On asymptotic expansions of ergodic integrals for $\mathbb{Z}^d$-extensions of translation flows. *

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Abstract

We obtain expansions of ergodic integrals for $\mathbb{Z}^d$-covers of compact self-similar translation flows, and as a consequence we obtain a form of weak rational ergodicity with optimal rates. As examples, we consider the so-called self-similar $(s, 1)$-staircase flows ($\mathbb{Z}$-extensions of self-similar translations flows of genus-2 surfaces), and a variation of the Ehrenfest wind-tree model.

Nous établissons des développements asymptotiques d'intégrales ergodiques pour des $\mathbb{Z}^d$-revêtements de flots directionnels compacts auto-similaires: De cela découle un résultat d'ergodicité rationnelle faible avec des taux optimaux. À titre d'exemple, nous considérons des flots sur des escaliers auto-similaires de type $(s, 1)$ (qui sont des $\mathbb{Z}$-extensions de flots directionnels auto-similaires sur des surfaces de genre 2) ainsi qu'une variante du modèle de vent dans les arbres des Ehrenfest.

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1 Introduction

Given a measure preserving flow $(\phi_t)_{t\in\mathbb{R}}$ on a measure space (X, μ), one is interested in describing the almost everywhere behaviour of its orbits. If the flow is ergodic and if $\mu(X) < \infty$, Birkhoff's Ergodic Theorem states that, for any integrable observable $G\colon X \to \mathbb{R}$, the time averages $\frac{1}{T}\int_0^T G\circ\phi_t \,\mathrm{d}t$ converge almost everywhere to $\frac{1}{\mu(X)}\int_X G \,\mathrm{d}\mu$. On the other hand, if $\mu(X) = \infty$, for any ergodic, conservative flow $(\phi_t)_{t\in\mathbb{R}}$ and for all integrable functions, its time averages converge to 0 almost everywhere. The situation does not improve even if we replace $\frac{1}{T}$ with any other normalizing family of functions $a(T)$, see Aaronson [1, Theorem 2.4.2]: for any non-negative integrable function G, either $\liminf_{T\to\infty}\int_0^T G\circ \phi_t \,\mathrm{d}t = 0$ or $\limsup_{T\to\infty}\int_0^T G\circ\phi_t \,\mathrm{d}t = \infty$ almost everywhere.

Nonetheless, one can still hope to describe the almost everywhere behaviour of the ergodic integrals in some weaker sense. In particular, for an integrable function $G\colon X \to \mathbb{R}$, we seek an expression of the form

$$\int_0^T G\circ\phi_t(x)\,\mathrm{d}t = a(T)\left(\int_X G\,\mathrm{d}\mu\right)\cdot \Phi_T(x)(1+o(1)), \tag{1}$$

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where $a(T)$ describes the "correct (almost everywhere) size" of the ergodic averages (which, at least for us, is $o(T)$) and $\Phi_T(x)$ is an "oscillating" term which, although does not converge almost everywhere, converges in some weaker sense (and, crucially, depends only on the point x and not on the function G).

In this paper, we consider a translation flow $(\phi_t)_{t\in\mathbb{R}}$ on a space X_Γ which is a $\mathbb{Z}^d$-cover of a compact translation surface X with the projection $p : X_\Gamma \to X$. Lebesgue measure m is infinite on X_Γ and invariant w.r.t. both the flow ϕ_t and the deck-transformations associated to the cover. Our main result is that, under certain assumptions described below, an expression as (1) holds for all continuous functions $G\colon X_\Gamma \to \mathbb{R}$ with compact support, with $a(T) \sim T(\log T)^{-d/2}$ and where $\sqrt{\log(\Phi_T \circ p)}$ converges in distribution to a Gaussian random variable.

Results of this type have been proved by many authors in several settings, including [24] for $\mathbb{Z}^d$ covers of horocycle flows, and [6] for $\mathbb{Z}$-covers of a translation torus. Furthermore, in [6], the authors used this result to prove temporal limit theorems for circle rotations $R_\theta : \mathbb{S}^1 \to \mathbb{S}^1$ for observables with $\int_{\mathbb{S}^1} G \,\mathrm{d}m = 0$ and specific (namely, quadratically irrational) rotation angles θ. This amounts to determining the asymptotics of $\sum_{i=0}^{n-1} G \circ R_\theta^i(x)$ for a fixed x and increasing time intervals $[0, n]$. The crucial idea in their proofs is renormalization, allowing one to speed up a translation flow ϕ_t on a $\mathbb{Z}$-cover X_Γ of a two-dimensional twice punctured torus X by means of a linear pseudo-Anosov automorphism ψ_Γ of X_Γ according to

$$\psi_\Gamma \circ \phi_t = \phi_{\lambda t} \circ \psi_\Gamma \quad \text{for every } t \in \mathbb{R}, \tag{2}$$

where $\lambda \in (-1, 1)$ is the contraction factor of ψ_Γ. Therefore the asymptotics of ergodic integrals $\int_0^T G \circ \phi_t \,dt$ for observables $G : X_\Gamma \to \mathbb{R}$ of compact support can be estimated using the asymptotics of $\int_{\mathcal{F}} G \circ \psi^k \,\mathrm{d}m$, where $\mathcal{F}$ is a fundamental domain and $T \approx \lambda^{-k}$.

The central result in [6] in our notation is

$$\int_0^T G \circ \phi_t(x)\,dt = \left(\int_{X_\Gamma} G \,\mathrm{d}m\right) \frac{(1+o(1))T}{\sigma\sqrt{2K}} \exp\left(-\frac{1+o(1)}{2\sigma^2}\frac{\left(\xi\circ\psi_\Gamma^K(x)\right)^2}{K}\right) \quad \text{as } T\to\infty, \tag{3}$$

where $K \sim \log^* T := \lceil -\frac{\log T}{\log \lambda} \rceil$ and x is such that it has zero average drift under iteration of ψ_Γ, and $\xi : X_\Gamma \to \mathbb{Z}$ is the projection on the $\mathbb{Z}$-part of the cover.

In this paper, we extend these results to (i) include higher order error terms of the asymptotics making the $o(1)$ terms in (3) explicit, and (ii) allow more general translation surfaces than tori. Our proofs continue to rely on the renormalization formula (2), hence restricting the direction of the translation flow to quadratically irrational slopes, but are on the whole simpler than those of [6], and pertain to $\mathbb{Z}^d$-covers as well. In fact, ergodicity of the flow seems to be a non-generic property; there are several results in the literature showing that for $\mathbb{Z}$- or $\mathbb{Z}^d$-extensions of many compact translation surfaces, the translation flow in a generic direction is non-ergodic, and even has uncountably many ergodic components, cf. [29, 30, 9, 17, 18]. The computational difficulty increases with d, of course, and there are few studied examples for $d \geq 2$. We present a variation of the Ehrenfest wind-tree model as an example of a $\mathbb{Z}^2$-cover where our results apply.

Phrased in dimension 1 (but see Theorem 2.8 for the precise formulation, also for $d = 2$), our main result reads as

Theorem 1.1. *Let $G \in C^1(X_\Gamma)$ be compactly supported. Then, there exist real bounded functions $g_{k,j}$ so that for all $N \geq 1$ and m-a.e. $x \in X_\Gamma$,*

$$\int_0^T G\circ\phi_t(x)\,dt = \frac{\int_{X_\Gamma} G\,\mathrm{d}m}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{\xi\left(\psi_\Gamma^K(x)\right)^2}{2\sigma^2 K}} \frac{T}{\sqrt{K}} \times \left(1 + \sum_{k=1}^{N}\frac{1}{K^k}\sum_{j=0}^{2k} g_{k,j}(x)\,\xi(\psi_\Gamma^K(x))^{2k-j} + O\left(\frac{1}{K^{N+1}}\right)\right)$$

as $T \to \infty$ and $K = \log^* T = \lceil -\frac{\log T}{\log \lambda} \rceil$.

The term $\frac{\xi(\psi_\Gamma^K(x))^2}{2\sigma^2 K}$ is oscillating and does not converge almost everywhere, but after integration over the space, it does lead to a form of weak rational ergodicity for C^1 observables with optimal rates, see Theorem 2.11. Weak rational ergodicity [2] means that there is a set $\mathcal{F} \subset X_\Gamma$ of positive finite measure (possibly but not necessarily a fundamental domain of the $\mathbb{Z}^d$-cover) such that

$$\lim_{T\to\infty} \frac{1}{a_T(\mathcal{F})} \int_0^T \mu(A \cap \phi_t(B))\, dt = \mu(A)\mu(B)$$

for all measurable sets $A, B \subset \mathcal{F}$, and $a_T(\mathcal{F}) := \int_0^T \mu(\mathcal{F} \cap \phi_t(\mathcal{F}))\, dt$ is called the *return sequence*. Further, as in [6] and [24], we obtain higher order rational ergodicity, see Theorem 2.14. The higher order rational ergodicity is defined as in the statement of Theorem 2.14 below.

The paper is organized as follows. Section 2 gives the core of the argument in an abstract setup, based on local limit laws of twisted transfer operators $\mathcal{L}_u$ acting on appropriate anisotropic Banach spaces. In Section 3 we formalize the concept of $\mathbb{Z}^d$-cover over a translation surface and study the automorphisms that commute with deck-transformations. We discuss the example of the $(s,1)$-staircase at length, which is the direct generalization of the model used in [6] (where $s = 2$). We give direct proofs of ergodicity of the pseudo-Anosov automorphism ψ_Γ and the translation flow ϕ_t although this also follows from the results of Section 2. Section 3 finishes with a version of the Ehrenfest wind-tree model, which is our example of a $\mathbb{Z}^2$-cover where the main theory applies. Section 4 gives an alternative approach to introducing a twisted transfer operator which is closer to the nature of $\mathbb{Z}^d$-covers of the compact translation surface.

2 Ergodic integrals and (weak) rational ergodicity via Local Limit Laws

We are interested in (weak) rational ergodicity (with optimal rates) of a translation flow ϕ_t defined on a $\mathbb{Z}^d$-cover X_Γ of the compact translation surface X that satisfies certain abstract assumptions. The main results of this section, Theorems 2.8 and 2.11, are a generalization of [6, Theorems 3.2 and 4.3]. The setting we require is as follows:

(H1) Let X be a compact surface and let X_Γ be a $\mathbb{Z}^d$-cover with projection $p : X_\Gamma \to X$ that is invariant under deck-transformations $p \circ \Delta_{\mathbf{n}} = p$. We assume that there exists a pseudo-Anosov automorphism $\psi_\Gamma : X_\Gamma \to X_\Gamma$ on the $\mathbb{Z}^d$-cover X_Γ that renormalizes the translation flow ϕ_t in the stable direction, that is $\psi_\Gamma \circ \phi_t = \phi_{\lambda t} \circ \psi_\Gamma$ for some $\lambda \in (-1, 1)$.

(H2) The pseudo-Anosov automorphism ψ_Γ commutes with the deck transformations, i.e., $\psi_\Gamma \circ \Delta_{\mathbf{n}} = \Delta_{\mathbf{n}} \circ \psi_\Gamma$ for all $\mathbf{n} \in \mathbb{Z}^d$ and $\psi = p \circ \psi_\Gamma \circ p^{-1} : X \to X$ is well-defined.

(H3) Upon the choice of a bounded fundamental domain $\mathcal{F}$ (i.e., X_Γ is the disjoint union $\bigsqcup_{\mathbf{n}\in\mathbb{Z}^d} \Delta_{\mathbf{n}}(\mathcal{F})$), we define $\xi : X_\Gamma \to \mathbb{Z}^d$ to be the $\mathbb{Z}^d$ component of $x \in X_\Gamma$, via $\xi(x) = \mathbf{n}$ if $x \in \Delta_{\mathbf{n}}(\mathcal{F})$. We can consider ψ_Γ as the lift of a pseudo-Anosov automorphism $\psi : X \to X$ defined via

$$\psi_\Gamma(x, \mathbf{n}) = (\psi(x), \mathbf{n} + F(x)), \qquad x \in X, \mathbf{n} \in \mathbb{Z}^d,$$

where $F(x) = \xi \circ \psi_\Gamma(x') - \xi(x')$, defined independently of a choice of $x' \in p^{-1}(x)$, is called the *Frobenius function*. We assume that $\int_X F\, \mathrm{d}m = 0$ (*no drift condition*) and that $F : X \to \mathbb{Z}^d$ is **not** a coboundary, i.e., $F \neq g \circ \psi - g$ for any $g : X \to \mathbb{Z}^d$.

Lebesgue measure m is invariant, for both the finite and the infinite measure preserving automorphism, ψ and ψ_Γ.

Remark 2.1. *The requirement that ψ is a linear pseudo-Anosov automorphism can be easily relaxed: much more general classes satisfy the spectral gap in the Banach spaces recalled in Section 2.3 below.*

Remark 2.2. *In Sections 3.2 and 3.3, we give examples where these hypotheses apply for $d = 1, 2$, respectively. That the Frobenius function has zero integral, but is not a coboundary is shown for the staircase example in Theorem 3.9 and for our wind-tree example in* **??**.

We are interested in a simple expression of the ergodic integral $\int_0^T G \circ \phi_t \, dt$ where $G \in C^1(X_\Gamma)$ is compactly supported. Hence $G(\cdot, r) := G \cdot \mathbb{1}_{\Delta_r(\mathcal{F})}$ is non-zero for at most finitely many $r \in \mathbb{Z}^d$ (regardless the exact choice of the fundamental domain $\mathcal{F}$), and we can write the ergodic integral as the sum of finitely many integrals $\int_0^T (G \cdot \mathbb{1}_{\Delta_r(\mathcal{F})}) \circ \phi_t \, dt$, accordingly.

We consider a Markov partition $\mathcal{P}$ for the automorphism ψ. Take a flow segment L connecting x and $\phi_T(x)$ and take $K \geq 1$ minimal such that $L_K := \psi_\Gamma^K(L)$ is contained in a single element $P \in \mathcal{P} \vee \psi_\Gamma^{-1}\mathcal{P}$. The minimality of K assures that the length of L_K is bounded away from 0.

Choose $N \in \mathbb{N}$ arbitrary; it will be the same N as the order to which the error terms in Theorem 2.8 are taken. Let $R = (N+2)\log^* T$, where as before $\log^* T = \lceil -\frac{\log T}{\log \lambda} \rceil$, and let $\mathcal{P}^R = \bigvee_{j=0}^R \psi_\Gamma^j \mathcal{P}$; its partition elements have stable length $O(\lambda^R)$. We can chop off tiny segments from the ends of L_K so as to make the remainder stretch exactly across a whole number of elements in $\mathcal{P}^R$. By the choice of R, the error we make here is $O(T/K^{N+2})$ and hence can be absorbed in the final big O term in Theorem 2.8.

Let ϕ_t^u be the flow on X_Γ in the unstable direction of ψ_Γ. Then there exists[1] T_0 such that $p(B_K) = X$ for $B_K := \bigcup_{|t| \leq T_0} \phi_t^u(L_K)$. Next we can select a union A_K of rectangles inside B_K such that $p : A_K \to X$ is a bijection (in particular, $m(A_K) = 1$), and at least one of these rectangles contains the (one-dimensional) interior of L_K in its interior. Now we choose our fundamental domain $\mathcal{F}$ such that $\Delta_\ell(\mathcal{F}) = A_K$ for some $\ell \in \mathbb{Z}^d$. Let $A := \psi_\Gamma^{-K}(A_K)$; it contains the (one-dimensional) interior of L and the width of A is $\leq 2T_0\lambda^K = O(1/T)$.

Let $\mathcal{L}_\Gamma : L^1(X_\Gamma) \to L^1(X_\Gamma)$ be the transfer operator associated with ψ_Γ defined via $\int_{X_\Gamma} \mathcal{L}_\Gamma v \, w \, \mathrm{d}m = \int_{X_\Gamma} v \, w \circ \psi_\Gamma \, \mathrm{d}m$ with $v \in L^1$ and $w \in L^\infty$ and compute that

$$\begin{aligned} \frac{1}{T}\int_0^T G \circ \phi_t(x) \, dt &= \int_{X_\Gamma} G \cdot \mathbb{1}_A \, \mathrm{d}m + O(\lambda^K) \\ &= \int_{X_\Gamma} G \cdot \mathbb{1}_{A_K} \circ \psi^K \, \mathrm{d}m + O(\lambda^K) = \int_{X_\Gamma} \mathcal{L}_\Gamma^K G \cdot \mathbb{1}_{A_K} \, \mathrm{d}m + O(\lambda^K). \end{aligned} \tag{4}$$

The strategy is to relate the behaviour of $\mathcal{L}_\Gamma^K$ with an operator (or conditional) local limit theorem in terms of the transfer operator $\mathcal{L} : L^1(X) \to L^1(X)$ for the automorphism ψ (defined via $\int_X \mathcal{L}v \, w \, \mathrm{d}m = \int_X v \, w \circ \psi \, \mathrm{d}m$ with $v \in L^1$ and $w \in L^\infty(X)$. Also we define the *twisted transfer operator* as

$$\mathcal{L}_u(v) = \mathcal{L}(e^{iuF}v).$$

The operator local limit theorem we are after is in the sense of [3, Section 6].

Lemma 2.3. *Let $v \in L^1(X)$ and $v(\cdot, r) \in L^1(X_\Gamma)$ be the lifted version supported on $\{\xi = r\}$, $r \in \mathbb{Z}^d$. For all $\ell, r \in \mathbb{Z}^d$ and for all $K \geq 1$,*

$$\mathcal{L}_\Gamma^K v(x,r) \, \mathbb{1}_{\{\xi = \ell\}} = \mathcal{L}^K v(x) \mathbb{1}_{\{F_K(x) = \ell - r\}} = \int_{[-\pi,\pi]^d} e^{-iu(\ell - r)} \mathcal{L}_u^K v(x) \, \mathrm{d}u.$$

for ergodic sums $F_K := \sum_{j=0}^{K-1} F \circ \psi_\Gamma^j$.

[1]This existence of such T_0 relies on the minimality of translation flow in the unstable direction of $\psi : X \to X$, and T_0 can be taken independent of K.

Proof. Let $v \in L^1(X)$, $w \in L^\infty(X)$ and $v(\cdot, r) = v \circ p$, $w(\cdot, \ell) = w \circ p$ be the versions supported on $\{\xi = r\}$ and $\{\xi = \ell\}$, respectively. Compute that

$$\begin{aligned}\int_{X_\Gamma} \mathcal{L}_\Gamma^K v(x,r) \mathbb{1}_{\{\xi=\ell\}} w(x,\ell)\, \mathrm{d}m(x) &= \int_{X_\Gamma} \mathcal{L}_\Gamma^K (\mathbb{1}_{\{X\times\{r\}\}} v)\, (\mathbb{1}_{\{X\times\{\ell\}\}} w)\, \mathrm{d}m \\ &= \int_{X_\Gamma} (\mathbb{1}_{\{X\times\{r\}\}} v(x,r))\, (\mathbb{1}_{\{X\times\{\ell\}\}} w(x,\ell)) \circ \psi_\Gamma^K(x)\, \mathrm{d}m \\ &= \int_X v\, w \circ \psi^K\, \mathbb{1}_{\{F_K=\ell-r\}}\, \mathrm{d}m \\ &= \int_X \mathcal{L}^K (v \mathbb{1}_{\{F_K=\ell-r\}})\, w(x)\, \mathrm{d}m,\end{aligned}$$

which gives the first equality in the statement. We can write the indicator function $\mathbb{1}_{\{F_K=\ell-r\}} = \int_{[-\pi,\pi]^d} e^{iu(F_K-(\ell-r))}\, du$, so

$$\begin{aligned}\int_X \mathcal{L}^K (v(x) \mathbb{1}_{\{F_K=\ell-r\}})\, w(x)\, \mathrm{d}m &= \int_X \int_{[-\pi,\pi]^d} \mathcal{L}^K \left(v e^{iu(F_K-(\ell-r))} \right)(x)\, du\; w(x)\, \mathrm{d}m \\ &= \int_{[-\pi,\pi]^d} \int_X e^{-iu(\ell-r)} \mathcal{L}_u^K v\, w\, \mathrm{d}m\, \mathrm{d}u.\end{aligned}$$

□

2.1 An (operator) local limit theorem (LLT) for F_K along with some consequences

Proposition 2.5 below is an asymptotic expansion operator LLT (in the sense of [3, Section 6]) for the ergodic sums F_K. This is entirely expected given the simple forms of the automorphism ψ and of the Frobenius function F. The expansion in Proposition 2.5 is a key ingredient in the proof of our main results Theorems 2.8 and 2.11. We recall that Theorem 2.8 is a precise version of Theorem 1.1, while Theorem 2.8 gives optimal rates in a form of weak rational ergodicity for C^1 functions.

