On asymptotic expansions of ergodic integrals for \mathbb{Z}^d -extensions of translation flows. *

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Abstract

We obtain expansions of ergodic integrals for \mathbb{Z}^d -covers of compact self-similar translation flows, and as a consequence we obtain a form of weak rational ergodicity with optimal rates. As examples, we consider the so-called self-similar (s, 1)-staircase flows (\mathbb{Z} -extensions of self-similar translations flows of genus-2 surfaces), and particular cases of the Ehrenfest wind-tree model.

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1 Introduction

Given a measure preserving flow $(\phi_t)_{t\in\mathbb{R}}$ on a measure space (X,μ) , one is interested in describing the almost everywhere behaviour of its orbits. If the flow is ergodic and if $\mu(X) < \infty$, Birkhoff's Ergodic Theorem states that, for any integrable observable $G: X \to \mathbb{R}$, the time averages $\frac{1}{T} \int_0^T G \circ \phi_t \, dt$ converge almost everywhere to $\frac{1}{\mu(X)} \int_X G \, d\mu$. On the other hand, if $\mu(X) = \infty$, for any ergodic, conservative flow $(\phi_t)_{t\in\mathbb{R}}$, its time averages for integrable functions converge to 0 almost everywhere. The situation does not improve even if we replace $\frac{1}{T}$ with any other normalizing family of functions a(T), see Aaronson [1, Theorem 2.4.2]: for any non-negative integrable function G, either $\lim \inf_{T\to\infty} \frac{1}{a(T)} \int_0^T G \circ \phi_t \, dt = 0$ or $\limsup_{T\to\infty} \frac{1}{a(T)} \int_0^T G \circ \phi_t \, dt = \infty$ almost everywhere. However, one can still hope to describe the almost everywhere behaviour of the ergodic integrals

However, one can still hope to describe the almost everywhere behaviour of the ergodic integrals in some weaker sense. In particular, for an integrable function $G: X \to \mathbb{R}$, we seek an expression of the form

$$\int_0^T G \circ \phi_t(x) \, \mathrm{d}t = a(T) \left(\int_X G \, \mathrm{d}\mu \right) \cdot \Phi_T(x) (1 + o(1)), \tag{1}$$

where a(T) describes the "correct (almost everywhere) size" of the ergodic averages (which, at least for us, is o(T)) and $\Phi_T(x)$ is an "oscillating" term which, although not convergent almost everywhere, converges in some weaker sense (and, crucially, depends only on the point x, not on the function G).

In this paper, we consider a translation flow $(\phi_t)_{t\in\mathbb{R}}$ on a space X_{Γ} which is a \mathbb{Z}^d -cover of a compact translation surface X with projection $p: X_{\Gamma} \to X$. The Lebesgue measure \boldsymbol{m} is infinite on X_{Γ} and

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invariant w.r.t. both the flow ϕ_t and the deck-transformations associated to the cover. Our main result is that, under certain assumptions described below, an expression as (1) holds for all continuous functions $G: X_{\Gamma} \to \mathbb{R}$ with compact support, with $a(T) \sim T(\log T)^{-d/2}$ and where $\sqrt{\log(\phi_T \circ p)}$ converges in distribution to a Gaussian random variable.

Results of this type have been proved by many authors in several settings, including [28] for \mathbb{Z}^d covers of horocycle flows, and [6] for \mathbb{Z} -covers of a translation torus. Furthermore, in [6], the authors used this result to prove temporal limit theorems for circle rotations $R_{\theta} : \mathbb{S}^1 \to \mathbb{S}^1$ for observables with $\int_{\mathbb{S}^1} G \, \mathrm{d} \boldsymbol{m} = 0$ and specific (namely, quadratically irrational) rotation angles θ . This amounts to determining the asymptotics of $\sum_{i=0}^{n-1} G \circ R_{\theta}^i(x)$ for a fixed x and increasing time intervals [0, n]. The crucial idea in their proofs is renormalization, allowing one to speed up a translation flow ϕ_t on a \mathbb{Z} -cover X_{Γ} of a two-dimensional twice punctured torus X in the contracting direction of a linear pseudo-Anosov lift¹ ψ_{Γ} of X_{Γ} according to

$$\psi_{\Gamma} \circ \phi_t = \phi_{\lambda t} \circ \psi_{\Gamma} \quad \text{for every } t \in \mathbb{R},\tag{2}$$

where $\lambda \in (0,1)$ is the contraction factor of ψ_{Γ} . Therefore the asymptotics of ergodic integrals $\int_0^T G \circ \phi_t \, dt$ for compactly supported observables $G : X_{\Gamma} \to \mathbb{R}$ can be estimated using the asymptotics of $\int_{\mathcal{F}} G \circ \psi^k \, \mathrm{d}\boldsymbol{m}$, where \mathcal{F} is a fundamental domain and $T \approx \lambda^{-k}$.

The central result in [6] in our notation is

$$\int_{0}^{T} G \circ \phi_{t}(x) \, \mathrm{d}t = \left(\int_{X_{\Gamma}} G \, \mathrm{d}\boldsymbol{m}\right) \, \frac{(1+o(1))T}{\sigma\sqrt{2K}} \exp\left(-\frac{1+o(1)}{2\sigma^{2}} \frac{\left(\xi \circ \psi_{\Gamma}^{K}(x)\right)^{2}}{K}\right) \quad \text{as } T \to \infty, \quad (3)$$

where $K \sim \log^* T := \left[-\frac{\log T}{\log \lambda} \right]$ and x is such that it has zero average drift under iteration of ψ_{Γ} , and $\xi : X_{\Gamma} \to \mathbb{Z}$ is the projection on the \mathbb{Z} -part of the cover.

In this paper, we extend these results to (i) include higher-order error terms of the asymptotics, making the o(1) terms in (3) explicit, and (ii) allow more general translation surfaces than tori. For instance, we cover a particular case of Ehrenfest's wind-tree model. Our proofs continue to rely on the renormalization formula (2), hence restricting the direction of the translation flow to quadratically irrational slopes, but are on the whole simpler than those of [6], and pertain to \mathbb{Z}^d -covers as well.

Ergodicity of the flow seems to be a non-generic property; there are several results in the literature showing that for \mathbb{Z} - or \mathbb{Z}^d -extensions of many compact translation surfaces, the translation flow in a generic direction is non-ergodic, and even has uncountably many ergodic components, cf. [33, 34, 8, 18, 19]. The landmark result of Fracek and Ulcigrai [18] proves the existence of uncountably many ergodic components for the square wind-tree model and Lebesgue a.e. direction.

In contrast, it was proved in [25] for wind-tree models with square obstacles, and more generally for rational rectangles, that in a dense set of directions (of Hausdorff dimension more than half), the billiard flow is ergodic.

Our result applies in particular to wind-tree models that have two finite-horizon directions, where the corresponding cylinders have commensurable moduli. It was proved in [26] that this is the case for most rational-length obstacles of the rectangular wind-tree model — and the result of [25] mentioned before is a consequence of this property. We generalize ergodicity results to wind-trees of different shapes and to \mathbb{Z}^d -covers for higher d, which has been little studied to our knowledge. In addition, we provide finer ergodicity results in these cases by describing the asymptotic behavior of Birkhoff integrals.

Phrased in dimension d = 1 (but see Theorem 3.3 for the precise formulation for d = 1 and d = 2), our main result reads as follows:

¹By this we mean the lift of a pseudo-Anosov automorphism; as X_{Γ} is not compact, and may have singularities with an infinite or ill-defined cone angle, X_{Γ} may not carry proper pseudo-Anosov automorphisms.

Theorem 1.1. Let $G \in C^1(X_{\Gamma})$ be compactly supported. Then, there exist real bounded functions $g_{k,j}$ so that for all $N \ge 1$ and \mathbf{m} -a.e. $x \in X_{\Gamma}$,

$$\int_0^T G \circ \phi_t(x) \, \mathrm{d}t = \frac{\int_{X_\Gamma} G \, \mathrm{d}\boldsymbol{m}}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{\xi\left(\psi_{\Gamma}^K(x)\right)^2}{2\sigma^{2K}}} \frac{T}{\sqrt{K}} \\ \times \left(1 + \sum_{k=1}^N \frac{1}{K^k} \sum_{j=0}^{2k} g_{k,j}(x) \,\xi(\psi_{\Gamma}^K(x))^{2k-j} + O\left(\frac{1}{K^{N+1}}\right)\right)$$

as $T \to \infty$ and $K = \log^* T = \left\lceil -\frac{\log T}{\log \lambda} \right\rceil$.

The term $\frac{\xi(\psi_{\Gamma}^{K}(x))^{2}}{2\sigma^{2}K}$ is oscillating and does not converge almost everywhere, but after integration over the space, it does lead to a form of weak rational ergodicity for C^{1} observables with optimal rates, see Theorem 3.5. Weak rational ergodicity [2] means that there is a set $\mathcal{F} \subset X_{\Gamma}$ of positive finite measure (possibly but not necessarily a fundamental domain of the \mathbb{Z}^{d} -cover) such that

$$\lim_{T \to \infty} \frac{1}{a_T(\mathcal{F})} \int_0^T \mu(A \cap \phi_t(B)) \, \mathrm{d}t = \mu(A)\mu(B)$$

for all measurable sets $A, B \subset \mathcal{F}$, and $a_T(\mathcal{F}) := \int_0^T \mu(\mathcal{F} \cap \phi_t(\mathcal{F})) dt$ is called the *return sequence*.

The paper is organized as follows. In Section 2 we formalize the concept of \mathbb{Z}^d -cover over a translation surface and study the automorphisms that commute with deck-transformations. We discuss the example of the (s, 1)-staircase at length, which is the direct generalization of the model used in [6] (where s = 2). We give direct proofs of ergodicity of the pseudo-Anosov lift ψ_{Γ} and the translation flow ϕ_t although this also follows from the results of Section 3. Section 2 finishes with a version of the Ehrenfest wind-tree model, as well as an example of the classical Ehrfest wind-tree model with $\frac{1}{2} \times \frac{1}{2}$ squares as obstacles; which is our examples of a \mathbb{Z}^2 -cover where the main theory applies. Section 3 gives the core of the argument in an abstract setup, based on local limit laws of twisted transfer operators \mathcal{L}_u acting on appropriate anisotropic Banach spaces. Finally, in the Appendix we review the tensor calculus we are using, and prove some technical lemma.

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2 Abelian covers and homogeneous automorphisms

A translation surface X is a connected topological surface with an atlas of charts $\phi_i : U_i \to X \setminus \Sigma$, $U_i \subset \mathbb{R}^2$ open and connected, such that each $\phi_j^{-1} \circ \phi_i$ is a translation from $U_i \cap \phi_i^{-1} \circ \phi_j(U_j)$ to its image. Here Σ is a discrete set of conical singularities, i.e., their cone angles differ from 2π (but are still finite) and potentially marked point, see Section 2.1. See [10] for an extensive monograph on translation surfaces, from which we have just paraphrased Definition 1.1.6.

A standard example is a right-angled polygon with pairwise identified sided via translations, see Figure 1 (left).

A \mathbb{Z}^d -cover X_{Γ} of a translation surface is a new (infinite) translation surface with a continuous projection $p: X_{\Gamma} \to X$ such that the quotient space $X_{\Gamma}/X \simeq \mathbb{Z}^d$. Any region \mathcal{F} such that copies by deck-transformations $\{\Delta_n\}_{n \in \mathbb{Z}^d}$ form a partition of X_{Γ} is called a *fundamental domain*. We can depict X_{Γ} as an infinite polygon (the countable union of copies of X) in \mathbb{R}^2 , but with sides identified in a different way from X, see Figure 1; a collection Γ of d independent primitive loops determines how these identifications are done, see Figure 1 (right).



Figure 1: The (3, 1)-rectangle (left) and the (3, 1)-staircase. On the Σ consists of a single point with cone angle 6π ; it splits into countable many singularities on the staircase, all with cone angle 6π .

A pseudo-Anosov diffeomorphism $\psi : X \to X$ is a bijection of X that is a diffeomorphism on $X \setminus \Sigma$, and admits a continuous splitting of the tangent bundle $T(X \setminus \Sigma)$ into a stable foliation $\{E^s(x)\}_{x \in X \setminus \Sigma}$ and an unstable foliation $\{E^u(x)\}_{x \in X \setminus \Sigma}$ such that for all $x \in X \setminus \Sigma$:

- the angle $\angle (E^s(x), E^u(x)) \ge \alpha;$
- $D\psi(x)(E^s(x)) = E^s(\psi(x))$ and $||D\psi^n(x)v|| \le Ce^{-cn}||v||$ for all $v \in E^s(x)$;
- $D\psi^{-1}(x)(E^u(x)) = E^u(\psi^{-1}(x))$ and $\|D\psi^{-n}(x)v\| \le Ce^{-cn}\|v\|$ for all $v \in E^u(x)$.

Here the constants $C, c, \alpha > 0$ are independent of $x \in X \setminus \Sigma$. The set Σ is discrete, and consists of common endpoints of so-called *prongs* of the stable and unstable foliations. In our case, Σ coincides with the set of conical singularities of the translation surface.

The lift ψ_{Γ} of a pseudo-Anosov diffeomorphism ψ is a bijection of the \mathbb{Z}^d -cover X_{Γ} such that $\psi \circ p = p \circ \psi_{\Gamma}$. It isn't fully a pseudo-Anosov diffeomorphism in its own right, partially because the singularities of X_{Γ} can be more complicated than conical singularities, e.g., they could have infinite cone angle, or no proper two-dimensional neighbourhood. However, ψ_{Γ} has enough hyperbolicity for the purpose in this paper.

2.1 Homological properties of \mathbb{Z}^d -covers

Let X be a compact translation surface of genus $g \ge 1$, and let $\Sigma \subset X$ be the finite set of singularities and marked points of X, whose cardinality we denote by $\kappa \ge 1$. The relative homology $H_1(X, \Sigma, \mathbb{Z})$ is a free abelian group of rank $2g + \kappa - 1$. The intersection form

$$\langle \cdot, \cdot \rangle \colon H_1(X, \Sigma, \mathbb{Z}) \times H_1(X \setminus \Sigma, \mathbb{Z}) \to \mathbb{Z}$$

is non-degenerate. Let us consider a homomorphism $\tilde{\zeta} : \pi_1(X \setminus \Sigma) \to \mathbb{Z}^d$. Using the action of the fundamental group π_1 on the universal cover of X, denoted by \tilde{X} , one can associate to the kernel of $\tilde{\zeta}$ a \mathbb{Z}^d -cover ker $\tilde{\zeta} \setminus \tilde{X}$. Notice that, as \mathbb{Z}^d is abelian, one can factor $\tilde{\zeta}$ by a morphism

$$\zeta: H_1(X \setminus \Sigma, \mathbb{Z}) \to \mathbb{Z}^d.$$

The projection of ζ on each coordinate of \mathbb{Z}^d in the canonical basis defines linear forms on $H_1(X \setminus \Sigma, \mathbb{Z})$. As the intersection form is non-degenerate, there exists a collection of independent primitive loops $\Gamma = \{\gamma_1, \ldots, \gamma_d\} \subset H_1(X, \Sigma, \mathbb{Z})$ such that

$$\zeta(\gamma) = (\langle \gamma_1, \gamma \rangle, \dots, \langle \gamma_d, \gamma \rangle).$$



Figure 2: The 0th step and its image under ψ_{Γ} with matrix $\begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}$.

Conversely, such a collection defines a \mathbb{Z}^d -cover X_{Γ} of X with projection $p: X_{\Gamma} \to X$. We denote its group of deck transformation by Deck, which is isomorphic to \mathbb{Z}^d . Thus we can label each element of Deck by $\Delta_{\mathbf{n}}$ with $\mathbf{n} \in \mathbb{Z}^d$.

As the intersection form is non-degenerate, there exists smooth 1-forms ω_i on X that vanish on Σ such that for all $\gamma \in H_1(X \setminus \Sigma, \mathbb{Z})$,

$$\int_{\gamma} \omega_i = \left< \gamma_i, \gamma \right>.$$

Notice that the ω_i s form a free family of vectors in $H^1(X, \Sigma, \mathbb{Z})$. We denote by $\mathcal{H}(\mathbb{Z})$ the \mathbb{Z} -module of $H^1(X, \Sigma, \mathbb{Z})$ generated by these vectors and $\mathcal{H}(\mathbb{R})$ the corresponding real sub-bundle of $H^1(X, \Sigma, \mathbb{R})$. Then the quotient $\mathcal{H}(\mathbb{R})/\mathcal{H}(\mathbb{Z})$ is isomorphic to the *d*-dimensional torus \mathbb{T}^d .

Let hol : $H_1(X, \Sigma, \mathbb{Z}) \to \mathbb{C}$ denote the holonomy map. As observed in [24], it is a necessary condition for the linear flow to be recurrent in almost every direction on the cover to have

$$hol(\gamma_i) = 0$$
 for all $i = 1, \dots, d$

This is commonly called a *no drift condition* and we assume it for the covers we consider. For \mathbb{Z} -covers, by [24, Proposition 10], the no drift condition as defined here is equivalent to the corresponding condition in (H3), namely that $\int_X F \, dm = 0$.

Proposition 2.1. Let $x_0 \in X_{\Gamma}$ be fixed. For any $\omega \in \mathcal{H}(\mathbb{R})$ and for all $x \in X_{\Gamma}$, the integral

$$\xi_{\omega}(x) = \int_{x}^{x_0} \omega \circ p$$

does not depend on the path chosen and is hence a well-defined smooth function on X_{Γ} .

Proof. It suffices to show that for any loop γ in the cover X_{Γ} , we have $\int_{[\gamma]} p^* \omega = 0$. By definition of the cover, $p_*[\gamma] = [p \circ \gamma] \in \ker \zeta$, so that

$$\int_{[\gamma]} p^* \omega = \int_{p_*[\gamma]} \omega = 0$$

which proves the claim.

Let $\mathcal{F} \subset X_{\Gamma}$ be a fundamental domain for the cover, namely a compact connected subset of X_{Γ} , whose boundary $\partial \mathcal{F}$ has measure zero, such that the restriction of the projection $p: X_{\Gamma} \to X$ to the interior of \mathcal{F} is injective and $p(\mathcal{F}) = X$.

Note that $|\xi_{\omega}(x) - \xi_{\omega}(y)| \leq 1$ for any $x, y \in \mathcal{F}$, and equality can hold only if $x, y \in \partial \mathcal{F}$.

The orbit of \mathcal{F} under deck transformations tessellates X_{Γ} , namely, for all $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$,

$$m[\Delta_{\mathbf{m}}(\mathcal{F}) \cap \Delta_{\mathbf{n}}(\mathcal{F})] = \delta_{\mathbf{m}\,\mathbf{n}}, \quad \text{and} \quad \bigcup_{\mathbf{n}\in\mathbb{Z}^d} \Delta_{\mathbf{n}}(\mathcal{F}) = X_{\Gamma}.$$

For a given fundamental domain, one can associate a \mathbb{Z}^d -coordinate function from $X_{\Gamma} \setminus \bigcup_{\mathbf{n} \in \mathbb{Z}^d} \partial \mathcal{F}$ to \mathbb{Z}^d , which associates to a point $x \in X_{\Gamma}$ the unique $\mathbf{n} \in \mathbb{Z}^d$ such that $\Delta_{-\mathbf{n}}(\mathcal{F}) \in \mathcal{F}$.

The following proposition shows that the functions ξ_{ω_i} form a smooth version of \mathbb{Z}^d -coordinates.

Proposition 2.2. Let $x_0 \in X_{\Gamma}$ and $\omega \in \mathcal{H}(\mathbb{R})$, they induce a simply connected fundamental domain

$$\mathcal{F} = \{ x : |\xi_{\omega_i}(x)| \le 1 \text{ for } 1 \le i \le d \}$$

and for any $x \in X_{\Gamma}$ such that x is not in the orbit by the deck transformation of $\partial \mathcal{F}$,

 $x \in \Delta_{\mathbf{n}}(\mathcal{F})$ if and only if $\lfloor \xi_{\omega_i}(x) \rfloor = n_i$ for all i = 1..., d.

