text{top}}(f) - \sum_{\log|\overline{\lambda}_i(z)| \geq \omega} (\log|\overline{\lambda}_i|(z) - \omega)$, where $\overline{\lambda}_i$ stands for the i-th upper Lyapunov exponent. The same result works for affine hyperbolic horseshoe maps (or non-affine but with bounded distortion). The global product structure of these maps is important to get this outcome, because the formula can be quite different too, as the following example shows.

Example 2.5 *Let $\overline{\mathbb{D}}$ be the closed unit disk, and let $f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ be given in polar coordinates by $f(r, \phi) \mapsto (r(2 - r), 3\phi \pmod{2\pi})$. The derivative of $r \mapsto r(2 - r)$ at $r = 0$ is 2. Therefore it takes $K_n(z) := \overline{B(z, e^{-\omega n})}$ about $\frac{\omega n + \log \varepsilon}{\log 2}$ iterates to reach size ε, and at this point, the ϕ-component will let $S(n, \varepsilon, K_n(z))$ grow, with exponential rate $\log 3$. It follows that $h_\omega(0) = \log 3(1 - \frac{\omega}{\log 2})$.*

It is maybe good at this point to make a comparison with the notion of local entropy of $f : X \to X$ used by Buzzi & Ruette [13, 15] and Newhouse [30]. In their definition, an (n, ε)-Bowen ball takes the place of our exponentially small ball $B(z, e^{-\omega n})$. Thus the *local entropy* is

$$h_{\text{loc}}(f) = \lim_{\varepsilon\to 0} h_{\text{loc}}(f, \varepsilon) \quad \text{for} \quad h_{\text{loc}}(f, \varepsilon) = \limsup_{n\to\infty} \frac{1}{n} \sup_{x\in X} \log S(n, \varepsilon, B_n(z, \varepsilon)). \tag{5}$$

In fact, Buzzi uses (n, ε)-spanning sets rather than (n, ε)-separating sets, but that is of no consequence. Also, by taking the supremum over $x \in X$, $h_{\text{loc}}(f)$ becomes a quantity independent of the point in space. Buzzi shows that $h_{\text{loc}}(f) \leq \frac{\dim M}{r} \log \inf_n \sqrt[n]{\|Df^n\|_\infty}$ if $f : M \to M$ is a C^r-smooth map on a manifold M, and hence $h_{\text{loc}}(f) = 0$ for C^∞ maps. But if we adjust Example 2.5 to $(r, \phi) = (r, 3\phi \pmod{2\pi})$, then the point 0 gives local entropy $\log 3$.

Compared to $h_\omega(z)$, for conformal maps (such as the C^2 expanding circle maps of Theorem 2.2), we find $h_{\text{loc}}(f) = h_\omega(z) = 0$ for $\omega = \lambda(z)$, but for non-conformal maps we don't expect $h_{\text{loc}}(f)$ and $h_\omega(z) = 0$ to coincide.

3 Local measure-theoretic entropy and pressure

3.1 Brin-Katok local measure entropy

For a continuous map $f : X \to X$ defined on a compact metric space (X, d) and f-invariant Borel probability measure μ, Brin & Katok [11] introduced notions of *local lower* (resp. *local upper*) *measure-theoretic entropy* w.r.t. μ as follows

$$\begin{cases} \underline{h}_\mu(f, x) := \lim_{\varepsilon \to 0} \liminf_{n \to \infty} -\frac{1}{n} \log \mu(B_n(x, \varepsilon)), \\ \overline{h}_\mu(f, x) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} -\frac{1}{n} \log \mu(B_n(x, \varepsilon)). \end{cases} \tag{6}$$

Theorem 3.1 ([11]) *For a continuous map $F : X \to X$ on a compact metric space (X, d) and Borel F-invariant probability measure μ, the equality $\underline{h}_\mu(f, x) = \overline{h}_\mu(f, x) := h_\mu(f, x)$ holds for μ-a.e. $x \in X$. Moreover, $h_\mu(f, x)$ is F-invariant and*

$$\int_X h_\mu(f, x)\, d\mu(x) = h_\mu(f).$$

In addition, for an ergodic measure μ the equality $\overline{h}_\mu(f, x) = h_\mu(f)$ holds for μ-a.e. $x \in X$. A variational principle for the Brin-Katok entropy was proved by Feng & Huang [18]:

$$h_{\text{top}}(f, K) = \sup\{\underline{h}_\mu(f) : \mu \text{ is ergodic and } f\text{-invariant and } \mu(K) = 1\}, \tag{7}$$

where $h_{\text{top}}(f, K)$ is as in (1) for a compact set K.