We first recall some facts on the spectral properties of $\mathcal{L}$ and its twisted version $\mathcal{L}_u f = \mathcal{L}(e^{iuF} f)$, $u \in \mathbb{R}^d$. Since $\psi : X \to X$ is an invertible map, we need adequate, anisotropic Banach spaces on which the corresponding transfer operator $\mathcal{L}$ can act. There are several choices in the literature, see the surveys [7, Section 2] and [10]. For the automorphism ψ it is convenient to work with a variant of the spaces introduced in [11] (see also [14] and references therein for generalizations applicable to billiards) applicable to a class of hyperbolic maps with singularities. The details on Banach spaces we shall use are deferred to Section 2.3.

Proposition 2.4. *(a) There exists anisotropic Banach spaces $\mathcal{B}, \mathcal{B}_w$ so that $C^1(X) \subset \mathcal{B} \subset \mathcal{B}_w \subset C^1(X)^*$ where $C^1(X)^*$ is the (topological) dual of $C^1(X)$. The transfer operator $\mathcal{L}$ acts continuously on $\mathcal{B}$ and $\mathcal{B}_w$. Moreover, $\mathcal{L}$ is quasicompact[2] when viewed as operator from $\mathcal{B}$ to $\mathcal{B}$. In particular,* 1 *is an isolated, simple eigenvalue in the spectrum of $\mathcal{L}$.*

(b) The derivatives $\frac{d^k}{du^k} \mathcal{L}_u f$ are linear operators on $\mathcal{B}$ with operator norm of $O(\|F\|_\infty^k)$.

(c) There exists $\delta > 0$ and a family of simple eigenvalues λ_u that is analytic in u for all $|u| < \delta$. Also, for all $|u| < \delta$ and $n \geq 1$

$$\mathcal{L}_u^n = \lambda_u^n \Pi_u + Q_u^n,$$

where Π_u is the family of spectral projections associated with λ_u with $\Pi_0 v = \int_X v\, \mathrm{d}m$, Π_u, Q_u are analytic when regarded as (family of) operators acting on $\mathcal{B}$, $\Pi_u Q_u = Q_u \Pi_u$ and $\|Q_u^n\|_{\mathcal{B}} \leq \delta_0^n$ for some $\delta_0 < 1$.

[2] the precise terminology is recalled and specified in Section 2.3

A classical argument (e.g., see the survey paper [10] and references therein) shows that an immediate consequence of Proposition 2.4(c) is the Central Limit Theorem (CLT) for the ergodic sums F_K:

$$\frac{F_K}{\sqrt{K}} \implies \chi, \tag{5}$$

where $\implies$ stands for convergence in distribution and χ is a Gaussian random variable with mean 0 (here we use the no drift condition in (H3)) and symmetric, non-degenerate, $d \times d$ covariance matrix $\Sigma^2 = \sum_{j\in\mathbb{Z}} \int_X F \otimes F \circ \psi^j \, \mathrm{d}m$. The speed of mixing of ψ ensures that this sum converges. The non-degeneracy of Σ^2 is ensured because F is not a coboundary, see (H3).

Throughout this section we let $\Pi_0^{(j)}$ be the j-th derivative in u of Π_u evaluated at $u = 0$. To simplify notation, from here onward, given $u \in \mathbb{R}^d$ we write $u^2 := u \otimes u$ for the tensor product and similarly for the product u^j, $j \geq 1$. Also we use the notation $\otimes^j(H_j u) = \sum_{k=1}^d (H_j u)_k^j$ for $u \in \mathbb{R}^d$ and a $d \times d$ matrix H_j.

Proposition 2.5. *Let $v(\cdot, r) = v \circ p$ be a function supported and C^1 on $\{\xi = r\}$ for some $r \in \mathbb{Z}^d$. Then*

*(a) If v is a **real** function then $\Pi_0^{(j)} v$ is real if j is even and $\Pi_0^{(j)} v$ is purely imaginary if j is odd.*

(b) Let δ and δ_0 be as in Proposition 2.4(c). Let Σ be the covariance matrix in (5). There exist real $d \times d$ matrices H_j, $j \geq 3$ so that for all $\ell, r \in \mathbb{Z}^d$ and for all $K \geq 1$,

$$\begin{aligned}&\mathcal{L}^K v(x,r)\mathbb{1}_{\{F_K(x)=\ell-r\}} + E_k v(x,r) \\ &= \frac{1}{\left(2\pi\sqrt{K}\right)^d} \int_{\left[-\delta\sqrt{K},\delta\sqrt{K}\right]^d} e^{-\frac{\langle \Sigma u, \Sigma u\rangle}{2}} e^{\sum_{j=3}^{\infty} \frac{i^j}{j!} \frac{\otimes^j(H_j u)}{K^{j/2}}} e^{iu\frac{\ell-r}{\sqrt{K}}} \times \sum_{j=0}^{\infty} \frac{1}{j!} \frac{u^j}{K^{j/2}} \Pi_0^{(j)} v(x,r)\, du,\end{aligned}$$

where E_k is an operator acting on $\mathcal{B}$ so that $\|E_k v\|_{\mathcal{B}} \leq C\delta_0^K \|v\|_{C^1}$ and so that $\left|\int_X E_k v \, \mathrm{d}m\right| \leq C'\delta_0^K \|v\|_{C^1}$ for some $C, C' > 0$.

The proof of Proposition 2.5 is deferred to Section 2.5. We note that the asymptotic expansion in the usual LLT follows immediately. That is, taking $\ell - r = M \in \mathbb{Z}^d$, $v \equiv 1$ in Proposition 2.5(b) and integrating over the space, we obtain that for all $n \geq 1$,

$$m\left(F_K(x) = M\right) = \frac{1}{\left(2\pi \det(\Sigma)\sqrt{K}\right)^d} \Phi\left(\frac{M}{\sqrt{K}}\right) + \sum_{j=1}^n \frac{C_j}{K^{(j+d)/2}} + o\left(\frac{C_{n+1}}{K^{(n+d)/2}}\right), \text{ as } K \to \infty, \tag{6}$$

where $C_j = C_j(M/\sqrt{K}) \in \mathbb{R}$ and Φ is the density of the Gaussian χ from (5). Given Proposition 2.5(b), the fact that the C_j are real follows as in the first displayed chain of equations in the proof of [28, Theorem 3.2]. In fact, C_j can be computed precisely as there, and they are variants of products of $\Phi^{(j)}\left(\frac{M}{\sqrt{K}}\right) \Pi_0^{(q)} 1$ for $j, q < [n/2]$, where $\Phi^{(j)}$ is the j-th derivative of Φ.

The expansion in (6) allows us to record a technical lemma that will play an important role in the proof of Theorem 2.11 below.

Lemma 2.6. *(i) Let $q \geq 0$ be an integer and $f_q : \mathbb{R}^d \to \mathbb{R}^d$, $f_q(x) = x^q g(x) e^{-\langle x,x\rangle}$ for a bounded function $g : \mathbb{R}^d \to \mathbb{R}^d$. Then, there exist real constants $C_{d,q}$ and $d_{j,q}$, so that for any $n \geq 1$,*

$$\int_{X_\Gamma} f_q\left(\frac{\xi\left(\psi_\Gamma^K\right)}{\det(\Sigma)\sqrt{2K}}\right) \mathrm{d}m = C_{d,q} + \sum_{j=1}^n \frac{d_{j,q}}{K^{j/2}} + o(K^{-n/2}), \quad \textit{as } K \to \infty.$$

(ii) Let $f : \mathbb{R}^d \to \mathbb{R}^d$, $f(x) = e^{-\langle x,x\rangle}$ and let χ be the d-dimensional Gaussian introduced in (5)*. Then, there exist real constants d_j, so that for any $n \geq 1$,*

$$\int_{X_\Gamma} f\left(\frac{\xi\left(\psi_\Gamma^K\right)}{\det(\Sigma)\sqrt{2K}}\right) \mathrm{d}m = \mathbb{E}(f_0(\chi)) + \sum_{j=1}^{n} \frac{d_j}{K^{j/2}} + o(K^{-n/2}), \qquad \textit{as } K \to \infty,$$

Proof. **Item (i)**. Recall that $F_K(x) = \xi \circ \psi_\Gamma^K(x') - \xi(x')$ for $x' \in p^{-1}(x)$. Therefore,

$$\begin{aligned}\int_{X_\Gamma} f_q\left(\frac{\xi\left(\psi_\Gamma^K\right)}{\det(\Sigma)\sqrt{2K}}\right) \mathrm{d}m &= \sum_{M\in\mathbb{Z}^d} \int_{\{F_K=M\}} f_q\left(\frac{\xi\left(\psi_\Gamma^K\right)}{\det(\Sigma)\sqrt{2K}}\right) \mathrm{d}m \\ &= \sum_{M\in\mathbb{Z}^d} f_q\left(\frac{M}{\det(\Sigma)\sqrt{2K}}\right) m\left(F_K(x) = M\right).\end{aligned}$$

This together with (6) gives

$$\int_{X_\Gamma} f_q\left(\frac{\xi\left(\psi_\Gamma^K\right)}{\det(\Sigma)\sqrt{2K}}\right) \mathrm{d}m = \frac{1}{K^{d/2}} \sum_{M\in\mathbb{Z}^d} H\left(\frac{M}{\sqrt{K}}\right) \qquad \text{as } K \to \infty, \tag{7}$$

for

$$H(x) = \left(\frac{\Phi(x)}{(2\pi\det(\Sigma))^d} + \sum_{j=1}^{n} \frac{C_j(x)}{K^{j/2}} + o\left(\frac{C_{n+1}(x)}{K^{n/2}}\right)\right) f_q\left(\frac{x}{\det(\Sigma)\sqrt{2}}\right) \qquad \text{and} \qquad C_j(x) \in \mathbb{R},$$

for $x = \frac{M}{\sqrt{K}}$. Next, for each $M \in \mathbb{Z}^d$, define functions $\tilde{C}_j$ on the unit cube $Q(M)$ centered at M in such a way that $\int_{Q(M)} \tilde{C}_j(x) f_q\left(\frac{x}{\det(\Sigma)\sqrt{2K}}\right) dx = C_j\left(\frac{M}{\sqrt{K}}\right) f_q\left(\frac{M}{\det(\Sigma)\sqrt{2K}}\right)$. Then

$$\begin{aligned}\sum_{M\in\mathbb{Z}^d} C_j\left(\frac{M}{\sqrt{K}}\right) f_q\left(\frac{M}{\det(\Sigma)\sqrt{2K}}\right) &= \int_{\mathbb{R}^d} \tilde{C}_j(x) f_q\left(\frac{x}{\det(\Sigma)\sqrt{2K}}\right) dx \\ &= \left(\sqrt{K}\right)^d \int_{\mathbb{R}^d} \tilde{C}_j(u\sqrt{K}) f\left(\frac{u}{\det(\Sigma)\sqrt{2}}\right) du,\end{aligned}$$

where we used the change of coordinates $u = x/\sqrt{K}$. A similar argument holds for the term $\Phi(x)/(2\pi\det(\Sigma))^2$. This shows that the sum scales as $(\sqrt{K})^d$, and since $\int_{\mathbb{R}^d} \tilde{C}_j(u\sqrt{K}) f\left(\frac{u}{\det(\Sigma)\sqrt{2}}\right) du < \infty$ due to the exponential factor in f, there are constants c_d such that $\sum_{M\in\mathbb{Z}^d} H\left(\frac{M}{\sqrt{K}}\right) = c_d \left(\sqrt{K}\right)^d$. We get Item (i) from this together with (7).

Item (ii) We just need to argue that the first term, that is $C_{d,0}$ in item (i), is exactly $\mathbb{E}(f_0(\chi))$. Apart from this constant the statement is as in item (i) for f_q with $q = 0$ and $g \equiv 1$. One could proceed via an exact calculation (using, for instance, the Euler-Maclaurin formula), but a quicker way is to recall (6) and note that by the Portmanteau Theorem,

$$\int_{X_\Gamma} f\left(\frac{\xi\left(\psi_\Gamma^K\right)}{\det(\Sigma)\sqrt{2K}}\right) \mathrm{d}m \to \mathbb{E}(f_0(\chi)), \text{ as } K \to \infty.$$

□

We have an direct consequence of Proposition 2.5 that gives an easy proof of Theorem 2.8 below.

Corollary 2.7. *Let $G(\cdot, r) = G \circ p$ (where p is as in (H1), that is $p \circ \Delta_n = p$) be a function supported and C^1 on $\{\xi = 0\}$. Let δ, $\delta_0 \in (0,1)$ be as in Proposition 2.5. Let Σ be the covariance matrix in* (5). *Then for all $K \geq 1$,*

$$\mathcal{L}^K G(x) 1_{\{F_K(x)=\xi(\psi_\Gamma^K(x))\}} + E_k v G(x)$$
$$= \frac{1}{\left(2\pi\sqrt{K}\right)^d} \int_{\left[-\delta\sqrt{K}, \delta\sqrt{K}\right]^d} e^{-\frac{\langle \Sigma u, \Sigma u\rangle}{2}} e^{\sum_{j=3}^{\infty} \frac{c_j}{j!} \frac{(iu)^j}{K^{j/2}}} e^{iu \frac{\xi(\psi_\Gamma^K(x))}{\sqrt{K}}} \times \sum_{j=0}^{\infty} \frac{1}{j!} \frac{u^j}{K^{j/2}} \Pi_0^{(j)} G(x)\, du,$$

where E_k is an operator acting on $\mathcal{B}$ so that $\|E_k G\|_{\mathcal{B}} \leq C\delta_0^K \|v\|_{C^1}$.

Proof. Let v so that $v(x, r) = G(x)$ if $r = 0$ and $v(x, r) = 0$, else. Take $\ell = 0$ and $x' \in p^{-1}(x)$ so that $r = \xi(\psi_\Gamma^K(x'))$. To justify this choice, just recall that $F_K(x) = \xi \circ \psi_\Gamma^K(x') - \xi(x')$. The conclusion follows by Proposition 2.5 (b). □

2.2 Main results

Let Σ be the covariance matrix in (5). For $d = 1$, we write $\Sigma = \sigma$. For $d = 2$, we diagonalize $\Sigma = AJA^{-1}$ for $J = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$ for a unitary matrix A, so $A^{-1} = A^*$ is the transpose of A.

Theorem 2.8. *Let $G \in C^1(X_\Gamma)$ be a compactly supported real function. Given the stable eigenvalue $\lambda \in (0,1)$ of ψ_Γ, let K be minimal such that $\psi_\Gamma^K(\bigcup_{t=0}^T \phi_t(x))$ is contained in a single element of $\mathcal{P} \vee \psi_\Gamma^{-1}\mathcal{P}$. (This holds for $K \approx \log^* T := \lceil \log_{\lambda^{-1}} T \rceil$.)*

I. Suppose $d = 1$. Then there exist real bounded functions $g_{k,j}$ so that for all $N \geq 1$,

$$\int_0^T G \circ \phi_t(x)\, dt = \frac{\int_{X_\Gamma} G \, \mathrm{d}m}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{\xi\left(\psi_\Gamma^K(x)\right)^2}{2\sigma^2 K}} \frac{T}{\sqrt{K}}$$
$$\times \left(1 + \sum_{k=1}^{N} \frac{1}{K^k} \sum_{j=0}^{2k} g_{k,j}(x) \xi(\psi_\Gamma^K(x))^{2k-j} + O\left(\frac{1}{K^{N+1}}\right)\right) \quad \text{as } T \to \infty.$$

II. Suppose $d = 2$. Then there exists a 2×2 matrix B, with real functions as entries, so that

$$\int_0^T G \circ \phi_t(x)\, dt = \frac{\int_{X_\Gamma} G \, \mathrm{d}m}{\det \Sigma} e^{-\frac{\left\|\Sigma^{-1}\xi\left(\psi_\Gamma^K(x)\right)\right\|^2}{2K}} \frac{T}{\sqrt{K}}$$
$$\times \left(A\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{K} B\begin{pmatrix}1\\1\end{pmatrix} + O\left(\frac{1}{K^2}\right)\right) \quad \text{as } T \to \infty.$$

Remark 2.9. *For $d = 2$, we can also go higher in the expansion, but since the calculations are tedious, we omit this. The assumption that $G \in C^1(X_\Gamma)$ can be relaxed to G being C^1 on partition elements of $\bigvee_{j=0}^R \psi_\Gamma^j \mathcal{P}$ for some finite R.*

To prove Theorem 2.8, we need the following integrals (which are computed in Section 2.5):

Lemma 2.10. *A. Assume $d = 1$. Given $\sigma, L \in \mathbb{R}$ and $j \in \{0, 1, 2, \dots\}$, write*

$$I_j(\sigma, L) = \int_{\mathbb{R}} e^{-\frac{\sigma^2}{2}u^2} e^{iLu} u^j \, du.$$

Then

$$I_0(\sigma, L) = \frac{\sqrt{2\pi}}{\sigma} e^{-\frac{L^2}{2\sigma^2}} \quad \text{and} \quad I_j(\sigma, L) = \frac{1}{\sigma^2}(iLI_{j-1} + (j-1)I_{j-2}).$$

B. Assume $d=2$. Given a 2×2 covariance matrix Σ, $L\in\mathbb{R}^2$ and $j\in\{0,1,2,\dots\}$, write

$$\vec{I}_j(\Sigma,L)=\int_{\mathbb{R}^2} e^{-\frac{1}{2}\langle\Sigma u,\Sigma u\rangle}e^{i\langle L,u\rangle}u^j\,du$$

for the vector $u^j=\binom{u_1^j}{u_2^j}$. Then

$$\vec{I}_0(\Sigma,L)=\frac{2\pi}{\sigma_1\sigma_2}e^{-\frac{1}{2}\langle\Sigma^{-1}L,\Sigma^{-1}L\rangle}\begin{pmatrix}A_{11}+A_{12}\\ A_{21}+A_{22}\end{pmatrix},$$

where $\Sigma=AJA^{-1}$, $J=\begin{pmatrix}\sigma_1&0\\0&\sigma_2\end{pmatrix}$, is the diagonalization of Σ with unitary matrix A. Also

$$\vec{I}_1(\Sigma,L)=\frac{2\pi i}{\sigma_1\sigma_2}e^{-\frac{1}{2}\langle\Sigma^{-1}L,\Sigma^{-1}L\rangle}\begin{pmatrix}\frac{1}{\sigma_1}(A^*L)_1\\ \frac{1}{\sigma_2}(A^*L)_2\end{pmatrix}$$

and

$$\vec{I}_2(\Sigma,L)=\frac{2\pi}{\sigma_1\sigma_2}e^{-\frac{1}{2}\langle\Sigma^{-1}L,\Sigma^{-1}L\rangle}\begin{pmatrix}\frac{b_{1,0}}{\sigma_1^2}\left(1-\frac{(A^*L)_1^2}{\sigma_1^2}\right)-b_{1,1}\frac{(A^*L)_1(A^*L)_2}{\sigma_1^2\sigma_2^2}+\frac{b_{1,2}}{\sigma_2^2}\left(1-\frac{(A^*L)_2^2}{\sigma_2^2}\right)\\ \frac{b_{2,0}}{\sigma_1^2}\left(1-\frac{(A^*L)_1^2}{\sigma_1^2}\right)-b_{2,1}\frac{(A^*L)_1(A^*L)_2}{\sigma_1^2\sigma_2^2}+\frac{b_{2,2}}{\sigma_2^2}\left(1-\frac{(A^*L)_2^2}{\sigma_2^2}\right)\end{pmatrix}$$

for real coefficients $b_{p,q}$ made explicit in the proof.