An automorphism of X_{Γ} is called *homogeneous* if it commutes with all deck transformations. For such a homogeneous automorphism ψ_{Γ} , and a given \mathbb{Z}^d -coordinate ξ , we can associate its *Frobenius* function

$$F(x) := \xi(\psi_{\Gamma}(x')) - \xi(x')$$

where $x' \in p^{-1}(x)$ and ξ is a \mathbb{Z}^d -coordinate for the cover. This is well defined since for all $\mathbf{n} \in \mathbb{Z}^d$, $\psi_{\Gamma} \circ \Delta_{\mathbf{n}} = \Delta_{\mathbf{n}} \circ \psi_{\Gamma}$ and $\xi \circ \Delta_{\mathbf{n}} = \xi + \mathbf{n}$ and a change in \mathbb{Z}^d -coordinates changes the Frobenius function by a ψ -coboundary.

We then can define the *average drift* of such an automorphism by

$$\delta(\psi_{\Gamma}) := \int_X F \, d\boldsymbol{m}$$

where \boldsymbol{m} is normalized Lebesgue measure on the surface. It is independent of the choice of \mathbb{Z}^d coordinates and if ϕ_{Γ} is another homogeneous automorphism, we have

$$\delta(\psi_{\Gamma} \circ \phi_{\Gamma}) = \delta(\psi_{\Gamma}) + \delta(\phi_{\Gamma}), \tag{4}$$

see [6, Lemma 2.2] for details.

2.2 Homogeneous pseudo-Anosov lifts

The next proposition determines when we can lift an automorphism on the base to an automorphism on the \mathbb{Z}^d -cover.

Proposition 2.3. Let $\psi : X \to X$ be a linear automorphism which preserves Σ and ψ_* its induced map on $H_1(X \setminus \Sigma, \mathbb{Z})$. If $\zeta \circ \psi_* = \zeta$ then ψ can be lifted to X_{Γ} and its lift is homogeneous.

Proof. Recall that, by definition, automorphisms are induced on the universal cover by acting on the set of paths up to homotopy. The condition of the proposition then implies that this induced map can be factored through a quotient by ker $\tilde{\zeta}$, leading to the creation of a lifted automorphism on the cover. Since the cover inherits its flat structure from the pulled-back structure on X, the lift is also linear.

Moreover, not only does ψ_* preserve the kernel of ζ , but it also preserves all homology classes modulo the kernel of ζ . As every deck transformation is determined by the action of an element γ in $\pi_1(X \setminus \Sigma)$ mod ker $\tilde{\zeta}$, one simply has to notice that the condition implies $\psi(\gamma) \equiv \gamma \mod \ker \tilde{\zeta}$. \Box

In a translation surface, a closed orbit in a given direction is contained in a cylinder formed by a union of closed orbits and whose boundary is a union of saddle connections in the surface. The length of those orbits is called the *width* of the cylinder and the distance across the cylinder in the orthogonal direction if called the *height*. The modulus of a cylinder if given by the ratio of width over height.

A direction of a translation surface is called $periodic^2$ if it can be decomposed as a union of cylinders in that direction and those cylinders have commensurable moduli (i.e., their pairwise ratios are rational). Notice that the actual period length of cylinders may be different and even not commensurable.

One feature of commensurable cylinder is that they have a common parabolic matrix acting trivially on them by a Dehn twist.



Figure 3: Dehn twist on an horizontal cylinder

Proposition 2.4. Consider a periodic direction of a translation surfaces X and the associated decomposition in a finite family of cylinders; let μ_1, \ldots, μ_n be their moduli and let η_1, \ldots, η_n be the homology classes of their core curves. Let $k_1, \ldots, k_n \in \mathbb{Z}$ such that $k_1 \cdot \mu_1 = \cdots = k_n \cdot \mu_n$.

There is a linear parabolic automorphism ψ on X, which acts on homology by the map defined for all $\gamma \in H_1(X \setminus \Sigma, \mathbb{R})$ by

$$\psi(\gamma) = \gamma + k_1 \cdot \langle \eta_1, \gamma \rangle \cdot \eta_1 + \dots + k_n \cdot \langle \eta_n, \gamma \rangle \cdot \eta_n.$$
(5)

Proof. Assume that the flow is in the horizontal direction. For a given cylinder let us denote by w, h and $\mu = w/h$ its width, height and modulus. The action of the parabolic matrix $\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$ defines an automorphism on this cylinder which topologically acts as a Dehn twist, meaning that it adds the homology class of the core curve of the cylinder each time a representing loop crosses the core curve positively and subtracts when it crosses it negatively.

By assumption, the moduli μ_1, \ldots, μ_n of the cylinders decomposing the surface are commensurable. Thus there exist integers k_1, \ldots, k_n such that the parabolic matrix

$$\begin{pmatrix} 1 & \mu_1 \\ 0 & 1 \end{pmatrix}^{k_1} = \dots = \begin{pmatrix} 1 & \mu_n \\ 0 & 1 \end{pmatrix}^{k_n}$$

²The name comes from periodicity of the Teichmüller flow in this direction.

defines an automorphism on the whole surface.

In this work, we will be interested in products of Dehn twists in transverse directions in order to produce lifted linear automorphisms which associated matrix is hyperbolic *i.e.* has trace larger than 2. Such automorphisms are extensively studied on compact translation surfaces and are called linear *pseudo-Anosov* maps. They preserve two transverse foliations in their contracting and expanding directions.

2.3 Staircases

Let us consider a partition \mathcal{P}_I of the interval I into subintervals I_1, I_2, \ldots, I_n and a permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$. Let us identify the vertical sides of the rectangle $I \times [0, 1]$ and, for all $1 \leq i \leq n$, interval I_i at the top with interval $I_{\sigma(i)}$ at the bottom of the rectangle.

This defines a genus g surface X with k marked points at the boundaries of the intervals in the partition. The numbers g and k depend on the permutation σ , but always satisfy dim $H_1(X, \Sigma, \mathbb{Z}) = 2g + k - 1 = n + 1$. For $1 \le i \le n$, let $\eta_i \in H_1(X, \Sigma, \mathbb{Z})$ be a relative homology class corresponding to the path along the interval I_i from left to right. We define the classes $\gamma_i = \eta_i - \eta_{\sigma(i)}$; they form a set Γ and let X_{Γ} be the corresponding \mathbb{Z}^d -cover.

An important example of compact translation surface cover in [6] and [18] is given by a rectangle with horizontal and vertical side length $s \in \mathbb{N}$ and 1 respectively, and identifications $(0, x) \sim (s, x)$, $(x, 0) \sim (x + s - 1, 1), (x, 1) \sim (x + s - 1, 0)$, for $x \in [0, 1]$ and $(x, 0) \sim (x, 1)$ for $x \in [1, s - 1]$. It is a particular case of the above setting, with n = 3, $|I_1| = |I_3| = 1$, $|I_2| = s - 2$, $\sigma = (13)$ and $\Gamma = \{\gamma_3 - \gamma_1\}$. The corresponding \mathbb{Z} -cover is called the (s, 1)-staircase.

We formulate the next lemma for this set of examples, although this should be possible to adapt this argument for general cases.

Lemma 2.5. Let X_{Γ} be the \mathbb{Z} -cover associated to the (s, 1)-staircase. There exists an homogeneous linear pseudo-Anosov automorphism ψ on X with zero average drift.

Proof. By Proposition 2.4, the action (from the right on row vectors) of the matrices

$$D_h = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$
 and $D_v = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

on the surface define parabolic automorphisms (in fact, Dehn twists) in the horizontal and vertical directions. Indeed, we note that the surface X can be decomposed into the union of two vertical cylinders (of widths 2 and 1 and heights 1 and s-2 respectively), or of a single horizontal cylinder (of width s and height 1). The homology class defining the cover can be represented by linear combination of horizontal paths (here $\gamma_3 - \gamma_1$), so the first automorphism preserves them. If η is the homology class corresponding to the vertical side, the second automorphism maps γ_k to $\gamma_k + 2 \cdot \eta$, for k = 1 or 3. Therefore, it maps $\gamma_3 - \gamma_1$ to itself and both D_h and D_v preserve $\gamma_3 - \gamma_1$ and so any hyperbolic composition of these two linear maps defines a linear pseudo-Anosov lift which preserves the cover morphism.

Notice moreover that their is a linear involution σ with derivative -Id on X which commutes with Dehn twists and such that $\sigma_*\Gamma = -\Gamma$. Hence, if we consider the Frobenius function corresponding to one of these Dehn twists, it satisfies $F \circ \sigma = -F$ and has thus zero average. This implies, by Equation (4), that the Frobenius function of a pseudo-Anosov automorphisms obtained as product of such Dehn twists also has zero average.

No claim is made that these are all the possible pseudo-Anosov automorphisms with these properties (up to homotopy); for example, there is an extra automorphism that turns the (s, 1)-staircase by π .

2.4 Wind-tree billiards

In a cover, and more generally in any infinite translation surface, a direction is said to have finite horizon if there is no infinite line in that direction. One can generalize the Lemma 2.5 to \mathbb{Z}^d -covers with finite horizon.

Lemma 2.6. If a \mathbb{Z}^d -cover X_{Γ} has finite horizon and is periodic in two distinct directions then there exists a pseudo-Anosov map ψ on X which preserves ζ .

Proof. Let us a consider a cylinder in X_{Γ} , its core curve γ is a loop, thus the image of the curve by the projection p must be in ker ζ . The projection of the cylinder to the base surface is again a cylinder whose core curve homology is an integer multiple of the class $[p_*\gamma]$. By (5), the Dehn twist acts trivially on $H_1(X \setminus \Sigma, \mathbb{Z})$ quotiented by ker ζ . Hence the Dehn twist on this cylinder preserves ζ .

If the cylinders have commensurable moduli, with common multiple μ , this implies that there exists a linear automorphism $\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$ in the corresponding direction on X which can be lifted to X_{Γ} . Thus if it is finite horizon and periodic in two distinct directions, the product of the corresponding two matrices is hyperbolic and the composition of the two Dehn twists produces the pseudo-Anosov automorphism we were looking for.

2.4.1 The wind-tree model with plus-shaped obstacles

For illustration, we present a variation of the Ehrenfest wind-tree model as an example of a \mathbb{Z}^2 -cover with finite horizon in vertical and horizontal directions. Whereas such directions for rectangular wind-tree is less straightforward to illustrate.



Figure 4: Left: Wind-tree model with plus-shapes wind-tree model (and finite horizon in horizontal and vertical directions). A fundamental domain in dotted lines. Right: Four copies forming a fundamental domain of the unfolded translation surface of the plus-shaped wind-tree model.

In the wind-tree model represented in Figure 4, the matrices

$$D_h = \begin{pmatrix} 1 & 12\\ 0 & 1 \end{pmatrix}$$
 and $D_v = \begin{pmatrix} 1 & 0\\ 6 & 1 \end{pmatrix}$

act as Dehn twists that commute with deck transformations. These two maps generate a group of automorphisms that contain infinitely many pseudo-Anosov which also commute with deck transformations.

The classical construction to study the flow of wind-tree models is to *unfolded* the billiard in the torus into 4 copies. So that each time the flow is bouncing on a side it is translated to the symmetric

surface with respect to this flow. The \mathbb{Z}^2 -cover is thus defined by the homology classes γ_v in red (in Figure 3) for the first coordinate and γ_h in blue for the second. These correspond to γ_1 and γ_2 in the first section.

Notice that for symmetric obstacles, one has two automorphisms τ_h and τ_v of the surface which exchange the two copies horizontally or vertically. Notice that these automorphisms commute with the vertical and horizontal Dehn twists and

$$(\tau_h)_*\gamma_h = \gamma_h, \quad (\tau_v)_*\gamma_v = \gamma_v, \quad (\tau_h)_*\gamma_v = -\gamma_v, \quad (\tau_v)_*\gamma_h = -\gamma_h.$$

Lemma 2.7. The Frobenius function for the Dehn twists on the plus-shaped wind-tree models have zero average drift.

Proof. Consider the horizontal Dehn twist D_h . It preserves the top two copies of the unfolded surface in Figure 4 (right) and also the bottom two. But as the τ_v automorphism sends γ_h to $-\gamma_h$, the Frobenius function for D_h satisfies $F \circ \tau_h = -F$. Thus its integral on the surface is zero. The argument for the vertical Dehn twist D_v is the same; it preserves the left two copies of the unfolded surface and also the right two.

2.4.2 The classical wind-tree model

In the classical Ehrenfest wind-tree model, the obstacles are $a \times b$ -rectangles centered and aligned with the lattice \mathbb{Z}^2 . Let us take $a = b = \frac{1}{2}$, see Figure 4. All lines in north-east and north-west direction have finite horizon. We can take cylinders in those direction, which (when lifted to the unfolding) have width $\frac{1}{4}\sqrt{2}$ and length $3\sqrt{2}$. Two of them cover the unfolded fundamental domain. When lifted to the \mathbb{Z}^2 -cover, they are still cylinders (i.e., they don't break up into strips, see Figure 4, and therefore the appropriate Dehn twists lift to the cover.

As can be seen from Figure 5, the unfolded fundamental domain has four singularities, with cone angle 6π . A Dehn twist that shears by twelve units, maps these two cylinders back to themselves in a way that that extends continuously over both cylinders. The affine parts of these Dehn twists, in the north-east and north-west directions, are represented by the matrices

$$\begin{pmatrix} 1 + \frac{t}{2} & -\frac{t}{2} \\ \frac{t}{2} & 1 - \frac{t}{2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 + \frac{t}{2} & \frac{t}{2} \\ -\frac{t}{2} & 1 - \frac{t}{2} \end{pmatrix}, \quad t = 12.$$

We now describe the action of the north-east and of the north-west Dehn twists D_{ne} and D_{nw} on the relative homology $H_1(X, \Sigma, \mathbb{Z})$ as follows: let us fix the basis $\gamma_1, \ldots, \gamma_{13}$ as in Figure 5 (note that the rank of $H_1(X, \Sigma, \mathbb{Z})$ is $2 \cdot 5 + 4 - 1 = 13$, where 5 is the genus of X and 4 is the number of singularities). With respect to this basis, the cover is generated by the pair $\Gamma = \{-\gamma_3 + \gamma_7 + \gamma_9 - \gamma_{13}, -\gamma_4 - \gamma_6 + \gamma_{10} + \gamma_{12}\}$. We define

$$\gamma_{ne} := \gamma_1 + \gamma_3 + \gamma_4 + \gamma_5 + \gamma_6 + \gamma_7 + \gamma_8 + \gamma_9 + \gamma_{10} + \gamma_{11} + \gamma_{12} + \gamma_{13}, \quad \text{and} \\ \gamma_{nw} := \gamma_1 + 6\gamma_2 - \gamma_3 + \gamma_4 + 5\gamma_5 + \gamma_6 - \gamma_7 + \gamma_8 - \gamma_9 + \gamma_{10} - \gamma_{11} + \gamma_{12} - \gamma_{13}.$$

The action of the north-east Dehn-twist on $H_1(X, \Sigma, \mathbb{Z})$ is given by (see as in Figure 6)

$$\begin{cases} \gamma_i \mapsto \gamma_i + \gamma_{ne} & \text{for } i \in \{1, 2, 4, 6, 8, 10, 12\}, \\ \gamma_j \mapsto \gamma_j - \gamma_{ne} & \text{for } j \in \{3, 5, 7, 9, 11, 13\}, \end{cases}$$



Figure 5: A fundamental domain for the surface X. The four colored dots represents the four singularities in Σ , each of angle 6π . The elements $\gamma_1, \ldots, \gamma_{13} \in H(X, \Sigma, \mathbb{Z})$ form a basis of the relative homology: the homology class of any oriented path connecting two elements of Σ can be written as a linear combination with integer coefficients of $\gamma_1, \ldots, \gamma_{13}$ (for example, the path marked as η_1 corresponds to the homology class $\gamma_2 + \gamma_4 + \gamma_5$, since the concatenation of the paths $\gamma_2, \gamma_4, \gamma_5$ and $-\eta_1$ bounds a disk and hence is trivial in homology).

so that

The north-west Dehn-twist acts on $H_1(X, \Sigma, \mathbb{Z})$ as

$$\begin{cases} \gamma_i \mapsto \gamma_i + \gamma_{nw} & \text{for } i \neq 2, \\ \gamma_2 \mapsto \gamma_2 - \gamma_{nw}, \end{cases}$$



Figure 6: Two copies of the fundamental domain for the surface X with a cylinder decomposition for the north-east Dehn twist D_{ne} (left) and for the north-west Dehn twist D_{nw} (right). The closed paths in red and green on the left correspond to the same element $\gamma_{ne} \in H_1(X, \Sigma, \mathbb{Z})$; similarly, the closed paths in blue and orange on the right correspond to the same element $\gamma_{nw} \in H_1(X, \Sigma, \mathbb{Z})$.

so that

Lemma 2.8. The Frobenius functions for the Dehn twists D_{ne} and D_{nw} have zero average drift.

Proof. The Dehn twist D_{ne} preserves the two north-east cylinders in the unfolded surface in Figure 6 (left), and each of these cylinders cover half of the area of each quarter of the unfolded surface. The horizontal and vertical automorphism exchanging these quarter act on the horizontal and vertical component of the Frobenius function as:

$$\begin{cases} (F \circ \tau_h)_h = -F_h \\ (F \circ \tau_h)_v = F_v \end{cases} \quad \text{and} \quad \begin{cases} (F \circ \tau_v)_h = F_h \\ (F \circ \tau_v)_v = -F_v \end{cases}$$

Thus its integral on the surface is zero. The argument for the north-west Dehn twist D_v is the same; it preserves the two north-west cylinders in the unfolded surface in Figure 6 (right).

2.5 Ergodic properties

Although the main result of this paper gives rational ergodicity of ϕ_t (with rates), we can use the more classical method of establishing the essential value 1 for a first return map of this flow. This in particular will enable us to prove that the Frobenius function is not a coboundary.

To simplify the exposition, rotating the coordinate axes, we can consider the vertical flow on a surface endowed with a pseudo-Anosov automorphism contracting the vertical direction and expanding the horizontal.

Let us choose a horizontal segment I in the surface X and \tilde{I} the union of lifts of I in X_{Γ} . If $T: I \to I$ and $\tilde{T}: \tilde{I} \to \tilde{I}$ are the first return maps of the vertical linear flow respectively on I and \tilde{I} , then there exists a map $f: I \to \mathbb{Z}^d$ such that we can express \tilde{T} as a skew-product

$$T(x, \mathbf{n}) = (T(x), \mathbf{n} + f(x))$$

where \widetilde{I} is identified with $I \times \mathbb{Z}$. Notice that T and \widetilde{T} are ergodic if and only if the linear flow respectively on X and X_{Γ} are ergodic.

The relevant object to study the orbits by \widetilde{T} is then the induced cocycle for f defined for $k \in \mathbb{Z}$ by

$$f_k(x) = f(x) + \dots + f(T^{k-1}x).$$

The ergodicity of these two maps can be linked with the following concept.

Definition 2.9. We call $e \in \mathbb{Z}^d$ is an essential value of (the induced cocycle by) f if for every measurable set K of positive measure, there exists $k \in \mathbb{Z}$ such that $K \cap T^{-k}(K) \cap \{x \in X : f_k(x) = e\}$ has positive measure.

Note that $0 \in \mathbb{Z}^d$ is always an essential value. The set of all finite essential values associated to f is denoted by Ess_f ; it forms a subgroup of \mathbb{Z}^d and it follows from [35] that the skew product \widetilde{T} is ergodic if and only if $\mathbb{Z}^d = \operatorname{Ess}_f$. Also, the map is recurrent if 0 can be obtained as essential value using elements $k \in \mathbb{Z} \setminus \{0\}$ (the Lebesgue measure \boldsymbol{m} is infinite on X_{Γ} , so recurrence does not immediately follow from the invariance of \boldsymbol{m} .)

We represent the given translation surface as zippered rectangles, defining the linear flow as a suspension flow of an interval exchange transformation (see for an introduction [36] or [37] from which Figure 7 is taken).