Ma & Wen [27] noticed relationship between topological entropy restricted to a set $Z \subset X$ and local measure entropies on Z.

Theorem 3.2 ([27]) *For a continuous map $f : X \to X$ on a compact metric space (X, d), a Borel probability measure μ, a Borel subset $Z \subset X$ and $s \geq 0$ we have:*

1. *If $\underline{h}_\mu(f, x) \leq s$ for all $x \in X$, then $h_{top}(f, Z) \leq s$,*

2. *$\overline{h}_\mu(f, x) \geq s$ for all $x \in X$ and $\mu(Z) > 0$, then $h_{top}(f, Z) \geq s$.*

In the next section we provide a generalization of Theorem 3.2.

3.2 Topological pressure and Carathéodory structures

We recall the notion of topological pressure, in the spirit of Carathéodory structures elaborated by Pesin [32], for a continuous map $f : X \to X$ defined on a compact metric space (X, d), and restricted to a subset $Z \subset X$. Fix a continuous map (called *potential*) $\phi : X \to \mathbb{R}$, $N \in \mathbb{N}$, $r > 0$ and $s \geq 0$, define

$$M_Z(s, r, \phi, N) := \inf \left\{ \sum_{j \in J} \exp[-s \cdot n_j + \sum_{m=0}^{n_j - 1} \phi(f^m(x_j))] : Z \subset \bigcup_{j \in J} B_{n_j}(x_j, r);\ n_j \geq N \right\},$$

where the infimum is taken over all finite or countable coverings of the Z by Bowen balls $\{B_{n_j}(x_j, r)\}_{j \in J}$ with $n_j \geq N$. Here, the Bowen balls are indexed by the elements of a finite or countable set J Using standard arguments (see [32]) one can easily show that for any $N \in \mathbb{N}$, the inequality $M_Z(s, r, \phi, N + 1) \geq M_Z(s, r, \phi, N)$ holds. Therefore, there exists a limit

$$M_Z(s, r, \phi) := \lim_{N \to \infty} M_Z(s, r, \phi, N).$$

The graph of the function $s \mapsto M_Z(s, r, \phi)$ is very similar to the graph of s-Hausdorff measure function, i.e., there exists a unique critical parameter s_0, where the function $s \mapsto M_Z(s, r, \phi)$ drops from ∞ to 0. Thus, we can define

$$M_Z(r, \phi) := \sup\{s \geq 0 : M_Z(s, r, \phi) = \infty\} = \inf\{s \geq 0 : M_Z(s, r, \phi) = 0\} = s_0.$$

Fix $r_1 \leq r_2$ and consider a cover $\{B_{n_j}(x_j, r_1)\}_{j \in J}$ of Z, with $n_j \geq N$. Then $\{B_{n_j}(x_j, r_2)\}_{j \in J}$ is a covering of Z, so $M_Z(s, r_1, \phi) \geq M_Z(s, r_2, \phi)$. Therefore

$$M_Z(r_1, \phi) = \inf\{s \geq 0 : M_Z(s, r_1, \phi) = 0\} \geq \inf\{s \geq 0 : M_Z(s, r_2, \phi) = 0\} = M_Z(r_2, \phi).$$

This proves that the function $r \to M_Z(r, \phi)$ is non-increasing, so there exists a limit

$$P_Z(f, \phi) := \lim_{r \to 0} M_Z(r, \phi).$$

The quantity $P_Z(f, \phi)$ is called the *topological pressure* of f, restricted to Z, with respect to the potential $\phi : X \to \mathbb{R}$. Basic properties of the topological pressure are presented in the following lemma.

Lemma 3.1 *For a continuous map $f : X \to X$ defined on a compact metric space (X, d), $r > 0$, $s \geq 0$, $N \in \mathbb{N}$ and a continuous potential $\phi : X \to \mathbb{R}$ we have:*

1. *If $Z_1 \subset Z_2 \subset X$, then $M_{Z_1}(s, r, \phi, N) \leq M_{Z_2}(s, r, \phi, N)$.*
2. *If $Z_1 \subset Z_2 \subset X$, then $P_{Z_1}(f, \phi) \leq P_{Z_2}(f, \phi)$.*
3. *If $Z = \bigcup_{k \in \mathbb{N}} Z_k$, then $P_Z(f, \phi) = \sup\{P_{Z_k}(f, \phi) : k \in \mathbb{N}\}$.*

Next, we give a modification of the classical covering lemma (see Section 2.8.4 in [19]).

