

ROTATED ODOMETERS AND ACTIONS ON ROOTED TREES

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ABSTRACT. A rotated odometer is an infinite interval exchange transformation (IET) obtained as a composition of the von Neumann-Kakutani map and a finite IET of intervals of equal length. In this paper, we consider rotated odometers for which the finite IET is of intervals of length 2^{-N} , for some $N \geq 1$. We show that every such system is measurably isomorphic to a \mathbb{Z} -action on a rooted tree, and that the unique minimal aperiodic subsystem of this action is always measurably isomorphic to the action of the adding machine. We discuss the applications of this work to the study of group actions on binary trees.

1. INTRODUCTION

In this paper, we consider infinite interval exchange transformations (IETs) obtained by precomposing the von Neumann-Kakutani map of an interval with a finite IET of equal length intervals, and study the dynamics of such systems.

Let \mathbf{a} be the von Neumann-Kakutani map, represented on the half-open unit interval $[0, 1)$ as

$$(1) \quad \mathbf{a}(x) = x - (1 - 3 \cdot 2^{-n}) \quad \text{if } x \in [1 - 2^{1-n}, 1 - 2^{-n}), \quad n \geq 1.$$

For $q \in \mathbb{N}$, divide the interval $I = [0, 1)$ into q half-open subintervals of length $\frac{1}{q}$. Let π be a permutation of q symbols and let R_π be the corresponding piecewise continuous map of the subintervals. The infinite IET $F_\pi : I \rightarrow I$ defined by $F_\pi = \mathbf{a} \circ R_\pi$ is called the *rotated odometer*. This generalizes the case when $R_\pi : x \mapsto x + p/q \pmod{1}$ is a circle rotation, and we keep the name for the general case.

It was shown in [4] that every rotated odometer (I, F_π, λ) with Lebesgue measure λ is measurably isomorphic to the first return map of a flow of rational slope on a certain infinite-type translation surface. The translation surfaces in question have interesting properties: they are non-compact surfaces of finite area, infinite genus and with a finite number of ends. The closure of such a surface contains a single *wild* singularity and possibly a finite number of cone angle singularities, see [5, 13] for definitions and details about translation surfaces of infinite type. On the other hand, one can consider (I, F_π, λ) as a perturbation of the von Neumann-Kakutani system (I, \mathbf{a}, λ) . A natural question is, what dynamical properties of (I, \mathbf{a}, λ) are preserved under such perturbation? For the case $q \neq 2^N$, $N \geq 1$, this question was partially answered in [4].

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Let I_{per} the set of periodic points in I and $I_{np} = I \setminus I_{per}$ be the non-periodic points. It was shown in [4] that the aperiodic subsystem (I_{np}, F_π) of the rotated odometer (I, F_π) can be embedded into the Bratteli-Vershik system on a suitable Bratteli diagram, which can be constructed using coding partitions. The ergodic measures and the spectrum of the Koopman operator for (I_{np}, F_π) can then be studied using the methods developed in the literature for stationary Bratteli diagrams, see [2, 6]. In [4] we investigated these questions for the case $q \neq 2^N$, $N \geq 1$. In particular, it was shown that (I_{np}, F_π) may be non-minimal with unique minimal set, and that it admits at most q invariant ergodic measures (examples of rotated odometers with 2 invariant ergodic measures are given too).

In this paper, we consider the case $q = 2^N$, $N \geq 1$, where it is possible to construct a different, simpler Cantor model for the dynamical system of a rotated odometer than in [4]. More precisely, we show that the rotated odometer (I, F_π, λ) is measurably isomorphic to a \mathbb{Z} -action on a rooted binary tree, and, using this model, we study the dynamical and ergodic properties of the system. We also discuss the applications of our results to the study of group actions on binary trees.

We now give an overview of the main steps in the procedure which builds a measurable isomorphism between (I, F_π, λ) and a \mathbb{Z} -action on a tree.

As a first step, we embed (I, F_π) into a dynamical system given by a homeomorphism of a Cantor set, that is, there exists a Cantor set I^* , a homeomorphism $F_\pi^* : I^* \rightarrow I^*$ and an injective map $\iota : I \rightarrow I^*$, such that the image $\iota(I)$ is dense in I^* and $\iota \circ F_\pi = F_\pi^* \circ \iota$. This procedure has an important difference with an embedding of (I, F_π) into a compact space (I^*, F_π) constructed in [4].