Figure 7: From a suspension flow over an interval exchange transformation to a zippered rectangles representation

This representation defines a fundamental domain which is a union of vertical rectangles and such that the singularity of the surface are in their vertical sides. We choose ξ to be constant in the interior of this domain. This domain has the nice feature that if there is an embedded rectangle in the surfaces X with vertices x, y at the bottom and $\phi_t(x), \phi_t(y)$ at the top then $\xi(x) - \xi(y) = \xi(\phi_t(x)) - \xi(\phi_t(y))$.

Theorem 2.10. If X_{Γ} is endowed with a lifted linear pseudo-Anosov automorphism ψ_{Γ} then the linear flow in the stable and unstable directions of the corresponding matrix is ergodic.

Proof. To any rectangle in this representation, on can associate a homology class by closing the curve going from bottom to top with a piece of the interval I. These homology classes form a basis of $H_1(X \setminus \Sigma, \mathbb{Z})$ which is dual for the intersection form to the basis of $H_1(X, \Sigma, \mathbb{Z})$ associated to the polygonal representation. Let $e \in \mathbb{Z}^d$ be the associated shift in ξ for the flow from bottom to top of a rectangle; it is equal to the value of ζ on the homology class. Thus such values e generate \mathbb{Z}^d since the homology class of rectangles form a basis of $H_1(X \setminus \Sigma, \mathbb{Z})$ and ζ is surjective, see [36, Section 4.5]. We will now prove that e is an essential value.

It suffices to show that there exists $\delta > 0$ such that for any arbitrary rectangle in X, there exists a δ proportion of points in this rectangle such that $\phi_T(x)$ is in the rectangle and $\xi(\phi_T(x)) - \xi(x) = e$.

When applying the pseudo-Anosov automorphism (or its inverse) dilating in the vertical direction by λ^{-1} , one gets an interval exchange on the contracted interval $I' \subset I$ by a factor λ . The heights of the rectangles are multiplied by λ and the flow visits several of the initial rectangle of the zippered rectangle decomposition over I before returning to I'. The rectangles over I' can then be decomposed along their height by the rectangles they visit, this is usually called Rokhlin towers. This decomposition enables us to follow the intersection number defining ξ .

Assume that, after we apply the automorphism n_0 times, the contracted interval $I^{(n_0)}$ is contained a single interval I_0 of I. Let T_0 be the height of the rectangle above I_0 . Assume $x \in I^{(n_0)}$ is in the image of the rectangle whose shift is e, if $T_1 = \lambda^{-n_0} T_e$ is the return time of the rectangle above x in $I^{(n_0)}$. We have $\xi(\phi_{T_1+t}(x)) - \xi(\phi_t(x)) = \xi(\phi_{T_1}(x)) - \xi(x)$ for all $0 \le t \le T_0$.

Consider \mathcal{R} the image of the rectangle over I which associated shift is e after $n + n_0$ iterations. Then by the previous remark, if we consider the subrectangle \mathcal{R}_0 of \mathcal{R} with the same base but with height $\lambda^{-n}T_0$, all points $x \in \mathcal{R}_0$ satisfy $\xi(\phi_{T_1}(x)) - \xi(x) = e$. Then, for an arbitrary rectangle, if δ is half of its measure, by unique ergodicity of the flow, if we take n large enough \mathcal{R}_0 intersects at least a proportion δ of the rectangle, and for all these points, we have $\xi(\phi_{T_1}(x)) - \xi(x) = e$. Thus e is an essential value.

Corollary 2.11. The Frobenius function F is not a coboundary.

Proof. Assume by contradiction that $F = g \circ \psi - g$ for some measurable function g. As F is constant on a finite partition into cylinder sets w.r.t. the Markov partition, F is Hölder continuous in the symbolic metric. By Livshits regularity (see [27, Theorem 19.2.1], which is stated for manifolds, but works in metric space too), g is also Hölder continuous in symbolic metric, and therefore bounded.

Assume by contradiction that F is a coboundary. Take \mathcal{R} a rectangle of positive measure in X_{Γ} . For each $x' \in \mathcal{R}$ and $n \in \mathbb{Z}$

$$\xi(\psi_{\Gamma}^{n}x') - \xi(x') = \sum_{j=0}^{n-1} F \circ \psi^{j}(x) = \sum_{j=0}^{n-1} g \circ \psi^{j+1}(x) - g \circ \psi^{j}(x) = g \circ \psi^{n}(x) - g(x) \le 2 \|g\|_{\infty}.$$

The images of ψ_{Γ} contains larger and larger sections of the flow that remains in a compact set. Thus $\bigcup_{n \in \mathbb{N}} \psi_{\Gamma}^n \mathcal{R}$ is a compact set of positive measure which is invariant by the flow. Hence the flow is not ergodic.

3 Ergodic integrals and (weak) rational ergodicity via Local Limit Laws

We are interested in (weak) rational ergodicity (with optimal rates) of a translation flow ϕ_t defined on a \mathbb{Z}^d -cover X_{Γ} , $d \in \{1,2\}$, of the compact translation surface X that satisfies certain abstract assumptions. Throughout the entire paper, we let $d \in \{1,2\}$. The main results of this section, Theorems 3.3 and 3.5, are a generalization of [6, Theorems 3.2 and 4.3]. The precise error rates for \mathbb{Z}^d -covers, d = 1, 2, are new. As far as we are aware, the treatment of \mathbb{Z}^2 covers is completely new. The setting we require is as follows:

- (H1) Let X be a compact, two dimensional, surface and let X_{Γ} be a \mathbb{Z}^d -cover with projection p: $X_{\Gamma} \to X$ that is invariant under deck-transformations $p \circ \Delta_{\mathbf{n}} = p$. We assume that there exists a linear pseudo-Anosov automorphism $\psi_{\Gamma} : X_{\Gamma} \to X_{\Gamma}$ on the \mathbb{Z}^d -cover X_{Γ} that renormalizes the translation flow ϕ_t in the stable direction, that is $\psi_{\Gamma} \circ \phi_t = \phi_{\lambda t} \circ \psi_{\Gamma}$ for some $\lambda \in (0, 1)$.³
- (H2) The linear pseudo-Anosov automorphism ψ_{Γ} commutes with the deck transformations, i.e., $\psi_{\Gamma} \circ \Delta_{\mathbf{n}} = \Delta_{\mathbf{n}} \circ \psi_{\Gamma}$ for all $\mathbf{n} \in \mathbb{Z}^d$ and $\psi = p \circ \psi_{\Gamma} \circ p^{-1} : X \to X$ is well-defined.
- (H3) Upon the choice of a bounded fundamental domain \mathcal{F} (i.e., X_{Γ} is the disjoint union $\bigsqcup_{\mathbf{n}\in\mathbb{Z}^d}\Delta_{\mathbf{n}}(\mathcal{F})$), we define $\xi: X_{\Gamma} \to \mathbb{Z}^d$ to be the \mathbb{Z}^d component of $x \in X_{\Gamma}$, via $\xi(x) = \mathbf{n}$ if $x \in \Delta_{\mathbf{n}}(\mathcal{F})$. We consider ψ_{Γ} as the lift of a pseudo-Anosov automorphism $\psi: X \to X$ defined via

$$\psi_{\Gamma}(x,\mathbf{n}) = (\psi(x),\mathbf{n} + F(x)), \qquad x \in X, \mathbf{n} \in \mathbb{Z}^d$$

where $F(x) = \xi \circ \psi_{\Gamma}(x') - \xi(x')$, defined independently of a choice of $x' \in p^{-1}(x)$, is called the *Frobenius function*. We assume that $\int_X F \, \mathrm{d}\boldsymbol{m} = 0$ (*no drift condition*) and that $F: X \to \mathbb{Z}^d$ is **not** a coboundary, i.e., $F \neq g \circ \psi - g$ for any $g: X \to \mathbb{Z}^d$.

The Lebesgue measure \boldsymbol{m} is invariant, for both the finite and the infinite measure preserving automorphism, ψ and ψ_{Γ} .

Remark 3.1. The requirement that ψ is (a two dimensional) linear automorphism can be relaxed. The reason this simplification is that it allows us to work with simpler anisotropic Banach spaces as described in Section 3.5 below.

Remark 3.2. In Sections 2.3 and 2.4, we give examples where these hypotheses apply for d = 1, 2, respectively. That the Frobenius function has zero integral, but is not a coboundary was shown in Corollary 2.11.

As expected (and clarified in Subsection 3.3), the Central Limit Theorem (CLT) for the ergodic sum F_K , as $K \to \infty$, holds. That is,

$$\frac{F_K}{\sqrt{K}} \implies \chi, \text{ as } K \to \infty, \tag{6}$$

where \implies stands for convergence in distribution and χ is a Gaussian random variable with mean 0 (here we use the no drift condition in (H3) and covariance matrix Σ^2 . Here Σ^2 is a symmetric, non-degenerate, $d \times d$ matrix $\Sigma^2 = \sum_{j \in \mathbb{Z}} \int_X F^T \otimes F \circ \psi^j \, \mathrm{d}\boldsymbol{m}$ (with F^T the transpose of F), and Σ will be its unique symmetric positive-definite square root. For d = 1, the matrix $\Sigma^2 = \sigma^2$ is a scalar. The speed of mixing of ψ ensures that the above sum converges. The non-degeneracy of Σ^2 is ensured because F is not a coboundary, see (H3).

We are interested in a simple expression of the ergodic integral $\int_0^T G_{\Gamma} \circ \phi_t \, dt$ where $G_{\Gamma} \in C^1(X_{\Gamma})$ is compactly supported. Hence $G(\cdot, r) := G_{\Gamma} \cdot \mathbb{1}_{\Delta_r(\mathcal{F})}$ is non-zero for at most finitely many $r \in \mathbb{Z}^d$ (regardless of the exact choice of the fundamental domain \mathcal{F}), and we can write the ergodic integral as the sum of finitely many integrals $\int_0^T (G_{\Gamma} \cdot \mathbb{1}_{\Delta_r(\mathcal{F})}) \circ \phi_t \, dt$, accordingly.

3.1 Main results

Let Σ^2 be the covariance matrix in (6). For d = 1, we write $\Sigma^2 = \sigma^2$. Recall that $\lambda \in (0, 1)$ is the stable eigenvalue of ψ_{Γ} .

³If the stable eigenvalue is negative, we take ψ_{Γ}^2 instead.

Theorem 3.3. Assume (H1)–(H3). Let $G_{\Gamma} \in C^{1}(X_{\Gamma})$ be a compactly supported real function. Choose $K \in \mathbb{N}$ so large that ψ_{Γ}^{K} maps the flow-line between $x' \in p^{-1}(x)$ and $\phi_{T}(x')$ into a single copy of the fundamental domain (this amounts to $K \approx \log^{*} T := \lceil \log_{\lambda^{-1}} T \rceil$).

I. Suppose d = 1. Then there exist real bounded functions $g_{k,j}$ so that for all $N \ge 1$,

$$\begin{split} \int_0^T G_{\Gamma} \circ \phi_t(x) \, \mathrm{d}t &= \frac{\int_{X_{\Gamma}} G_{\Gamma} \, \mathrm{d}\boldsymbol{m}}{\sigma \sqrt{2\pi}} \cdot e^{-\frac{\xi \left(\psi_{\Gamma}^K(x)\right)^2}{2\sigma^{2_K}}} \frac{T}{\sqrt{K}} \\ & \times \left(1 + \sum_{k=1}^N \frac{1}{K^k} \sum_{j=0}^{2k} g_{k,j}(x) \xi (\psi_{\Gamma}^K(x))^{2k-j} + O\left(\frac{1}{K^{N+1}}\right) \right) \quad \text{as } T \to \infty. \end{split}$$

II. Suppose d = 2. Then there exist real bounded functions $g_1 : X \to \mathbb{R}^2$, $g_2 : X \to GL(\mathbb{R}, 2)$ and a real constant g'_2 , so that as $T \to \infty$,

$$\begin{split} \int_0^T G_{\Gamma} \circ \phi_t(x) \, \mathrm{d}t &= \frac{\int_{X_{\Gamma}} G_{\Gamma} \, \mathrm{d}\boldsymbol{m}}{2\pi\sqrt{\det \Sigma^2}} e^{-\frac{1}{2}\langle \Sigma^{-1}\frac{\xi(\psi_{\Gamma}^K(x))}{\sqrt{K}}, \Sigma^{-1}\frac{\xi(\psi_{\Gamma}^K(x))}{\sqrt{K}}\rangle} \frac{T}{K} \\ & \times \left(1 + \frac{\langle g_1(x), \xi(\psi_{\Gamma}^K(x')) \rangle}{K} + \frac{\langle \xi(\psi_{\Gamma}^K(x), g_2(x)\xi(\psi_{\Gamma}^K(x)) + g_2'}{K^2} + O\left(\frac{1}{K^3}\right)\right) + \frac{\langle \xi(\psi_{\Gamma}^K(x), g_2(x)\xi(\psi_{\Gamma}^K(x)) + g_2') \rangle}{K^2} + O\left(\frac{1}{K^3}\right) \right) + \frac{\langle \xi(\psi_{\Gamma}^K(x), g_2(x)\xi(\psi_{\Gamma}^K(x)) + g_2') \rangle}{K^2} + O\left(\frac{1}{K^3}\right) = 0 \end{split}$$

Remark 3.4. The functions $g_{k,j}$ in the case d = 1, and g_1, g_2 in the case d = 2 are described precisely inside the proof. For d = 2, we can also go higher in the expansion, but since the calculations are tedious, we omit this.

The assumption that G is real valued (compactly supported) can be relaxed to G (compactly supported) taking values in \mathbb{C} or even \mathbb{C}^d . We would still need to separate the (real or imaginary) components, and for \mathbb{C}^d , the vector valued functions $g_{k,j}$ need to be adjusted, which is a tedious exercise, even for d = 1.

Using Theorem 3.3 we obtain expansion in weak rational ergodicity for 'good' functions.

Theorem 3.5. Assume the setup of Theorem 3.3, and let $\mathcal{F} \subset X_{\Gamma}$ be a fundamental doain. Let $\chi \simeq \mathcal{N}(0, \Sigma^2)$ be a d = 1, 2-dimensional Gaussian random variable.

(i) Suppose d = 1. Then, there exist real constants $d_{k,j}$ so that for all $N \ge 1$,

$$\int_{\mathcal{F}} \int_0^T G_{\Gamma} \circ \phi_t(x) \, \mathrm{d}t \, \mathrm{d}\boldsymbol{m} = \frac{\int_{X_{\Gamma}} G_{\Gamma} \, \mathrm{d}\boldsymbol{m}}{\sigma\sqrt{2\pi}} \frac{T}{\sqrt{K}} \left(\mathbb{E}(e^{-\frac{\chi^2}{2}}) + \sum_{k=1}^N \sum_{j=0}^{2k} \frac{d_{k,j}}{\sqrt{K}^{2k+j}} + O\left(\frac{1}{K^{N+1}}\right) \right).$$

(ii) Suppose d = 2. Then, there are real constants d_1, d_2 so that

$$\int_{\mathcal{F}} \int_0^T G_{\Gamma} \circ \phi_t(x) \, \mathrm{d}t \, \mathrm{d}\boldsymbol{m} = \frac{\int_{X_{\Gamma}} G_{\Gamma} \, \mathrm{d}\boldsymbol{m}}{\sqrt{\det \Sigma^2}} \frac{T}{K} \left(\mathbb{E}(e^{-\frac{\chi^2}{2}}) + \frac{d_1}{K} + \frac{d_2}{K^2} + O\left(\frac{1}{K^3}\right) \right).$$

Remark 3.6. Weak rational ergodicity without rates follows immediately since convergence for all $L^1(X_{\Gamma})$ -functions is an immediate consequence of Theorem 3.3 and the ratio ergodic theorem, see [1].

3.2 Strategy of the proof

Following the approach in [20], in particular [20, Eq. (2.4) and (2.6)], (see also [9, Eq. (4)] for the same expression), we first exploit the commutation relation (2), which is part of our assumption (H1).

For any $v_{\Gamma} \in \mathcal{C}^0(X_{\Gamma})$, for every $x \in X_{\Gamma}$, for all T > 0 and all integers $K \ge 0$, we compute that

$$\begin{split} \int_0^T v_{\Gamma} \circ \phi_t(x) \, \mathrm{d}t &= \int_0^T v_{\Gamma} \circ \psi_{\Gamma}^{-K} \circ \psi_{\Gamma}^K \circ \phi_t(x) \, \mathrm{d}t = \frac{1}{\lambda^k} \int_0^T v_{\Gamma} \circ \psi_{\Gamma}^{-K} \circ \phi_{\lambda^K r} \circ \psi_{\Gamma}^K(x) \, \lambda^K \, \mathrm{d}r \\ &= \frac{1}{\lambda^K} \int_0^{\lambda^K T} v_{\Gamma} \circ \psi_{\Gamma}^{-K} \circ \phi_r \circ \psi_{\Gamma}^K(x) \, \mathrm{d}r. \end{split}$$

Let $\mathcal{L}_{\Gamma} : L^{1}(X_{\Gamma}) \to L^{1}(X_{\Gamma})$ be the transfer operator associated with ψ_{Γ} defined via $\int_{X_{\Gamma}} \mathcal{L}_{\Gamma} v_{\Gamma} w_{\Gamma} d\boldsymbol{m} = \int_{X_{\Gamma}} v_{\Gamma} w_{\Gamma} \circ \psi_{\Gamma} d\boldsymbol{m}$ with $v_{\Gamma} \in L^{1}(X_{\Gamma})$ and $w_{\Gamma} \in L^{\infty}(X_{\Gamma})$. Since the map ψ_{Γ} is invertible and preserves \boldsymbol{m} , we also have $\mathcal{L}_{\Gamma} v_{\Gamma} = v_{\Gamma} \circ \psi_{\Gamma}^{-1}$. Thus,

$$\int_0^T v_{\Gamma} \circ \phi_t(x) \, \mathrm{d}t = \frac{1}{\lambda^K} \int_0^{\lambda^k T} \mathcal{L}_{\Gamma}^K v_{\Gamma} \circ \phi_r \circ \psi_{\Gamma}^K(x) \, \mathrm{d}r.$$
(7)

The strategy is to relate the behaviour of \mathcal{L}_{Γ}^{K} with an operator (or conditional) local limit theorem in terms of the transfer operator $\mathcal{L}: L^{1}(X) \to L^{1}(X)$ for the automorphism ψ (defined via $\int_{X} \mathcal{L}v w \, \mathrm{d}\boldsymbol{m} = \int_{X} v \, w \circ \psi \, \mathrm{d}\boldsymbol{m}$ with $v \in L^{1}(X)$ and $w \in L^{\infty}(X)$. Also we define the *twisted transfer operator* as

$$\mathcal{L}_u v = \mathcal{L}(e^{iuF}v),$$

where uF indicates the scalar product if u and F are vectors. The operator local limit theorem we are after is in the sense of [3, Section 6].

The first lemma below makes the relation between $\mathcal{L}_{\Gamma}^{K}v_{\Gamma}$, for compactly supported functions v_{Γ} , and $\mathcal{L}_{u}^{K}v$ precise. Recall (from (H1)) that $p: X_{\Gamma} \to X$.

Lemma 3.7. Let $v \in L^1(X)$ and let $v_{\Gamma}(\cdot) = v(\cdot, r) = v \circ p \in L^1(X_{\Gamma})$ be the lifted version supported on $\{\xi = r\}, r \in \mathbb{Z}^d$. For all $\ell, r \in \mathbb{Z}^d$, for all $K \ge 1$ and for all $x \in X$,

$$\mathcal{L}_{\Gamma}^{K}v(x,r)\,\mathbb{1}_{\{\xi=\ell\}} = \mathcal{L}^{K}v(x)\mathbb{1}_{\{F_{K}(x)=\ell-r\}} = \int_{[-\pi,\pi]^{d}} e^{-iu(\ell-r)}\mathcal{L}_{u}^{K}v(x)\,\mathrm{d}u$$

for ergodic sums $F_K := \sum_{j=0}^{K-1} F \circ \psi_{\Gamma}^j$.