Lemma 3.2 *[Lemma 1 in [27]] Let $f : X \to X$ be a continuous map on a compact metric space. Let $r > 0$ and $\mathcal{B}(r) = \{B_n(x, r) : x \in X, n \in \mathbb{N}\}$. For any family $\mathcal{F} \subset \mathcal{B}(r)$ there exists a subfamily $\mathcal{G} \subset \mathcal{F}$ consisting of disjoint Bowen balls such that*

$$\bigcup_{B_n(x,r) \in \mathcal{F}} B_n(x, r) \subset \bigcup_{B_n(x,r) \in \mathcal{G}} B_n(x, 3 \cdot r).$$

3.3 Local measure-theoretic pressures

For a continuous map $f : X \to X$ defined on a compact metric space (X, d), a Borel probability measure μ defined on X, a continuous potential $\phi : X \to \mathbb{R}$ and $x \in X$, we define the *upper local measure-theoretic pressure* of f at x, with respect to μ, by

$$\overline{P}_\mu(f, \phi, x) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \left(\sum_{m=0}^{n-1} \phi(f^m(x)) - \log(\mu(B_n(x, \varepsilon))) \right). \tag{8}$$

Similarly, we define the *lower local measure-theoretic pressure* of f at x, with respect to μ, by

$$\underline{P}_\mu(f, \phi, x) := \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \left(\sum_{m=0}^{n-1} \phi(f^m(x)) - \log(\mu(B_n(x, \varepsilon))) \right).$$

Remark 3.1 *Notice that local measure-theoretic pressures, calculated with respect to the potential $\phi \equiv 0$, coincide with local measure-theoretic Brin-Katok entropies. Moreover, local measure-theoretic pressures and topological pressure are related. We have the following partial variational principle, extending Ma & Wen's result Theorem 3.2.*

Theorem 3.3 *Given a continuous map $f : X \to X$ on a compact metric space, a Borel subset $Z \subset X$ and a Borel probability measure μ on X, we have for any $s \geq 0$ and continuous $\phi : X \to \mathbb{R}$:*

1) If $\overline{P}_\mu(f, \phi, x) \leq s$ for all $x \in Z$, then $P_Z(f, \phi) \leq s$.

2) If $\mu(Z) > 0$ and $\underline{P}_\mu(f, \phi, x) \geq s$ for all $x \in Z$, then $P_Z(f, \phi) \geq s$.

Proof. (1) Fix a Borel subset $Z \subset X$, a Borel probability measure μ, a continuous potential $\phi : X \to \mathbb{R}$ and $\varepsilon > 0$. Assume that there exists $s \geq 0$ such that $\overline{P}_\mu(f, \phi, x) \leq s$ for every $x \in Z$. Define a sequence of sets $(Z_k)_{k \in \mathbb{N}}$ by

$$Z_k := \left\{ x \in Z : \limsup_{n \to \infty} \frac{1}{n} \left(\sum_{m=0}^{n-1} \phi(f^m(x)) - \log(\mu(B_n(x, r))) \right) \leq s + \varepsilon \text{ for all } r \in (0, \frac{1}{3k}) \right\}.$$

Then $Z = \bigcup_{k \in \mathbb{N}} Z_k$. Now we fix $k \in \mathbb{N}$ and $r \in (0, \frac{1}{3k})$. The definition of Z_k yields that for any $x \in Z_k$, there exists a strictly increasing sequence $(n_j(x))_{j \in \mathbb{N}}$ with

$$\sum_{m=0}^{n_j(x)-1} \phi(f^m(x)) - \log(\mu(B_{n_j(x)}(x, r))) \leq (s + \varepsilon) \cdot n_j(x).$$

Notice that Z_k is contained in the union of elements of the family

$$\mathcal{F}_N := \{B_{n_j(x)}(x, r)) : x \in Z_k,\ n_j(x) \geq N, j \in J\}.$$

By Lemma 3.2, there exists a subfamily $\mathcal{G}_N$ of $\mathcal{F}_N$, by pairwise disjoint Bowen balls,

$$\mathcal{G}_N := \{B_{n_i(x_i)}(x_i, r)) : x_i \in Z_k,\ n_i(x) \geq N, i \in I\},$$

where $I \subset J$ such that $Z_k \subset \bigcup_{i \in I} B_{n_i(x_i)}(x_i, 3 \cdot r)$ and

$$\mu(B_{n_i(x_i)}(x_i, r))) \geq \exp\left(-(s + \varepsilon) n_i(x_i) + \sum_{m=0}^{n_i(x_i)-1} \phi(f^m(x_i)) \right).$$