Indeed, to define the compact space I^* in [4] we employ a technique standard in the study of finite IETs, see for instance [9]. Namely, we create gaps in I by doubling points in the orbits of discontinuities of F_π . Periodic points in [4] have half-open neighborhoods where each point is periodic with the same period as x , and no points in this neighborhood get doubled. Consequently I^* is not totally disconnected. However, the closure of $\iota(I_{np})$ is always a Cantor set.

In this paper I^* is constructed by simply doubling every dyadic rational $p/2^m$, $m \geq 1$, $0 < p < 2^m$, thus repeating the construction of the middle-third Cantor set, if we think of the middle interval as collapsed to a point. The compact space I^* obtained this way is always totally disconnected. The discontinuity points of (I, F_π) are among the doubled points, which implies that F_π extends to a homeomorphism F_π^* of I^* . The embedding ι is a measurable map with respect to the Lebesgue measure λ on I and the measure μ on I^* defined in Section 2.1.

We next build a tree model.

Definition 1.1. *A rooted binary tree T consists of the set $V = \bigsqcup_{i \geq 0} V_i$ of vertices and the set $E = \bigsqcup_{i \geq 1} E_i$ of edges, which satisfy the following properties for all $i \geq 0$:*

- (1) *The cardinality $|V_i| = 2^i$.*

- (2) Every vertex in V_i is connected by edges to precisely two vertices in V_{i+1} .
- (3) Every vertex in V_{i+1} is connected by an edge to precisely one vertex in V_i .

We modify the binary tree to obtain a *grafted* binary tree as follows.

Definition 1.2. For $N \geq 1$, a *grafted binary tree* T_N consists of the set $V = \bigsqcup_{i \geq 0} V_i$ of vertices and the set $E = \bigsqcup_{i \geq 1} E_i$ of edges, such that:

- (1) $|V_0| = 1$, $|V_1| = 2^N$ and for $i \geq 2$ we have $|V_i| = 2^{N+i-1}$.
- (2) The root $v_0 \in V_0$ is connected by edges to 2^N vertices in V_1 .
- (3) For $i \geq 1$, every vertex in V_i is connected by edges to precisely 2 vertices in V_{i+1} , and to a single vertex in V_{i-1} .

In the notation of Definition 1.2, we have $T_1 = T$, where T is the binary tree of Definition 1.1.

We introduce a labelling of vertices in V . Write $\mathcal{A}_k = \{0, 1, \dots, 2^k - 1\}$, for $k \geq 1$, and consider the tree T_N . The root $v_0 \in V_0$ is not labelled, vertices in V_1 are labelled by digits in \mathcal{A}_N , and for $i \geq 1$, if $v \in V_i$ is labelled by a word $w_1 w_2 \dots w_i$ where $w_1 \in \mathcal{A}_N$ and $w_i \in \mathcal{A}_1$ for $i \geq 2$, then the two vertices in V_{i+1} connected to v are labelled by $w_1 \dots w_i 0$ and $w_1 \dots w_i 1$.

Definition 1.3. An infinite path in the tree T_N is an infinite sequence in the product space

$$(2) \quad \partial T_N = \{(w_i) = w_1 w_2 \dots \mid w_1 \in \mathcal{A}_N, w_i \in \mathcal{A}_1, i \geq 2\} = \mathcal{A}_N \times \prod_{i \geq 2} \mathcal{A}_{1,i}, \quad \mathcal{A}_{1,i} = \mathcal{A}_1 \text{ for } i \geq 2.$$

The space ∂T_N is called the boundary of the tree T_N .

Since N and the cardinality of \mathcal{A}_1 are finite, ∂T_N is a Cantor set.

Definition 1.4. An automorphism $g : T_N \rightarrow T_N$ is a map of T_N which restricts to bijective maps on the sets V and E of vertices and edges respectively, and which preserves the structure of the tree. That is, if $v_1 \dots v_i \in V_i$ is a vertex, then for any vertex $v_1 \dots v_i w \in V_{i+1}$, where $w \in \{0, 1\}$, we have that $g(v_1 \dots v_i)$ is a subword of $g(v_1 \dots v_i w)$. In other words, two vertices in V_i and V_{i+1} are joined by an edge if and only if their images under g are joined by an edge.