Proof. Let $v \in L^1(X)$, $w \in L^{\infty}(X)$ and $v(\cdot, r) = v \circ p$, $w(\cdot, \ell) = w \circ p$ be the versions supported on $\{\xi = r\}$ and $\{\xi = \ell\}$, respectively. Compute that

$$\begin{split} \int_{X_{\Gamma}} \mathcal{L}_{\Gamma}^{K} v(x,r) \mathbb{1}_{\{\xi=\ell\}} w(x,\ell) \, \mathrm{d}\boldsymbol{m}(x) &= \int_{X_{\Gamma}} \mathcal{L}_{\Gamma}^{K} (\mathbb{1}_{\{X \times \{r\}\}} v) \, (\mathbb{1}_{\{X \times \{\ell\}\}} w) \, \mathrm{d}\boldsymbol{m} \\ &= \int_{X_{\Gamma}} (\mathbb{1}_{\{X \times \{r\}\}} v(x,r)) \, (\mathbb{1}_{\{X \times \{\ell\}\}} w(x,\ell)) \circ \psi_{\Gamma}^{K}(x) \, \mathrm{d}\boldsymbol{m} \\ &= \int_{X} v \, w \circ \psi^{K} \, \mathbb{1}_{\{F_{K}=\ell-r\}} \, \mathrm{d}\boldsymbol{m} \\ &= \int_{X} \mathcal{L}^{K} (v \mathbb{1}_{\{F_{K}=\ell-r\}}) \, w(x) \, \mathrm{d}\boldsymbol{m}, \end{split}$$

which gives the first equality in the statement. We can write the indicator function $\mathbb{1}_{\{F_K=\ell-r\}} = \int_{[-\pi,\pi]^d} e^{iu(F_K-(\ell-r))} du$, so

$$\int_{X} \mathcal{L}^{K}(v(x)\mathbb{1}_{\{F_{K}=\ell-r\}}) w(x) \, \mathrm{d}\boldsymbol{m} = \int_{X} \int_{[-\pi,\pi]^{d}} \mathcal{L}^{K}\left(ve^{iu(F_{K}-(\ell-r))}\right)(x) \, \mathrm{d}u \, w(x) \, \mathrm{d}\boldsymbol{m}$$
$$= \int_{[-\pi,\pi]^{d}} \int_{X} e^{-iu(\ell-r)} \mathcal{L}_{u}^{K} v \, w \, \mathrm{d}\boldsymbol{m} \, \mathrm{d}u.$$

Given Lemma 3.7, our task comes down to obtain a precise expansion of $\mathcal{L}^{K}v(x)\mathbb{1}_{\{F_{K}(x)=\ell-r\}}$ in powers of K and combined it with (7). As explained in subsection 3.3 below, $\ell-r$ in $\mathcal{L}^{K}v(x)\mathbb{1}_{\{F_{K}(x)=\ell-r\}}$ will be replaced by $\xi(\psi_{\Gamma}^{K}(x'))$, which in the end, will give the form of Theorem 3.3.

3.3 An (operator) local limit theorem (LLT) for F_K

Proposition 3.11 below is an asymptotic expansion operator LLT (in the sense of [3, Section 6]) for the ergodic sums F_K . The expansion in Proposition 3.11 is a key ingredient in the proof of our main results Theorems 3.3 and 3.5. We recall that Theorem 3.3 is a version of Theorem 1.1, including a precise statement for d = 2, while Theorem 3.5 gives optimal rates in a form of weak rational ergodicity for C^1 functions.

We first recall some facts on the spectral properties of \mathcal{L} and its twisted version $\mathcal{L}_u f = \mathcal{L}(e^{iuF}f)$, $u \in \mathbb{R}^d$. Since $\psi : X \to X$ is an invertible map, we need adequate, anisotropic Banach spaces on which the corresponding transfer operator \mathcal{L} can act. The details on Banach spaces we shall use are deferred to Section 3.5. The first proposition summarizes all that we need to use in terms of Banach spaces (regardless the particular of these spaces) to prove the main results.

- **Proposition 3.8.** (a) There exist anisotropic Banach spaces $\mathcal{B}, \mathcal{B}_w$ so that $C^1(X) \subset \mathcal{B} \subset \mathcal{B}_w \subset C^1(X)^*$ where $C^1(X)^*$ is the (topological) dual of $C^1(X)$. The transfer operator \mathcal{L} acts continuously on \mathcal{B} and \mathcal{B}_w . Moreover, \mathcal{L} is quasicompact⁴ when viewed as operator from \mathcal{B} to \mathcal{B} . In particular, 1 is an isolated, simple eigenvalue in the spectrum of \mathcal{L} .
 - (b) The derivatives $\frac{d^k}{du^k} \mathcal{L}_u f$ are linear operators on \mathcal{B} with operator norm of $O(||F||_{\infty}^k)$.
 - (c) There exist $\delta > 0$ and a family of simple eigenvalues λ_u that is analytic in u for all $|u| < \delta$. Also, for all $|u| < \delta$ and $n \ge 1$,

$$\mathcal{L}_u^n = \lambda_u^n \Pi_u + Q_u^n$$

where Π_u is the family of spectral projections associated with λ_u with $\Pi_0 v = \int_X v \, \mathrm{d}\boldsymbol{m}$, Π_u, Q_u are analytic when regarded as (family of) operators acting on \mathcal{B} , $\Pi_u Q_u = Q_u \Pi_u$ and $\|Q_u^n\|_{\mathcal{B}} \leq \delta_0^n$ for some $\delta_0 < 1$.

(d) There exists $\delta_1 \in (0,1)$ so that $\|\mathcal{L}^n_u\|_{\mathcal{B}} \leq \delta_1^n$ for all $n \geq 1$.

The proof of Proposition 3.8 is provided in Section 3.5 (the headers of the subsections indicate which item of the proposition is proved). Proposition 3.8 is known in various settings similar to the one here (see, for instance, the survey paper [11] and references therein).

We recall that throughout, $d \in \{1, 2\}$. Throughout this section we let $\Pi_0^{(j)}, \lambda_0^{(j)}$ denote the *j*-th derivative in u of Π_u, λ_u evaluated at u = 0. From here onward, given $u \in \mathbb{R}^d$ we write $u^{\otimes j} := u \otimes \cdot \otimes u$ for the *j*-fold tensor product of u with itself. We define the *-product u * v on column vectors $u \in \mathbb{C}^d$ and $v \in \mathbb{C}^{d'}$, where we assume that d' is a multiple of d, or vice versa. The meaning of these type of products is clarified in Appendix A.

Remark 3.9. If d = d' then $u * v = uv = \sum_{i=1}^{d} u_i v_i$ is the usual scalar product.

By Proposition 3.8(c), for $|u| < \delta$ and for any $v \in \mathcal{B}$,

$$\mathcal{L}_{u}^{n}v = \lambda_{u}^{n}\Pi_{u}v + Q_{u}^{n}v = \left(\sum_{m=0}^{\infty}\lambda_{0}^{(m)} \ast u^{\otimes m}\right) \times \left(\sum_{j=0}^{\infty}\Pi_{0}^{(j)}v \ast u^{\otimes m}\right) + Q_{u}^{n}v.$$
(8)

Remark 3.10. Throughout we restrict to v taking real values. In this case we note that when d = 1 $\Pi_0^{(j)}v, \lambda_0^{(j)}$ are scalars, and when d = 2, are column vectors with 2^j entries. A similar statement holds for $\lambda_0^{(j)}, j \ge 0$. This is in the sense of the terminology clarified in Appendix A. Clearly, when j = 0, $\Pi_0^{(0)}v = \Pi_0 v = \int_X v \, \mathrm{d}\boldsymbol{m}, \, \lambda_0^{(0)} = \lambda_0 = 1$ are scalars.

 $^{^{4}}$ the precise terminology is recalled and specified in Section 3.5

Recall $u \in \mathbb{R}^d$, $d \in \{1, 2\}$. A classical (not necessarily short) argument which dates back to [31, 23] (see also [3, 22], shows that provided that λ_u is twice differentiable at 0 (so, much weaker than analyticity of λ ensured by Proposition 3.8(c)), then

$$\lambda_u - 1 = \frac{1}{2} \Sigma^2 * u^{\otimes 2} (1 + o(1)), \lambda_u = e^{-\frac{1}{2} \Sigma^2 * u^{\otimes 2}} (1 + o(1))$$
(9)

where $\langle \cdot, \cdot \rangle$ is the usual scalar product and where Σ is the unique positive definite symmetric square root of the non-degenerate $d \times d$ covariance matrix introduced in (6).

An immediate consequence of (9) and (8) is that $\mathbb{E}\left(e^{iuF_{\kappa}}\right) = e^{-\frac{1}{2}\Sigma^{2}*u^{\otimes 2}(1+o(1))}$, as $u \to 0$. A classical argument based on the Levy continuity theorem (see, for instance, the survey [22]) shows that CLT stated in (6) holds.

In the setup of the current section, a refined version of the CLT (6) holds, namely a Local Limit Theorem (LLT). This means that for $M \in \mathbb{Z}^d$,

$$\boldsymbol{m}\left(F_{K}(x)=M\right) = \frac{1}{\left(2\pi\sqrt{K}\right)^{d}} \Phi\left(\frac{M}{\sqrt{K}}\right) (1+o(1)), \text{ as } K \to \infty,$$
(10)

where Φ is the density of the Gaussian random variable χ in (6).

In the sequel we shall exploit, and prove, a stronger version of (10), namely an operator LLT with precise expansion, as in Proposition 3.11 below. This type of expansion for LLT is, essentially, contained inside [32, Proof of Theorem 3.2], where different Banach spaces are used.

Before the statement, recall Remark 3.10 on the meaning of $\Pi_0^{(j)}v$ and $\lambda_0^{(j)}$. With the conventions on tensors, see Appendix A and specifically (24), we have $A_u := \frac{1}{2}\Sigma^2 * u^{\otimes 2} = \frac{1}{2} \langle \Sigma u, \Sigma u \rangle$. Recalling (9), and using the analyticity of λ_u , we can write

$$\lambda_u^n = e^{-\frac{n}{2}\langle \Sigma u, \Sigma u \rangle} \left(1 + \sum_{m=1}^\infty \frac{1}{m!} \left(\frac{\lambda^n}{A^n} \right)_0^{(m)} * u^{\otimes m} \right),\tag{11}$$

where $\left(\frac{\lambda^n}{A^n}\right)_0^{(m)}$ is the *m*-th derivative of $\frac{\lambda_u^n}{A_u^n}$ evaluated at 0, which is a column vector with d^m entries, d = 1, 2.

Recall that F takes values in \mathbb{Z}^d , d = 1, 2.

Proposition 3.11. Let $v \in C^1(X)$. Then

(a) If v is a **real** function then $\Pi_0^{(j)}v$ is a column vector with d^j entries which are if j is even and purely imaginary entries if j is odd.

Moreover, $\left(\frac{\lambda}{A}\right)_{0}^{(m)}$ is a column vector with d^{m} real entries if m is even and $\left(\frac{\lambda}{A}\right)_{0}^{(m)}$ is is a column vector with d^{m} purely imaginary entries if m is odd.

(b) Let δ , δ_0 and δ_1 be as in Proposition 3.8(c). Set $\delta_2 = \max\{\delta_0, \delta_1\}$. Let Σ^2 be the covariance matrix in (6). Then for all $x \in X$ and for all $\ell, r \in \mathbb{Z}^d$,

$$\mathcal{L}^{K} v(x) \mathbb{1}_{\{F_{K}(x)=\ell-r\}} + E_{K} v(x)$$

$$= \frac{1}{\left(2\pi\sqrt{K}\right)^{d}} \int_{\left[-\delta\sqrt{K},\delta\sqrt{K}\right]^{d}} e^{-iu\frac{\ell-r}{\sqrt{K}}} e^{-\frac{\langle\Sigma u,\Sigma u\rangle}{2}} \left(1 + \sum_{m=1}^{\infty} \frac{1}{m!} \left(\frac{\lambda^{K}}{A^{K}}\right)_{0}^{(m)} * \frac{u^{\otimes m}}{K^{m/2}}\right)$$

$$\times \left(\int_{X} v \, \mathrm{d}\boldsymbol{m} + \sum_{j=1}^{\infty} \frac{1}{j!} \Pi_{0}^{(j)} v(x) * \frac{u^{\otimes j}}{K^{j/2}}\right) \, du,$$

where E_K is an operator acting on \mathcal{B} so that $||E_K v||_{\mathcal{B}} \leq C\delta_2^K ||v||_{C^1}$ and so that $\left|\int_X E_K v \, \mathrm{d}\boldsymbol{m}\right| \leq C'\delta_2^K ||v||_{C^1}$ for some C, C' > 0.

The proof of Proposition 3.11(a) is deferred to Section 3.8. Here we provide the argument for Proposition 3.11(b).

Proof of Proposition 3.11(b). Recall d = 1, 2. By Lemma 3.7 (second equality there) and Proposition 3.8(c) and (d),

$$\mathcal{L}^{K}v(x)\mathbb{1}_{\{F_{K}(x)=\ell-r\}} = \int_{[-\pi,\pi]^{d}} e^{-iu(\ell-r)} \mathcal{L}_{u}^{K}v(x) \, du$$

$$= \int_{[-\delta,\delta]^{d}} e^{-iu(\ell-r)} (\lambda_{u}^{K}\Pi_{u} + Q_{u}^{K})v(x) \, du + O(\delta_{1}^{K})$$

$$= \int_{[-\delta,\delta]^{d}} e^{-iu(\ell-r)} \lambda_{u}^{K}\Pi_{u}v(x) \, du + O(\delta_{2}^{K}).$$
(12)

By equation (11), $\lambda_u^K = e^{-\frac{K}{2} \langle \Sigma u, \Sigma u \rangle} \left(1 + \sum_{j=1}^{\infty} \frac{1}{j!} \left(\frac{\lambda^K}{A^K} \right)_0^{(j)} * u^{\otimes j} \right)$. We already know that $\Pi_u v = 0$

 $\int_X v \, \mathrm{d}\boldsymbol{m} + \sum_{j=1}^{\infty} \Pi_0^{(j)} v * u^{\otimes j}.$ Putting these two expressions together and using a change of coordinates $u \to \frac{u}{2\pi\sqrt{K^d}}$ in (12) gives the conclusion.

To clarify that integral in Proposition 3.11(b) leads to a real scalar (when v takes real values), we rewrite it in a more transparent way and record this as a lemma.

Lemma 3.12. Assume the setup of Proposition 3.11. Let Φ is the density of the Gaussian random variable χ in (6). Let d = 1, 2 and set $I_j(\Sigma, L) = \int_{\mathbb{R}^d} e^{-iuL} e^{-\frac{\langle \Sigma u, \Sigma u \rangle}{2}} u^{\otimes j} du$ for $j \ge 1$ and $L \in \mathbb{R}^d$. Then for any $n \geq 1$,

$$\int_X v \, \mathrm{d}\boldsymbol{m} \int_X v \, \mathrm{d}\boldsymbol{m}$$

$$\mathcal{L}^{K}v(x)\mathbb{1}_{\{F_{K}(x)=\ell-r\}} = \frac{\int_{X} v \,\mathrm{d}\boldsymbol{m}}{\left(2\pi\sqrt{K}\right)^{d}} \Phi\left(\frac{M}{\sqrt{K}}\right) + \sum_{j=1}^{N} \frac{1}{j!} \frac{C_{j}(v)}{K^{(j+d)/2}} + E_{K,N}v(x,r)$$

for real bounded functions $C_1(v) = I_1\left(\Sigma, \frac{\ell-r}{\sqrt{K}}\right) * \Pi_0^{(j)}v(x) + I_1\left(\Sigma, \frac{\ell-r}{\sqrt{K}}\right) * \left(\frac{\lambda}{A}\right)_0^{(j)}$

$$C_{j}(v) = I_{j}\left(\Sigma, \frac{\ell - r}{\sqrt{K}}\right) * \Pi_{0}^{(j)}v(x) + I_{j}\left(\Sigma, \frac{\ell - r}{\sqrt{K}}\right) * \left(\frac{\lambda}{A}\right)_{0}^{(j)} + I_{j}\left(\Sigma, \frac{\ell - r}{\sqrt{K}}\right) * \left(\sum_{r_{1}+r_{2}=j} \frac{1}{r_{1}!r_{2}!} \Pi_{0}^{(r_{1})}v(x) \otimes \left(\frac{\lambda}{A}\right)_{0}^{(r_{2})}\right) \quad for \ 2 \le j \le N,$$

and $E_{K,N}$ is an operator acting on \mathcal{B} so that $||E_{K,N}v||_{\mathcal{B}} = o\left(K^{-(N+d)/2}\right)||v||_{C^1}$ and $\left|\int_X E_{K,N}v \,\mathrm{d}\boldsymbol{m}\right| = o\left(K^{-(N+d)/2}\right)||v||_{C^1}$ $o(K^{-(N+d)/2}).$

Proof. Truncating each sum inside the integral in Proposition 3.11(b) at $N \ge 1$ and using the information on the operator $E_{K,N}$, we obtain

$$\mathcal{L}^{K}v(x)\mathbb{1}_{\{F_{K}(x)=\ell-r\}} = \frac{1}{\left(2\pi\sqrt{K}\right)^{d}} \int_{\left[-\delta\sqrt{K},\delta\sqrt{K}\right]^{d}} e^{-iu\frac{\ell-r}{\sqrt{K}}} e^{-\frac{\langle\Sigma u,\Sigma u\rangle}{2}} \left(1 + \sum_{m=1}^{N} \frac{1}{m!} \left(\frac{\lambda^{K}}{A^{K}}\right)_{0}^{(m)} * \frac{u^{\otimes m}}{K^{m/2}}\right) \times \left(\int_{X} v \,\mathrm{d}\boldsymbol{m} + \sum_{j=1}^{N} \frac{1}{j!} \Pi_{0}^{(j)}v(x) * \frac{u^{\otimes j}}{K^{j/2}}\right) \,\mathrm{d}\boldsymbol{u} + E_{K,N}v(x),$$

where $E_{K,N}$ is an operator as in the statement of the corollary.

The density of the Gaussian can be written as $\Phi(L) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-iuL} e^{-\frac{\langle \Sigma u, \Sigma u \rangle}{2}} du$, L vector in \mathbb{R}^d , due to the Fourier inversion formula (i.e., the inverse Fourier transform of the characteristic function). Note that

$$\Phi^{(j)}(L) = (-i)^j \int_{\mathbb{R}^d} e^{-iuL} e^{-\frac{\langle \Sigma u, \Sigma u \rangle}{2}} u^{\otimes j} \, du.$$

So, $i^j \Phi^{(j)}(L) = I_j(\Sigma, L)$.