Therefore,

$$
\begin{aligned}
M_{Z_k}(s+\varepsilon, 3\cdot r, \phi, N) &\leq \sum_{i\in I} \exp\left(-(s+\varepsilon)n_i(x_i) + \sum_{m=0}^{n_i(x_i)-1} \phi(f^m(x_i))\right) \\
&\leq \sum_{i\in I} \mu(B_{n_i(x_i)}(x_i, r))) \leq 1,
\end{aligned}
$$

where the last inequality follows by the disjointness of Bowen balls in the family $\mathcal{G}_N$. Passing to the limit $N \to \infty$, we get

$$M_{Z_k}(s+\varepsilon, 3\cdot r, \phi) \leq 1.$$

This means that $M_{Z_k}(3\cdot r, \phi) \leq s+\varepsilon$. As $r \to 0$, we obtain $P_{Z_k}(f,\phi) \leq s+\varepsilon$. Finally, by Lemma 3.1, we obtain

$$P_Z(f,\phi) = \sup\{P_{Z_k}(f,\phi) : k \in \mathbb{N}\} \leq s+\varepsilon$$

and consequently

$$P_Z(f,\phi) = \sup\{P_{Z_k}(f,\phi) : k \in \mathbb{N}\} \leq s,$$

since $\varepsilon > 0$ is arbitrarily small.

(2) Fix $Z \subset X$ such that $\mu(Z) > 0$ and $s \geq 0$. Assume that $\underline{P}_\mu(f,\phi,x) \geq s$ for all $x \in Z$. Fix $\varepsilon > 0$ and for any $k \in \mathbb{N}$ define

$$V_k := \left\{ x \in Z : \liminf_{n\to\infty} \frac{1}{n}\left(\sum_{m=0}^{n-1} \phi(f^m(x)) - \log(\mu(B_n(x,r)))\right) \geq s-\varepsilon, \text{ for every } r \in (0, \frac{1}{k}) \right\}.$$

Since the sequence of sets $(V_k)_{k\in\mathbb{N}}$ increases to Z, by continuity of the Borel probability measure μ we get

$$\lim_{k\to\infty} \mu(V_k) = \mu(Z) > 0.$$

There exists $k_0 \in \mathbb{N}$ such that $\mu(V_{k_0}) > \frac{1}{2}\mu(Z) > 0$. Now define a sequence of sets $(V_{k_0,N})_{N\in\mathbb{N}}$ by

$$V_{k_0,N} := \left\{ x \in Z : \frac{1}{n}\left(\sum_{m=0}^{n-1} \phi(f^m(x)) - \log(\mu(B_n(x,r)))\right) \geq s-\varepsilon, \text{ for all } n \geq N, r \in \left(0, \frac{1}{k_0}\right) \right\}.$$

Notice that the sequence $(V_{k_0,N})_{N\in\mathbb{N}}$ increases to V_{k_0}, so by continuity of the Borel probability measure μ, there exists $N_0 \in \mathbb{N}$ such that $\mu(V_{k_0,N_0}) > \frac{1}{2}\mu(V_{k_0}) > 0$. By definition of V_{k_0,N_0}, for any $x \in V_{k_0,N_0}$, $n_j \geq N_0$ and $r \in (0, \frac{1}{k_0})$, we have

$$\sum_{m=0}^{n_j-1} \phi(f^m(x)) - \log(\mu(B_{n_j}(x,r))) > (s-\varepsilon)\cdot n_j,$$

so

$$\mu(B_{n_j}(x,r)) < \exp\left[-(s-\varepsilon)\cdot n_j + \sum_{m=0}^{n_j-1} \phi(f^m(x))\right].$$

Next, consider a countable cover $\mathcal{C}$ of V_{k_0,N_0} by Bowen balls, defined by

$$\mathcal{C} := \left\{ B_{n_j}(y_j, r) : y_j \in V_{k_0,N_0},\ n_j \geq N_0,\ r \in (0, \frac{1}{k_0}),\ B_{n_j}(y_j, r) \cap V_{k_0,N_0} \neq \emptyset, j \in J \right\}.$$