Proof of Theorem 2.8. Let $G(\cdot,0)=G\circ p$ be supported on $\{\xi=0\}$. Take $K\geq 1$ minimal such that the ψ_Γ^K-image of the segment $[x,\phi_T(x)]$ is contained in at most two adjacent elements $P\in\mathcal{P}\vee\psi_\Gamma^{-1}\mathcal{P}$. We will argue as if only one P is needed; the other case can be easily derived by cutting the segment into two pieces. We have $K\approx\log^* T$ as in the statement of the theorem.

Take $\ell=0$ and $r=\xi\circ\psi_\Gamma^K(x,0)$. By (4) and Lemma 2.3 and the fact that $A_K=\{\xi=\ell\}$,

$$\frac{1}{T}\int_0^T G\circ\phi_t(x)\,dt=\int_X\mathcal{L}^K G(x)\mathbb{1}_{\{F_K=\xi\circ\psi^K\}}\,\mathrm{d}m+O(1/T).$$

This together with Corollary 2.7 yields

$$\begin{aligned}\int_0^T G\circ\phi_t(x)\,dt+O(1)+O(\delta_0^K)&=\frac{T}{2\pi\sqrt{K}}\int_X G(x)\,\mathrm{d}m\\ &\times\int_{-\delta\sqrt{K}}^{\delta\sqrt{K}}\left(e^{-\frac{\langle\Sigma u,\Sigma u\rangle}{2}}e^{\sum_{j=3}^\infty\frac{c_j}{j!}\frac{(iu)^j}{K^{j/2}}}e^{iu\frac{\xi(\psi_\Gamma^K(x))}{\sqrt{K}}}\cdot\sum_{j=0}^\infty\frac{i^j}{j!}\frac{\otimes^j(H_ju)}{K^{j/2}}\Pi_0^{(j)}\right)du.\end{aligned}\tag{8}$$

It remains to analyze the last integral in (8). First, note that for $N\geq 1$,

$$e^{\sum_{j=3}^\infty\frac{i^j}{j!}\frac{\otimes^j(H_ju)}{K^{j/2}}}=1+\sum_{j=1}^N\frac{i^j}{(j+2)!}\frac{\otimes^{j+2}(H_{j+2}u)}{K^{j/2+1}}+O\left(\frac{|u|^{N+3}}{K^{N/2+1}}\right).\tag{9}$$

Next, by Proposition 2.5 (a), for any $N\geq 1$

$$\sum_{j=0}^\infty\frac{1}{j!}\frac{u^j}{K^{j/2}}\Pi_0^{(j)}G(x)=\int_{X_\Gamma}G\,\mathrm{d}m+\sum_{j=1}^N\frac{d_{j,G}(x)}{j!}\frac{u^j}{K^{j/2}}+O\left(\frac{|u|^{N+1}}{K^{(N+1)/2}}\right),$$

where $d_{j,G}$ is a bounded function that, by Proposition 2.5, is real for even j and purely imaginary for odd j. Thus,

$$\sum_{j=0}^\infty\frac{1}{j!}\frac{u^j}{K^{j/2}}\Pi_0^{(j)}G(x)=\int_{X_\Gamma}G\,\mathrm{d}m\left(1+\sum_{j=1}^N\frac{d'_{j,G}(x)}{j!}\frac{u^j}{K^{j/2}}\right)+O\left(\frac{|u|^{N+1}}{K^{(N+1)/2}}\right),$$

where $d'_{j,G}$ is a bounded function which is real if j is even and purely imaginary if j is odd. Together with (9), this gives

$$e^{\sum_{j=3}^{\infty} \frac{i^j}{j!} \frac{\otimes^j (H_j u)}{K^{j/2}}} \times \sum_{j=0}^{\infty} \frac{1}{j!} \frac{u^j}{K^{j/2}} \Pi_0^{(j)} G(x) = \int_{X_\Gamma} G \, \mathrm{d}m \left(1 + \sum_{j=1}^{N} \frac{e_{j,G}(x)\, u^j}{K^{j/2}} \right) + O\left(\frac{|u|^{N+1}}{K^{(N+1)/2}} \right),$$

where $e_{j,G}$ is a bounded function which is real if j is even and purely imaginary if j is odd. This is because when two series of functions with even real parts and odd imaginary parts are multiplied, the product also has an even real part and an odd imaginary part. Plugging this in (8), we obtain

$$\int_0^T G \circ \phi_t(x)\, dt = \frac{\int_X G \, \mathrm{d}m \, T}{2\pi\sqrt{K}} \int_{-\delta\sqrt{K}}^{\delta\sqrt{K}} e^{iu \frac{\xi(\psi_\Gamma^K(x))}{\sqrt{K}}} e^{-\frac{\sigma^2 u^2}{2}} \left(1 + \sum_{j=1}^{N} \frac{e_{j,G}(x)\, u^j}{K^{j/2}} \right) du$$
$$+ O\left(\frac{T}{\sqrt{K}} \frac{\int_{-\delta\sqrt{K}}^{\delta\sqrt{K}} |u|^{N+1} e^{-\frac{\sigma^2 u^2}{2}} \, du}{K^{(N+1)/2}} \right). \tag{10}$$

We are left with computing the integrals in (10). Set $I_j := \int_{-\infty}^{\infty} e^{iLt - \frac{\langle \Sigma u, \Sigma u \rangle}{2}} u^j \, dt$, with $L = L(x) := \frac{\xi(\psi_\Gamma^K(x))}{\sqrt{K}}$. From Lemma 2.10, we obtain $I_0 = \frac{\sqrt{2\pi}}{\sigma} e^{-\frac{L^2}{2\Sigma^2}}$ and $I_j = \frac{1}{\sigma^2}(iLI_{j-1} + (j-1)I_{j-2})$ for $j \geq 1$.

Item I., $d = 1$. Recall that in this case we write $\Sigma = \sigma$. For $j = 1, 2, 3, 4$,

$$I_1 = iL\frac{\sqrt{2\pi}}{\sigma^3} e^{-\frac{L^2}{2\sigma^2}}, \qquad I_2 = \frac{\sqrt{2\pi}}{\sigma^3} e^{-\frac{L^2}{2\sigma^2}} \left(1 - \frac{L^2}{\sigma^2} \right).$$
$$I_3 = iL\frac{\sqrt{2\pi}}{\sigma^5} e^{-\frac{L^2}{2\sigma^2}} \left(3 - \frac{L^2}{\sigma^2} \right), \qquad I_4 = \frac{\sqrt{2\pi}}{\sigma^5} e^{-\frac{L^2}{2\sigma^2}} \left(3 - 6\frac{L^2}{\sigma^2} + \frac{L^4}{\sigma^4} \right), \ldots$$

Observe that I_j is real if j is even and purely imaginary if j is odd. There for real coefficients $c_{p,j}$ such that the I_j's, written as power series in $L = \frac{\xi(\psi_\Gamma^K(x))}{\sqrt{K}}$ become

$$I_j = (iL)^{j \bmod 2} \sum_{p=0}^{\lfloor j/2 \rfloor} c_{p,j} L^{2p} e^{-\frac{L^2}{2\sigma^2}} = i^{j \bmod 2} \cdot \sum_{p=0}^{\lfloor j/2 \rfloor} f_{p,j} \frac{(\xi(\psi_\Gamma^K(x))^{2p + (j \bmod 2)}}{K^{2p + (j \bmod 2))/2}} e^{-\frac{L^2}{2\sigma^2}}, \tag{11}$$

Going back to (10), the integrals $\int_{-\delta\sqrt{K}}^{\delta\sqrt{K}} e^{iu \frac{\xi(\psi_\Gamma^K(x))}{\sqrt{K}}} e^{-\frac{\sigma^2 u^2}{2}} u^j \, du$ can be approximated by I_j: the change from $\pm\delta\sqrt{K}$ to $\pm\infty$ gives an error that is negligible compared to other error estimates. The rest of the argument comes down to combining the series $1 + \sum_{j=1}^{N} \frac{e_{j,G}(x)}{K^{j/2}} u^j$ from (10) with (11). Recall that the bounded functions $e_{j,G}$ are real if j is even and purely imaginary if j is odd, and this combines with $i^{j \bmod 2}$ to a real coefficient. Taking $f_{k,j} = i^{j \bmod 2} e_{j,G} c_{(2k-j-(j \bmod 2))/2,j}$, we get

$$I_0 + \sum_{j=1}^{N} \frac{i^{j \bmod 2} e_{j,G}(x)}{K^{j/2}} I_j = \sum_{j=0}^{N} \sum_{p=0}^{\lfloor j/2 \rfloor} f_{p,j} i^{j \bmod 2} e_{j,G}(x) \frac{(\xi(\psi_\Gamma^K(x))^{2p+(j \bmod 2)}}{K^{(2p+j+(j \bmod 2))/2}} e^{-\frac{L^2}{2\sigma^2}}$$
$$= \sum_{k=0}^{N} \left(\sum_{j=0}^{2k} f_{k,j} \xi(\psi_\Gamma^K(x))^{2k-j} \right) e^{-\frac{1}{2\sigma^2} \frac{\xi\left(\psi_\Gamma^K(x)\right)^2}{K}} \left(\frac{1}{K} \right)^k + O\left(\left(\frac{1}{K} \right)^{N+1} \right),$$

where we introduced a new summation index $2k = 2p + j + (j \bmod 2)$ and switched the order of the sums. The terms in the inner brackets can all be computed explicitly. We just give the first two as illustration:

$$I_0 + \frac{e_{1,G}(x)}{K^{1/2}} I_1 + \cdots = \left(\frac{\sqrt{2\pi}}{\sigma} + \frac{\sqrt{2\pi}}{\sigma^3} \frac{e_{2,G}(x) + i e_{1,G}(x) \xi(\psi_\Gamma^K(x))}{K} + O\left(\frac{1}{K^2} \right) \right) e^{-\frac{L^2}{2\sigma^2}}.$$

Putting everything together gives

$$\int_0^T G\circ\phi_t(x)\,dt = \frac{\int_X G\,\mathrm{dm}}{\sigma\sqrt{2\pi}} e^{-\frac{\xi\left(\psi_\Gamma^K(x)\right)^2}{2\sigma^2 K}}\,\frac{T}{\sqrt{K}}$$
$$+\frac{\int_X G\,\mathrm{dm}}{\sigma^3\sqrt{2\pi}}\left(e_{2,G}(x)+ie_{1,G}(x)\xi(\psi_\Gamma^K(x))\right)\,e^{-\frac{\xi\left(\psi_\Gamma^K(x)\right)^2}{2\sigma^2 K}}\,\frac{T}{(\sqrt{K})^3}$$
$$+\frac{\int_X G\,\mathrm{dm}}{2\pi}\sum_{k=2}^N\left(\sum_{j=0}^{2k} f_{k,j}(x)\xi(\psi_\Gamma^K(x))^{2k-j}\right)\,e^{-\frac{\xi\left(\psi_\Gamma^K(x)\right)^2}{2\sigma^2 K}}\,\frac{T}{(\sqrt{K})^{2k+1}}$$
$$+\ O\left(\frac{T}{(\sqrt{K})^{2N+3}}\right),$$

for real bounded functions $f_{k,j}$. Item I. follows with $g_{k,j}=f_{k,j}\int_X G\,\mathrm{d}m$.

Item II., $d=2$. Using the 2-dimensional integrals in Lemma 2.10, the same proof can be used to give an expansion for $\int_0^T G\circ\phi_t(x)\,dt$ when ξ takes value in $\mathbb{Z}^2$. The result up to the "linear" term is

$$\int_0^T G\circ\phi_t(x)\,dt = \frac{\int_X G\,\mathrm{dm}}{\det(\Sigma)} e^{-\frac{\left\langle \Sigma^{-1}\xi\left(\psi_\Gamma^K(x)\right),\Sigma^{-1}\xi\left(\psi_\Gamma^K(x)\right)\right\rangle}{2\sigma_1^2 K}}\times$$
$$\left(\begin{pmatrix}A_{11}+A_{12}\\ A_{21}+A_{22}\end{pmatrix}\frac{T}{\sqrt{K}}+\begin{pmatrix}e_{2,G}(\frac{b_{1,0}}{\sigma_1^2}+\frac{b_{1,2}}{\sigma_2^2})+\frac{ie_{1,G}}{\sigma_1}\xi_1\left(\psi_\Gamma^K(x)\right)\\ e_{2,G}(\frac{b_{2,0}}{\sigma_1^2}+\frac{b_{2,2}}{\sigma_2^2})+\frac{ie_{1,G}}{\sigma_2}\xi_2\left(\psi_\Gamma^K(x)\right)\end{pmatrix}\frac{T}{(\sqrt{K})^3}\right)+O\left(\frac{T}{(\sqrt{K})^5}\right)$$

for the unitary matrix A and coefficients $b_{p,q}$ as in Lemma 2.10. The conclusion follows with

$$A\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}A_{11}+A_{12}\\ A_{21}+A_{22}\end{pmatrix}\quad\text{and}\quad B\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}e_{2,G}(\frac{b_{1,0}}{\sigma_1^2}+\frac{b_{1,2}}{\sigma_2^2})+\frac{ie_{1,G}}{\sigma_1}\xi_1\left(\psi_\Gamma^K(x)\right)\\ e_{2,G}(\frac{b_{2,0}}{\sigma_1^2}+\frac{b_{2,2}}{\sigma_2^2})+\frac{ie_{1,G}}{\sigma_2}\xi_2\left(\psi_\Gamma^K(x)\right)\end{pmatrix}.$$

□

Using Theorem 2.8 and Lemma 2.6 we obtain *expansion in weak rational ergodicity for 'good' functions.*

Theorem 2.11. *Assume the setup of Theorem 2.8.*

(i) Suppose $d=1$. Then, there exist real constants $d_{k,j}$ so that for all $N\geq 1$,

$$\int_{X_\Gamma}\int_0^T G\circ\phi_t(x)\,dt\ \mathrm{dm} = \frac{\int_{X_\Gamma} G\,\mathrm{dm}}{\sigma\sqrt{2\pi}}\,\frac{T}{\sqrt{K}}\mathbb{E}(e^{-\frac{\chi^2}{2}})+\sum_{k=1}^N\sum_{j=0}^{2k}\frac{d_{k,j}\,T}{\sqrt{K}^{2k+1+j}}+\ O\left(\frac{T}{\sqrt{K}^{2N+3}}\right).$$

(ii) Suppose $d=2$. Then, there exists a 2×2 matrix B_0 with real entries so that

$$\int_{X_\Gamma}\int_0^T G\circ\phi_t(x)\,dt\ \mathrm{dm} = \frac{\int_{X_\Gamma} G\,\mathrm{dm}}{\det\Sigma}\,\frac{T}{\sqrt{K}}A\begin{pmatrix}1\\ [1mm]1\end{pmatrix}\mathbb{E}(e^{-\frac{\chi^2}{2}})+\frac{T}{\sqrt{K}^3}B_0\begin{pmatrix}1\\ [1mm]1\end{pmatrix}+O\left(\frac{T}{K^2}\right).$$

Remark 2.12. *Weak rational ergodicity without rates follows immediately since convergence for all $L^1(X_\Gamma)$-functions is an immediate consequence of Theorem 2.8 and the ratio ergodic theorem, see [1].*

Proof. We prove Item (i). item (ii) follows similarly.

By Theorem 2.8,

$$\begin{aligned}\int_{X_\Gamma}\int_0^T G\circ\phi_t(x)\,dt\;\mathrm{d}m &= \frac{\int_{X_\Gamma} G\;\mathrm{d}m}{\sigma\sqrt{2\pi}}\cdot\int_{X_\Gamma} e^{-\frac{\xi\left(\psi_\Gamma^K(x)\right)^2}{2\sigma^2 K}}\,\frac{T}{\sqrt{K}}\;\mathrm{d}m\\ &+\frac{\int_{X_\Gamma} G\;\mathrm{d}m}{2\pi}\sum_{k=1}^{N}\frac{T}{(\sqrt{K})^{2k+1}}\left(\sum_{j=0}^{2k}\int_{X_\Gamma} g_{k,j}(x)\xi(\psi_\Gamma^K(x))^{2k-j}\,e^{-\frac{\xi\left(\psi_\Gamma^K(x)\right)^2}{2\sigma^2 K}}\;\mathrm{d}m\right)\\ &+\;O\left(\frac{T}{(\sqrt{K})^{2N+3}}\right).\end{aligned} \tag{12}$$

Since $g_{k,j}$ are bounded functions, Lemma 2.6(ii) with $f(x)=e^{-\frac{x^2}{2}}$ ensures that

$$\int_{X_\Gamma} g_{k,j}(x)\xi(\psi_\Gamma^K(x))^{2k-j}\,e^{-\frac{\xi\left(\psi_\Gamma^K(x)\right)^2}{2\sigma^2 K}}\;\mathrm{d}m=\mathbb{E}(f_0(\chi))+\sum_{j=1}^{n}\frac{d_j}{K^{j/2}}+o(K^{-n/2}),$$

for some real constants d_j. To deal with the sum in (12), we apply Lemma 2.6(i) with $f_{2k-j}(x)=g_{k,j}(x)x^{2k-j}e^{-\frac{x^2}{2}}$. This ensures that the sum $\sum_{j=0}^{2k}$ of integrals in (12) convergence to $\sum_{j=0}^{2k} d_{k,j}$ for real constants $d_{k,j}$. □

To obtain higher order rational ergodicity from Theorem 2.8, we need the almost sure behaviour of $\frac{\xi(\psi_\Gamma^K(x))}{\sqrt{K}}$. The following almost sure invariance principle (ASIP) follows immediately from [21, Proposition 2.6] using the Banach spaces in Section 2.3 (and it has been recorded in, for instance, [10]). The precise exponent is due to the fact that the Frobenius function F is bounded.

Lemma 2.13. *There exist i.i.d. Gaussian random variables X_j defined on the probability space (X,m) (or any other copy of it) with mean zero and covariance matrix Σ so that almost surely as $n\to\infty$,*

$$n^{-1/2}\left(F_n-\sum_{j=1}^{n}X_j\right)=O\left(n^{-(1/4-\varepsilon)}\right),$$

for any $\varepsilon>0$. The i.i.d. Gaussian sum is a Brownian motion at integers.