Applying the formula for $\Phi^{(j)}$ inside the integral above with $L = \frac{\ell - r}{\sqrt{K}} \in \mathbb{R}^d$, d = 1, 2, we obtain

$$\mathcal{L}^{K}v(x)\mathbb{1}_{\{F_{K}(x)=\ell-r\}} = \frac{\int_{X} v \, \mathrm{d}\boldsymbol{m}}{\left(2\pi\sqrt{K}\right)^{d}} \Phi\left(\frac{\ell-r}{\sqrt{K}}\right) + \frac{\int_{X} v \, \mathrm{d}\boldsymbol{m}}{\left(2\pi\sqrt{K}\right)^{d}} \sum_{m=1}^{N} \frac{1}{m!} \frac{i^{m}}{K^{m/2}} \Phi^{(m)}\left(\frac{\ell-r}{\sqrt{K}}\right) * \left(\frac{\lambda^{K}}{A^{K}}\right)_{0}^{(m)} + \frac{1}{\left(2\pi\sqrt{K}\right)^{d}} \sum_{j=1}^{N} \frac{1}{j!} \frac{i^{j}}{K^{j/2}} \Phi^{(m)}\left(\frac{\ell-r}{\sqrt{K}}\right) * \Pi_{0}^{(j)}v(x) + \frac{1}{\left(2\pi\sqrt{K}\right)^{d}} \sum_{m=1}^{N} \sum_{j=1}^{N} \frac{1}{m!j!} \frac{i^{m+j}}{K^{(m+j)/2}} \Phi^{(m+j)}\left(\frac{\ell-r}{\sqrt{K}}\right) * \left(\Pi_{0}^{(j)}v(x) \otimes \left(\frac{\lambda^{K}}{A^{K}}\right)_{0}^{(m)}\right) + E_{K,N}v(x).$$

Note that

$$\begin{split} \sum_{m=1}^{N} \sum_{j=1}^{N} \frac{1}{m! j!} \frac{i^{m+j}}{K^{(m+j)/2}} \Phi^{(m+j)} \left(\frac{\ell-r}{\sqrt{K}}\right) * \Pi_{0}^{(j)} v(x) \otimes \left(\frac{\lambda^{K}}{A^{K}}\right)_{0}^{(m)} \\ &= \sum_{j=2}^{2N} \frac{1}{j!} \frac{i^{j}}{K^{(j)/2}} \Phi^{(j)} \left(\frac{\ell-r}{\sqrt{K}}\right) * \left(\sum_{r_{1}+r_{2}=j} \frac{1}{r_{1}! r_{2}!} \Pi_{0}^{(r_{1})} v(x) \otimes \left(\frac{\lambda^{K}}{A^{K}}\right)_{0}^{(r_{2})}\right) \\ &= \sum_{j=2}^{N} \frac{1}{j!} \frac{i^{j}}{K^{(j)/2}} \Phi^{(j)} \left(\frac{\ell-r}{\sqrt{K}}\right) * \left(\sum_{r_{1}+r_{2}=j} \frac{1}{r_{1}! r_{2}!} \Pi_{0}^{(r_{1})} v(x) \otimes \left(\frac{\lambda^{K}}{A^{K}}\right)_{0}^{(r_{2})}\right) + O\left(\frac{1}{K^{(N+1)/2}}\right). \end{split}$$

Recall that $i^{j}\Phi^{(j)}\left(\left(\frac{\ell-r}{\sqrt{K}}\right)\right) = I_{j}\left(\Sigma, \frac{\ell-r}{\sqrt{K}}\right)$. Writing the above three sums in a single sum gives the expression of $C_{j}(v)$, as in the statement of the lemma (after multiplication with $(2\pi\sqrt{K})^{-d}$).

It remains to justify that $C_i(v)$ are real bounded functions.

 $C_j(v)$ are real or complex functions. Recall d = 1, 2. Recall from Remark 3.9 that when the dimension of (tensor) vectors are the same, the operation * gives a scalar. By Lemma B with $L = \frac{\ell - r}{\sqrt{K}} \in \mathbb{R}^d$, d = 1, 2 (in Appendix B), the integrals I_j are column vectors with d^j entries. We already know from Remark 3.10 that $\Pi_0^{(j)}$ and $\left(\frac{\lambda^K}{A^K}\right)_0^{(j)}$ are column vectors with d^j entries. So the operation * between I_j and $\Pi_0^{(j)}$, and $\left(\frac{\lambda^K}{A^K}\right)_0^{(j)}$ gives a bounded real or complex function. The same applies to $I_j\left(\frac{\ell-r}{\sqrt{K}}\right)*\left(\Pi_0^{(r_1)}v(x)\otimes\left(\frac{\lambda^K}{A^K}\right)_0^{(r_2)}\right)$, since $r_1 + r_2 = j$.

 $C_j(v)$ are real bounded functions. By Proposition 3.11(a), $\Pi_0^{(j)}$ is a column vector with d^j real entries if j is even and with purely imaginary if j is odd. The same applies to $\left(\frac{\lambda^K}{A^K}\right)_0^{(j)}$. Finally, by Lemma B.1 in Appendix B, I_j have real entries if j is even and purely imaginary entries if j is odd. The boundedness of these functions follows from the analyticity of Π_t and λ_t , ensured by Proposition 3.8 (b), (c).

The asymptotic expansion in the usual LLT follows immediately. That is, taking $\ell - r = M \in \mathbb{Z}^d$,

 $v \equiv 1$ in Lemma 3.12, and integrating over the space, we obtain that for all $N \geq 1$,

$$\boldsymbol{m}\left(F_{K}(x)=M\right) = \frac{1}{\left(2\pi\sqrt{K}\right)^{d}} \Phi\left(\frac{M}{\sqrt{K}}\right) + \sum_{j=1}^{N} \frac{C_{j}}{K^{(j+d)/2}} + o\left(\frac{C_{N+1}}{K^{(N+d)/2}}\right), \quad \text{as } K \to \infty,$$
(13)

where C_j are real constants.

The expansion in (13) allows us to record a technical lemma that will play an important role in the proof of Theorem 3.5 below. Before the statement we recall that $F_K(x) = \xi \circ \psi_{\Gamma}^K(x') - \xi(x')$ for $x' \in p^{-1}(x)$.

Lemma 3.13. (i) Let $q \ge 0$ be an integer and $f_q : X \times \mathbb{R}^d \to \mathbb{R}$, $f_q(x, w) = e^{-\frac{1}{2} \langle w, w \rangle} g(x) * w^{\otimes q}$ for a bounded function $g : X \to \mathbb{R}^{d^q}$. Then there exist real constants $C_{d,q}$ and $d_{j,q}$, so that for any $N \ge 1$,

$$\int_{X_{\Gamma}} f_q\left(p(x'), \frac{\xi\left(\psi_{\Gamma}^K(x')\right)}{\sqrt{K}}\right) \,\mathrm{d}\boldsymbol{m} = C_{d,q} + \sum_{j=1}^N \frac{d_{j,q}}{K^{j/2}} + o(K^{-N/2}), \qquad as \ K \to \infty$$

(ii) Let $f : \mathbb{R}^d \to \mathbb{R}$, $f(w) = e^{-\frac{1}{2}\langle w, w \rangle}$ and let χ be the d-dimensional Gaussian introduced in (6). Then, there exist real constants d_j , so that for any $N \ge 1$,

$$\int_{X_{\Gamma}} f\left(\frac{\xi\left(\psi_{\Gamma}^{K}(x')\right)}{\sqrt{K}}\right) \,\mathrm{d}\boldsymbol{m} = \mathbb{E}(f(\chi)) + \sum_{j=1}^{N} \frac{d_{j}}{K^{j/2}} + o(K^{-N/2}), \quad as \ K \to \infty,$$

Proof. Item (i). Since $F_K(x) = \xi \circ \psi_{\Gamma}^K(x') - \xi(x')$ for $x' \in p^{-1}(x)$,

$$\int_{X_{\Gamma}} f_q\left(p(x'), \frac{\xi\left(\psi_{\Gamma}^K(x')\right)}{\sqrt{K}}\right) d\boldsymbol{m} = \sum_{M \in \mathbb{Z}^d} \int_{\{F_K \circ p = M\}} f_q\left(p(x'), \frac{\xi\left(\psi_{\Gamma}^K(x')\right)}{\sqrt{K}}\right) d\boldsymbol{m}$$
$$= \sum_{M \in \mathbb{Z}^d} a_M e^{-\frac{1}{2K}\langle M, M \rangle} \boldsymbol{m}\left(F_K = M\right),$$

where a_M is chosen by the intermediate value theorem for integrals. Due to the boundedness of g and exponential factor, $a_M = O(M^q)$, and hence finite for each M, even though $\mathbf{m}(\{x' \in X_{\Gamma} : F_K \circ p = M\}) = \infty$.

This together with (13) gives

$$\int_{X_{\Gamma}} f_q\left(p(x'), \frac{\xi\left(\psi_{\Gamma}^K(x')\right)}{\sqrt{K}}\right) \,\mathrm{d}\boldsymbol{m} = \frac{1}{K^{d/2}} \sum_{M \in \mathbb{Z}^d} a_M e^{-\frac{1}{2K}\langle M, M \rangle} \,H\left(\frac{M}{\sqrt{K}}\right) \quad \text{as } K \to \infty, \tag{14}$$

for

$$H\left(\frac{M}{\sqrt{K}}\right) = (2\pi)^{-d} \Phi\left(\frac{M}{\sqrt{K}}\right) + \sum_{j=1}^{n} \frac{C_j}{K^{j/2}} + o\left(\frac{C_{n+1}}{K^{n/2}}\right), \quad \text{for } C_j \in \mathbb{R}$$

Next, for each $M \in \mathbb{Z}^d$, define functions $a_M : Q(M) \to \mathbb{R}$ on the unit cube Q(M) centered at $M \in \mathbb{Z}^2$ in such a way that $\int_{Q(M)} a_M(w) e^{-\frac{1}{2K} \langle w, w \rangle} H\left(\frac{w}{\sqrt{K}}\right) dw = a_M e^{-\frac{1}{2K} \langle M, M \rangle} H\left(\frac{M}{\sqrt{K}}\right)$, and set $a = \sum_{M \in \mathbb{Z}^d} \tilde{a}_M \cdot \mathbb{1}_{Q(M)}$. Then

$$\sum_{M \in \mathbb{Z}^d} a_M e^{-\frac{1}{2K} \langle M, M \rangle} H\left(\frac{M}{\sqrt{K}}\right) = \int_{\mathbb{R}^d} a(w) e^{-\frac{1}{2K} \langle w, w \rangle} H\left(\frac{w}{\sqrt{K}}\right) dw$$
$$= K^{d/2} \int_{\mathbb{R}^d} a(v\sqrt{K}) e^{-\frac{1}{2} \langle v, v \rangle} H(v) dv,$$

where we used the change of coordinates $v = w/\sqrt{K}$. Since $\int_{\mathbb{R}^d} a(w\sqrt{K})e^{-\frac{1}{2}\langle v,v\rangle}H(v) dv < \infty$ due to the exponential factor, the sum scales as $K^{d/2}$. Insert this estimate into (14) to find constants $C_{q,d}$ and $d_{j,d} \in \mathbb{R}$ such that item (i) holds.

Item (ii) We just need to argue that the first term, that is $C_{d,0}$ in item (i), is exactly $\mathbb{E}(f_0(\chi))$. Apart from this constant the statement is as in item (i) for f_q with q = 0 and $g \equiv 1$. One could proceed via an exact calculation (using, for instance, the Euler-Maclaurin formula), but a quicker way is to recall (13) and note that by the Portmanteau Theorem,

$$\int_{X_{\Gamma}} f\left(\frac{\xi\left(\psi_{\Gamma}^{K}(x')\right)}{\sqrt{K}}\right) \, \mathrm{d}\boldsymbol{m} \to \mathbb{E}(f(\chi)), \quad \text{as } K \to \infty.$$

We record an immediate consequence of Lemma 3.12 that will be instrumental in the proof of Theorem 3.3. Recall that $F_K(x) = \xi \circ \psi_{\Gamma}^K(x') - \xi(x') \in \mathbb{Z}^d$, d = 1, 2, for $x' \in p^{-1}(x)$.

Corollary 3.14. Set $I_j(\Sigma, L) = \int_{\mathbb{R}^d} e^{-iuL} e^{-\frac{\langle \Sigma u, \Sigma u \rangle}{2}} u^{\otimes j} du$ for $j \ge 0$ and $L \in \mathbb{R}^d$. Let $G \in C^1(X)$. Let $x' \in p^{-1}(x)$. Then

$$\mathcal{L}^{K}G(x)\mathbb{1}_{\{F_{K}(x)=\xi(\psi_{\Gamma}^{K}(x'))\}} = \frac{\int_{X} G \,\mathrm{d}\boldsymbol{m}}{\left(2\pi\sqrt{K}\right)^{d}} I_{0}\left(\frac{\xi(\psi_{\Gamma}^{K}(x'))}{\sqrt{K}}\right) + \sum_{j=1}^{N} \frac{1}{j!} \frac{C_{j}(G,\xi(\psi_{\Gamma}^{K}(x')))}{K^{(j+d)/2}} + E_{K,N}G(x),$$

for real bounded functions $C_1(v) = I_1\left(\Sigma, \frac{\ell-r}{\sqrt{K}}\right) * \Pi_0^{(j)}G + I_1\left(\Sigma, \frac{\ell-r}{\sqrt{K}}\right) * \left(\frac{\lambda}{A}\right)_0^{(j)}$,

$$C_{j}(v) = I_{j}\left(\Sigma, \frac{\ell - r}{\sqrt{K}}\right) * \Pi_{0}^{(j)}G + I_{j}\left(\Sigma, \frac{\ell - r}{\sqrt{K}}\right) * \left(\frac{\lambda^{K}}{A^{K}}\right)_{0}^{(j)} + I_{j}\left(\Sigma, \frac{\ell - r}{\sqrt{K}}\right) * \left(\sum_{r_{1}+r_{2}=j} \frac{1}{r_{1}!r_{2}!} \Pi_{0}^{(r_{1})}v(x, r) \otimes \left(\frac{\lambda^{K}}{A^{K}}\right)_{0}^{(r_{2})}\right) \quad for \ 2 \le j \le N,$$

and $E_{K,N}$ is an operator acting on \mathcal{B} so that $||E_{K,N}G||_{\mathcal{B}} = o\left(K^{-(N+d)/2}||G||_{C^1}\right)$ and $\left|\int_X E_{K,N}G \,\mathrm{d}\boldsymbol{m}\right| = o\left(K^{-(N+d)/2}\right)$.

Proof. We want to apply Lemma 3.12 with v = G and suitable choice of $\ell, r \in \mathbb{Z}^d$.

Take $\ell = 0$ and $x' \in p^{-1}(x)$ so that $r = -\xi(\psi_{\Gamma}^{K}(x'))$. To justify this choice, just recall that $F_{K}(x) = \xi \circ \psi_{\Gamma}^{K}(x') - \xi(x')$. The conclusion follows from Lemma 3.12 with G instead of v and $\ell - r = \xi(\psi_{\Gamma}^{K}(x'))$.

3.4 Proof of the main results

Before proceeding to the proof of Theorem 3.3 we record one more technical lemma. Recall the notation of equation (7).

Lemma 3.15. Consider the operator $E_{K,N}$ defined in Corollary 3.14. Then

$$\left|\frac{1}{\lambda^K} \int_0^{\lambda^K T} E_{K,N} G \circ \phi_r(x_K) \, \mathrm{d}r\right| = o\left(\frac{T}{K^{(N+d)/2}} \|G\|_{C^1}\right).$$

We can now proceed to

Proof of Theorem 3.3. Let us assume without loss of generality that G_{Γ} is supported on $\{\xi = 0\}$. To emphasize that, we write $G_{\Gamma}(x) = G(x,0)$, so $G(\cdot,0) = G \circ p$ for a unique $G \in C^{1}(X)$ and $\int_{X} G \, \mathrm{d}\boldsymbol{m} = \int_{X_{\Gamma}} G_{\Gamma} \, \mathrm{d}\boldsymbol{m}$. By equation (7) and Lemma 3.7 (first equality there with $\ell = 0$ and $r = -\xi(\psi_{\Gamma}^{K}(x')))$,

$$\int_0^T G \circ \phi_t(x) \, \mathrm{d}t = \frac{1}{\lambda^K} \int_0^{\lambda^k T} \mathcal{L}_{\Gamma}^K(G(x,0)) \circ \phi_r \circ \psi_{\Gamma}^K(x') \, \mathrm{d}r$$
$$= \frac{1}{\lambda^K} \int_0^{\lambda^k T} \left(\mathcal{L}^K G(x) \mathbb{1}_{\{F_K(x) = \xi(\psi_{\Gamma}^K(x'))\}} \right) \circ \phi_r \circ \psi_{\Gamma}^K(x') \, \mathrm{d}r. \tag{15}$$

By Corollary 3.14,

$$\mathcal{L}^{K}G(x)\mathbb{1}_{\{F_{K}(x)=\xi(\psi_{\Gamma}^{K}(x'))\}} = \frac{\int_{X} G \,\mathrm{d}\boldsymbol{m}}{\left(2\pi\sqrt{K}\right)^{d}}I_{0}\left(\frac{\xi(\psi_{\Gamma}^{K}(x'))}{\sqrt{K}}\right) + \sum_{j=1}^{N}\frac{1}{j!}\frac{I_{j}\left(\Sigma,\frac{\xi(\psi_{\Gamma}^{K}(x'))}{\sqrt{K}}\right)}{K^{(j+d)/2}} * e_{j,G}(x) + o\left(K^{-(N+d)/2}\right),$$
(16)

where $e_{j,G} = \left(\Pi_0^{(j)}G + \left(\frac{\lambda^K}{A^K}\right)_0^{(j)}\right)$ and $e_{j,G} = \left(\Pi_0^{(j)}G + \left(\frac{\lambda^K}{A^K}\right)_0^{(j)} + \sum_{r_1+r_2=j} \frac{1}{r_1!r_2!}\Pi_0^{(r_1)}G \otimes \left(\frac{\lambda^K}{A^K}\right)_0^{(r_2)}\right), \quad j \ge 2.$

are column vectors with d^j entries which are real if j is even and purely imaginary entries if j is odd. Each such entry is bounded, due to Proposition 3.8 (b), (c). As in the proof of Lemma 3.12, the operation * between I_j and $e_{j,G}$ produces a bounded function.

We are left with describing the integrals $I_j\left(\Sigma, \frac{\xi(\psi_{\Gamma}^K(x'))}{\sqrt{K}}\right)$ and in the end combining with (15). Recall d = 1, 2 and write $I_j(\Sigma, L) = \int_{\mathbb{R}^d} e^{iuL - \frac{\langle \Sigma u, \Sigma u \rangle}{2}} u^j du$, with $L = L(x') := \frac{\xi(\psi_{\Gamma}^K(x'))}{\sqrt{K}}$.

Item I., d = 1. Recall that when d = 1, we write $\Sigma = \sigma$. Describing the integrals I_j and obtaining a close expression of $\mathcal{L}^K G(x) \mathbb{1}_{\{F_K(x) = \xi(\psi_{i}^K(x'))\}}$.

From Lemma B.1 in Appendix B, we obtain $I_0(\sigma, L) = \frac{\sqrt{2\pi}}{\sigma} e^{-\frac{L^2}{2\sigma^2}}$ and $I_j(\sigma, L) = \frac{1}{\sigma^2} (iLI_{j-1}(\sigma, L) + (j-1)I_{j-2}(\sigma, L))$ for $j \ge 1$. By Lemma B.1, $I_j(\sigma, L)$ is real if j is even and purely imaginary if j is odd. Thus, for real coefficients $c_{p,j}$ we can write

$$I_{j}\left(\sigma, \frac{\xi(\psi_{\Gamma}^{K}(x'))}{\sqrt{K}}\right) = i^{j \mod 2} \cdot \sum_{p=0}^{\lfloor j/2 \rfloor} c_{p,j} \frac{(\xi(\psi_{\Gamma}^{K}(x))^{2p+(j \mod 2)})}{K^{(2p+(j \mod 2))/2}} e^{-\frac{\xi\left(\psi_{\Gamma}^{K}(x')\right)^{2}}{2\sigma^{2}K}}.$$
 (17)

Recall from (16) that the bounded functions $e_{j,G}$ are real if j is even and purely imaginary if j is odd. This combines with $i^{j \mod 2}$ to get a real coefficient. Let $f_{k,j} = i^{j \mod 2} e_{j,G} c_{(2k-j-(j \mod 2))/2,j}$. Combining (16) and (17),

$$I_{0} + \sum_{j=1}^{N} \frac{i^{j \mod 2} e_{j,G}(x)}{K^{j/2}} I_{j} = \sum_{j=0}^{N} \sum_{p=0}^{\lfloor j/2 \rfloor} c_{p,j} i^{j \mod 2} e_{j,G}(x) \frac{(\xi(\psi_{\Gamma}^{K}(x))^{2p+(j \mod 2)}}{K^{(2p+j+(j \mod 2))/2}} e^{-\frac{\xi(\psi_{\Gamma}^{2\sigma^{2}K}(x'))^{2}}{K}} = \sum_{k=0}^{N} \left(\sum_{j=0}^{2k} f_{k,j}(x) \xi(\psi_{\Gamma}^{K}(x))^{2k-j} \right) e^{-\frac{\xi(\psi_{\Gamma}^{K}(x'))^{2}}{2\sigma^{2}K}} \left(\frac{1}{K}\right)^{k} + O\left(\left(\frac{1}{K}\right)^{N+1}\right)$$

where we introduced a new summation index $2k = 2p + j + (j \mod 2)$ and switched the order of the sums. The terms in the inner brackets can all be computed explicitly. We just give the first two as illustration:

$$I_0 + \frac{e_{1,G}(x)}{K^{1/2}}I_1 + \dots = \left(\frac{\sqrt{2\pi}}{\sigma} + \frac{\sqrt{2\pi}}{\sigma^3}\frac{e_{2,G}(x) + ie_{1,G}(x)\xi(\psi_{\Gamma}^K(x))}{K} + O\left(\frac{1}{K^2}\right)\right)e^{-\frac{\xi(\psi_{\Gamma}^K(x))^2}{2\sigma^2K}}.$$

Putting all the above together gives, for any $x \in X$ and $x' \in p^{-1}(x)$,

$$\begin{split} \mathcal{L}^{K}G(x)\mathbb{1}_{\{F_{K}(x)=\xi(\psi_{\Gamma}^{K}(x'))\}} &= \frac{\int_{X} G \,\mathrm{d}\boldsymbol{m}}{\sigma\sqrt{2\pi}} e^{-\frac{\xi\left(\psi_{\Gamma}^{K}(x')\right)^{2}}{2\sigma^{2}K}} \frac{1}{\sqrt{K}} \\ &+ \frac{1}{\sigma^{3}\sqrt{2\pi}} \left(e_{2,G}(x) + ie_{1,G}(x)\xi(\psi_{\Gamma}^{K}(x))\right) \, e^{-\frac{\xi\left(\psi_{\Gamma}^{K}(x')\right)^{2}}{2\sigma^{2}K}} \frac{1}{(\sqrt{K})^{3}} \\ &+ \frac{1}{2\pi} \sum_{k=2}^{N} \left(\sum_{j=0}^{2k} f_{k,j}(x)\xi(\psi_{\Gamma}^{K}(x'))^{2k-j}\right) \, e^{-\frac{\xi\left(\psi_{\Gamma}^{K}(x')\right)^{2}}{2\sigma^{2}K}} \frac{1}{(\sqrt{K})^{2k+1}} \\ &+ O\left(\frac{1}{(\sqrt{K})^{2N+3}}\right), \end{split}$$

for real bounded functions $f_{k,j}$.