Notice that for any $\delta \in (0, \frac{1}{2}\mu(V_{k_0,N_0}))$, there exists a countable family J_δ such that

$$\begin{aligned} M_{V_{k_0,N_0}}(s-\varepsilon, r, \phi, N) &\geq -\delta + \sum_{j \in J_\delta} \exp\left(-(s-\varepsilon) \cdot n_j + \sum_{m=0}^{n_j - 1} \phi(f^m(x_j)) \right) \\ &\geq -\delta + \sum_{j \in J_\delta} \mu(B_{n_j}(y_j, r)) \geq -\delta + \mu(V_{k_0,N_0}) \geq \frac{1}{2}\mu(V_{k_0,N_0}). \end{aligned}$$

Since $V_{k_0,N_0} \subset Z$, for any potential $\phi : X \to \mathbb{R}$ and $N \in \mathbb{N}$, Lemma 3.1 gives

$$M_Z(s-\varepsilon, r, \phi, N) \geq M_{V_{k_0,N_0}}(s-\varepsilon, r, \phi, N) \geq \frac{1}{2}\mu(V_{k_0,N_0}) > 0.$$

Passing to the limit $N \to \infty$, we get $M_Z(s-\varepsilon, r, \phi) > 0$. This means that $M_Z(r, \phi) > s - \varepsilon$. Finally passing to the limit $r \to 0$ and taking into account that ε is arbitrary small, we obtain $P_Z(f, \phi) \geq s$. The proof is complete. $\square$

3.4 The translocal version of measure-theoretic local pressure

We suggest for translocal version for the upper measure-theoretic local pressure $\overline{P}_\mu(f, \phi, z)$ from (8) is the following:

$$\overline{P}_{\mu,\omega}(f, \phi, z) := \limsup_{n \to \infty} \frac{1}{n} \left(\sum_{m=0}^{n-1} \phi(f^m(z)) - \log \mu(B(z, e^{-\omega n})) \right),$$

and an analogous replacement for lower measure-theoretic local pressure.

$$\underline{P}_{\mu,\omega}(f, \phi, z) := \liminf_{n \to \infty} \frac{1}{n} \left(\sum_{m=0}^{n-1} \phi(f^m(z)) - \log \mu(B(z, e^{-\omega n})) \right),$$

The topological translocal pressure itself we will define in the spirit of [32]. That is, define

$$M_{f,Z,\omega}(s, \phi, N) := \inf \left\{ \sum_{j \in J} \exp[-s \cdot n_j + \sum_{m=0}^{n_j - 1} \phi(f^m(x_j))] : Z \subset \bigcup_{j \in J} B(x_j, e^{-\omega n_j});\ n_j \geq N \right\},$$

where the infimum is taken over all finite or countable coverings of the Z by metric balls $\{B(x_j, e^{-\omega n_j})\}_{j \in J}$ with $n_j \geq N$. Let

$$\overline{M}_{f,Z,\omega}(s, \phi) := \limsup_{N \to \infty} M_{f,Z,\omega}(s, \phi, N),$$

$$\underline{M}_{f,Z,\omega}(s, \phi) := \liminf_{N \to \infty} M_{f,Z,\omega}(s, \phi, N).$$

Lemma 3.3 *The graph of the function $s \mapsto \overline{M}_{f,Z,\omega}(s,r,\phi)$ has a unique critical parameter s_0, where the function $s \mapsto \overline{M}_{f,Z,\omega}(s,r,\phi)$ drops from ∞ to 0. The graph of the function $s \mapsto \underline{M}_{f,Z,\omega}(s,r,\phi)$ has a unique critical parameter s_1, where the function $s \mapsto \underline{M}_{f,Z,\omega}(s,r,\phi)$ drops from ∞ to 0.*

Proof. Proof of the lemma is similar to the proof of Proposition 1.2 in [32]. $\square$
Thus, we define

$$\overline{P}_{Z,\omega}(f,\phi) := \sup\{s \geq 0 : \overline{M}_{f,Z,\omega}(s,\phi) = \infty\} = \inf\{s \geq 0 : \overline{M}_{f,Z,\omega}(s,\phi) = 0\} = s_0,$$

$$\underline{P}_{Z,\omega}(f,\phi) := \sup\{s \geq 0 : \underline{M}_{f,Z,\omega}(s,\phi) = \infty\} = \inf\{s \geq 0 : \underline{M}_{f,Z,\omega}(s,\phi) = 0\} = s_1.$$

A metric space X is called boundedly compact if all bounded closed subsets of X are compact. In particular $\mathbb{R}^n$ and Riemannian manifolds (see [22, p.9]) are boundedly compact.