We denote by $Aut(T_N)$ the group of automorphisms of T_N . It is straightforward to see that every automorphism $g \in Aut(T_N)$ induces a homeomorphism of the boundary ∂T_N .

A *cylinder*, or a *cylinder set* $[w_1 w_2 \dots w_i]$ in T_N , $i \geq 1$, is the set of all infinite paths starting with the finite sequence $w_1 w_2 \dots w_i$. The *Bernoulli measure* μ_N on ∂T_N is the standard measure in which every cylinder $[w_1 w_2 \dots w_i]$ has the mass 2^{-N-i+1} . It is straightforward that μ_N is preserved under every automorphism of T_N .

Theorem 1.5. Let $q = 2^N$, let π be a permutation on q symbols and let (I, F_π, λ) be a rotated odometer with Lebesgue measure λ . Then there exists an automorphism $\tilde{F}_\pi \in Aut(T_N)$ and a

measurable isomorphism

$$\phi : (I, F_\pi, \lambda) \rightarrow (\partial T_N, \tilde{F}_\pi, \mu_N)$$

such that $\tilde{F}_\pi \circ \phi = \phi \circ F_\pi$.

Theorem 1.5 is proved in Section 2.3.

A consequence of Theorem 1.5 is the following description of the dynamics of (I, F_π, λ) in the case $q = 2^N$, $N \geq 1$, which is more precise than the result of [4].

Theorem 1.6. *Let $q = 2^N$ for some $N \geq 1$, and let (I, F_π) be a rotated odometer. There exists a decomposition $I = I_{per} \cup I_{np}$ with the following properties:*

- (i) *Every point in I_{per} is periodic, the restriction $F_\pi : I_{per} \rightarrow I_{per}$ is well-defined and invertible.*
- (ii) *If I_{per} is non-empty, then I_{per} is a finite union of half-open maximal periodic intervals $[x, y)$, $x, y \in I$. Thus the set of periods of points in (I, F_π) is finite.*
- (iii) *The set I_{np} contains 0 and $F_\pi : I_{np} \rightarrow I_{np}$ is well-defined and invertible at every point in $I_{np} \setminus \{0\}$.*
- (iv) *The aperiodic system (I_{np}, F_π) is minimal.*

The difference with the general case $q \geq 2$ in [4] is that there I_{per} can be an infinite union of half-open intervals, while for $q = 2^N$, I_{per} is at most a *finite* union of half-open intervals. It follows that the set of periods which occur in (I, F_π) is finite, which need not be the case in [4]. Another difference is that for $q = 2^N$ the aperiodic subsystem (I_{np}, F_π) is always minimal, while this need not hold for $q \neq 2^N$. Theorem 1.6 is proved in Section 2.3.

Since I_{per} is a finite union of half-open intervals, I_{np} is also a finite union of half-open intervals, and its Lebesgue measure $\lambda(I_{np}) > 0$. We normalise $\lambda_{np}(U) = \lambda(U)/\lambda(I_{np})$ for every $U \subset I_{np}$. The *dyadic adding machine* $\mathbf{a} : \{0, 1\}^{\mathbb{N}} \rightarrow \{0, 1\}^{\mathbb{N}}$ is a well-known example of a minimal \mathbb{Z} -action on the space of one-sided infinite sequences of 0's and 1's. For a finite set $S = \{0, \dots, r-1\}$, $r \geq 1$, we define the adding machine $\mathbf{a}_S : S \times \{0, 1\}^{\mathbb{N}} \rightarrow S \times \{0, 1\}^{\mathbb{N}}$ as the addition of 1 in S with infinite carry to the right. In other words,

$$(3) \quad \mathbf{a}_S(s, x) = \begin{cases} (s+1, x) & \text{if } s < r-1, \\ (0, \mathbf{a}(x)) & \text{if } s = r-1, \end{cases} \text{ where } \mathbf{a} \text{ is the dyadic adding machine,}$$

The adding machine \mathbf{a}_S preserves the obvious Bernoulli measure μ_S .

Since $\{0, 1\}^{\mathbb{N}}$ is homeomorphic to ∂T_1 , there is a conjugate action on ∂T_1 which we also call the adding machine and denote by a . A recursive definition of the adding machine on the boundary ∂T_N of the grafted tree T_N is given in Example 2.3.