Concluding the argument in the case d = 1, combining with (15). We use the above displayed equation for expression of $\mathcal{L}^{K}G(x)\mathbb{1}_{\{F_{K}(x)=\xi(\psi_{\Gamma}^{K}(x'))\}}$ inside (15) to get

$$\begin{split} \int_0^T G_{\Gamma} \circ \phi_t(x) \, dt &= \frac{\int_{X_{\Gamma}} G_{\Gamma} \, \mathrm{d}\boldsymbol{m}}{\sigma \sqrt{2\pi}} \frac{1}{\sqrt{K}} \frac{1}{\lambda^K} \int_0^{\lambda^K T} e^{-\frac{\xi \left(\psi_{\Gamma}^K(x')\right)^2}{2\sigma^2 \kappa}} \circ \phi_r \circ \psi_{\Gamma}^K(x') \, dr. \\ &+ \frac{1}{2\pi} \sum_{k=1}^N \frac{1}{(\sqrt{K})^{2k+1}} \left(\sum_{j=0}^{2k} \frac{1}{\lambda^K} \int_0^{\lambda^K T} f_{k,j}(x) \xi (\psi_{\Gamma}^K(x'))^{2k-j} e^{-\frac{\xi \left(\psi_{\Gamma}^K(x')\right)^2}{2\sigma^2 \kappa}} \circ \phi_r \circ \psi_{\Gamma}^K(x') \, dr \right) \\ &+ \frac{1}{\lambda^K} \int_0^{\lambda^K T} E_{K,N} G \circ \phi_r \circ \psi_{\Gamma}^K(x') \, dr, \end{split}$$

where $||E_{K,N}|| = O\left(\frac{1}{(\sqrt{K})^{2N+3}}\right)$ and $x' \in p^{-1}(x)$.

Let $x_K = \psi_{\Gamma}^K(x)$. Recall that one main assumption of the theorem we prove here is that we choose $K \in \mathbb{N}$ so large that ψ_{Γ}^K maps the flow-line between x' and $\phi_T(x')$ into a single copy of the fundamental domain. Thus, $\xi\left(\psi_{\Gamma}^K(\phi_r(x_K))\right) = \xi\left(\psi_{\Gamma}^K(x')\right)$ is constant for all $0 \le r \le \lambda^K T$. Thus,

$$\frac{1}{\lambda^K} \int_0^{\lambda^K T} e^{-\frac{\xi \left(\psi_{\Gamma}^K(x')\right)^2}{2\sigma^2 K}} \cdot \phi_r \circ \psi_{\Gamma}^K(x) \, dr = e^{-\frac{\xi \left(\psi_{\Gamma}^K(x')\right)^2}{2\sigma^2 K}} \cdot \left(\frac{1}{\lambda^K} \int_0^{\lambda^K T} 1 \, dr\right) = e^{-\frac{\xi \left(\psi_{\Gamma}^K(x')\right)^2}{2\sigma^2 K}} \cdot T$$

By a similar argument,

$$\frac{1}{\lambda^{K}} \int_{0}^{\lambda^{K}T} f_{k,j}(x) \xi(\psi_{\Gamma}^{K}(x'))^{2k-j} e^{-\frac{\xi(\psi_{\Gamma}^{K}(x'))^{2}}{2\sigma^{2}K}} \circ \phi_{r} \circ \psi_{\Gamma}^{K}(x') dr$$
$$= \xi(\psi_{\Gamma}^{K}(x'))^{2k-j} e^{-\frac{\xi(\psi_{\Gamma}^{K}(x'))^{2}}{2\sigma^{2}K}} \frac{1}{\lambda^{K}} \int_{0}^{\lambda^{K}T} f_{k,j}(\phi_{r}(x_{K}) dr.$$

Since the functions f_{kj} are bounded, $f_{k,j}^*(x) := \frac{1}{\lambda^K} \int_0^{\lambda^K T} f_{k,j}(\phi_r(x_K) dr) dr$ are bounded as well.

Thus,

$$\begin{split} \int_{0}^{T} G_{\Gamma} \circ \phi_{t}(x) \, \mathrm{d}t &= \frac{\int_{X_{\Gamma}} G \, \mathrm{d}m}{\sigma \sqrt{2\pi}} e^{-\frac{\xi \left(\psi_{\Gamma}^{K}(x')\right)^{2}}{2\sigma^{2}K}} \frac{T}{\sqrt{K}} \\ &+ \frac{1}{2\pi} \sum_{k=2}^{N} \left(\sum_{j=0}^{2k} f_{k,j}^{*}(x) \xi (\psi_{\Gamma}^{K}(x'))^{2k-j} \right) \, e^{-\frac{\xi \left(\psi_{\Gamma}^{K}(x')\right)^{2}}{2\sigma^{2}K}} \frac{T}{(\sqrt{K})^{2k+1}} \\ &+ \frac{1}{\lambda^{K}} \int_{0}^{\lambda^{K}T} E_{K,N} G \circ \phi_{r}(x_{K}) \, \mathrm{d}r. \end{split}$$

By Lemma 3.15, $\left|\frac{1}{\lambda^{K}}\int_{0}^{\lambda^{K}T} E_{K,N}G \circ \phi_{r}(x_{K}) \mathrm{d}r\right| = O\left(\frac{T}{(\sqrt{K})^{2N+3}} \|G\|_{C^{1}}\right)$. Item I. follows with $g_{k,j} = \frac{f_{k,j}\sigma}{\int_{\sqrt{2\pi}X}G \mathrm{d}\boldsymbol{m}}$.

Item II, d = 2. Similarly to the case d = 1, we first want a precise expression of $\mathcal{L}^{K}G(x)\mathbb{1}_{\{F_{K}(x)=\xi(\psi_{\Gamma}^{K}(x'))\}}$. In this case, both $I_{j}(\Sigma, L)$ and $e_{j,G}$ in (16) are column vectors with 2^{j} entries. We already know that the product gives a bounded function.

that the product gives a bounded function. We diagonalize $\Sigma = AJA^{-1}$ for $J = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$ for a unitary matrix A, so $A^{-1} = A^*$ is the transpose of A. Note that $\sqrt{\det \Sigma^2} = \sigma_1 \sigma_2$.

of A. Note that $\sqrt{\det \Sigma^2} = \sigma_1 \sigma_2$. Lemma B.1 B computes the integrals I_0, I_1, I_2 precisely. In particular, $I_0(\Sigma, L) = \frac{2\pi}{\sigma_1 \sigma_2} e^{-\frac{1}{2} \langle \Sigma^{-1}L, \Sigma^{-1}L \rangle}$, so when inserted in (16) with $L = \xi(\psi_{\Gamma}^K(x'))/\sqrt{K}$ and after the integration $\frac{1}{\lambda^K} \int_0^{\lambda^K T}$ to pick up an extra factor T, we obtain as constant term

$$\frac{\int_{X_{\Gamma}} G_{\Gamma} \,\mathrm{d}\boldsymbol{m}}{2\pi\sqrt{\det\Sigma^2}} \cdot e^{-\frac{1}{2}\langle \Sigma^{-1}\frac{\xi(\psi_{\Gamma}^K(x'))}{\sqrt{K}}, \Sigma^{-1}\frac{\xi(\psi_{\Gamma}^K(x'))}{\sqrt{K}}\rangle} \frac{T}{K}$$

Also,

$$I_1(\Sigma, L) = \frac{2\pi i}{\sigma_1 \sigma_2} e^{-\frac{1}{2} \langle \Sigma^{-1}L, \Sigma^{-1}L \rangle} A \begin{pmatrix} \frac{1}{\sigma_1^2} (A^*L)_1 \\ \frac{1}{\sigma_2^2} (A^*L)_2 \end{pmatrix}$$

and *-multiplication with the purely imaginary vector $e_{1,G}$ produces a scalar linear form which we can denote as $\langle g_1(x), \cdot \rangle$. Then applying integration $\frac{1}{\lambda^K} \int_0^{\lambda^K T}$ to (16) gives as linear term

$$\frac{\int_{X_{\Gamma}} G_{\Gamma} \,\mathrm{d}\boldsymbol{m}}{2\pi\sqrt{\det\Sigma^{2}}} \cdot e^{-\frac{1}{2}\langle \Sigma^{-1}\frac{\xi(\psi_{\Gamma}^{K}(x'))}{\sqrt{K}}, \Sigma^{-1}\frac{\xi(\psi_{\Gamma}^{K}(x'))}{\sqrt{K}}\rangle} \frac{T}{K} \frac{\langle g_{1}(x), \xi(\psi_{\Gamma}^{K}(x'))\rangle}{K}.$$

For the quadratic term, we have $I_2(\Sigma, L) = \frac{2\pi}{\sigma_1 \sigma_2} e^{-\frac{1}{2} \langle \Sigma^{-1}L, \Sigma^{-1}L \rangle} (Q(L \otimes L) + Q')$, *-multiplication with the real vector $e_{2,G}$ produces a scalar quadratic form which we can denote as $\langle \cdot, g'_2(x) \cdot \rangle + g'_2(x)$. Replacing L by $\xi(\psi_{\Gamma}^K(x'))/\sqrt{K}$ and then applying integration $\frac{1}{\lambda^K} \int_0^{\lambda^K T}$ to (16) gives as quadratic term

$$\frac{\int_{X_{\Gamma}} G_{\Gamma} \,\mathrm{d}\boldsymbol{m}}{2\pi\sqrt{\det\Sigma^{2}}} \cdot e^{-\frac{1}{2}\langle \Sigma^{-1} \frac{\xi(\psi_{\Gamma}^{K}(x'))}{\sqrt{K}}, \Sigma^{-1} \frac{\xi(\psi_{\Gamma}^{K}(x'))}{\sqrt{K}}\rangle} \frac{T}{K} \frac{\langle \xi(\psi_{\Gamma}^{K}(x')), g_{2}(x)\xi(\psi_{\Gamma}^{K}(x'))\rangle + g_{2}'}{K^{2}},$$

as required.

Using Theorem 3.3 and Lemma 3.13, we complete

Proof of Theorem 3.5. We prove Item (i). Item (ii) follows similarly.

By Theorem 3.3, the following holds for bounded functions $g_{k,j}$.

$$\int_{\mathcal{F}} \int_{0}^{T} G \circ \phi_{t}(x) \, \mathrm{d}t \, \mathrm{d}\boldsymbol{m} = \frac{\int_{X_{\Gamma}} G \, \mathrm{d}\boldsymbol{m}}{\sigma \sqrt{2\pi}} \frac{T}{\sqrt{K}} \cdot \int_{X_{\Gamma}} e^{-\frac{\xi \left(\psi_{\Gamma}^{K}(x')\right)^{2}}{2\sigma^{2}K}} \, \mathrm{d}\boldsymbol{m} \\ + \sum_{k=1}^{N} \frac{T}{(\sqrt{K})^{2k+1}} \left(\sum_{j=0}^{2k} \int_{X_{\Gamma}} g_{k,j}(x) \xi (\psi_{\Gamma}^{K}(x'))^{2k-j} e^{-\frac{\xi \left(\psi_{\Gamma}^{K}(x)\right)^{2}}{2\sigma^{2}K}} \, \mathrm{d}\boldsymbol{m} \right) + O\left(\frac{T}{(\sqrt{K})^{2N+3}}\right).$$
(18)

We want to apply Lemma 3.13 with $f_q(x,w) = e^{-\frac{w^2}{2}}w^q g_{k,j,q}(x)$, with $q \in \{0,\ldots,2k-j\}$, w replaced by $\frac{\xi(\psi_{\Gamma}^K(x'))}{\sigma\sqrt{K}}$ and $g_{k,j,0} = 1$ and $g_{k,j,q} = g_{k,j}$ for $q \neq 0$. By Lemma 3.13 (ii) (so, the case q = 0),

$$\int_{\mathcal{F}} e^{-\frac{\xi \left(\psi_{\Gamma}^{K}(x')\right)^{2}}{2\sigma^{2}K}} \, \mathrm{d}\boldsymbol{m} = \mathbb{E}(f_{0}(\chi)) + \sum_{j=1}^{n} \frac{d_{j}}{K^{j/2}} + o(K^{-n/2}),$$

for real constants d_i .

To deal with the sum in (18), we apply Lemma 3.13(i) with $f_q(x,w) = e^{-\frac{w^2}{2}} w^q g_{k,j,q}(x)$, with $q \in \{0, \ldots, 2k - j\}$ described in the last to previous paragraph. This ensures that the sum $\sum_{j=0}^{2k}$ of the integrals in (18) convergence to $\sum_{j=0}^{2k} d_{k,j}$ for real constants $d_{k,j}$.

3.5Banach spaces and Proof of Proposition 3.8

There are several choices in the literature for the Banach spaces we can use, see the surveys [7, Section 2] and [11]. For the automorphism ψ it is convenient to work with a variant of the spaces introduced in [12] (see also [15] and references therein for generalizations applicable to billiards) applicable to a class of hyperbolic maps with singularities. A possible alternative choice would be the anisotropic Banach spaces considered in [17], which are a great tool for studying the Ruelle spectrum for general pseudo Anosov maps. In the current setup we are only interested in the spectral gap of the transfer operator of the above mentioned simple automorphism (along properties of the twist). This is why we use the spaces in [11], which among others, allows us to use some facts already established for this class of automorphisms.

We find it convenient to work with a slight modification of the Banach spaces considered in [11] for the purpose of obtaining limit theorem via spectral methods for a general class of baker maps⁵. For a similar (simplified) variation of the spaces in [12] of the Banach spaces in [11] we refer to [29], which focused on some two-dimensional, non-uniformly hyperbolic versions of Pomeau-Manneville maps.

The automorphism ψ resembles a baker map except for the existence of singular (and potentially and marked) points, see Section 2.1. For a baker map the singularities are given by the set of discontinuity points. In the setup of ψ , we say that a point $s \in X$ is singular if the cone angle at s is not 2π . The difference in the type of singularities introduces a difference in the class of admissible leaves. In all other aspects the variant of the Banach spaces in [11] remain the same in the set of ψ . We summarize below the ingredients of these Banach spaces, using the notation of [11], as to emphasize that the case of the automorphism ψ (regarding the spectral gap for \mathcal{L}) is one of the easiest possible examples that the spaces introduced in [12] can treat.

We remark that Proposition 3.8, of which proof we sketch below, is not new with us, and that our only tasks is collect the statements scattered throughout [11] and similar papers mentioned below.

3.5.1**Definitions of Banach spaces**

Although, the presence of a (natural) Markov partition is not a crucial element in the construction in [11] for baker type maps, it does simplify the writing. The presence of this type of Markov structure

⁵The baker map itself is defined as $b(x, y) = (2x \mod 1, \frac{1}{2}(y + \lfloor 2x \rfloor))$ on the unit square.

considerably simplifies the description of admissible leaves. In particular, it allows us to define admissible leaves as full unstable segments. For the same reason, that of simplicity, we will take advantage of the Markov partition \mathcal{P} .

We define the set \mathcal{W}^s of *admissible leaves* as the set of stable segments W that exactly stretch across an element $P \in \mathcal{P}$ such that its (one dimensional) interior is contained in the interior of P. Note that, for any such $W \in \mathcal{W}^s$, the stable segment $\psi^{-1}W$ can be decomposed into a finite union of elements of \mathcal{W}^s .

Any $W \in \mathcal{W}^s$ has an affine parametrization $\{\chi_W(r) : r \in [0, l]\}$, where l is the length of W. Then, for any measurable function $h : W \to \mathbb{C}$, we write

$$\int_W h \,\mathrm{d}\boldsymbol{m} = \int_0^l h \circ \chi_W(r) \,\mathrm{d}r.$$

It is usual to construct \mathcal{P} based on stable and unstable manifold of integers lattice points.

Let $\alpha \in [0,1]$. For any $W \in \mathcal{W}^s$, we let $C^{\alpha}(W, \mathbb{C})$ denote the Banach space of complex-valued functions W with Hölder exponent α , equipped with the norm

$$|h|_{C^{\alpha}(W,\mathbb{C})} = \sup_{z \in W} |h(z)| + \sup_{z,w \in W} \frac{|h(z) - h(w)|}{|z - w|^{\alpha}}.$$

Such a set is a collection of local unstable manifolds that do not contain a singularity point. From here onward all required definitions are as in [11, Section 2.2].

We say that $\varphi \in C^{\alpha}(X, \mathbb{C})$ if it is $C^{\alpha}(W, \mathbb{C})$ for all $W \in \mathcal{W}^{s}$. Given $h \in C^{1}(X, \mathbb{C})$, define the *weak* norm by

$$\|h\|_{\mathcal{B}_w} := \sup_{W \in \Sigma} \sup_{|\phi|_{C^1(W,\mathbb{C})} \le 1} \int_W h\phi \, \mathrm{d}\boldsymbol{m}.$$

Given $\alpha \in [0, 1)$, define the strong stable norm by

$$\|h\|_s := \sup_{W \in \Sigma} \sup_{|\phi|_{C^{\alpha}(W,\mathbb{C})} \le 1} \int_W h\phi \, \mathrm{d}\boldsymbol{m}.$$

For any two aligned⁶ admissible leaves $W_1, W_2 \in \mathcal{W}^s$ in the same atom of \mathcal{P} , let $d(W_1, W_2)$ denote the distance in the unstable direction between W_1 and W_2 . In other words, if $W_i = \{\chi_i(r) : r \in [0, l]\}$, then $d(W_1, W_2)$ is the length of the segment in the unstable direction connecting $\chi_1(r)$ to $\chi_2(r)$.

With the same notation as above, for two functions $\varphi_i \in C^1(W_i, \mathbb{C})$, with i = 1, 2, we also define

$$d_0(\varphi_1,\varphi_2) = \sup_{r \in [0,l]} |\varphi_1 \circ \chi_r(x_1) - \varphi_2 \circ \chi_r(x_2)|.$$

Next define the strong unstable norm by

$$\|h\|_{u} := \sup_{W_{1}, W_{2} \in \Sigma} \sup_{|\varphi_{i}|_{C^{1}} \leq 1, d_{0}(\varphi_{1}, \varphi_{2}) = 0} \frac{1}{d(W_{1}, W_{2})^{1-\alpha}} \left| \int_{W_{1}} h\varphi_{1} \, \mathrm{d}\boldsymbol{m} - \int_{W_{2}} h\varphi_{2} \, \mathrm{d}\boldsymbol{m} \right|.$$

Finally, the strong norm is defined by $\|\varphi\|_{\mathcal{B}} = \|\varphi\|_s + \|\varphi\|_u$. These norms are exactly those of [11, Section 2.3].