Lemma 3.4 *(Vitali Covering Lemma, Theorem 2.1 in [28]). Let X be a boundedly compact metric space and a family of closed balls $\mathcal{B} = \{\overline{B(x,r)} : x \in X, r > 0\}$ in X such that*

$$\sup\{diam(B(x,r)) : B(x,r) \in \mathcal{B}\} < \infty.$$

Then, there is a finite or countable sequence $B(x_i,r_i) \in \mathcal{B}$, indexed by $i \in I$, of disjoint balls such that

$$\bigcup_{B(x,r)\in\mathcal{B}} \subset \bigcup_{i\in I} \overline{B(x_i, 5\cdot r_i)}.$$

Properties analogous to those of Lemma 3.1 hold for this version of topological pressure.

Theorem 3.4 *Given a continuous map $f : X \to X$ on a compact metric space, a Borel subset $Z \subset X$ and a Borel probability measure μ on X, we have for any $s \geq 0$ and continuous $\phi : X \to \mathbb{R}$:*

1) If $\overline{P}_{\mu,\omega}(f,\phi,x) \leq s$ for all $x \in Z$, then $\overline{P}_{Z,\omega}(f,\phi) \leq s$.

2) If $\mu(Z) > 0$ and $\underline{P}_{\mu}(f,\phi,x) \geq s$ or all $x \in Z$, then $\underline{P}_{Z,\omega}(f,\phi) \geq s$.

Proof. (1) Fix a Borel subset $Z \subset X$, a Borel probability measure μ, a continuous potential $\phi : X \to \mathbb{R}$ and $\varepsilon > 0$. Assume that there exists $s \geq 0$ such that $\overline{P}_{\mu,\omega}(f,\phi,x) \leq s$ for every $x \in Z$. Therefore, for $x \in Z$, there exists a strictly increasing sequence $(n_j(x))_{j\in\mathbb{N}}$ with

$$\sum_{m=0}^{n_j(x)-1} \phi(f^m(x)) - \log(\mu(B(x, e^{-\omega n_j(x)}))) \leq (s+\varepsilon)\cdot n_j(x).$$

Notice that Z is contained in the union of elements of the family

$$\mathcal{F}_N := \{B(x_j, e^{-\omega n_j(x_j)}) : x_j \in Z,\ n_j(x_j) \geq N, j \in J\}.$$

Choose n_0 minimal such that $e^{\omega n_0} \geq 5$. By Lemma 3.4 there exists a subfamily $\mathcal{G}_N$ of $\mathcal{F}_N$, indexed by elements of a set I, by pairwise disjoint metric balls,

$$\mathcal{G}_N := \{B(x_i, e^{-\omega n_i(x_i)})) : x_i \in Z,\ n_i(x_i) \geq N, i \in I\},$$

where $I \subset J$, such that $Z \subset \bigcup_{i\in I} B(x_i, e^{-\omega(n_i(x_i)-n_0)}))$ and

$$\mu(B(x_i, e^{-\omega n_i(x_i)}))) \geq \exp\left(-(s+\varepsilon)n_i(x_i) + \sum_{m=0}^{n_i(x_i)-1} \phi(f^m(x_i))\right).$$

Therefore,

$$\begin{aligned}
M_{Z,\omega}(s+\varepsilon, \phi, N+n_0) &\leq \sum_{i\in I} \exp\left(-(s+\varepsilon)[n_i(x_i)-n_0] + \sum_{m=0}^{n_i(x_i)-n_0-1} \phi(f^m(x_i))\right) \\
&\leq \exp[(s+\varepsilon)\cdot n_0] \sum_{i\in I} \exp\left(-(s+\varepsilon)n_i(x_i) + \sum_{m=0}^{n_i(x_i)-1} \phi(f^m(x_i))\right) \\
&\leq \exp[(s+\varepsilon)\cdot n_0] \sum_{i\in I} \mu(B(x_i, e^{-\omega n_i(x_i))})) \leq \exp[(s+\varepsilon)\cdot n_0],
\end{aligned}$$

where the last inequality follows by the disjointness of metric balls in the family $\mathcal{G}_N$. Taking the lim sup as $N \to \infty$, we get

$$\overline{M}_{Z,\omega}(s+\varepsilon, \phi) \leq \exp[(s+\varepsilon)\cdot n_0] < \infty$$

and therefore

$$\overline{P}_{Z,\omega}(f, \phi) \leq s+\varepsilon.$$

Since $\varepsilon > 0$ is arbitrarily small, $\overline{P}_{Z,\omega}(f,\phi) \leq s$.