Using Theorem 2.8 and Lemma 2.13 (the precise error term is irrelevant) we can easily obtain higher order rational ergodicity (in the sense of [16, 4]).

Theorem 2.14. *Assume the setup of Theorem 2.8 with $G\in L^1(X_\Gamma)$. Set $a(T)=\frac{\int_{X_\Gamma} G\;\mathrm{d}m}{\sigma\sqrt{2\pi}}\frac{T}{\sqrt{K}}$. Then we have* higher order rational ergodicity*, i.e.,*

$$\lim_{N\to\infty}\frac{1}{\log\log N}\int_3^N\frac{1}{T\log T}\left(\frac{1}{a(T)}\int_0^T G\circ\phi_t(x)\,dt\right)=1\qquad\text{for } m\text{-a.e.}x.$$

Proof. By Remark 2.12, the convergence (that is, just the first term) in Theorem 2.8 holds for all $G\in L^1(X_\Gamma)$. Using Lemma 2.13 we approximate almost surely the first term in Theorem 2.8. That is, we can replace (almost surely), $e^{-\frac{\xi(\psi_\Gamma^K(x))^2}{2\sigma^2 K}}$ with $e^{-\frac{\left(\sum_{j=1}^K X_j\right)^2}{2\Sigma^2 K}}(1+o(1))$. Hence,

$$\frac{1}{T\log T}\left(\frac{1}{a(T)}\int_0^T G\circ\phi_t(x)\,dt\right)=e^{-\frac{\left(\sum_{j=1}^K X_j\right)^2}{2\det(\Sigma)^2 K}}(1+o(1)),\ \text{almost surely.} \tag{13}$$

By the almost sure Central Limit Theorem for i.i.d. random variables (see [8] and references therein), the following holds almost surely, for all a:

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{k=1}^{N}\frac{1}{k}\mathbb{1}_{\left\{\frac{\left(\sum_{j=1}^{k}X_j\right)^2}{2k\det(\Sigma)^2}<a\right\}}=\Phi(a).$$

where Φ is the distribution of the standard Gaussian. A version of this statement holds with Lipschitz functions instead of indicator functions, see [27, Theorem 2.4] (with $d_k=1$, there). That is, given a Lipschitz function f,

$$\frac{1}{\log N}\sum_{k=1}^{N}\frac{1}{k}f\left(\frac{\left(\sum_{j=1}^{k}X_j\right)^2}{2k\det(\Sigma)^2}\right)\to\int f\,d\Phi.$$

Taking $f(x)=e^{-\langle x,x\rangle}$,

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{k=1}^{N}\frac{1}{k}e^{-\frac{\left(\sum_{j=1}^{k}X_j\right)^2}{2k\det(\Sigma)^2}}=\int e^{-\langle x,x\rangle}\,d\Phi. \tag{14}$$

The conclusion from (14) together with (13) taking $k=\log^* T$ and recalling the definition of $a(T)$. □

Remark 2.15. *Alternatively, in the previous proof one could try to use the argument in [24, Proof of Lemma 2].*

2.3 Banach spaces

We find it convenient to work with a slight modification of the Banach spaces considered in [10] for the purpose of obtaining limit theorem via spectral methods for a general class of baker maps[3]. For a very similar (simplified) variation of the spaces in [11] of the Banach spaces in [10] we refer to [25], which focused on some two dimensional, non-uniformly hyperbolic versions of Pomeau-Manneville maps.

The automorphism ψ resembles a baker map except for the type of singularities. For a baker map the singularities are given by the set of discontinuity points. In the setup of ψ, we say that a point $s\in X$ is singular if the cone angle at s is not 2π. The difference in the type of singularities introduces a difference in the class of admissible leaves. In all other aspects the variant of the Banach spaces in [10] remain the same in the set of ψ. We summarize below the ingredients of these Banach spaces, using the notation from [10], as to emphasize that the case of the automorphism ψ (regarding the spectral gap for $\mathcal{L}$) is one of the easiest possible examples that the spaces introduced in [11] can treat.

2.3.1 Definitions of Banach spaces

Although, the presence of a (natural) Markov partition is not a crucial element of the construction in [10] for baker type maps, it does simplify the writing. The presence of this type of Markov structure considerably simplifies the description of admissible leaves. In particular, it allows us to define admissible leaves as full unstable segments. For the same reason, that of simplicity, we will take advantage of the Markov partition $\mathcal{P}$ and more precisely of $\mathcal{P}^R:=\bigvee_{j=0}^{R}\psi^j\mathcal{P}$.

We define the set $\mathcal{W}^s$ of *admissible leaves* as the set of stable segments W that exactly stretch across an element $P\in\mathcal{P}^R$ such that its (one dimensional) interior is contained in the interior of P. Note that, for any such $W\in\mathcal{W}^s$, the stable segment $\psi^{-1}W$ can be decomposed into a finite union of elements of $\mathcal{W}^s$.

[3]The baker map itself is defined as $b(x,y)=(2x \bmod 1, \frac{1}{2}(y+\lfloor 2x\rfloor))$ on the unit square.

Any $W \in \mathcal{W}^s$ has an affine parametrization $\{\chi_W(r) : r \in [0, l]\}$, where l is the length of W. Then, for any measurable function $h : W \to \mathbb{C}$, we write

$$\int_W h \, \mathrm{d}m = \int_0^l h \circ \chi_W(r) \, \mathrm{d}r.$$

It is usual to construct $\mathcal{P}$ based on stable and unstable manifold of integers lattice points. The slopes of these manifold belong to $\mathbb{Q}[\lambda]$ and λ^{-1} is a Pisot number, so by the Garsia separation principle [19], there is a constant $C > 0$, independent of R, such that the stable length of all the elements of $\mathcal{P}^R$ is between $|\lambda|^R/C$ and $C|\lambda|^R$. Therefore we use $|\lambda|^{-R}$ to normalize the integrals over leaves $W \in \mathcal{W}^s$.

Let $\alpha \in [0, 1]$. For any $W \in \mathcal{W}^s$, we let $C^\alpha(W, \mathbb{C})$ denote the Banach space of complex-valued functions W with Hölder exponent α, equipped with the norm

$$|h|_{C^\alpha(W,\mathbb{C})} = \sup_{z \in W} |h(z)| + \sup_{z,w \in W} \frac{|h(z) - h(w)|}{|z - w|^\alpha}.$$

Such a set is a collection of local unstable manifolds that do not containing a singularity point. From here onward all required definitions are as in [10, Section 2.2].

We say that $\varphi \in C^\alpha(X, \mathbb{C})$ if it is $C^\alpha(W, \mathbb{C})$ for all $W \in \mathcal{W}^s$. Given $h \in C^1(X, \mathbb{C})$, define the *weak norm* by

$$\|h\|_{\mathcal{B}_w} := \sup_{W \in \Sigma} \sup_{|\phi|_{C^1(W,\mathbb{C})} \leq 1} \lambda^{-R} \int_W h\phi \, \mathrm{d}m.$$

Given $\alpha \in [0, 1)$, define the *strong stable norm* by

$$\|h\|_s := \sup_{W \in \Sigma} \sup_{|\phi|_{C^\alpha(W,\mathbb{C})} \leq 1} \lambda^{-R} \int_W h\phi \, \mathrm{d}m.$$

For any two *aligned*[4] admissible leaves $W_1, W_2 \in \mathcal{W}^s$ *in the same atom of* $\mathcal{P}^R$, let $d(W_1, W_2)$ denote the distance *in the unstable direction* between W_1 and W_2. In other words, if $W_i = \{\chi_i(r) : r \in [0, l]\}$, then $d(W_1, W_2)$ is the length of the segment in the unstable direction connecting $\chi_1(r)$ to $\chi_2(r)$.

With the same notation as above, for two functions $\varphi_i \in C^1(W_i, \mathbb{C})$, with $i = 1, 2$, we also define

$$d_0(\varphi_1, \varphi_2) = \sup_{r \in [0,l]} |\varphi_1 \circ \chi_r(x_1) - \varphi_2 \circ \chi_r(x_2)|.$$

Next define the *strong unstable norm* by

$$\|h\|_u := \sup_{W_1, W_2 \in \Sigma} \sup_{|\varphi_i|_{C^1} \leq 1, \, d_0(\varphi_1, \varphi_2) = 0} \frac{|\lambda|^{-R}}{d(W_1, W_2)^{1-\alpha}} \left| \int_{W_1} h\varphi_1 \, \mathrm{d}m - \int_{W_2} h\varphi_2 \, \mathrm{d}m \right|.$$

Finally, the *strong norm* is defined by $\|\varphi\|_{\mathcal{B}} = \|\varphi\|_s + \|\varphi\|_u$. These norms are exactly those of [10, Section 2.3] but normalized with λ^{-R} because our use of the finer partition $\mathcal{P}^R$.

Define the weak space $\mathcal{B}_w$ to be the completion of $C^1(X)$ in the weak norm and define $\mathcal{B}$ to be the completion of $C^1(X)$ in the strong norm.

By [10, Lemma 2.4] (see also [25, Lemma 7.2]),

Lemma 2.16. *We have the following sequence of continuous, injective embeddings: $C^1(X) \subset \mathcal{B} \subset \mathcal{B}_w \subset (C^1(X))^*$. Moreover, the unit ball of $\mathcal{B}$ is relatively compact in $\mathcal{B}_w$.*

[4]i.e., the one is obtained from the other by a translation in the unstable direction.

2.3.2 Well-definedness of $\mathcal{L}$ on $\mathcal{B}$ and boundedness of $\mathcal{L}$ on $\mathcal{B}$ and $\mathcal{B}_w$

Recall that ψ is piecewise affine (so ψ is $C^1(W)$, for any $W \in \mathcal{W}^s$) and note that for any $\alpha \in [0,1]$, for any $W \in \mathcal{W}^s$ and for any $\varphi \in C^\alpha(W, \mathbb{C})$, $\varphi \circ \psi \in C^\alpha(X, \mathbb{C})$. Moreover, for any $n \geq 1$, $\psi^{-n}(W)$ consists of a union of leaves in $\mathcal{W}^s$ and the transfer operator $\mathcal{L}$ of ψ is defined as

$$\mathcal{L}h(\varphi) = h(\varphi \circ \psi), \quad \text{for all } h \in C^\alpha(\mathcal{W}^s) \text{ and } \varphi \in (C^\alpha(\mathcal{W}^s))^*. \tag{15}$$

Recall that Lebesgue measure m is invariant for ψ. We identify h with the measure $\mathrm{d}\mu = h\, \mathrm{d}m$. Then $h \in C^1(\mathcal{W}^s) \subset (C^1(\mathcal{W}^s))^*$ and $\mathcal{L}h$ is associated with the measure having density

$$\mathcal{L}h(x) = \frac{h \circ \psi^{-1}(x)}{J_\psi(\psi^{-1}(x))} = h \circ \psi^{-1}(x), \tag{16}$$

where J_ψ is the Jacobian of ψ with respect to m, which is equal to 1 (since the contraction and expansion are the same).

In general, it is not true that for systems with discontinuities, $\mathcal{L}(C^1(X)) \subset C^1(X)$ and hence it is not obvious that $\mathcal{L}$ is well defined on $\mathcal{B}$: see, for instance, [10, Footnote 13]. However, in the current setup of ψ, similar to the first line of the proof of [10, Lemma 4.1], $\mathcal{L}h \in C^1(\mathcal{W}^s)$ (since for any $W \in \Sigma$, $\psi^{-1}W$ is an exact union of leaves in $\mathcal{W}^s$). Hence, $\mathcal{L}(C^1(X)) \subset C^1(X)$ and **$\mathcal{L}$ is well defined on $\mathcal{B}$.**

Also, by [10, Lemma 4.1], $\mathcal{L}$ **acts continuously on $\mathcal{B}$ and $\mathcal{B}_w$** and the proof of [10, Theorem 2.5] (for baker type maps) goes word for word the same in the setup of ψ. This yields

Lemma 2.17. *[10, Theorem 2.5] The operator $\mathcal{L}$ is quasi-compact as an operator on $\mathcal{B}$. That is, its spectral radius is 1 and its essential spectral radius is strictly less than 1. Moreover, 1 is a simple eigenvalue, and all other eigenvalues have modulus strictly less than 1.*

The computations on the relevant Lasota-Yorke inequalities are almost the same as in [10, Proposition 4.2], the fact that we use a finer partition $\mathcal{P}^R$ has no effect on these inequalities and hence the bound on the essential spectral radius is independent of R. We use here that due to the uniform expansion factor λ^{-1}, the preimage $\psi^{-n}(W)$ is a union of admissible leaves of total length equal to $|\lambda|^{-n}|W|$.

So far, we have summarized all the required ingredients for the proof of Proposition 2.4(a).

2.3.3 Analyticity of the twisted transfer operator $\mathcal{L}_u f = \mathcal{L}(e^{iuF} f)$, $f \in \mathcal{B}$. Proof of Proposition 2.4(b)

The Frobenius function $F : X \to \mathbb{Z}$, $x \mapsto \xi \circ \psi_\Gamma(x') - \xi(x')$ for $x' \in p^{-1}(x)$, is not globally C^1, hence the simple argument of [10, Lemma 4.8] cannot go through. However, F is constant on each element of the partition $\mathcal{P}^R \vee \psi^{-1}\mathcal{P}^R$ (hence C^∞ on each element of $\mathcal{P}^R \vee \psi^{-1}\mathcal{P}^R$). As a consequence, the argument for the analyticity of the twisted transfer operator is a much more simplified version of the argument used in the proof of [15, Lemma 3.9] (essentially a consequence of the arguments used in [12, 13, 14]).

Lemma 2.18. *Let $u \in \mathbb{R}^d$, $f \in \mathcal{B}$ and $m \geq 1$. Then $\frac{d^k}{du^k}\mathcal{L}_u f$ is a linear operator on $\mathcal{B}$ with operator norm of $O(\|F\|_\infty^k)$.*

Proof. Using (16), compute that

$$\frac{d^k}{du^k}\mathcal{L}_u f = i^k \mathcal{L}(F^k e^{iuF} f) = i^k (F^k e^{iuF}) \circ \psi^{-1} \mathcal{L} f.$$

Since F is locally constant and since each element of $\mathcal{P}^R \vee \psi^{-1}\mathcal{P}^R$ contains no singularities in its interior, a simplified version[5] of the argument used in [13, Lemma 3.7] (see also [15, Lemma 3.3])

[5] The (serious) simplification comes from the simple form of admissible leaves and the fact that the Jacobian is constant.

shows that for any $f \in \mathcal{B}$, $f\,F \in \mathcal{B}$ and that for some $C > 0$,

$$\|f\,F\|_{\mathcal{B}} \le C\|f\|_{\mathcal{B}} \sup_{P_i \in \mathcal{P}^R} \|F\|_{C^\alpha(P_i)}. \tag{17}$$

Thus,

$$\left\| \frac{d^k}{du^k} \mathcal{L}_u f \right\|_{\mathcal{B}} \le C \sup_{P_i \in \mathcal{P}^R} \|(F^k e^{iuF}) \circ \psi^{-1}\|_{C^\alpha(P_i)} \|f\|_{\mathcal{B}} \le C\|F\|_\infty^k \|f\|_{\mathcal{B}}.$$

□

2.4 Spectrum of $\mathcal{L}_u$ and leading eigenvalue and proof of Proposition 2.4(c)

We already know (see Lemma 2.17) that 1 is a simple isolated eigenvalue of $\mathcal{L}_0$. Since $u \to \mathcal{L}_u$ is analytic (see Lemma 2.18), there exists $\delta > 0$ and a simple family of simple eigenvalues λ_u, analytic in $u \in (0, \delta)$ with $\lambda_0 = 1$. Also, standard perturbation theory ensures that for all $u \in (0, \delta)$,

$$\mathcal{L}_u^n = \lambda_u^n \Pi_u + Q_u^n, \tag{18}$$

where Π_u is the spectral projection onto the one-dimensional eigenspace associated to λ_u with $\Pi_0 f = \int_X f \; dm$ and where $\|Q_u^n\| \le \theta^n$ for some $\theta \in (0,1)$, and $Q_u \Pi_u = \Pi_u Q_u$, (see, for instance, [22, Section 2]). By Lemma 2.18 and standard perturbation theory, Π_u, Q_u are also analytic in $u \in (0, \delta)$.

To obtain limit theorems, one still needs to understand the expansion of λ_u in u. Again, due to the fact that F is globally bounded and also locally constant, the following expansion follows by standard arguments exploiting (17) among others. We recall this briefly.

Let $v_u = \frac{\Pi_u 1}{\int \Pi_u 1 \; dm}$ be the normalized eigenvector associated with λ_u, i.e., $\int_X v_u \; dm = 1$. Since $\mathcal{L}_u$ is analytic, so is v_u and we write $v_0^{(j)}$ for the j-th derivative evaluated at $u = 0$. A simple calculation starting from $\int_X \mathcal{L}_u v_u \; dm = \lambda_u \int_X v_u \; dm = \lambda_u$ shows that we can write

$$\begin{aligned} 1 - \lambda_u &= \int_X (1 - e^{iuF}) \; dm + \int_X (1 - e^{iuF})(v_u - v_0) \; dm \\ &= -\sum_{j=1}^\infty \frac{(iu)^j}{j!} \mathbb{E}(F^j) + \int_X \left(\sum_{j=1}^\infty \frac{(iu)^j}{j!} F^j \right) \left(\sum_{j=1}^\infty \frac{(iu)^j}{j!} v_0^{(j)} \right) \; dm. \end{aligned} \tag{19}$$

This together with (17) and a standard argument (see, for instance, [14, Proof of Theorem 2.6 (b)] who work with the same Banach spaces, and more generally the survey [22]), gives

$$1 - \lambda_u = 1 - \frac{1}{2}\langle \Sigma u, \Sigma u \rangle (1 + o(1)), \tag{20}$$

where $\Sigma^2 = \sum_{j \in \mathbb{Z}} \int_X F \otimes F \circ \psi^j \; dm$.

Equations (27) and (20), together with standard arguments (via the so-called Aaronson-Denker-Nagaev-Guivarch's method: see [3, 22]) ensure CLT for F_n, that is equation (5).

For the proof of the local limit theorem Proposition 2.5, we need to control the spectrum of $\mathcal{L}_u$ for $u \in (\delta, \pi]$. The following lemma gives the required control.

Lemma 2.19. *[15, Lemma C.1] Let $u \in (-\pi, \pi]$, $h \in \mathcal{B}$ and $\eta \in \mathbb{C}$ be such that $\mathcal{L}_u h = \eta h$ in $\mathcal{B}$ and $|\eta| \ge 1$. Then either $h \equiv 0$ or $u \in 2\pi\mathbb{Z}$ and h is m-a.s. constant.*

The proof of Lemma 2.19 goes word for word as [15, Proof of Lemma C.1], except for differences in notations. The differences in the definitions of the norms used in Subsection 2.3 are irrelevant for this argument. Lemma 2.19 ensures that there exists $\delta_0 \in (0, 1)$ so that

$$\|\mathcal{L}_u^n\|_{\mathcal{B}} \le \delta_0^n \quad \text{for all } |u| \ge \delta \text{ and } n \ge 1. \tag{21}$$

2.5 Remaining proofs of Section 2

We first record a technical lemma that will be instrumental in the proof of item (a) of Proposition 2.5. For $m \in \mathbb{Z}$, $n \geq 0$ and $r_1, \dots, r_n \in \mathbb{N}$, define

$$G(m, n, r_1, \dots, r_n) = (y-1)^{-m}(y-\mathcal{L}_0)^{-1}\mathcal{L}_0(F^{r_1}) \cdots (y-\mathcal{L}_0)^{-1}\mathcal{L}_0(F^{r_n})(y-\mathcal{L}_0)^{-1}. \tag{22}$$

As F takes values in $\mathbb{Z}^d$, note that the algebraic powers F^{r_j} need to be interpreted coordinate-wise.