Define the weak space \mathcal{B}_w to be the completion of $C^1(X)$ in the weak norm and define \mathcal{B} to be the completion of $C^1(X)$ in the strong norm.

Lemma 3.16. [11, Lemma 2.4] (see also [29, Lemma 7.2]) We have the following sequence of continuous, injective embeddings: $C^1(X) \subset \mathcal{B} \subset \mathcal{B}_w \subset (C^1(X))^*$. Moreover, the unit ball of \mathcal{B} is relatively compact in \mathcal{B}_w .

⁶i.e., the one is obtained from the other by a translation in the unstable direction.

3.5.2 Well-definedness and boundedness of \mathcal{L} on \mathcal{B} and \mathcal{B}_w . Proof of Proposition 3.8(a)

Recall that ψ is piecewise affine (so ψ is $C^1(W)$, for any $W \in \mathcal{W}^s$) and note that for any $\alpha \in [0, 1]$, for any $W \in \mathcal{W}^s$ and for any $\varphi \in C^{\alpha}(W, \mathbb{C}), \ \varphi \circ \psi \in C^{\alpha}(X, \mathbb{C})$. Moreover, for any $n \ge 1, \ \psi^{-n}(W)$ consists of a union of leaves in \mathcal{W}^s and the transfer operator \mathcal{L} of ψ is defined as

$$\mathcal{L}h(\varphi) = h(\varphi \circ \psi), \quad \text{for all } h \in C^{\alpha}(\mathcal{W}^s) \text{ and } \varphi \in (C^{\alpha}(\mathcal{W}^s))^*.$$
(19)

Recall that Lebesgue measure \boldsymbol{m} is invariant for ψ . We identify h with the measure $d\mu = h d\boldsymbol{m}$. Then $h \in C^1(\mathcal{W}^s) \subset (C^1(\mathcal{W}^s))^*$ and $\mathcal{L}h$ is associated with the measure having density

$$\mathcal{L}h(x) = \frac{h \circ \psi^{-1}(x)}{J_{\psi}(\psi^{-1}(x))} = h \circ \psi^{-1}(x),$$
(20)

where J_{ψ} is the Jacobian of ψ with respect to \boldsymbol{m} , which is equal to 1 (since the contraction and expansion are the same).

In general, it is not true that for systems with discontinuities, $\mathcal{L}(C^1(X)) \subset C^1(X)$ and hence it is not obvious that \mathcal{L} is well defined on \mathcal{B} : see, for instance, [11, Footnote 13]. However, in the current setup of ψ , similar to the first line of the proof of [11, Lemma 4.1], $\mathcal{L}h \in C^1(\mathcal{W}^s)$ (since for any $W \in \Sigma$, $\psi^{-1}W$ is an exact union of leaves in \mathcal{W}^s). Hence, $\mathcal{L}(C^1(X)) \subset C^1(X)$ and \mathcal{L} is well defined on \mathcal{B} .

Also, by [11, Lemma 4.1], \mathcal{L} acts continuously on \mathcal{B} and \mathcal{B}_w and the proof of [11, Theorem 2.5] (for baker type maps) goes word for word the same in the setup of ψ . This yields

Lemma 3.17. [11, Theorem 2.5] The operator \mathcal{L} is quasi-compact as an operator on \mathcal{B} . That is, its spectral radius is 1 and its essential spectral radius is strictly less than 1. Moreover, 1 is a simple eigenvalue, and all other eigenvalues have modulus strictly less than 1.

Lemmas 3.16 and 3.17 is exactly the content of Proposition 3.8(a).

3.5.3 Analyticity of the twisted transfer operator $\mathcal{L}_u f = \mathcal{L}(e^{iuF}f), f \in \mathcal{B}$. Proof of Proposition 3.8(b)

The Frobenius function $F: X \to \mathbb{Z}, x \mapsto \xi \circ \psi_{\Gamma}(x') - \xi(x')$ for $x' \in p^{-1}(x)$, is not globally C^1 , hence the simple argument of [11, Lemma 4.8] cannot go through. However, F is constant on each element of the partition $\mathcal{P}^R \lor \psi^{-1} \mathcal{P}^R$ (hence C^{∞} on each element of $\mathcal{P}^R \lor \psi^{-1} \mathcal{P}^R$). As a consequence, the argument for the analyticity of the twisted transfer operator is a much simplified version of the argument used in the proof of [16, Lemma 3.9] (essentially a consequence of the arguments used in [13, 14, 15]).

Lemma 3.18. Let $u \in \mathbb{R}^d$, $f \in \mathcal{B}$ and $m \geq 1$. Then $\frac{d^k}{du^k} \mathcal{L}_u f$ is a linear operator on \mathcal{B} with operator norm of $O(\|F\|_{\infty}^k)$.

Proof. Using (20), compute that

$$\frac{d^k}{du^k}\mathcal{L}_u f = i^k \mathcal{L}(F^k e^{iuF} f) = i^k (F^k e^{iuF}) \circ \psi^{-1} \mathcal{L} f.$$

Since F is locally constant and since each element of $\mathcal{P} \vee \psi^{-1} \mathcal{P}$ contains no singularities in its interior, a simplified version⁷ of the argument used in [14, Lemma 3.7] (see also [16, Lemma 3.3]) shows that for any $f \in \mathcal{B}$, $f F \in \mathcal{B}$ and that for some C > 0,

$$\|fF\|_{\mathcal{B}} \le C \|f\|_{\mathcal{B}} \sup_{P_i \in \mathcal{P}^R} \|F\|_{C^{\alpha}(P_i)}.$$
(21)

Thus,

$$\left\|\frac{d^k}{du^k}\mathcal{L}_u f\right\|_{\mathcal{B}} \le C \sup_{P_i \in \mathcal{P}^R} \|(F^k e^{iuF}) \circ \psi^{-1}\|_{C^{\alpha}(P_i)} \|f\|_{\mathcal{B}} \le C \|F\|_{\infty}^k \|f\|_{\mathcal{B}}.$$

 $^{^{7}}$ The (serious) simplification comes from the simple form of admissible leaves and the fact that the Jacobian is constant.

3.6 Spectrum of \mathcal{L}_u and leading eigenvalue. Proof of Proposition 3.8(c)

We already know (see Lemma 3.17) that 1 is a simple isolated eigenvalue of \mathcal{L}_0 . Since $u \to \mathcal{L}_u$ is analytic (see Lemma 3.18), there exists $\delta > 0$ and a simple family of simple eigenvalues λ_u , analytic in $u \in (0, \delta)$ with $\lambda_0 = 1$. Standard perturbation theory (see [30] and, for instance, [22, Section 2])) ensures that for all $u \in (0, \delta)$,

$$\mathcal{L}_u^n = \lambda_u^n \Pi_u + Q_u^n, \tag{22}$$

where Π_u is the spectral projection onto the one-dimensional eigenspace associated to λ_u with $\Pi_0 f = \int_X f \, \mathrm{d}\boldsymbol{m}$ and where $\|Q_u^n\| \leq \theta^n$ for some $\theta \in (0, 1)$, and $Q_u \Pi_u = \Pi_u Q_u$. By Lemma 3.18 and standard perturbation theory (see [30]), all the eigen-elements are again analytic. That is, Π_u, Q_u are also analytic in $u \in (0, \delta)$.

3.7 Spectrum of \mathcal{L}_u for $u \in (\delta, \pi]$. Proof of Proposition 3.8(d)

The following lemma gives the required control.

Lemma 3.19. [16, Lemma C.1] Let $u \in (-\pi, \pi]$, $h \in \mathcal{B}$ and $\eta \in \mathbb{C}$ be such that $\mathcal{L}_u h = \eta h$ in \mathcal{B} and $|\eta| \geq 1$. Then either $h \equiv 0$ or $u \in 2\pi\mathbb{Z}$ and h is m-a.s. constant.

The proof of Lemma 3.19 goes word for word as [16, Proof of Lemma C.1], except for differences in notations. The differences in the definitions of the norms used in Subsection 3.5 are irrelevant for this argument. Lemma 3.19 ensures that there exists $\delta_1 \in (0, 1)$ so that $\|\mathcal{L}^n_u\|_{\mathcal{B}} \leq \delta_1^n$ for all $|u| > \delta$ and all $n \geq 1$.

3.8 Proof of Proposition 3.11(a)

We first record a technical lemma that will be instrumental in the proof of item (a) of Proposition 3.11. For integers $m, n \ge 0$ and $r_1, \ldots, r_n \in \mathbb{N}$, define the operator

$$G(m, n, r_1, \dots, r_n) = (y-1)^{-m} (y-\mathcal{L}_0)^{-1} \mathcal{L}_0(F^{\otimes r_1}) \otimes \dots \otimes (y-\mathcal{L}_0)^{-1} \mathcal{L}_0(F^{\otimes r_n}) \otimes (y-\mathcal{L}_0)^{-1}.$$
 (23)

As F takes values in \mathbb{Z}^d , the tensor products $F^{\otimes r_j}$ are vectors with d^{r_j} components.

Lemma 3.20. For all $m, n \ge 0, r_1, \ldots, r_n \in \mathbb{N}$ and real-valued $v \in \mathcal{B}$, the contour integral

$$\int_{|y-1|=\delta} G(m,n,r_1,\ldots,r_n) v \, dy \quad is \ purely \ imaginary$$

Proof. We will use induction on m, n, starting with n = 0. If m = n = 0, then

$$\int_{|y-1|=\delta} G(0,0)v \, dy = \int_{|y-1|=\delta} (y - \mathcal{L}_0)^{-1} v \, dy = 2\pi i \,\Pi_0 v \text{ is purely imaginary}$$

Now if the statement holds for n = 0 and all $0 \le m' < m$, then

$$\begin{split} \int_{|y-1|=\delta} G(m,0)v\,dy &= \int_{|y-1|=\delta} (y-1)^{-m} (y-\mathcal{L}_0)^{-1} v\,dy \\ &= \int_{|y-1|=\delta} (y-1)^{-m} (y-\mathcal{L}_0)^{-1} (v-\int_X v\,dm)\,dy \\ &+ \int_{|y-1|=\delta} (y-1)^{-m} (y-\mathcal{L}_0)^{-1} \int_X v\,dm\,dy \\ &= \int_{|y-1|=\delta} (y-1)^{-m} \left[(y-\mathcal{L}_0)^{-1} - (1-\mathcal{L}_0)^{-1} \right] (v-\int_X v\,dm)\,dy \\ &+ \int_{|y-1|=\delta} (y-1)^{-m} (1-\mathcal{L}_0)^{-1} (v-\int_X v\,d\mu)\,dy \\ &+ \int v_X \,d\mu \int_{|y-1|=\delta} (y-1)^{-(m+1)} \,dy. \end{split}$$

The third integral vanishes otherwise because m > 0. For the second integral, write $v_1 = (1 - \mathcal{L}_0)^{-1}(v - \int_X v \, d\mu)$, so $v_1 \in \mathcal{B}$ is real with $\int_X v_1 \, d\mathbf{m} = 0$. Hence the integral is purely imaginary for the same reason as the second integral. For the first integral we use the resolvent identity:

$$\int_{|y-1|=\delta} (y-1)^{-m} \left[(y-\mathcal{L}_0)^{-1} - (1-\mathcal{L}_0)^{-1} \right] \left(v - \int_X v \, \mathrm{d}\boldsymbol{m} \right) \, dy$$

= $-\int_{|y-1|=\delta} (y-1)^{1-m} (y-\mathcal{L}_0)^{-1} (1-\mathcal{L}_0)^{-1} (v - \int_X v \, \mathrm{d}\boldsymbol{m}) \, dy$
= $\lambda \int_{|y-1|=\delta} (y-1)^{1-m} (y-\mathcal{L}_0)^{-1} v_1 \, dy = -\int_{|y-1|=\delta} G(m-1,0) v_1 \, dy$

This is purely imaginary by the induction hypothesis and since $m \ge 1$.

Now we continue with the induction step over n; in this case $G(m, n, r_1, \ldots, r_n)$ contains n + 1 factors $(y - \mathcal{L}_0)^{-1}$, and therefore it has a pole at 1 of order $\leq n + 1$. Our induction hypothesis is that $\int_{|y-1|=\delta} G(m', n', r_1, \ldots, r_{n'}) v \, dy$ is purely imaginary for every real-valued $v \in \mathcal{B}$ when $0 \leq n' < n$ and $m' \geq 0$ or when n' = n and $n \leq m' < m$. Then

$$\int_{|y-1|=\delta} G(m,n,r_1,\dots,r_n) v \, dy = \int_{|y-1|=\delta} G(m,n,r_1,\dots,r_n) (v - \int_X v \, \mathrm{d}\boldsymbol{m}) \, dy \\ + \int_{|y-1|=\delta} G(m,n-1,r_1,\dots,r_{n-1}) \mathcal{L}_0(F^{\otimes r_n}) \otimes (y-\mathcal{L})^{-1} \int_X v \, \mathrm{d}\boldsymbol{m} \, dy.$$

The second integral is equal to $\int_X v \, \mathrm{d}\boldsymbol{m} \cdot \int_{|y-1|=\delta} G(m+1, n-1, r_1, \dots, r_{n-1}) \mathcal{L}_0(F^{\otimes r_n}) \, dy$ and thus purely imaginary by the induction hypothesis. We rewrite the first integral to

$$\int_{|y-1|=\delta} G(m, n-1, r_1, \dots, r_{n-1}) \mathcal{L}_0(F^{\otimes r_n}) \otimes \left[(y - \mathcal{L}_0)^{-1} - (1 - \mathcal{L}_0)^{-1} \right] (v - \int_X v \, \mathrm{d}\boldsymbol{m}) \, dy \\ + \int_{|y-1|=\delta} G(m, n-1, r_1, \dots, r_{n-1}) \mathcal{L}_0(F^{\otimes r_n}) \otimes (1 - \mathcal{L}_0)^{-1} (v - \int_X v \, \mathrm{d}\boldsymbol{m}) \, dy.$$

With $v_2 = \mathcal{L}_0(F^{\otimes r_n})(1-\mathcal{L})^{-1}(v-\int_X v \, \mathrm{d}\boldsymbol{m})$, the second term becomes $\int_{|y-1|=\delta} G(m, n-1, r_1, \dots, r_{n-1})v_2 \, dy$, which is purely imaginary by induction. The resolvent identity applied to the first term gives

$$-\int_{|y-1|=\delta} G(m-1, n-1, r_1, \dots, r_{n-1}) \mathcal{L}_0(F^{\otimes r_n}) \otimes (y-\mathcal{L}_0)^{-1} (v-\int_X v \, \mathrm{d}\boldsymbol{m}) \, dy$$
$$= -\int_{|y-1|=\delta} G(m-1, n, r_1, \dots, r_n) (v-\int_X v \, \mathrm{d}\boldsymbol{m}) \, dy.$$

This is purely imaginary by the induction hypothesis. If, however, m = n, then the integrand contains a factor $(y-1)^{n+1}$, which removes the pole (of order $\leq n+1$) of the remaining part of the integrand, and hence Cauchy's Theorem gives again that the integral vanishes. This completes the induction and the entire proof.

We can now complete

Proof of Proposition 3.11(a). Recall that $\Pi_u = \frac{1}{2\pi i} \int_{|y-1|=\delta} (y-\mathcal{L}_u)^{-1} dy$ is the eigenprojection w.r.t. the leading eigenvalue. Clearly $\Pi_0 v$ is real for a real $v \in \mathcal{B}$. Taking the *j*-th derivative w.r.t. *u* and then evaluating at u = 0, gives $2\pi i^{j+1}$ times the contour integral of a linear combination of terms of the form (23). These integrals are all purely imaginary by Lemma 3.20, so the *j*-th derivative produces alternatingly real and purely imaginary outcomes. So, $\Pi_0^{(j)}v$ has only real entries if j is even and purely imaginary entries if j is odd.

It remains to look at the *j*-th derivatives $\left(\frac{\lambda}{A}\right)_0^{(j)}$ of $\frac{\lambda_u}{A_u}$ evaluated at 0. We recall that λ_u is the eigenvalue and $A_u = e^{-\langle \Sigma u, \Sigma u \rangle}$. Let $v_u = \frac{\Pi_u 1}{\int \Pi_u 1 \, \mathrm{d}\boldsymbol{m}}$ be the normalized eigenvector associated with λ_u , i.e., $\int_X v_u \, \mathrm{d}\boldsymbol{m} = 1$. Since Π_u

is analytic, so is v_u . $\Pi_0^{(j)}$ has only real entries if j is even and purely imaginary entries if j is odd, the same applies to $v_0^{(j)}v$ has only real entries if j is even and purely imaginary entries if j is odd.

A simple calculation starting from $\int_X \mathcal{L}_u v_u \, \mathrm{d}\boldsymbol{m} = \lambda_u \int_X v_u \, \mathrm{d}\boldsymbol{m} = \lambda_u$ shows that we can write

$$1 - \lambda_u = \int_X (1 - e^{iuF}) \, \mathrm{d}\boldsymbol{m} + \int_X (1 - e^{iuF})(v_u - v_0) \, \mathrm{d}\boldsymbol{m}$$
$$= -\sum_{j=1}^\infty \frac{i^j u^{\otimes j}}{j!} * \mathbb{E}(F^{\otimes j}) + \int_X \left(\sum_{j=1}^\infty \frac{i^j u^{\otimes j}}{j!} F^{\otimes j}\right) * \left(\sum_{j=1}^\infty \frac{i^j u^{\otimes j}}{j!} v_0^{(j)}\right) \, \mathrm{d}\boldsymbol{m}.$$

Using that $v_0^{(m)}$ has only real entries if m is even and purely imaginary entries if m is odd, we can see that the same applies to every term of product of sums $\left(\sum_{j=1}^{\infty} \frac{i^j u^{\otimes j}}{j!} F^{\otimes j}\right) * \left(\sum_{j=1}^{\infty} \frac{i^j u^{\otimes j}}{j!} v_0^{(j)}\right)$. Clearly, every j term in the sum $\sum_{j=1}^{\infty} \frac{i^j u^{\otimes j}}{j!} \mathbb{E}(F^{\otimes j})$ has real entries if j is even or purely imaginary entries if j is odd. Thus, $(\lambda)_{0}^{(j)}$ has real entries if j is even or purely imaginary entries if j is odd.

Finally, dividing λ_u by $A_u = e^{-\frac{1}{2}\langle \Sigma u, \Sigma u \rangle}$ makes no difference since A_u is real.

3.9Proof of Lemma 3.15

Note that

$$\left|\frac{1}{\lambda^K}\int_0^{\lambda^K T} E_{K,N}G \circ \phi_r(x_K) \, dr\right| = \left|\frac{1}{\lambda^K}\int_{\tilde{W}} E_{K,N}G\right|,$$

where \tilde{W} is the segment connecting x_K and $\phi_{\lambda^K T}(x_K)$.

Recalling the definition of weak norm $\|h\|_{\mathcal{B}_w}$ in subsection 3.5.1, we see that

$$\left|\frac{1}{\lambda^K}\int_{\tilde{W}} E_{K,N}G \circ \phi_r(x_K) \, dr\right| \leq \frac{|\tilde{W}|}{\lambda^K} \|E_{K,N}G\|_{\mathcal{B}_w}.$$

Recalling $C^1 \subset \mathcal{B} \subset \mathcal{B}_w$, $||E_{K,N}G||_{\mathcal{B}_w} \leq ||E_{K,N}||_{\mathcal{B}} ||G||_{C^1}$, and Lemma 3.15 follows.