Corollary 1.7. *Let $q = 2^N$ for some $N \geq 1$, and let (I, F_π) be a rotated odometer. The aperiodic system $(I_{np}, F_\pi, \lambda_{np})$ is measurably isomorphic to the action of the adding machine on $S \times \{0, 1\}^{\mathbb{N}}$, for $|S| \leq 2^N$, with Bernoulli measure μ_S .*

A rotated odometer need not be conjugate to an automorphism of the binary tree T , since F_π may be such that the permutation π does not respect the structure of the binary tree, see Remark 3.1. Therefore T_N cannot be substituted by T_1 in Theorem 1.5.

Theorem 1.5 has applications in the study of group actions on binary trees. Infinite IETs and actions of self-similar groups on binary trees are related. For instance, the famous Grigorchuk group was initially defined as a group of infinite IETs of the unit interval, see [7, Section 2]. Actions of self-similar groups on binary rooted trees are an active topic of research in Geometric Group Theory [1, 7, 11], and they also have applications in the study of arboreal representations of absolute Galois groups of number fields [8, 10]. We now present a corollary of Theorem 1.5 for the actions of groups on binary trees.

To this end, let $T_1 = T$ be the binary tree. An automorphism $g \in \text{Aut}(T)$ is of finite order if $g^m = \text{id}$ for some $m \geq 1$. For instance, if g_i interchanges 0's and 1's in the i -th coordinate w_i , then g_i has order 2. Another example of an element of order 2 is g_{even} , which interchanges 0 and 1 in w_i for every even i , and of course one can construct many more examples. The adding machine (3) is an automorphism of T of infinite order.

Let $G \subset \text{Aut}(T)$ be a profinite group such that G acts transitively on ∂T . Given $g \in \text{Aut}(T)$, the restriction $g|_{V_n}$ is a permutation of a finite set V_n , and so it can be written as a product of cycles. Let $(x_i) = x_1x_2\cdots \in \partial T$, then $x_1\cdots x_n$ is a vertex in V_n . Denote by $g_{n,x_1\cdots x_n}$ the cycle containing $x_1\cdots x_n$, then one can ask how the sequence of cycles $\{g_{n,x_1\cdots x_n}\}$ behaves as n increases. It is conjectured in [3], that when G is a representation of the absolute Galois group of a number field, elements with a certain type of cycle structure are dense in G . To the best of our knowledge, this conjecture is solved only in a few cases.

As a rule, given $g \in \text{Aut}(T)$, it is not immediate to determine the cycle structure of g , except in a few simple cases when g is periodic or when g acts transitively on every level V_n , $n \geq 1$. The theorem below allows us to determine the cycle structure for compositions of the adding machine and some periodic elements of $\text{Aut}(T)$.

Theorem 1.8. *Let μ_1 be the Bernoulli measure on ∂T , and let λ be Lebesgue measure on the half-open unit interval I . Let $g \in \text{Aut}(T)$ be such that there exists $m \geq 1$ such that for every $i > m$ and every sequence $w_1w_2\cdots \in \partial T$ the action of g leaves w_i unchanged (which implies that g has finite order). Let $a \in \text{Aut}(T)$ be the adding machine. Then the following is true:*

- (1) *For some permutation π on 2^m intervals, there exists a rotated odometer (I, F_π) and an injective measure-preserving map $\phi : (I, \lambda) \rightarrow (\partial T, \mu)$, such that $\phi \circ F_\pi = (a \circ g) \circ \phi$.*
- (2) *Consequently, $a \circ g$ has infinite order, there is a clopen subset $U \subset \partial T$ such that the restriction $\langle a \circ g \rangle|_U$ is minimal, and there is an $n_0 \geq 0$ such that every $x \in \partial T \setminus U$ is periodic of period 2^k for some $k \leq n_0$.*

The realization of a tree automorphism as an interval exchange transformation in Theorem 1.8 relies on the fact that, under the hypotheses of the theorem, g respects the embedding of an interval into the boundary of a tree T in Theorem 1.6. This means, in particular, that the orbits of points which do not have preimages under ϕ consist of points which also do not have preimages under ϕ . This condition need not hold for a general finite order automorphism of T . We discuss this and the possibility of generalizing Theorem 1.8 to a larger class of tree automorphisms in Remark 3.3.