Lemma 2.20. *For all $m, n \geq 0$, $r_1, \dots, r_n \in \mathbb{N}$ and real-valued $v \in \mathcal{B}$, the contour integral*

$$\int_{|y-1|=\delta} G(m, n, r_1, \dots, r_n) v \, dy \quad \textit{is purely imaginary.}$$

Proof. We will use induction on m, n, starting with $n = 0$. If $m = n = 0$, then

$$\int_{|y-1|=\delta} G(0,0) v \, dy = \int_{|y-1|=\delta} (y-\mathcal{L}_0)^{-1} v \, dy = 2\pi i \, \Pi_0 v \quad \text{is purely imaginary.}$$

Now if the statement holds for $n = 0$ and all $0 \leq m' \leq m$, then

$$\begin{aligned}
\int_{|y-1|=\delta} G(m,0) v \, dy &= \int_{|y-1|=\delta} (y-1)^{-m}(y-\mathcal{L}_0)^{-1} v \, dy \\
&= \int_{|y-1|=\delta} (y-1)^{-m}(y-\mathcal{L}_0)^{-1}(v - \int v \, dm) \, dy \\
&\quad + \int_{|y-1|=\delta} (y-1)^{-m}(y-\mathcal{L}_0)^{-1} \int v \, dm \, dy \\
&= \int_{|y-1|=\delta} (y-1)^{-m} \left[(y-\mathcal{L}_0)^{-1} - (1-\mathcal{L}_0)^{-1}\right] (v - \int v \, dm) \, dy \\
&\quad + \int_{|y-1|=\delta} (y-1)^{-m}(1-\mathcal{L}_0)^{-1}(v - \int v \, d\mu) \, dy \\
&\quad + \int v \, d\mu \int_{|y-1|=\delta} (y-1)^{-(m+1)} \, dy.
\end{aligned}$$

The third integral equals $2\pi i \int v \, d\mu$ if $m = 0$ and vanishes otherwise. For the second integral we call $v_1 = (1-\mathcal{L}_0)^{-1}(v - \int v \, d\mu)$; then $v_1 \in \mathcal{B}$ is real with $\int v_1 \, dm = 0$. Hence the integral is purely imaginary for the same reason as the second integral. For the first integral we use the resolvent identity:

$$\begin{aligned}
\int_{|y-1|=\delta} & (y-1)^{-m} \left[(y-\mathcal{L}_0)^{-1} - (1-\mathcal{L}_0)^{-1}\right] (v - \int v \, dm) \, dy \\
&= -\int_{|y-1|=\delta} (y-1)^{1-m}(y-\mathcal{L}_0)^{-1}(1-\mathcal{L}_0)^{-1}(v - \int v \, dm) \, dy \\
&= \int_{|y-1|=\delta} (y-1)^{1-m}(y-\mathcal{L}_0)^{-1} h_1 \, dy = -\int_{|y-1|=\delta} G(m-1, 0) h_1 \, dy.
\end{aligned}$$

This is purely imaginary by the induction hypothesis, except when $m = 0$. When $m = 0$ the factor $y - 1$ removes the simple pole of $G(m-1, 0) = (y - \mathcal{L}_0)^{-1}$, and the integral vanishes by Cauchy's Theorem.

Now we continue with the induction step over n; in this case $G(m, n, r_1, \dots, r_n)$ contains $n+1$ factors $(y-\mathcal{L}_0)^{-1}$, and therefore it has a a pole at 1 of order $\leq n+1$. Our induction hypothesis is that $\int_{|y-1|=\delta} G(m', n', r_1, \dots, r_{n'}) v \, dy$ is purely imaginary for every real-valued $v \in \mathcal{B}$ when $0 \leq n' < n$

and $m' \geq -n$ or when $n' = n$ and $-n \leq m' < m$. Then

$$\begin{aligned}\int_{|y-1|=\delta} G(m,n,r_1,\dots,r_n) v\, dy &= \int_{|y-1|=\delta} G(m,n,r_1,\dots,r_n)(v - \int v\, dm)\, dy \\ &\quad + \int_{|y-1|=\delta} G(m,n-1,r_1,\dots,r_{n-1})\mathcal{L}_0(F^{r_n})(y-\mathcal{L})^{-1} \int v\, dm\, dy.\end{aligned}$$

The second integral is equal to $\int v\, dm \int_{|y-1|=\delta} G(m+1,n-1,r_1,\dots,r_{n-1})\mathcal{L}_0(F^{r_n})\, dy$ and thus purely imaginary by the induction hypothesis. We rewrite the first integral to

$$\begin{aligned}&\int_{|y-1|=\delta} G(m,n-1,r_1,\dots,r_{n-1})\mathcal{L}_0(F^{r_n}) \left[(y-\mathcal{L}_0)^{-1} - (1-\mathcal{L}_0)^{-1}\right](v - \int v\, dm)\, dy \\ &\quad + \int_{|y-1|=\delta} G(m,n-1,r_1,\dots,r_{n-1})\mathcal{L}_0(F^{r_n})(1-\mathcal{L}_0)^{-1}(v - \int v\, dm)\, dy.\end{aligned}$$

Writing $h_2 = \mathcal{L}_0(F^{r_n})(1-\mathcal{L})^{-1}(v - \int v\, dm)$, we get the second $\int_{|y-1|=\delta} G(m,n-1,r_1,\dots,r_{n-1})h_2\, dy$, which is purely imaginary by induction. The resolvent identity applied to the first term gives

$$\begin{aligned}&-\int_{|y-1|=\delta} G(m-1,n-1,r_1,\dots,r_{n-1})\mathcal{L}_0(F^{r_n})(y-\mathcal{L}_0)^{-1}(v - \int v\, dm)\, dy \\ &\quad = -\int_{|y-1|=\delta} G(m-1,n,r_1,\dots,r_n)(v - \int v\, dm)\, dy.\end{aligned}$$

This is purely imaginary by the induction hypothesis. If, however, $m = -n$, then the integrand contains a factor $(y-1)^{n+1}$, which removes the pole (of order $\leq n+1$) of the remaining part of the integrand, and hence Cauchy's Theorem gives again that the integral vanishes. This completes the induction and the entire proof. □

We can now complete

Proof of Proposition 2.5. **Item (a)** Recall that $\Pi_u = \frac{1}{2\pi i}\int_{|y-1|=\delta}(y-\mathcal{L}_u)^{-1}\, dy$ is the eigenprojection w.r.t. the leading eigenvalue. Clearly $\Pi_0 v$ is real for a real $v \in \mathcal{B}$. Taking the k-th derivative w.r.t. u and then evaluated at $u = 0$, gives $2\pi i^{j+1}$ times the contour integral of a linear combination of terms of the form (22). These integrals are all purely imaginary by Lemma 2.20, so the k-th derivative produces alternatingly real and purely imaginary outcomes.

Item (b) From Lemma 2.3 we have

$$\begin{aligned}\mathcal{L}^K v(x,r)1\!\!1_{\{F_K(x)=\ell-r\}} &= \int_{[-\pi,\pi]^d} e^{-iu(\ell-r)}\mathcal{L}_u^K v(x,r)\, du \\ &= \int_{[-\pi,\pi]^d} e^{-iu(\ell-r)}(\lambda_u^K \Pi_u + Q_u^K)v(x,r)\, du. \qquad (23)\end{aligned}$$

The proof is standard, but we recall the main ingredients for completeness.

First, it follows from (a) that $v_0^{(j)}$ is purely imaginary for odd j and real for even j. Thus, $\int_X \left(\sum_{j=1}^\infty \frac{(iu)^j}{j!} F^j\right)\left(\sum_{j=1}^\infty \frac{(iu)^j}{j!} v_0^{(j)}\right) dm$ has the same property. Using this information in (19), we see that the eigenvalue

$$\lambda_u^K = e^{-\frac{1}{2}\langle \Sigma u, \Sigma u\rangle} e^{\sum_{j=3}^\infty \frac{i^j}{j!}\otimes^j (H_j u)}. \qquad (24)$$

for real matrices H_j. By (21), we have $\|\mathcal{L}_u^K\|_{\mathcal{B}} \leq \delta_0^K$ for $|u| \geq \delta$, so that part of the integral can be captured in an operator E_k satisfying $\|E_k v\|_{\mathcal{B}} \leq \|v\|_{C^1(X)}$.

After a change of coordinates $u \mapsto u/2\pi\sqrt{K}$ in (23), the relevant integral reduces to one over $\frac{1}{2\pi\sqrt{K})^d}\int_{[-\delta\sqrt{K},\delta\sqrt{K}]^d}$. This together with (24), (27) and the analyticity of Π_u gives the statement for $v \in \mathcal{B}$. The statement for $v \in C^1$ follows from Lemma 2.16. □

Proofs of Lemma 2.10. **Item A., d=1**. The integrals $I_j = I_j(\sigma, L)$ can be computed via integration by parts, namely for $j \geq 1$ we have (taking into account that integrals over odd real or imaginary parts of the integrand vanish):

$$\begin{aligned} I_j &= \int_{-\infty}^{\infty} u e^{-\frac{\sigma^2}{2}u^2} e^{iLu} u^{j-1}\, du \\ &= \int_{-\infty}^{\infty} \frac{1}{\sigma^2} e^{-\frac{\sigma^2}{2}u^2} e^{iLu} \left(iLu^{j-1} + (j-1)u^{j-2}\right) du = \frac{1}{\sigma^2}(iLI_{j-1} + (j-1)I_{j-2}). \end{aligned}$$

For $j = 1$ we get $I_1 = \frac{iL}{\sigma^2} I_0$ and I_0 is computed via a change of coordinates:

$$\begin{aligned} I_0 &= \int_{-\infty}^{\infty} e^{iLu - \frac{\sigma^2}{2}u^2}\, du = \int_{-\infty}^{\infty} e^{-(\frac{\sigma u}{\sqrt{2}} - \frac{iuL}{\sqrt{2}\sigma})^2} e^{-\frac{L^2}{2\sigma^2}}\, du \\ &= e^{-\frac{L^2}{2\sigma^2}} \frac{\sqrt{2}}{\sigma} \int_{-\infty}^{\infty} e^{-u^2}\, du = \frac{\sqrt{2\pi}}{\sigma} e^{-\frac{L^2}{2\sigma^2}}. \end{aligned}$$

Item B., $d = 2$. Using diagonalization and the change of coordinates $u = Av$ (so $\langle \Sigma u, \Sigma u\rangle = \sigma_1^2 v_1^2 + \sigma_2^2 v_2^2$ and $\langle L, u\rangle = \langle A^* L, v\rangle$), we get

$$\begin{aligned} \vec{I}_0(\Sigma, L) &= \int_{-\infty}^{\infty} e^{-\frac{1}{2}\sigma_1^2 v_1^2} e^{i(A^*L)_1 v_1}\, dv_1 \int_{-\infty}^{\infty} e^{-\frac{1}{2}\sigma_1^2 v_2^2} e^{i(A^*L)_2 v_2}\, dv_2 \cdot A \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= I_0(\sigma_1, (A^*L)_1) \cdot I_0(\sigma_2, (A^*L)_2) \cdot A \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \frac{2\pi}{\sigma_1\sigma_2} e^{-\frac{1}{2}\left(\frac{1}{\sigma_1^2}(A^*L)_1^2 + \frac{1}{\sigma_2^2}(A^*L)_2^2\right)} \begin{pmatrix} A_{11} + A_{12} \\ A_{21} + A_{22} \end{pmatrix}. \end{aligned}$$

Using the same change of coordinates, we get

$$\begin{aligned} \vec{I}_1(\Sigma, L) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\sigma_1^2 v_1^2} e^{i(A^*L)_1 v_1} e^{-\frac{1}{2}\sigma_1^2 v_2^2} e^{i(A^*L)_1 v_2} \cdot A \cdot A^* \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} dv_2\, dv_1 \\ &= \begin{pmatrix} I_1(\sigma_1, (A^*L)_1) \cdot I_0(\sigma_2, (A^*L)_2) \\ I_0(\sigma_1, (A^*L)_1) \cdot I_1(\sigma_2, (A^*L)_2) \end{pmatrix} \\ &= \frac{2\pi i}{\sigma_1\sigma_2} e^{-\frac{1}{2}\left(\frac{1}{\sigma_1^2}(A^*L)_1^2 + \frac{1}{\sigma_2^2}(A^*L)_2^2\right)} \begin{pmatrix} \frac{1}{\sigma_1}(A^*L)_1 \\ \frac{1}{\sigma_2}(A^*L)_2 \end{pmatrix}. \end{aligned}$$

For $\vec{I}_j(\Sigma, L)$, $j \geq 3$, the same methods works, but the computations are getting increasingly lengthy. To explain a bit about $j = 2$, the change of coordinates now leads to the factor

$$A \begin{pmatrix} (A^*v)_1^2 \\ (A^*v)_2^2 \end{pmatrix} = \begin{pmatrix} b_{1,0}v_1^2 + b_{1,1}v_1v_2 + b_{1,2}v_2^2 \\ b_{2,0}v_1^2 + b_{2,1}v_1v_2 + b_{2,2}v_2^2 \end{pmatrix},$$

for $b_{1,0} = A_{11}^3 + A_{12}^3$, $b_{1,1} = 2(A_{11}^2 A_{21} + A_{12}^2 A_{22})$, $b_{1,2} = A_{12}A_{21}^2 + A_{11}A_{22}^2$, $b_{2,0} = A_{21}A_{11}^2 + A_{22}A_{12}^2$, $b_{2,1} = 2(A_{21}^2 A_{11} + A_{22}^2 A_{12})$, and $b_{2,2} = A_{21}^3 + A_{22}^3$. After this decomposition, we can factor the integrals and express them as combinations of $I_j(\sigma_1, (A^*L)_1)$ and $I_j(\sigma_2(A^*L)_2)$:

$$\vec{I}_2(\Sigma, L) = \begin{pmatrix} b_{1,0}I_2(\sigma_1, (A^*L)_1)I_0(\sigma_2, (A^*L)_2) + b_{1,1}I_1(\sigma_1, (A^*L)_1)I_1(\sigma_2, (A^*L)_2) \\ + b_{1,2}I_0(\sigma_1, (A^*L)_1)I_2(\sigma_2, (A^*L)_2) \\ \\ b_{2,0}I_2(\sigma_1, (A^*L)_1)I_0(\sigma_2, (A^*L)_2) + b_{2,1}I_1(\sigma_1, (A^*L)_1)I_1(\sigma_2, (A^*L)_2) \\ + b_{2,2}I_0(\sigma_1, (A^*L)_1)I_2(\sigma_2, (A^*L)_2) \end{pmatrix}.$$

Filling in the 1-dimensional integrals $I_0(\sigma, L) = \frac{\sqrt{2\pi}}{\sigma} e^{-\frac{L^2}{2\sigma^2}}$, $I_1(\sigma, L) = \frac{\sqrt{2\pi}iL}{\sigma^3} e^{-\frac{L^2}{2\sigma^2}}$ and $I_2(\sigma, L) = \frac{\sqrt{2\pi}}{\sigma^3}(1 - \frac{L^2}{\sigma^2})e^{-\frac{L^2}{2\sigma^2}}$, and using the fact that $\frac{1}{2\sigma_1^2}(A^*L)_1^2 + \frac{1}{2\sigma_2^2}(A^*L)_2^2)$ can be rewritten as $\langle \Sigma^{-1}L, \Sigma^{-1}L\rangle$, gives the result. $\square$

3 Abelian covers and homogeneous automorphisms

Let X be a compact translation surface of genus $g \geq 1$, and let $\Sigma \subset X$ be the finite set of singularities and marked points of X, whose cardinality we denote by $\kappa \geq 1$. The relative homology $H_1(X, \Sigma, \mathbb{Z})$ is a free abelian group of rank $2g + \kappa - 1$. The intersection form

$$\langle \cdot, \cdot \rangle : H_1(X, \Sigma, \mathbb{Z}) \times H_1(X \setminus \Sigma, \mathbb{Z}) \to \mathbb{Z}$$

is non-degenerate. Let us consider a homomorphism $\tilde{\zeta} : \pi_1(X \setminus \Sigma) \to \mathbb{Z}^d$. Using the action of the fundamental group π_1 on the universal cover of X, denoted by $\widetilde{X}$, one can associate to the kernel of $\tilde{\zeta}$ a $\mathbb{Z}^d$-cover $\ker \tilde{\zeta} \setminus \widetilde{X}$. Notice that, as $\mathbb{Z}^d$ is abelian, one can factor $\tilde{\zeta}$ by a morphism

$$\zeta : H_1(X \setminus \Sigma, \mathbb{Z}) \to \mathbb{Z}^d.$$

The projection of ζ on each coordinate of $\mathbb{Z}^d$ in the canonical basis defines linear forms on $H_1(X \setminus \Sigma, \mathbb{Z})$. As the intersection form is non-degenerate, there exists a collection of independent primitive loops $\Gamma = \{\gamma_1, \ldots, \gamma_d\} \subset H_1(X, \Sigma, \mathbb{Z})$ such that

$$\zeta(\gamma) = \left(\langle \gamma_1, \gamma \rangle, \ldots, \langle \gamma_d, \gamma \rangle \right).$$

Conversely, such a collection defines a $\mathbb{Z}^d$-cover X_Γ of X with projection $p : X_\Gamma \to X$. We denote its group of deck transformation by Deck, which is isomorphic to $\mathbb{Z}^d$. Thus we can label each element of Deck by $\Delta_{\mathbf{n}}$ with $\mathbf{n} \in \mathbb{Z}^d$.

As the intersection form is non-degenerate, there exists smooth 1-forms ω_i on X that vanish on Σ such that for all $\gamma \in H_1(X \setminus \Sigma, \mathbb{Z})$,

$$\int_\gamma \omega_i = \langle \gamma_i, \gamma \rangle.$$

Notice that the ω_is form a free family of vectors in $H^1(X, \Sigma, \mathbb{Z})$. We denote by $\mathcal{H}(\mathbb{Z})$ the $\mathbb{Z}$-module of $H^1(X, \Sigma, \mathbb{Z})$ generated by these vectors and $\mathcal{H}(\mathbb{R})$ the corresponding real sub-bundle of $H^1(X, \Sigma, \mathbb{R})$. Then the quotient $\mathcal{H}(\mathbb{R})/\mathcal{H}(\mathbb{Z})$ is isomorphic to the d-dimensional torus $\mathbb{T}^d$.

Let $\mathrm{hol} : H_1(X, \Sigma, \mathbb{Z}) \to \mathbb{C}$ denote the holonomy map. As observed in [23], it is a necessary condition for the linear flow to be recurrent in almost every direction on the cover to have

$$\mathrm{hol}(\gamma_i) = 0 \qquad \text{for all } i = 1, \ldots, d.$$

This is commonly called a *no drift condition* and we assume it for the covers we consider. For $\mathbb{Z}$-covers, by [23, Proposition 10], the no drift condition as defined here is equivalent to the corresponding condition in (H3), namely that $\int_X F \,\mathrm{d}m = 0$.