A On tensor products \otimes and *

We briefly review tensor products \otimes and "scalar" product *. If A is an $c \times d$ matrix, and A' an $c' \times d'$ -matrix, then $A \otimes A'$ is an $cc' \times dd'$ -matrix defined as

$$A \otimes A' = \begin{pmatrix} a_{11} & \dots & a_{1d} \\ \vdots & \ddots & \vdots \\ a_{c1} & \dots & a_{cd} \end{pmatrix} \otimes \begin{pmatrix} a'_{11} & \dots & a'_{1d'} \\ \vdots & \ddots & \vdots \\ a'_{c'1} & \dots & a'_{c'd'} \end{pmatrix}$$
$$= \begin{pmatrix} a_{11} \begin{pmatrix} a'_{11} & \dots & a'_{1d'} \\ \vdots & \ddots & \vdots \\ a'_{c'1} & \dots & a'_{c'd'} \end{pmatrix} & \dots & a_{1d} \begin{pmatrix} a'_{11} & \dots & a'_{1d'} \\ \vdots & \ddots & \vdots \\ a'_{c'1} & \dots & a'_{c'd'} \end{pmatrix} \\ & \vdots & \ddots & \vdots \\ a_{c1} \begin{pmatrix} a'_{11} & \dots & a'_{1d'} \\ \vdots & \ddots & \vdots \\ a'_{c'1} & \dots & a'_{c'd'} \end{pmatrix} & \dots & a_{cd} \begin{pmatrix} a'_{11} & \dots & a'_{1d'} \\ \vdots & \ddots & \vdots \\ a'_{c'1} & \dots & a'_{c'd'} \end{pmatrix} \end{pmatrix}.$$

Note that this tensor product is not commutative: $A \otimes A'$ is only isomorphic but in general not equal to $A' \otimes A$.

Throughout, our vectors $u \in \mathbb{C}^d$ will be considered as column vectors, also if they appear as the arguments of (scaler) functions $f : \mathbb{C}^d \to \mathbb{C}$. Now the *j*-fold tensor $u^{\otimes j}$ product of u with itself is defined inductively:

$$u^{\otimes 0} = 1 \in \mathbb{C}, \quad u^{\otimes 1} = u = \begin{pmatrix} u_1 \\ \vdots \\ u_d \end{pmatrix}, \quad u^{\otimes 2} = u \otimes u = \begin{pmatrix} u_1 u_1 \\ u_1 u_2 \\ \vdots \\ u_1 u_d \\ \vdots \\ \vdots \\ u_d u_1 \\ u_d u_2 \\ \vdots \\ u_d u_d \end{pmatrix} \quad \text{and} \quad u^{\otimes j} = u \otimes u^{\otimes j-1}.$$

Thus $u^{\otimes j}$ is a column vector with d^j entries.

It goes similarly with derivatives of scalar functions $f : \mathbb{C}^d \to \mathbb{C}$:

$$f^{(0)} = f, \qquad f' = f^{(1)} = \nabla f = \begin{pmatrix} \frac{\partial f}{\partial u_1} \\ \vdots \\ \frac{\partial f}{\partial u_d} \end{pmatrix}, \quad f^{(2)} = \begin{pmatrix} \frac{\partial^2 f}{\partial u_1 \partial u_d} \\ \vdots \\ \frac{\partial^2 f}{\partial u_1 \partial u_d} \\ \vdots \\ \frac{\partial^2 f}{\partial u_d \partial u_1} \\ \frac{\partial^2 f}{\partial u_d \partial u_2} \\ \vdots \\ \frac{\partial^2 f}{\partial u_d \partial u_d} \end{pmatrix} \text{ and } f^{(j)} = \begin{pmatrix} \frac{\partial}{\partial u_1} \\ \frac{\partial}{\partial u_2} \\ \vdots \\ \frac{\partial}{\partial u_d} \\ \frac{\partial^2 f}{\partial u_d \partial u_d} \end{pmatrix}$$

Next we define the *-product u * v on matrices of the same size as $A * A' = \sum_{i,j} a_{ij}a'_{ij}$, if $u, u' \in \mathbb{C}^d$ are column vectors, then $u * u = \sum_{i=1}^d u_i u'_i$ is the usual scalar product. We can extend this to $c \times d$ -matrix A and $c' \times d'$ -matrix A' provided cd = c'd' and c' is a multiple of c (or vice versa). For this, we divide A into $d/d' c \times d'$ -matrices and stack them up to a single $c' \times d'$ -matrix an then *-multiply with A'. For example

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \end{pmatrix} * \begin{pmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \\ a'_{31} & a'_{32} \\ a'_{41} & a'_{42} \\ a'_{51} & a'_{52} \\ a'_{61} & a'_{62} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{13} & a_{14} \\ a_{23} & a_{24} \\ a_{15} & a_{16} \\ a_{26} & a_{26} \end{pmatrix} * \begin{pmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \\ a'_{31} & a'_{32} \\ a'_{41} & a'_{42} \\ a'_{51} & a'_{52} \\ a'_{61} & a'_{62} \end{pmatrix}$$
$$= a_{11}a'_{11} + a_{12}a'_{12} + a_{21}a'_{21} + a_{22}a'_{22} + a_{13}a'_{31} + a_{14}a'_{32} \\ + a_{23}a'_{41} + a_{24}a'_{42} + a_{15}a'_{16} + a_{51}a'_{52} + a_{25}a'_{26} + a_{61}a'_{62}.$$

For example, for a symmetric matrix $\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$ and vector $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, we get

$$\Sigma^{2} * u^{\otimes 2} = \begin{pmatrix} \sigma_{11}^{2} + \sigma_{12}^{2} & 2\sigma_{12}^{2} \\ 2\sigma_{12}^{2} & \sigma_{12}^{2} + \sigma_{22}^{2} \end{pmatrix} * \begin{pmatrix} u_{1}^{2} \\ u_{1}u_{2} \\ u_{2} \end{pmatrix}$$
$$= (\sigma_{11}^{2} + \sigma_{12}^{2})u_{1}^{2} + (\sigma_{11}\sigma_{12} + 2\sigma_{12}^{2} + \sigma_{12}\sigma_{22})u_{1}u_{2} + (\sigma_{12}^{2} + \sigma_{22}^{2})u_{2}^{2}$$
$$= \begin{pmatrix} \sigma_{11}u_{1} + \sigma_{21}u_{2} \\ \sigma_{12}u_{1} + \sigma_{22}u_{2} \end{pmatrix} * \begin{pmatrix} \sigma_{11}u_{1} + \sigma_{21}u_{2} \\ \sigma_{12}u_{1} + \sigma_{22}u_{2} \end{pmatrix} = \langle \Sigma u, \Sigma u \rangle,$$
(24)

as used in Section 3

We can extend * even further to column vectors $u \in \mathbb{C}^d$ and $v \in \mathbb{C}^{d'}$, where we assume that d' is a multiple of d. To compute u * v, divide v into d'/d blocks of height d, multiply the k-th entry of uwith the k-th block, and add up these blocks. For example,

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} * \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{pmatrix} = u_1 \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + u_2 \cdot \begin{pmatrix} v_4 \\ v_5 \\ v_6 \end{pmatrix} = \begin{pmatrix} u_1 v_1 + u_2 v_4 \\ u_1 v_2 + u_2 v_5 \\ u_1 v_3 + u_2 v_6 \end{pmatrix},$$

and if d is a multiple of d', we divide u into d/d' block, etc. Thus * acts commutatively on column vectors. Also, if d = d', then $u * v = \sum_{i=1}^{d} u_i v_i$ is the usual scalar product. In this specific case d = d', Leibniz rule works: $\nabla(u * v) = \nabla u * v + u * \nabla v$, but in general, the dimensions don't match. However, we have the Taylor formula for a C^{∞} scalar function $f : \mathbb{C}^d \to \mathbb{C}$ and column vector

 $u \in \mathbb{C}^d$:

$$f(u) = \sum_{j=0}^{\infty} \frac{1}{j!} f^{(j)}(0) * u^{\otimes j} = 1 \cdot f(0) \cdot 1 + 1 \cdot \begin{pmatrix} \frac{\partial f(0)}{\partial u_1} \\ \frac{\partial f(0)}{\partial u_2} \\ \vdots \\ \frac{\partial f(0)}{\partial u_2} \end{pmatrix} * \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_d \end{pmatrix} + \frac{1}{2} \cdot \begin{pmatrix} \frac{\partial^2 f(0)}{\partial u_1 \partial u_2} \\ \vdots \\ \frac{\partial^2 f(0)}{\partial u_1 \partial u_d} \\ \vdots \\ \vdots \\ \frac{\partial^2 f(0)}{\partial u_d \partial u_1} \\ \frac{\partial^2 f(0)}{\partial u_d \partial u_d} \end{pmatrix} * \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ \frac{\partial^2 f(0)}{\partial u_d \partial u_1} \\ \frac{\partial^2 f(0)}{\partial u_d \partial u_d} \end{pmatrix} + \dots$$

Integrals used in the proof of mains results Β

In the proof of Theorem 3.3, we need the following lemma.

A. Assume d = 1. Given $\sigma, L \in \mathbb{R}$ and $j \in \{0, 1, 2, ...\}$, write Lemma B.1.

$$I_j(\sigma,L) = \int_{\mathbb{R}} e^{-\frac{\sigma^2}{2}u^2} e^{iLu} u^j \, du.$$

Then

$$I_0(\sigma,L) = \frac{\sqrt{2\pi}}{\sigma} e^{-\frac{L^2}{2\sigma^2}}, \quad I_1 = \frac{i\sqrt{2\pi}L}{\sigma^3} e^{-\frac{L^2}{2\sigma^2}}, \quad and \quad I_j(\sigma,L) = \frac{1}{\sigma^2} (iLI_{j-1} + (j-1)I_{j-2}).$$

B. Assume d = 2. Given a 2×2 covariance matrix Σ^2 , $L \in \mathbb{R}^2$ and $j \in \{0, 1, 2, ...\}$, write

$$I_j(\Sigma, L) = \int_{\mathbb{R}^2} e^{-\frac{1}{2} \langle \Sigma u, \Sigma u \rangle} e^{i \langle L, u \rangle} u^{\otimes j} \, du,$$

where $u^{\otimes j}$ is the *j*-fold tensor product of the vector $u = \binom{u_1}{u_2}$ with itself. Then $\vec{I_j}$ is an alternatingly real and purely imaginary vector with 2^j components. Specifically:

$$I_{0}(\Sigma,L) = \frac{2\pi}{\sigma_{1}\sigma_{2}}e^{-\frac{1}{2}\langle\Sigma^{-1}L,\Sigma^{-1}L\rangle} \quad and \quad I_{1}(\Sigma,L) = \frac{2\pi i}{\sigma_{1}\sigma_{2}}e^{-\frac{1}{2}\langle\Sigma^{-1}L,\Sigma^{-1}L\rangle}A\begin{pmatrix}\frac{1}{\sigma_{1}^{2}}(A^{*}L)_{1}\\\frac{1}{\sigma_{2}^{2}}(A^{*}L)_{2}\end{pmatrix}$$

where $\Sigma = AJA^{-1}$, $J = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$, is the diagonalization of Σ with unitary matrix A. Also

$$I_2(\Sigma, L) = \frac{2\pi}{\sigma_1 \sigma_2} e^{-\frac{1}{2} \langle \Sigma^{-1} L, \Sigma^{-1} L \rangle} (Q(L \otimes L) + Q')$$

with the quadratic form Q and vector $Q' \in \mathbb{R}^{d^2}$ made explicit in the proof.

Proofs of Lemma B.1. Item A., d=1. The integrals $I_j = I_j(\sigma, L)$ can be computed via integration by parts, namely for $j \ge 1$ we have (taking into account that integrals over odd real or imaginary parts of the integrand vanish):

$$I_{j} = I_{j}(\sigma, L) = \int_{-\infty}^{\infty} u e^{-\frac{\sigma^{2}}{2}u^{2}} e^{iLu} u^{j-1} du$$

=
$$\int_{-\infty}^{\infty} \frac{1}{\sigma^{2}} e^{-\frac{\sigma^{2}}{2}u^{2}} e^{iLu} \left(iLu^{j-1} + (j-1)u^{j-2} \right) du = \frac{1}{\sigma^{2}} (iLI_{j-1} + (j-1)I_{j-2}).$$

This recursion show that I_j is alternatingly real and purely imaginary. We compute I_0 via a change of coordinates:

$$I_{0} = \int_{-\infty}^{\infty} e^{iLu - \frac{\sigma^{2}}{2}u^{2}} du = \int_{-\infty}^{\infty} e^{-(\frac{\sigma u}{\sqrt{2}} - \frac{iL}{\sqrt{2}\sigma})^{2}} e^{-\frac{L^{2}}{2\sigma^{2}}} du$$
$$= e^{-\frac{L^{2}}{2\sigma^{2}}} \frac{\sqrt{2}}{\sigma} \int_{-\infty}^{\infty} e^{-u^{2}} du = \frac{\sqrt{2\pi}}{\sigma} e^{-\frac{L^{2}}{2\sigma^{2}}}.$$
(25)

Then the recursion gives

$$I_{1} = \frac{iL}{\sigma^{2}}I_{0} = \frac{i\sqrt{2\pi}L}{\sigma^{3}}e^{-\frac{L^{2}}{2\sigma^{2}}}, \qquad I_{2} = \frac{1}{\sigma^{2}}(iLI_{1} + I_{0}) = \frac{\sigma^{2} - L^{2}}{\sigma^{5}}\sqrt{2\pi}e^{-\frac{L^{2}}{2\sigma^{2}}},$$
$$I_{3} = iL\frac{\sqrt{2\pi}}{\sigma^{5}}e^{-\frac{L^{2}}{2\sigma^{2}}}\left(3 - \frac{L^{2}}{\sigma^{2}}\right), \qquad I_{4} = \frac{\sqrt{2\pi}}{\sigma^{5}}e^{-\frac{L^{2}}{2\sigma^{2}}}\left(3 - 6\frac{L^{2}}{\sigma^{2}} + \frac{L^{4}}{\sigma^{4}}\right),$$

and so on.

Item B., d = 2. Using diagonalization and the unitary change of coordinates u = Av (so $\langle \Sigma u, \Sigma u \rangle = \langle AJA^{-1}u, AJA^{-1}u \rangle = \langle JA^{-1}u, JA^{-1}u \rangle = \sigma_1^2 v_1^2 + \sigma_2^2 v_2^2$ and $\langle L, u \rangle = \langle A^*L, v \rangle$), we get

$$I_{0}(\Sigma,L) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}\sigma_{1}^{2}v_{1}^{2}} e^{i(A^{*}L)_{1}v_{1}} dv_{1} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\sigma_{1}^{2}v_{2}^{2}} e^{i(A^{*}L)_{2}v_{2}} dv_{2}$$

$$= I_{0}(\sigma_{1}, (A^{*}L)_{1}) \cdot I_{0}(\sigma_{2}, (A^{*}L)_{2}) = \frac{2\pi}{\sigma_{1}\sigma_{2}} e^{-\frac{1}{2}\left(\frac{1}{\sigma_{1}^{2}}(A^{*}L)_{1}^{2} + \frac{1}{\sigma_{2}^{2}}(A^{*}L)_{2}^{2}\right)}.$$

Since $\frac{1}{\sigma_1^2}(A^*L)_1^2 + \frac{1}{\sigma_2^2}(A^*L)_2^2) = \langle J^{-1}A^{-1}L, J^{-1}A^{-1}L \rangle = \langle A^{-1}\Sigma^{-1}LA^{-1}\Sigma^{-1}L \rangle = \langle \Sigma^{-1}L, \Sigma^{-1}L \rangle$, the result follows.

Using the same change of coordinates, we get

$$\begin{split} I_{1}(\Sigma,L) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\sigma_{1}^{2}v_{1}^{2}} e^{i(A^{*}L)_{1}v_{1}} e^{-\frac{1}{2}\sigma_{1}^{2}v_{2}^{2}} e^{i(A^{*}L)_{1}v_{2}} \cdot A\binom{v_{1}}{v_{2}} dv_{2} dv_{1} \\ &= A\binom{I_{1}(\sigma_{1},(A^{*}L)_{1}) \cdot I_{0}(\sigma_{2},(A^{*}L)_{2})}{I_{0}(\sigma_{1},(A^{*}L)_{1}) \cdot I_{1}(\sigma_{2},(A^{*}L)_{2})} \\ &= \frac{2\pi i}{\sigma_{1}\sigma_{2}} e^{-\frac{1}{2}\left(\frac{1}{\sigma_{1}^{2}}(A^{*}L)_{1}^{2} + \frac{1}{\sigma_{2}^{2}}(A^{*}L)_{2}^{2}\right)} A\binom{\frac{1}{\sigma_{1}^{2}}(A^{*}L)_{1}}{\left(\frac{1}{\sigma_{2}^{2}}(A^{*}L)_{2}\right)} = \frac{2\pi i}{\sigma_{1}\sigma_{2}} e^{-\frac{1}{2}\langle\Sigma^{-1}L,\Sigma^{-1}L\rangle} A\binom{\frac{1}{\sigma_{1}^{2}}(A^{*}L)_{1}}{\left(\frac{1}{\sigma_{2}^{2}}(A^{*}L)_{2}\right)}. \end{split}$$

For $I_j(\Sigma, L)$, $j \ge 2$, the same methods works, but the computations are getting increasingly lengthy. To explain a bit about j = 2, the change of coordinates now leads to the factor

$$(Av) \otimes (Av) = \begin{pmatrix} (A_{11})^2 v_1^2 + 2A_{11}A_{12}v_1v_2 + (A_{12})^2 v_2^2 \\ A_{11}A_{21}v_1^2 + (A_{11}A_{22} + A_{12}A_{21})v_1v_2 + A_{12}A_{22}v_2^2 \\ A_{21}A_{11}v_1^2 + (A_{21}A_{12} + A_{22}A_{11})v_1v_2 + A_{22}A_{12}v_2^2 \\ (A_{21})^2 v_1^2 + 2A_{21}A_{22}v_1v_2 + (A_{22})^2 v_2^2 \end{pmatrix}$$

Multiplying this with $e^{-\frac{1}{2}\sigma_1^2 v_1^2} e^{i(A^*L)_1} e^{-\frac{1}{2}\sigma_1^2 v_2^2} e^{i(A^*L)_1 v_2}$ and integrating over \mathbb{R}^2 means replacing v_1^2 by $I_2(\sigma_1, (A^*L)_1)I_0(\sigma_2, (A^*L)_2) = \frac{\sigma_1^2 - (A^*L)_1^2}{\sigma_1^5 \sigma_2}$, etc. This way we get

$$I_2(\Sigma, L) = \frac{2\pi}{\sigma_1 \sigma_2} e^{-\frac{1}{2} \langle \Sigma^{-1} L, \Sigma^{-1} L \rangle} (Q(L \otimes L) + Q')$$

for

$$Q(L \otimes L) + Q' = \begin{pmatrix} (A_{11})^2 \frac{\sigma_1^2 - (A^*L)_1^2}{\sigma_1^4} - 2A_{11}A_{12} \frac{(A^*L)_1}{\sigma_1^2} \frac{(A^*L)_2}{\sigma_2^2} + (A_{12})^2 \frac{\sigma_2^2 - (A^*L)_2^2}{\sigma_2^4} \\ A_{11}A_{21} \frac{\sigma_1^2 - (A^*L)_1^2}{\sigma_1^4} - (A_{11}A_{22} + A_{12}A_{21}) \frac{(A^*L)_1}{\sigma_1^2} \frac{(A^*L)_2}{\sigma_2^2} + A_{12}A_{22} \frac{\sigma_2^2 - (A^*L)_2^2}{\sigma_2^4} \\ A_{21}A_{11} \frac{\sigma_1^2 - (A^*L)_1^2}{\sigma_1^4} - (A_{21}A_{12} + A_{22}A_{11}) \frac{(A^*L)_1}{\sigma_1^2} \frac{(A^*L)_2}{\sigma_2^2} + A_{22}A_{12} \frac{\sigma_2^2 - (A^*L)_2^2}{\sigma_2^4} \\ (A_{21})^2 \frac{\sigma_1^2 - (A^*L)_1^2}{\sigma_1^5} - 2A_{21}A_{22} \frac{(A^*L)_1}{\sigma_1^2} \frac{(A^*L)_2}{\sigma_2^2} + (A_{22})^2 \frac{\sigma_2^2 - (A^*L)_2^2}{\sigma_1^4} \end{pmatrix}$$

In general, the terms in $I_j(\Sigma, L)$ are all scalars of the form cI_aI_b , where $c \in \mathbb{R}$ and a + b = j, so these are alternatingly real and purely imaginary in j.

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