(2) Fix $Z \subset X$ such that $\mu(Z) > 0$ and $s \geq 0$. Assume that $\underline{P}_{\mu,\omega}(f,\phi,x) \geq s$ for all $x \in Z$. Fix $\varepsilon > 0$ and for any $N \in \mathbb{N}$, define

$$V_N := \left\{ x \in Z : \frac{1}{n}\left(\sum_{m=0}^{n-1} \phi(f^m(x)) - \log(\mu(B(x, e^{-\omega n})))\right) \geq s - \varepsilon, \text{ for all } n \geq N \right\}.$$

Notice that the sequence $(V_N)_{N\in\mathbb{N}}$ increases to Z, so by continuity of the Borel probability measure μ there exists $N_0 \in \mathbb{N}$ such that $\mu(V_{N_0}) > \frac{1}{2}\mu(Z) > 0$. By definition of V_{N_0}, for any $x \in V_{N_0}$ and $n_j \geq N_0$, we have

$$\sum_{m=0}^{n_j-1} \phi(f^m(x)) - \log(\mu(B(x_j, e^{-\omega n_j}))) > (s-\varepsilon)\cdot n_j,$$

so

$$\mu(B(x, e^{-\omega n_j}))) < \exp\left[-(s-\varepsilon)\cdot n_j + \sum_{m=0}^{n_j-1} \phi(f^m(x))\right].$$

Next, consider a countable cover $\mathcal{C}$ of V_{N_0} by metric balls, defined by

$$\mathcal{C} := \left\{ B_{n_j}(y_j, e^{-\omega n_j}) : y_j \in V_{N_0},\ n_j \geq N_0,\ B(y_j, e^{-\omega n_j}) \cap V_{N_0} \neq \emptyset, j \in J \right\}.$$

Notice that for any $\delta \in (0, \frac{1}{2}\mu(V_{N_0}))$, there exists a countable family J_δ such that

$$\begin{aligned} M_{V_{N_0},\omega}(s-\varepsilon,\phi,N) &\geq -\delta + \sum_{j\in J_\delta} \exp\left(-(s-\varepsilon)\cdot n_j + \sum_{m=0}^{n_j-1}\phi(f^m(x_j))\right) \\ &\geq -\delta + \sum_{j\in J_\delta}\mu(B(x_j, e^{-\omega n_j}))) \geq -\delta + \mu(V_{N_0}) \geq \frac{1}{2}\mu(V_{N_0}). \end{aligned}$$

Since $V_{k_0,N_0} \subset Z$, for any potential $\phi : X \to \mathbb{R}$ and $N \in \mathbb{N}$, we get

$$M_{Z,\omega}(s-\varepsilon,\phi,N) \geq M_{V_{N_0}}(s-\varepsilon,\phi,N) \geq \frac{1}{2}\mu(V_{N_0}) > 0.$$

Taking the lim inf as $N \to \infty$, we get $\underline{M}_{Z,\omega}(s-\varepsilon,\phi) > 0$. This means that $\underline{M}_{Z,\omega}(\phi) > s-\varepsilon$. Finally taking into account that ε is arbitrary small, we obtain $\underline{P}_{Z,\omega}(f,\phi) \geq s$. The proof is complete. $\square$

4 Discussion on local entropy

In the literature on dynamical systems there are several attempts to define local entropy. Bowen [9] introduced the notion of ε-entropy to get an upper estimate of the difference between $h_\mu(f)$ and the entropy of a partition with diameter less than ε. Another notion of local entropy (called conditional topological entropy) was given by Misiurewicz [29], who proved that vanishing of his local entropy implies existence of maximal measure. Finding (uniqueness of) measures of maximal entropy was also the reason for Newhouse [30] and Buzzi & Ruette [13, 15] to define a version the local entropy $h_{\text{loc}}(f)$, see (5). Buzzi & Ruette [15] proved that for C^1 maps $f : [0,1] \to [0,1]$ with critical set $Crit(g) = \{x \in [0,1] : f|_U$ is not monotone on each neighborhood $U \ni x\}$, if $h_{\text{top}}(g) > h_{\text{top}}(g, Crit(g)) + h_{\text{loc}}(f)$, then f admits a measure of maximal entropy.