Proposition 3.1. *Let $x_0 \in X_\Gamma$ be fixed. For any $\omega \in \mathcal{H}(\mathbb{R})$ and for all $x \in X_\Gamma$, the integral*

$$\xi_\omega(x) = \int_x^{x_0} \omega \circ p$$

does not depend on the path chosen and is hence a well-defined smooth function on X_Γ.

Proof. It suffices to show that for any loop γ in the cover X_Γ, we have $\int_{[\gamma]} p^*\omega = 0$. By definition of the cover, $p_*[\gamma] = [p \circ \gamma] \in \ker \zeta$, so that

$$\int_{[\gamma]} p^*\omega = \int_{p_*[\gamma]} \omega = 0,$$

which proves the claim. □

Let $\mathcal{F} \subset X_\Gamma$ be a fundamental domain for the cover, namely a compact connected subset of X_Γ, whose boundary $\partial\mathcal{F}$ has measure zero, such that the restriction of the projection $p\colon X_\Gamma \to X$ to the interior of $\mathcal{F}$ is injective and $p(\mathcal{F}) = X$.

Note that $|\xi_\omega(x) - \xi_\omega(y)| \leq 1$ for any $x, y \in \mathcal{F}$, and equality can hold only if $x, y \in \partial\mathcal{F}$.

The orbit of $\mathcal{F}$ under deck transformations tessellates X_Γ, namely, for all $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$,

$$m[\Delta_{\mathbf{m}}(\mathcal{F}) \cap \Delta_{\mathbf{n}}(\mathcal{F})] = \delta_{\mathbf{m}\,\mathbf{n}}, \qquad \text{and} \qquad \bigcup_{\mathbf{n}\in\mathbb{Z}^d} \Delta_{\mathbf{n}}(\mathcal{F}) = X_\Gamma.$$

For a given fundamental domain, one can associate a $\mathbb{Z}^d$-coordinate function from X_Γ minus the points in the orbit of by the deck transformation of $\partial\mathcal{F}$ to $\mathbb{Z}^d$. Which associate to a point $x \in X_\Gamma$ the unique $\mathbf{n} \in \mathbb{Z}^d$ such that $\Delta_{-\mathbf{n}}(\mathcal{F}) \in \mathcal{F}$.

The following proposition shows that the functions ξ_{ω_i} form a smooth version of $\mathbb{Z}^d$-coodinates.

Proposition 3.2. *Let $x_0 \in X_\Gamma$ and $\omega \in \mathcal{H}(\mathbb{R})$, they induce a simply connected fundamental domain*

$$\mathcal{F} = \{x \mid |\xi_{\omega_i}(x)| \leq 1 \text{ for } 1 \leq i \leq d\}$$

and for any $x \in X_\Gamma$ such that x is not in the orbit by the deck transformation of $\partial\mathcal{F}$,

$$x \in \Delta_{\mathbf{n}}(\mathcal{F}) \qquad \text{if and only if} \qquad \lfloor \xi_{\omega_i}(x) \rfloor = n_i \qquad \text{for all } i = 1 \ldots, d.$$

An automorphism of X_Γ is called *homogeneous* if it commutes with all deck transformations. For such a homogeneous automorphism ψ_Γ, and a given $\mathbb{Z}^d$-coordinate ξ, we can associate its *Frobenius* function

$$F(x) := \xi(\psi_\Gamma(x')) - \xi(x')$$

where $x' \in p^{-1}(x)$ and ξ is a $\mathbb{Z}^d$-coordinate for the cover. This is well defined since for all $\mathbf{n} \in \mathbb{Z}^d$, $\psi_\Gamma \circ \Delta_{\mathbf{n}} = \Delta_{\mathbf{n}} \circ \psi_\Gamma$ and $\xi \circ \Delta_{\mathbf{n}} = \xi + \mathbf{n}$ and a change in $\mathbb{Z}^d$-coordinates changes the Frobenius function by a ψ-coboundary.

We then can define the *average drift* of such an automorphism by

$$\delta(\psi_\Gamma) := \int_X F\, dm$$

where m is normalized Lebesgue measure on the surface. It is independent of the choice of $\mathbb{Z}^d$-coordinate and if ϕ_Γ is another homogeneous automorphism, we have

$$\delta(\psi_\Gamma \circ \phi_\Gamma) = \delta(\psi_\Gamma) + \delta(\phi_\Gamma) \tag{25}$$

(see Lemma 2.2 in [6] for details).

3.1 Lifted homogeneous pseudo-Anosov automorphisms

The next proposition determines when we can lift an automorphism on the base to an automorphism on the $\mathbb{Z}^d$-cover.

Proposition 3.3. *Let $\psi : X \to X$ be a linear automorphism which preserves Σ and ψ_* its induced map on $H_1(X \setminus \Sigma, \mathbb{Z})$. If $\zeta \circ \psi_* = \zeta$ then ψ can be lifted to X_Γ and its lift is homogeneous.*

Proof. Recall that, by definition, automorphisms are induced on the universal cover by acting on the set of paths up to homotopy. The condition of the proposition then implies that this induced map can be factored through a quotient by $\ker \widetilde{\zeta}$, leading to the creation of a lifted automorphism on the cover.

Since the cover inherits its flat structure from the pulled-back structure on X, the lift is also linear.

Moreover, not only does ψ_* preserve the kernel of ζ, but it also preserves all homology classes modulo the kernel of ζ. As every deck transformation is determined by the action of an element γ in $\pi_1(X \setminus \Sigma)$ mod $\ker \widetilde{\zeta}$, one simply has to notice that the condition implies $\psi(\gamma) \equiv \gamma$ mod $\ker \widetilde{\zeta}$. □

In a translation surface, a closed orbit in a given direction is contained in a cylinder formed by a union of closed orbits and whose boundary is a union of saddle connections in the surface. The length of those orbits is called the *width* of the cylinder and the distance across the cylinder in the orthogonal direction if called the *height*. The modulus of a cylinder if given by the ratio of width over height.

A direction of a translation surface is called *periodic*[6] if it can be decomposed as a union of cylinders in that direction and those cylinders have commensurable moduli (i.e., their pairwise ratios are rational). Notice that the actual period length of cylinders may be different and even not commensurable.

One feature of commensurable cylinder is that they have a common parabolic matrix acting trivially on them by a Dehn twist.

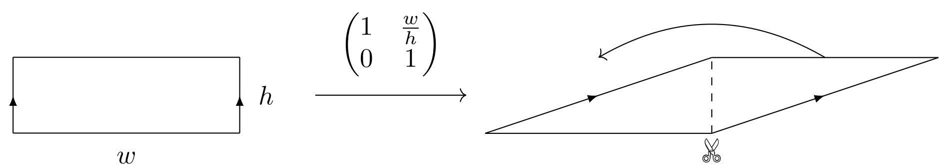

Figure 1: Dehn twist on an horizontal cylinder

Proposition 3.4. *Consider a periodic direction of a translation surfaces X and the associated decomposition in a finite family of cylinders; let $\mu_1, \dots, \mu_n$ be their moduli and let $\eta_1, \dots, \eta_n$ be the homology classes of their core curves. Let $k_1, \dots, k_n \in \mathbb{Z}$ such that $k_1 \cdot \mu_1 = \cdots = k_n \cdot \mu_n$.*
There is a linear parabolic automorphism ψ on X, which acts on homology by the map defined for all $\gamma \in H_1(X \setminus \Sigma,\ \mathbb{R})$ by

$$\psi(\gamma) = \gamma + k_1 \cdot \langle \eta_1, \gamma \rangle \cdot \eta_1 + \cdots + k_n \cdot \langle \eta_n, \gamma \rangle \cdot \eta_n. \tag{26}$$

Proof. Assume that the flow is in the horizontal direction. For a given cylinder let us denote by w, h and $\mu = w/h$ its width, height and modulus. The action of the parabolic matrix $\left(\begin{smallmatrix} 0 & \mu \\ 0 & 1 \end{smallmatrix}\right)$ defines an automorphism on this cylinder which topologically acts as a Dehn twist, meaning that it adds the homology class of the core curve of the cylinder each time a representing loop crosses the core curve positively and subtracts when it crosses it negatively.

By assumption, the moduli $\mu_1, \dots, \mu_n$ of the cylinders decomposing the surface are commensurable. Thus there exist integers $k_1, \dots, k_n$ such that the parabolic matrix

$$\begin{pmatrix} 1 & \mu_1 \\ 0 & 1 \end{pmatrix}^{k_1} = \ \dots \ = \begin{pmatrix} 1 & \mu_n \\ 0 & 1 \end{pmatrix}^{k_n}$$

defines an automorphism on the whole surface. □

In this work, we will be interested in products of Dehn twists in transverse directions in order to produce lifted linear automorphisms which associated matrix is hyperbolic *i.e.* has trace larger than 2. Such automorphisms are extensively studied on compact translation surfaces and are called linear *pseudo-Anosov* maps. They preserve two transverse foliations in their contracting and dilating directions which are the key objects for our renormalization argument.

3.2 Staircases

Let us consider of partition $\mathcal{P}_I$ of the interval I into subintervals $I_1, I_2, \dots, I_n$ and a permutation $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$. Let us identify in the rectangle $I \times [0, 1]$ the vertical sides and each of intervals I_i on top to the $I_{\sigma(i)}$ at bottom of the rectangle.

[6] The name comes from periodicity of the Teichmüller flow in this direction.

This defines a surface X which is topologically a genus g surface with k marked points at the boundaries of the intervals in the partition. The numbers g and k depend on the permutation σ, but always satisfies $2g-1+k=n+1$ and $\dim H_1(X,\Sigma,\mathbb{Z}) = 2g+k-1$. Let $\eta_k \in H_1(X,\Sigma,\mathbb{Z})$ be a relative homology class corresponding to the path along the interval I_k from left to right. We define the classes $\gamma_k = \eta_k - \eta_{\sigma(k)}$ which form a set Γ and we let X_Γ be the corresponding $\mathbb{Z}^d$-cover.

The main example of a compact translation surface X in [6] is a rectangle with side lengths $s \in \mathbb{N}$ and 1, and identifications $(0,x) \sim (s,x)$, $(x,0) \sim (x+s-1,1)$, $(x,1) \sim (x+s-1,0)$, for $x \in [0,1]$ and $(x,0) \sim (x,1)$ for $x \in [1,s-1]$, so $|I_k| = 1$ for al k, $\sigma = (1s)$ and $\Gamma = \{\gamma_4 - \gamma_2\}$. The corresponding $\mathbb{Z}$-covers are called *staircases*.

We formulate the next lemma for the $(s,1)$-staircase, although the adaptations for the more general case are easy to make.

Lemma 3.5. *Let X_Γ be the $\mathbb{Z}$-cover associated to the $(s,1)$-staircase. There exists an homogeneous linear pseudo-Anosov automorphism ψ on X with zero average drift.*

Proof. By Proposition 3.4, the action (from the right on row vectors) of the matrices

$$D_h = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad D_v = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

on the surface define parabolic automorphisms (in fact, Dehn twists) in the horizontal and vertical directions. Indeed, we note that the surface X can be decomposed into the union of two vertical cylinders (of widths 2 and 1 and heights 1 and $s-2$ respectively), or of a single horizontal cylinder (of width s and height 1). The homology class defining the cover can be represented by linear combination of horizontal paths (here $\gamma_4 - \gamma_2$), so the first automorphism preserves them. The second automorphism maps γ_k to $\gamma_k + 2|\gamma_k| \cdot \gamma_1$, where $|\gamma_k|$ is the length of γ_k. Therefore, it maps $\gamma_4 - \gamma_2$ to $(\gamma_4 + 2\gamma_1) - (\gamma_2 + 2\gamma_1) = \gamma_4 - \gamma_2$. Therefore, both D_h and D_v preserve $\gamma_4 - \gamma_2$ and so any hyperbolic composition of these two linear maps defines a linear pseudo-Anosov automorphism which preserves the cover morphism.

Notice moreover that their is a linear involution σ with derivative $-Id$ on X which commutes with Dehn twists and such that $\sigma_*\Gamma = -\Gamma$. Hence, if we consider the Frobenius fonction corresponding to one of these Dehn twist, it satisfies $F \circ \sigma = -F$ and has thus zero average. Which implies by Equation (25) that the Frobenius function of pseudo-Anosov obtained as product of these Dehn twists are also zero average. □

No claim is made that these are all the possible pseudo-Anosov automorphisms with these properties (up to homotopy); for example, we don't consider orientation reversing automorphisms.

3.3 Wind-tree billiards

In a cover, and more generally in any infinite translation surface, a direction is said to have finite horizon if there is no infinite line in that direction. One can generalize the Lemma 3.5 to $\mathbb{Z}^d$-covers with finite horizon.

Lemma 3.6. *If a $\mathbb{Z}^d$-cover X_Γ has finite horizon and is periodic in two distinct directions then there exists a pseudo-Anosov map ψ on X which preserves ζ.*

Proof. Let us a consider a cylinder in X_Γ, its core curve γ is a loop, thus the image of the curve by the projection p must be in $\ker\zeta$. The projection of the cylinder to the base surface is again a cylinder whose core curve homology is an integer multiple of the class $[p_*\gamma]$. By (26), the Dehn twist acts trivially on $H_1(X \setminus \Sigma, \mathbb{Z})$ quotiented by $\ker\zeta$. Hence the Dehn twist on this cylinder preserves ζ.

If the cylinders have commensurable moduli, with common multiple μ, this implies that there exists a linear automorphism $\left(\begin{smallmatrix} 1 & \mu \\ 0 & 1 \end{smallmatrix}\right)$ in the corresponding direction on X which can be lifted to X_Γ. Thus if it is finite horizon and periodic in two distinct directions, the product of the corresponding two matrices is hyperbolic and the composition of the two Dehn twists produces the pseudo-Anosov automorphism we were looking for. ☐

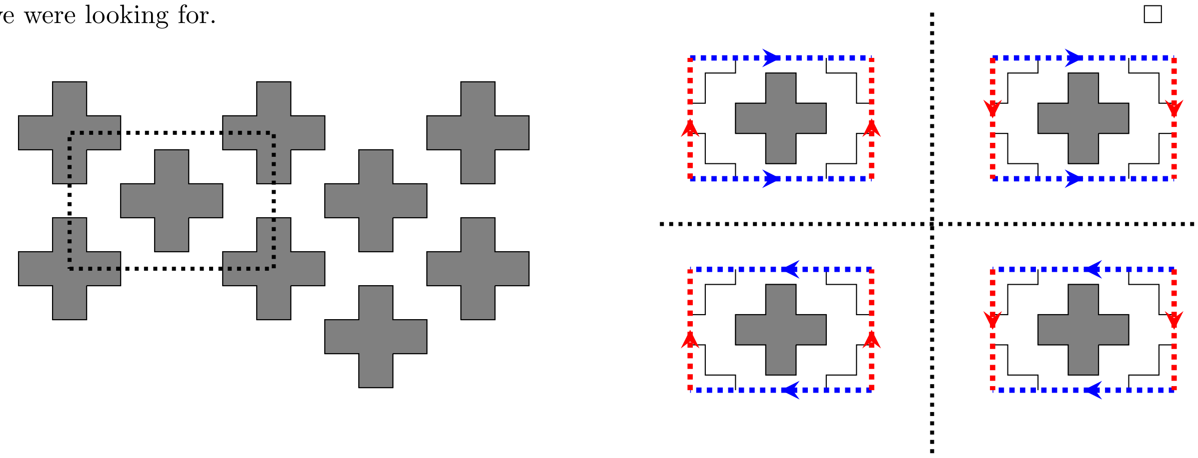

Figure 2: Left: Wind-tree model with plus-shapes wind-tree model (and finite horizon in horizontal and vertical directions). A fundamental domain in dotted lines. Right: Four copies forming a fundamental domain of the folded out translation surface of the plus-shaped wind-tree model.

In the wind-tree model represented in Figure 2, the matrices

$$D_h = \begin{pmatrix} 1 & 12 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad D_v = \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix}$$

act as Dehn twists that commute with deck transformations. These two maps generate a group of automorphisms that contain infinitely many pseudo-Anosov which also commute with deck transformations.

The classical construction to study the flow of wind-tree models is to *fold-out* the billard in the torus into 4 copies. So that each time the flow is bouncing on a side it is translated to the symmetric surface with respect to this flow. The $\mathbb{Z}^2$-cover is thus defined by the homology classes γ_v in red (in Figure 1) for the first coordinate and γ_h in blue for the second. These correspond to γ_1 and γ_2 in the first section.

Notice that for symmetric obstacles, one has two automorphisms τ_h and τ_v of the surface which exchange the two copies horizontally or vertically. Notice that these automorphisms commute with the vertical and horizontal Dehn twists and

$$(\tau_h)_*\gamma_h = \gamma_h, \quad (\tau_v)_*\gamma_v = \gamma_v, \quad (\tau_h)_*\gamma_v = -\gamma_v, \quad (\tau_v)_*\gamma_h = -\gamma_h.$$

Lemma 3.7. *Frobenius function for Dehn twists on symmetric wind-tree models have zero average drift.*

Proof. Consider the horizontal Dehn twist D_h. It preserves the top two copies of the folded out surface and the bottom two. But as the τ_v automorphism sends γ_h to $-\gamma_h$, the Frobenius fonction for D_h satisfies $F \circ \tau_h = -F$. Thus its integral is zero on the surface. ☐

3.4 Ergodic properties

Although the main result of this paper gives rational ergodicity of ϕ_t (with rates), we can use the more classical method of establishing the essential value 1 for a first return map of this flow. This in particular will enable us to prove that the Frobenius function is not a coboundary.

To simplify the exposition, rotationg the coordinate axes, we can consider the vertical flow on a surface endowed with a pseudo-Anosov automorphism contracting the vertical direction and dilating the horizontal.

Let us choose a horizontal segment I in the surface X and $\widetilde{I}$ the union of lifts of I in X_Γ. If $T : I \to I$ and $\widetilde{T} : \widetilde{I} \to \widetilde{I}$ are the first return maps of the vertical linear flow respectively on I and $\widetilde{I}$, then there exists a map $f : I \to \mathbb{Z}^d$ such that we can express $\widetilde{T}$ as a skew-product

$$\widetilde{T}(x, \mathbf{n}) = (T(x), \mathbf{n} + f(x))$$

where $\widetilde{I}$ is identified with $I \times \mathbb{Z}$. Notice that T and $\widetilde{T}$ are ergodic if and only if the linear flow respectively on X and X_Γ are ergodic.

The relevant object to study the orbits by $\widetilde{T}$ is then the induced cocycle for f defined for $k \in \mathbb{Z}$ by

$$f_k(x) = f(x) + \cdots + f(T^{k-1}x).$$

The ergodicity of these two maps can be linked with the following concept.

Definition 3.8. *We call $e \in \mathbb{Z}^d$ is an essential value of (the induced cocycle by) f if for every positive measure set $K \in \mathcal{B}$ there exists $k \in \mathbb{Z}$ such that $K \cap T^{-k}(K) \cap \{x \in X : f_k(x) = e\}$ has positive measure.*

Note that $0 \in \mathbb{Z}^d$ is always an essential value. The set of all finite essential values associated to f is denoted by Ess_f; it forms a subgroup of $\mathbb{Z}^d$ and it follows from [31] that the skew product $\widetilde{T}$ is ergodic if and only if $\mathbb{Z} = \mathrm{Ess}_f$. Also, the map is recurrent if 0 can be obtained as essential value using elements $k \in \mathbb{Z} \setminus \{0\}$ (Lebesgue measure m is infinite on X_Γ, so recurrence does not immediately follow from the invariance of m.)

We represent the given translation surface as zippered rectangles, defining the linear flow as a suspension flow of an interval exchange transformation (see for an introduction [33] from which Figure 3 is taken).

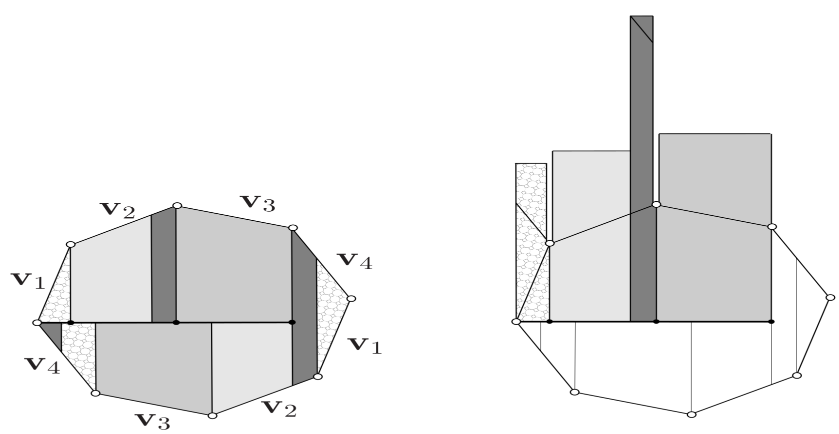

Figure 3: From a suspension flow over an interval exchange transformation to a zippered rectangles representation

This representation defines a fundamental domain which is a union of vertical rectangles and such that the singularity of the surface are in their vertical sides. We choose ξ to be constant in the interior of this domain. This domain has the nice feature that if there is an embedded rectangle in the surfaces X with vertices x, y at the bottom and $\phi^t(x), \phi^t(y)$ at the top then $\xi(x) - \xi(y) = \xi\left(\phi^t(x)\right) - \xi\left(\phi^t(y)\right)$.

Theorem 3.9. *If X_Γ is endowed with a lifted linear pseudo-Anosov automorphism ψ_Γ then the linear flow in the stable and unstable directions of the corresponding matrix is ergodic.*

Proof. To any rectangle in this representation, on can associate a homology class by closing the curve going from bottom to top with a piece of the interval I. These homology classes form a basis of $H_1(X \setminus \Sigma, \mathbb{Z})$ which is dual for the intersection form to the basis of $H_1(X, \Sigma, \mathbb{Z})$ associated to the polygonal representation. Let $e \in \mathbb{Z}^d$ be the associated shift in ξ for the flow from bottom to top of a rectangle; it is equal to the value of ζ on the homology class. Thus such values e generate $\mathbb{Z}^d$ since the homology class of rectangles form a basis of $H_1(X \setminus \Sigma, \mathbb{Z})$ and ζ is surjective, see [32, Section 4.5]. We will now prove that e is an essential value.

It suffices to show that there exists $\delta > 0$ such that for any arbitrary rectangle in X, there exists a δ proportion of points in this rectangle such that $\phi^T(x)$ is in the rectangle and $\xi(\phi^T(x)) - \xi(x) = e$.

Consider an embedded rectangle $\mathcal{R}$ of height T in the surface such that its base is horizontal and for all points x_0 in the base, $\xi\left(\phi^T(x_0)\right) - \xi(x_0) = e$. Up to narrowing the base of $\mathcal{R}$, one can assume that for all $0 \le t \le T$ the flow starting at any point in $\mathcal{R}$ does not meet a singularity. In other words, it is contained in another embedded rectangle of height $2T$.

For all points $x \in \mathcal{R}$ there exists $0 \le t \le T$ such that $x = \phi^t(x_0)$ for some x_0 in the base of $\mathcal{R}$, and thus $\xi\left(\phi^T(x)\right) - \xi(x) = \xi\left(\phi^{T+t}(x_0)\right) - \xi\left(\phi^t(x_0)\right) = e$.

The automorphism sends embedded rectangles to embedded rectangles and dilate the vertical directions by a factor $\lambda^{-1} > 1$. If we take δ to be half the area of $\mathcal{R}$, by unique ergodicity of the flow, for any arbitrary rectangle in X a proportion δ of points intersect the images $\psi^n\mathcal{R}$ for n is large enough. For all these points, we have $\xi(\phi^{\lambda^n T}(x)) - \xi(x) = e$. □

Corollary 3.10. *The Frobenius function F is not a coboundary.*

Proof. Assume by contradiction that F is a coboundary. Take $\mathcal{R}$ a rectangle of positive measure in X_Γ such that $\xi(x) = 0$, for all $x \in \mathcal{R}$. Then $\|\xi(\psi_\Gamma^n x)\| \le \|F\|_\infty$ for all $n \in \mathbb{N}$. The images of ψ_Γ contains larger and larger sections of the flow that remains in a compact set. Thus $\bigcup_{n\in\mathbb{N}} \psi_\Gamma^n \mathcal{R}$ is a compact set of positive measure which is invariant by the flow . Hence the flow is not ergodic. □

4 An alternative approach to twisted transfer operators

In this section, we describe an alternative approach to introducing a twisted transfer operator instead of exploiting Lemma 2.3. The core ideas are indeed the same, but the language is closer to the one used in the beginning of Section 3 and provides a more geometric description of some of the objects considered in Section 2 (e.g., the covariance matrix Σ, see Lemma 4.9 below).

4.1 Hilbert spaces with twists

We define an analogue of the Fourier transform for function on a $\mathbb{Z}^d$-cover. For any $\alpha \in \mathbb{T}^d$, let us define the set

$$L^2(X_\Gamma, \alpha) := \left\{ f : X_\Gamma \to \mathbb{C} : f \circ \Delta_{\mathbf{n}} = e^{-2\pi i \alpha\cdot\mathbf{n}} f \text{ for all } \mathbf{n} \in \mathbb{Z}^d, \text{ and } \int_X |f|^2 \, \mathrm{d}m < \infty \right\}$$

equipped with the inner product $\int_X f \cdot \overline{g} \, \mathrm{d}m$, which turns $L^2(X, \alpha)$ into a Hilbert space. Notice that the inner product is well-defined: if $f, g \colon X_\Gamma \to \mathbb{C}$ satisfy $f \circ \Delta_{\mathbf{n}} = e^{-2\pi i\alpha\cdot\mathbf{n}} f$ and $g \circ \Delta_{\mathbf{n}} = e^{-2\pi i\alpha\cdot\mathbf{n}} g$, then $f \cdot \overline{g}$ is Deck-invariant and hence a well-defined function on X (alternatively, one could take the integral over a fundamental domain $\mathcal{F}$ and observe that the definition does not depend on the choice of $\mathcal{F}$). For any $\ell \ge 0$, we also let

$$\mathcal{C}^\ell(X_\Gamma, \alpha) := L^2(X_\Gamma, \alpha) \cap \mathcal{C}^\ell(X_\Gamma).$$

Let $\mathcal{C}_c^\ell(X_\Gamma)$ denote the space of $\mathcal{C}^\ell$-functions on X_Γ with compact support. For every fixed $\alpha \in \mathbb{T}^d$ and $f \in \mathcal{C}_c^\ell(X_\Gamma)$, we define

$$\pi_\alpha(f)(x) = \sum_{\mathbf{n}\in\mathbb{Z}^d} e^{2\pi i\alpha\cdot\mathbf{n}} \cdot f \circ \Delta_{\mathbf{n}}(x).$$

For every fixed $x \in X_\Gamma$, the sum in the right hand side above is finite since deck transformations act properly discontinuously on X_Γ.

Lemma 4.1. *For every integer $\ell \geq 0$, $\pi_\alpha \colon \mathcal{C}_c^\ell(X_\Gamma) \to \mathcal{C}^\ell(X, \alpha)$. Moreover, for every $f \in \mathcal{C}_c^\ell(X_\Gamma)$ and for any $x \in X_\Gamma$, we have*

$$f(x) = \int_{\mathbb{T}^d} \pi_\alpha(f)(x)\, \mathrm{d}\alpha.$$

Proof. Let $\alpha \in \mathbb{T}^d$ and $f \in \mathcal{C}_c^\ell(X_\Gamma)$. It is clear that $\pi_\alpha(f)$ is a $\mathcal{C}^\ell$-function since locally the sum is finite. There exists a constant $C > 0$ such that $f \circ \Delta_{\mathbf{n}}(x) = 0$ for all $x \in \mathcal{F}$ and $\|\mathbf{n}\| \geq C$ (the choice of the specific norm $\|\cdot\|$ on $\mathbb{Z}^d$ is irrelevant). By the Cauchy-Schwartz inequality, for all $x \in \mathcal{F}$,

$$\begin{aligned}
|\pi_\alpha(f)(x)|^2 &= \left| \sum_{\mathbf{n}\in\mathbb{Z}^d} e^{2\pi i\alpha\cdot\mathbf{n}} \cdot f\circ\Delta_{\mathbf{n}}(x) \frac{\|\mathbf{n}\|^{d+1}}{\|\mathbf{n}\|^{d+1}} \right|^2 \\
&\leq \left(\sum_{\mathbf{n}\in\mathbb{Z}^d} \frac{1}{\|\mathbf{n}\|^{2d+2}} \right) \left(\sum_{\mathbf{n}\in\mathbb{Z}^d} |f|^2 \circ \Delta_{\mathbf{n}}(x) \cdot \|\mathbf{n}\|^{2d+2} \right) \\
&\leq C^{2d+2} \left(\sum_{\mathbf{n}\in\mathbb{Z}^d} \frac{1}{\|\mathbf{n}\|^{2d+2}} \right) \left(\sum_{\mathbf{n}\in\mathbb{Z}^d} |f|^2 \circ \Delta_{\mathbf{n}}(x) \right).
\end{aligned}$$

Note that the first series in the last term above converges. Therefore, up to increasing the constant C, since the orbit of $\mathcal{F}$ under deck transformations tessellates X_Γ, we deduce

$$\int_{\mathcal{F}} |\pi_\alpha(f)|^2 \,\mathrm{d}m \leq C \sum_{\mathbf{n}\in\mathbb{Z}^d} \int_{\mathcal{F}} |f|^2 \circ \Delta_{\mathbf{n}} \,\mathrm{d}m = C \int_{X_\Gamma} |f|^2 \,\mathrm{d}m,$$

which is finite by assumption. Moreover

$$\begin{aligned}
\pi_\alpha(f) \circ \Delta_{\mathbf{m}} &= \sum_{\mathbf{n}\in\mathbb{Z}^d} e^{2\pi i\alpha\cdot\mathbf{n}} \cdot f \circ \Delta_{\mathbf{n}} \circ \Delta_{\mathbf{m}} \\
&= \sum_{\mathbf{n}\in\mathbb{Z}^d} e^{2\pi i\alpha\cdot(\mathbf{n}+\mathbf{m})} \cdot e^{-2\pi i\alpha\cdot\mathbf{m}} \cdot f \circ \Delta_{\mathbf{m}+\mathbf{n}} = e^{-2\pi i\alpha\cdot\mathbf{m}} \cdot \pi_\alpha(f),
\end{aligned}$$

which proves that $\pi_\alpha(f) \in \mathcal{C}^\ell(X, \alpha)$. Finally, since $\int_{\mathbb{T}^d} e^{2\pi i\alpha\cdot\mathbf{n}}\,\mathrm{d}\alpha = \delta_0(\mathbf{n})$, we obtain

$$\int_{\mathbb{T}^d} \pi_\alpha(f)(x)\,\mathrm{d}\alpha = \sum_{\mathbf{n}\in\mathbb{Z}^d} f\circ\Delta_{\mathbf{n}}(x) \int_{\mathbb{T}^d} e^{2\pi i\alpha\cdot\mathbf{n}}\,\mathrm{d}\alpha = f(x).$$

□

Remark 4.2. *It is possible to show that $L^2(X_\Gamma)$ is unitarily equivalent to the direct integral*

$$\int_{\mathbb{T}^d}^{\oplus} L^2(X_\Gamma, \alpha)\,\mathrm{d}\alpha$$

of the Hilbert spaces $L^2(X, \alpha)$. We omit the proof of this fact since we are not going to use it.

By Proposition 3.1, for every $\omega \in \mathcal{H}(\mathbb{R})$, the function $\xi_\omega(x) = \int_x^{x_0} \omega \circ p$ is well defined on X_Γ. Let us define

$$E_\omega(x) = e^{2\pi i \xi_\omega(x)}.$$

Lemma 4.3. *For every $f \in \mathcal{C}_c^\ell(X_\Gamma)$ and $\omega \in \mathcal{H}(\mathbb{R})$, we have*

$$\pi_\alpha(f) = \pi_0(f \cdot E_{-\omega}) \cdot E_\omega,$$

where α is the element of $\mathbb{T}^d$ identified to the class $\omega + \mathcal{H}(\mathbb{R})$ in $\mathcal{H}(\mathbb{R})/\mathcal{H}(\mathbb{Z})$.

This shows that we can write

$$E_\omega(x) = e^{2\pi i \alpha \cdot \xi(x)} \quad \text{for } \xi = (\xi_{\omega_1}, \dots, \xi_{\omega_d}).$$

Proof. Note that $\Delta_{\mathbf{n}}^* p^* \omega = p^* \omega$; therefore

$$\begin{aligned} E_\omega(\Delta_{\mathbf{n}}(x)) &= e^{2\pi i \int_{\Delta_{\mathbf{n}}(x)}^{x_0} p^*\omega} = e^{2\pi i \int_{\Delta_{\mathbf{n}}(x_0)}^{x_0} p^*\omega} \cdot e^{2\pi i \int_{\Delta_{\mathbf{n}}(x)}^{\Delta_{\mathbf{n}}(x_0)} p^*\omega} \\ &= e^{-2\pi i \alpha \cdot \mathbf{n}} \cdot e^{2\pi i \int_x^{x_0} \Delta_{\mathbf{n}}^* p^* \omega} = e^{-2\pi i \alpha\cdot\mathbf{n}} \cdot e^{2\pi i \xi_\omega(x)} = e^{-2\pi i \alpha \cdot \mathbf{n}} \cdot E_\omega(x). \end{aligned}$$

Thus $E_{-\omega} \circ \Delta_{\mathbf{n}} = e^{2\pi i \alpha\cdot\mathbf{n}} \cdot E_{-\omega}$. We can now compute

$$\pi_0(f \cdot E_{-\omega})(x) = \sum_{\mathbf{n}\in\mathbb{Z}^d} (f \cdot E_{-\omega}) \circ \Delta_{\mathbf{n}}(x) = \sum_{\mathbf{n}\in\mathbb{Z}^d} f \circ \Delta_{\mathbf{n}}(x) \cdot e^{2\pi i \alpha\cdot\mathbf{n}} \cdot E_{-\omega}(x) = \pi_\alpha(f) \cdot E_{-\omega}(x).$$

□

Lemma 4.4. *Let $\omega \in \mathcal{H}$, and let $\alpha \in \mathbb{T}^d$ be identified to the class $\omega + \mathcal{H}(\mathbb{Z})$. For any $\eta \in \mathbb{T}^d$, the linear map $f \mapsto f \cdot E_\omega$ is a unitary equivalence between $L^2(X, \eta)$ and $L^2(X_\Gamma, \eta + \alpha)$. Moreover, it is a linear isomorphism between $\mathcal{C}^\ell(X_\Gamma, \eta)$ and $\mathcal{C}^\ell(X_\Gamma, \eta+\alpha)$ for every $\ell \geq 0$.*

Proof. Since $E_\omega \cdot E_{-\omega} = 1$, the map is invertible. It is also clear that

$$\langle f \cdot E_\omega, g \cdot E_\omega \rangle = \int_{\mathcal{F}} f \cdot E_\omega \cdot \overline{g \cdot E_\omega} \, \mathrm{d}m = \int_{\mathcal{F}} f \cdot \overline{g} \, \mathrm{d}m,$$

which shows that it is a unitary operator between the L^2 spaces, moreover the map $f \mapsto f \cdot E_\omega$ is clearly continuous between $\mathcal{C}^0(M, \eta)$ and $\mathcal{C}^0(M, \eta + \alpha)$.

Let us now prove the second claim for $\ell = 1$; the general case $\ell \geq 2$ is left to the reader. Let V be a smooth vector field of unit norm and fix $x \in X_\Gamma$. Then,

$$\begin{aligned} |V(f \cdot E_\omega)(x)| &= |V(f \cdot e^{2\pi i \xi_\omega})(x)| = |Vf(x)| + |f(x)| \cdot 2\pi |V\xi_\omega(x)| \\ &\leq \|Vf\|_\infty + \|f\|_\infty \cdot 2\pi |\omega_x(V)| \leq \|f\|_{\mathcal{C}^1} \cdot (1 + \|\omega\|_\infty). \end{aligned}$$

This shows that $\|f \cdot E_\omega\|_{\mathcal{C}^1} \leq (1 + \|\omega\|_\infty)\|f\|_{\mathcal{C}^1}$ and therefore completes the proof. □

4.2 Twisted transfer operators

We now introduce the action of the linear pseudo-Anosov ψ on X. We recall that the linear flow $(\phi_t)_{t\in\mathbb{R}}$ and the lift ψ_Γ of ψ satisfy the commutation relation (2), i.e., $\psi_\Gamma \circ \phi_t = \phi_{\lambda t} \circ \psi_\Gamma$ for every $t \in \mathbb{R}$, since the flow $(\phi_t)_{t\in\mathbb{R}}$ is in the stable direction of ψ.

Let us define the ergodic averages $A_{x,T}(f)$ of $f \in \mathcal{C}^0(X_\Gamma)$ as

$$A_{x,T}(f) = \frac{1}{T} \int_0^T f \circ \phi_t(x) \, \mathrm{d}t.$$

From (2), we deduce the following result.

Lemma 4.5. *For every $f \in \mathcal{C}^0(X_\Gamma)$, for almost every $x \in X_\Gamma$, for all $T > 0$ and all integers $k \geq 0$, we have*

$$A_{x,T}(f) = A_{\psi_\Gamma^k(x),\lambda^k T}(f \circ \psi_\Gamma^{-k}).$$

Proof. The commutation relation (2) yields

$$\begin{aligned}\frac{1}{T}\int_0^T f \circ \phi_t(x)\,\mathrm{d}t = \frac{1}{T}\int_0^T f \circ \psi_\Gamma^{-k} \circ \psi_\Gamma^k \circ \phi_t(x)\,\mathrm{d}t &= \frac{1}{\lambda^k T}\int_0^T f \circ \psi_\Gamma^{-k} \circ \phi_{\lambda^k r} \circ \psi_\Gamma^k(x)\,\lambda^k\,\mathrm{d}r \\ &= \frac{1}{\lambda^k T}\int_0^{\lambda^k T} f \circ \psi_\Gamma^{-k} \circ \phi_r \circ \psi_\Gamma^k(x)\,\mathrm{d}r.\end{aligned}$$

□

From Proposition 3.3, we can define the operator

$$\mathcal{L}\colon \mathcal{C}^1(X,\alpha) \to \mathcal{C}^1(X,\alpha) \qquad \text{as} \qquad \mathcal{L}(f) = f \circ \psi^{-1},$$

which is a linear isomorphism. We define the dual operator

$$\mathcal{L}^* : \mathcal{C}^1(X_\Gamma,\alpha)^* \to \mathcal{C}^1(X_\Gamma,\alpha)^* \qquad \text{as} \qquad [\mathcal{L}^*(\Phi)](f) = \Phi(\mathcal{L}^{-1}f) \text{ for any } f \in \mathcal{C}^1(X_\Gamma,\alpha).$$

Note that, for any $\alpha \in \mathbb{T}^d$, we have an embedding $\mathcal{C}^1(X_\Gamma,\alpha) \to \mathcal{C}^1(X_\Gamma,\alpha)^*$ given by $f \mapsto \Phi_f$ where $\Phi_f(h) := \int_{\mathcal{F}} h \cdot \bar{f}\,\mathrm{d}m$ is the inner product in $L^2(X,\alpha)$. It is easy to check that $\mathcal{L}\Phi_f^* = \Phi_{\mathcal{L}f}$.