In 1993, Blanchard [6] introduced a notion of entropy pairs, in order to localize "where" in the system entropy is generated. This was generalized to entropy n-tuples and entropy sets by Huang & Ye [24] and Blanchard & Huang [7].

Definition 4.1 *Let $f : X \to X$ be a continuous map on a compact space. A set $E \subset X$ is an* entropy set *if at least two of these points in E are distinct, and if for each open cover $\mathcal{U} = \{U_1, \dots, U_n\}$ of X with the property that for each i there is $x \in E$ such that $x \notin \overline{U_i}$, the topological entropy $h_{top}(T,\mathcal{U})$ (in the sense of Adler-Konheim-McAndrew [2]) of the cover is positive. An entropy set of n points is an* entropy n-tuple*, so an* entropy pair *is an entropy 2-tuple.*

Dou et al. [16] characterize maximal entropy sets, and prove that E is an entropy set if and only if each n-tuple in E with at least two distinct point is an entropy n-tuple.

The survey paper of Glasner & Ye [21] is largely on the relation between the topological and ergodic approach to dynamical systems from the global point of view. In particular, they discuss topological and measure entropy pairs and n-tuples and they present the relation between the topological entropy of open covers and the measure-theoretic entropy of finer partitions. The second part of the survey is on local properties of dynamical systems. They introduce n-tuples for an invariant measure, and establish their relation to topological entropy n-tuples. Glasner & Ye [21] discuss in detail the notion of entropy set, in order to find where the entropy is concentrated. In particular, it was shown that any topological dynamical system with positive entropy admits an entropy set with infinitely many points. For any topological dynamical system, there exists compact countable subset such that its Bowen entropy is equal to the entropy of the original system.

Glasner & Ye discuss notions of completely positive entropy and uniform positive entropy, introduced by Blanchard [5] in a topological context: A topological dynamical system has *completely positive entropy* if every non-trivial factor has positive topological entropy, and it has *uniform positive entropy* if the topological entropy of every non-dense finite open cover is positive. Uniform positive entropy implies weak mixing as well as completely positive entropy, which in turn implies the existence of a fully supported invariant measure.

The paper by Garcia-Ramos & Li [20] is also a survey about recent developments in the local entropy theory for topological dynamical systems; they cover similar material as Glasner & Ye, but for general (continuous) group actions.

In 1992, Thieullen [36] introduced a local form of entropy which he called α-entropy. In his definition, the ε in the Bowen (n, ε)-balls is replaced by an n-dependent quantity $e^{-\alpha n}$, but unlike our translocal entropy, he still uses Bowen balls. Ben Ovaria & Rodriguez-Hertz [4] introduced *neutralized* (local) entropy as

$$\mathcal{E}_\mu(x) := \lim_{\alpha\to 0} \limsup_{n\to\infty} -\frac{1}{n} \log \mu(B_n(x, e^{-\alpha n})); \tag{9}$$

this is α-entropy where eventually the limit $\alpha \to 0$ is taken. Ben Ovadia & Rodriguez-Hertz study the role it can play in hyperbolic dynamics and compare it with the Brin-Katok local entropy, see (6). They show that the neutralized local entropy for $C^{1+\beta}$ diffeomorphism $f : M \to M$ of a compact closed manifold M, calculated with respect to an f-invariant probability measure μ, coincides with Brin-Katok local entropy almost everywhere.

In a recent paper, Yang, Chen & Zhou [38], extending (7), obtained a variational principle for neutralized entropy of a continuous map $f : X \to X$ on a compact metric space. For a non-empty compact $K \subset X$ they prove the equality

$$h^{\widetilde{B}}_{\text{top}}(f, K) = \limsup_{\alpha\to 0} \{\underline{h}^{\widetilde{BK}}_\mu(f, \alpha) : \mu \text{ is } f\text{-invariant and } \mu(K) = 1\},$$

where $h^{\widetilde{B}}_{\text{top}}(f, K)$ denotes a neutralized version of the Bowen topological entropy of f (defined via a Carathéodory construction), restricted to K, and $\underline{h}^{\widetilde{BK}}_\mu(f, \alpha)$ is the *lower neutralized Brin-Katok local entropy* of μ, defined as

$$\underline{h}^{\widetilde{BK}}_\mu(f, \alpha) := \int_X \liminf_{n\to\infty} \; -\frac{1}{n} \log \mu(B_n(x, e^{-n\alpha})) \; d\mu(x).$$

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