By Lemma 4.4, the spaces $\mathcal{C}^1(X_\Gamma,\alpha)$ are all isomorphic to $\mathcal{C}^1(X_\Gamma,0) \simeq \mathcal{C}^1(X)$. In particular, let ω represent the class α in $\mathcal{H}(\mathbb{R})$, the dual map $I_\omega\colon \mathcal{C}^1(X_\Gamma,\alpha)^* \to \mathcal{C}^1(X)^*$ defined by

$$[I_\omega(\Phi)](f) = \Phi(f \cdot E_\omega)$$

for any $f \in \mathcal{C}^1(X)$, is an isomorphism.

We define $\widehat{\mathcal{L}}_\alpha$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{C}^1(X_\Gamma,\alpha)^* & \xrightarrow{\mathcal{L}^*} & \mathcal{C}^1(X_\Gamma,\alpha)^* \\ \big\downarrow{\scriptstyle I_\omega} & & \big\downarrow{\scriptstyle I_\omega} \\ \mathcal{C}^1(X)^* & \xrightarrow{\widehat{\mathcal{L}}_\alpha} & \mathcal{C}^1(X)^* \end{array}$$

In other words, $\widehat{\mathcal{L}}_\alpha = I_\omega \circ \mathcal{L}^* \circ I_{-\omega}$.

Lemma 4.6. *For any $\Phi \in \mathcal{C}^1(X)^*$, we have $\widehat{\mathcal{L}}_\alpha(\Phi) = \mathcal{L}(e^{2\pi i F_\omega} \cdot \Phi)$, where*

$$F_\omega(x) := \xi_\omega \circ \psi_\Gamma(x) - \xi_\omega(x) = \int_{\psi_\Gamma(x)}^x \omega \circ p.$$

The function F_ω is Deck-*invariant.*

Proof. Let $\Phi \in \mathcal{C}^1(X)^*$ and fix a test function $a \in \mathcal{C}^1(X)$. Let F_ω be defined as in the statement of the lemma. Straightforward computations give us

$$\begin{aligned}[(I_\omega \circ \mathcal{L}^* \circ I_{-\omega})(\Phi)](a) &= [(\mathcal{L} \circ I_{-\omega})(\Phi)](a \cdot E_\omega) = [I_\omega(\Phi)](a \circ \psi_\Gamma \cdot e^{2\pi i \xi_\omega \circ \psi}) \\ &= \Phi\left(a \circ \psi_\Gamma \cdot e^{2\pi i (\xi_\omega \circ \psi_\Gamma - \xi_\omega)}\right) = [e^{2\pi i (\xi_\omega \circ \psi_\Gamma - \xi_\omega)} \cdot \Phi](a \circ \psi) = [\mathcal{L}(e^{2\pi i F_\omega} \cdot \Phi)](a).\end{aligned}$$

Finally, as in the proof of Lemma 4.3, for any $\Delta_{\mathbf{n}} \in$ Deck, we have

$$F_\omega \circ \Delta_{\mathbf{n}}(x) = \int_{\psi(\Delta_{\mathbf{n}}(x))}^{\Delta_{\mathbf{n}}(x)} p^*\omega = \int_{\Delta_{\mathbf{n}}(\psi(x))}^{D(x)} p^*\omega = \int_{\psi(x)}^x p^*\omega = F_\omega(x),$$

which proves the invariance of F_ω by deck transformations. □

Recall that $A_{x,T}(f)$ is the ergodic average at x up to time T of a continuous function f on X_Γ.

Lemma 4.7. *Let $x \in X_\Gamma$ and $T > 0$ be fixed. For every $\alpha \in \mathbb{T}^d$, we have $A_{x,T} \in \mathcal{C}^1(X_\Gamma, \alpha)^*$ and, for every $f \in \mathcal{C}^1_c(X_\Gamma)$, we have*

$$A_{x,T}(f) = \int_{\mathbb{T}^d} A_{x,T}(f_\alpha)\,\mathrm{d}\alpha \quad \text{for } f_\alpha := \pi_\alpha(f).$$

Proof. Fix $f \in \mathcal{C}^1_c(X_\Gamma)$. Then, there exists a constant $C(f)$ such that $\sum_{\mathbf{n}\in\mathbb{Z}^d} |f| \circ \Delta_{\mathbf{n}}(x) \leq C(f)$ for all $x \in X_\Gamma$. This implies that $\|f_\alpha\|_\infty \leq C(f)$ as well. By Lemma 4.1, for any fixed $r \in \mathbb{R}$,

$$f \circ \phi_t(x) = \int_{\mathbb{T}^d} f_\alpha \circ \phi_t(x)\,\mathrm{d}\alpha,$$

and, by the Fubini-Tonelli Theorem,

$$A_{x,T}(f) = \frac{1}{T}\int_0^T \int_{\mathbb{T}^d} f_\alpha \circ \phi_t(x)\,\mathrm{d}\alpha\,\mathrm{d}t = \int_{\mathbb{T}^d} \frac{1}{T}\int_0^T f_\alpha \circ \phi_t(x)\,\mathrm{d}t\,\mathrm{d}\alpha.$$

□

By Lemma 4.5 and Lemma 4.7, for any $k \geq 0$ and $f \in \mathcal{C}^1_c(X_\Gamma)$ we have

$$A_{x,T}(f) = \int_{\mathbb{T}^d} A_{\psi^k(x),\lambda^k T}(\mathcal{L}^k f_\alpha)\,\mathrm{d}\alpha,$$

where $f_\alpha \in C^1(X_\Gamma, \alpha)$. By definition of $\widehat{\mathcal{L}}_\alpha$ and by Lemma 4.3, we obtain

$$\begin{aligned} A_{x,T}(f) &= \int_{\mathbb{T}^d} A_{\psi^k(x),\lambda^k T}\left(\mathcal{L}^k f_\alpha\right)\mathrm{d}\alpha = \int_{\mathbb{T}^d} A_{\psi^k(x),\lambda^k T}\left[\mathcal{L}^k\left(\pi_0(f\cdot E_{-\omega})\cdot E_\omega\right)\right]\mathrm{d}\alpha \\ &= \int_{\mathbb{T}^d} A_{\psi^k(x),\lambda^k T}\left[\widehat{\mathcal{L}}^k_\alpha\left(\pi_0(f\cdot E_{-\omega})\right)\cdot E_\omega\right]\mathrm{d}\alpha. \end{aligned}$$

Let $\xi(x) \in \mathbb{Z}^d$ be such that $x \in \Delta_{\xi(x)}(\mathcal{F})$, as in Proposition 3.2, namely

$$\xi(x) = (\lfloor \xi_{\omega_1}(x)\rfloor, \dots, \lfloor \xi_{\omega_d}(x)\rfloor).$$

Since the flow ϕ_t commutes with the deck-transformations, we have that $A_{x,T}(f) = A_{\Delta_{\mathbf{n}}^{-1}x,T}(f\circ\Delta_{\mathbf{n}})$ for every $\Delta_{\mathbf{n}} \in \mathrm{Deck}$. From this, it follows that

$$A_{\psi^k(x),\lambda^k T}[\widehat{\mathcal{L}}^k_\alpha(\pi_0(f\cdot E_{-\omega}))\cdot E_\omega] = A_{\Delta^{-1}_{\xi\circ\psi^k(x)}\circ\psi^k(x),\lambda^k T}[\widehat{\mathcal{L}}^k_\alpha(\pi_0(f\cdot E_{-\omega}))\cdot E_\omega\circ\Delta_{\xi\circ\psi^k(x)}],$$

where we used the fact that the function $\widehat{\mathcal{L}}^k_\alpha(\pi_0(f\cdot E_{-\omega}))$ is Deck-invariant by definition. The point $\Delta^{-1}_{\xi\circ\psi^k(x)}\circ\psi^k(x)$ is in $\mathcal{F}$ and, as in the proof of Lemma 4.3, we also have

$$E_\omega \circ \Delta_{\xi\circ\psi^k(x)} = e^{-2\pi i\alpha\xi\circ\psi^k(x)} E_\omega.$$

With a little abuse of notation, we will write E_α for E_ω, where $\alpha \in \mathbb{T}^d = \mathcal{H}(\mathbb{R})/\mathcal{H}(\mathbb{Z})$ is the class $\omega + \mathcal{H}(\mathbb{Z})$. We have proved the following result.

Proposition 4.8. *Let $f \in \mathcal{C}^1_c(X_\Gamma)$, $x \in \mathcal{F}$, and $T \geq 1$. Let $k \geq 0$ so that the vector $\xi\circ\psi^k(\phi_t(x)) \in \mathbb{Z}^d$ is constant for all $|t| \leq \lambda^k T$. Then, there exists a point $y = y(x,k) \in \mathcal{F}$ such that*

$$A_{x,T}(f) = \int_{\mathbb{T}^d} e^{-2\pi i\alpha\xi\circ\psi^k(x)} A_{y,\lambda^k T}[\widehat{\mathcal{L}}^k_\alpha(g_\alpha)\cdot E_\alpha]\,\mathrm{d}\alpha,$$

where

$$g_\alpha = \pi_0(f\cdot E_{-\alpha}) \in \mathcal{C}^1(X).$$

4.3 Banach spaces and twisted ergodic averages

We now consider the action of $\widehat{\mathcal{L}}_\alpha$ on the Banach spaces $\mathcal{C}^1(X) \subset \mathcal{B}_w \subset \mathcal{B} \subset \mathcal{C}^1(X)^*$ as defined in Section 2.3. We recall the spectral properties of $\widehat{\mathcal{L}}_\alpha$, described in that section, that we will need. As before, we are working under the assumption that F_ω has zero integral and is not a coboundary.

1. The family of operators $\alpha \mapsto \widehat{\mathcal{L}}_\alpha$ is analytic in α.
2. There exists $\delta > 0$ such that, for all $\alpha \in B(0,\delta) \subset \mathbb{T}^d$, there is a family of simple eigenvalues λ_α, with $\lambda_0 = 1$, which are analytic in α, and a decomposition

$$\widehat{\mathcal{L}}_\alpha^n = \lambda_\alpha^n \Pi_\alpha + Q_\alpha^n, \tag{27}$$

   where

   (a) Π_α is the spectral projection onto the one-dimensional eigenspace associated to λ_α and $\Pi_0 f = \int_X f \, \mathrm{d}m$,

   (b) $\|Q_\alpha^n\| \le \delta_0^n$ for some $\delta_0 \in (0,1)$,

   (c) $Q_\alpha \Pi_\alpha = \Pi_\alpha Q_\alpha = 0$,

   (d) both Π_α and Q_α are analytic in $B(0,\delta)$.

3. The only solutions to the eigenvalue equation $\widehat{\mathcal{L}}_\alpha h = \eta h$, for $h \in \mathcal{B}$ and $\eta \in \mathcal{B}$ with $|\eta| \ge 1$, are $\alpha = 0$ and h constant.

Let $S_n F_\omega(x) = F_\omega(x) + F_\omega \circ \psi(x) + \cdots + F_\omega \circ \psi^{n-1}(x) = \int_{\psi^n(x)}^x p^*\omega$ denote the Birkhoff sums of F_ω. The first and second derivatives of λ_α at $\alpha = 0$ can be computed explicitly.

Lemma 4.9. *For any $\omega, \eta \in \mathcal{H}(\mathbb{R})$, the first and second derivatives of λ_ω at 0 are given by*

$$D_\omega \lambda_0 = 2\pi i \int_X F_\omega \, \mathrm{d}m = 0, \quad \textit{and}$$
$$D_\eta \, D_\omega \lambda_0 = -4\pi^2 \lim_{n\to\infty} \int_X \frac{1}{n} S_n F_\omega \cdot S_n F_\eta \, \mathrm{d}m.$$

In particular, 0 is a stationary point and $(\eta,\omega) \mapsto D_\eta \, D_\omega \lambda_0$ is negative definite.

By Proposition 4.8 we are lead to consider the averages

$$A_{x,T}^\alpha(f) = \frac{1}{T} \int_0^T (f \cdot E_\alpha) \circ \phi_r(x) \, \mathrm{d}r.$$

As discussed in Section 2, we can restrict to the case where $x \in \mathcal{F} = X$ and $T > 0$ are such that the segment $t \mapsto \phi_t(x)$, for $t \in [0,T]$, stretches along an element of the Markov partition.

Lemma 4.10. *Let $x \in \mathcal{F}$ and $T > 0$ be as above. Then, the linear functional $A_{x,T}^\alpha$ defined on $\mathcal{C}^1(X)$ extends to a linear continuous operator on $\mathcal{B}_w$, whose norm is bounded by some constant $C > 0$ independent of T and x.*

Proof. From the definition of the weak norm, for every $f \in \mathcal{C}^1(X)$, we have

$$|A_{x,T}^\alpha(f)| \le C \, \|f\|_{\mathcal{B}_w},$$

where $C = 1 + 2\pi|\alpha|$ is a bound on the Lipschitz norms of E_α. □

Assume that $\alpha \mapsto g_\alpha$ is an analytic family of $\mathcal{C}^1$ functions. Let $\eta \in \mathcal{H}(\mathbb{R})$, with $\|\eta\| = 1$, and let us compute the directional derivative: from Lemma 4.10 we get

$$|D_\eta A^\alpha_{x,T}(g_\alpha)| = |A^\alpha_{x,T}(D_\eta g_\alpha + 2\pi i \xi_\eta \, g_\alpha)| \le C \|(D_\eta g_\alpha + 2\pi i \xi_\eta \, g_\alpha)\|_{\mathcal{B}_w},$$

therefore, up to increasing the constant C, we get

$$|D_\eta A^\alpha_{x,T}(g_\alpha)| \le C \max\{\|D_\eta g_\alpha\|_{\mathcal{B}_w}, \|g_\alpha\|_{\mathcal{B}_w}\}.$$

By induction, we obtain the following result.

Lemma 4.11. *Let $\alpha \mapsto g_\alpha$ be an analytic family on an open neighbourhood U of $0 \in \mathbb{T}^d$. Then, for every $T \ge 1$ and $x \in X$ as above, the function $\alpha \mapsto A^\alpha_{x,T}(g_\alpha)$ is smooth and, for any $\eta_1, \dots, \eta_n \in \mathcal{H}(\mathbb{R})$ with $\|\eta_i\| = 1$, we can bound*

$$|D_{\eta_1} \dots D_{\eta_n} A^\alpha_{x,T,R}(g_\alpha)| \le C \cdot \max_{0 \le j \le n} \|D_{\eta_j} \dots D_{\eta_n} g_\alpha\|_{\mathcal{B}_w},$$

for some constant $C > 0$ independent of T and x.

We conclude with the following consequence of the results we proved so far.

Proposition 4.12. *Let $f \in \mathcal{C}^1_c(X_\Gamma)$, $x \in \mathcal{F}$, and $T \ge 1$. Let $k \ge 0$ so that the vector $\xi \circ \psi^k(\phi_r(x)) \in \mathbb{Z}^d$ is constant for all $|r| \le \lambda^k T$. Then, there exist $\delta > 0$, a neighbourhood $U \subset \mathbb{T}^d$ of 0 and a smooth function $F = F_{f,x,T} \colon U \to \mathbb{C}$ such that*

$$\left| A_{x,T}(f) - \int_U e^{-2\pi i \alpha \xi \circ \psi^k(x) + k \log \lambda_\alpha} F(\alpha) \, \mathrm{d}\alpha \right| \le C(f) \delta^k,$$

for some constant $C(f)$ depending on f only. Moreover, the function F, defined in (28) *below, satisfies*

$$F(0) = \int_{X_\Gamma} f \, \mathrm{d}m, \qquad \text{and} \qquad \|F\|_{\mathcal{C}^N(U)} \le C(f)^N \, \|f\|_{\mathcal{C}^N(X_\Gamma)}.$$

If f is real-valued, then the derivatives $F^{(j)}(0)$ of F at 0 are real if j is even and are purely imaginary if j is odd.

Proof. By Proposition 4.8, we have

$$A_{x,T}(f) = \int_{\mathbb{T}^d} e^{-2\pi i \alpha \xi \circ \psi^k(x)} A_{y,\lambda^k T}[\widehat{\mathcal{L}}^k_\alpha(g_\alpha) \cdot E_\alpha] \, \mathrm{d}\alpha,$$

for some point $y \in \mathcal{F}$, where

$$g_\alpha = \pi_0(f \cdot E_{-\alpha}) \in \mathcal{C}^1(X).$$

Again, as discussed in Section 2, we can restrict to the case where y and $\lambda^k T$ are such that the segment $t \mapsto \phi_t(x)$ stretches along an element of the Markov partition $\mathcal{P}^R$ and, hence, by Lemma 4.10, $A_{y,\lambda^k T}$ is a continuous linear functional on $\mathcal{B}$. Using (27) and the related properties, we deduce that

$$\left| A_{x,T}(f) - \int_U e^{-2\pi i \alpha \xi \circ \psi^k(x)} A^\alpha_{y,\lambda^k T,R}[\lambda^k_\alpha \Pi_\alpha(g_\alpha)] \, \mathrm{d}\alpha \right| \le C(f) \delta^k,$$

for some $\delta \in (0,1)$ and $U \subset B(0, \delta)$. We then define

$$F(\alpha) = F_{f,x,T}(\alpha) = A^\alpha_{y,\lambda^k T}[\Pi_\alpha(g_\alpha)]. \tag{28}$$

Then,

$$F(0) = A^0_{y,\lambda^k T}[\Pi_0(g_0)] = \frac{1}{\lambda^k T} \int_0^{\lambda^k T} \left(\int_X g_0 \, \mathrm{d}m \right) \circ \phi_r(y) \, \mathrm{d}r = \int_X g_0 \, \mathrm{d}m$$
$$= \int_X \pi_0(f \cdot E_0) \, \mathrm{d}m = \int_X \pi_0(f) \, \mathrm{d}m = \int_{X_\Gamma} f \, \mathrm{d}m.$$

The claim on the smoothness of F and on its derivatives follows by Lemma 4.11. In order to compute its derivatives, we notice that $E_0^{(j)}$ is real if j is even and purely imaginary if j is odd, and the same holds for $g_0^{(j)}$, if f is real-valued. The claim on its derivatives at 0 follows then from Proposition 2.5. □

We have now shown that the averages $A_{x,T}(f)$ can be reduced to an integral of the form

$$\int_U e^{-2\pi i\alpha\xi\circ\psi^k(x)+k\log\lambda_\alpha} F(\alpha)\,\mathrm{d}\alpha,$$

for a smooth function F over a neighbourhood $U \subset \mathbb{T}^d$ of 0. This type of integral can be estimated effectively in the same way as we did in Section 2.

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