Remark 1.9. In the literature, an *odometer* in $Aut(T)$ is sometimes defined as any $h \in Aut(T)$ such that the action of the cyclic group $\langle h \rangle$ is transitive on each V_n , $n \geq 1$. Every such h is conjugate to the adding machine in Example 2.3 by some $g \in Aut(T)$ [12]. We stress that Theorem 1.8 only holds for the adding machine and need not hold for an odometer h . To this end we show in Remark 3.2 that it is possible to find $h \in Aut(T)$ such that the action of the cyclic subgroup $\langle h \rangle$ on ∂T is minimal, and a periodic $g \in Aut(T)$, such that the product $h \circ g$ has finite order. There exists an infinite IET that is measurably isomorphic to the action of such $\langle h \rangle$ on ∂T , but this IET will not be the rotated odometer of the form defined at the beginning of the introduction.

We finish with a sample open question motivated by applications to actions on binary trees. Consider compositions of the adding machine with a periodic element which does not satisfy the hypotheses of Theorem 1.8 but which respects the embedding of I in Theorem 1.6, see Remark 3.3 for the justification of such an assumption. It may be possible to solve the following problem by considering a sequence $\{F_{\pi_i}\}_{i \geq 1}$ of rotated odometers, where each π_i is a (possibly different) permutation of a finite number of symbols.

Problem 1.10. *Let $g \in Aut(T)$ be periodic such that for any $i \geq 1$ there is $j > i$ and $w_1 \cdots w_j \cdots \in \partial T$ such that $g(w_j) \neq w_j$, and such that g preserves the embedding ϕ in Theorem 1.6. Find a model for the action of the product $a \circ g$, where a is the adding machine, in terms of rotated odometers. What are the topological properties of infinite translation surfaces, which admit flows whose first return map is measurably isomorphic to such systems?*

The paper is organized as follows. In Section 2 we develop a tree model for rotated odometers and prove Theorem 1.5. In Section 3 we discuss the dynamics of rotated odometers and prove Theorems 1.6 and 1.8 and Corollary 1.7.

2. THE TREE MODEL

In this section we build a tree model for a rotated odometer with $q = 2^N$, and prove Theorem 1.5.

2.1. Embedding into a Cantor set. Set $C = \{p2^{-n} \mid n \geq 1, 0 < p < 2^n\}$; these dyadic rationals are used as cut-points. For each point $x \in C$ we add a double point x^- to I , and define

$$I^* = I \cup \{x^- \mid x \in C\} \cup \{1\}.$$

The subset $I \cup \{1\}$ of I^* has total order $<$ induced from \mathbb{R} . We extend this order to I^* by defining $x^- < x$ if $x \in C$, and $y < x^-$ if $y \in I \setminus C$, $x \in C$ and $y < x$. Since there are no points between x^- and x in I^* , adding x^- to I can be thought of as creating a gap. We give I^* an order topology with open sets

$$\mathcal{B} = \{(a, b) \mid a, b \in I^*\} \bigsqcup \{[0, b) \mid b \in I^*\} \cup \{(a, 1] \mid a \in I^*\}.$$

It is straightforward that the sets $\{[x, y^-] \mid x, y \in C\}$ are clopen in this topology. Since C is dense in I , every point $z \in I^*$ has a system of decreasing clopen neighborhoods

$$C(z, n) = \{[p_n 2^{-n}, (p_n + 1)2^{-n}] \mid n \geq 0, 0 \leq p_n < 2^n\}.$$

Recall that a metric d on a space X is an *ultrametric* if it satisfies the following stronger form of the triangle inequality,

$$d(x, y) = \max\{d(x, z), d(z, y)\} \quad \text{for all } x, y, z \in X.$$

We put an ultrametric on I^* by declaring that

$$d(z_1, z_2) = \frac{1}{2^r}, \quad r = \max\{n \geq 0 \mid C(z_1, n) = C(z_2, n)\}.$$

Then I^* is a compact totally disconnected perfect metric space, that is, I^* is a Cantor set.

Define a measure μ on I^* by setting for each clopen set $\{[x, y^-] \mid x, y \in C\}$

$$\mu([x, y^-]) = y - x,$$

and denote by $\iota : I \rightarrow I^*$ the inclusion map. Clearly $\mu(I^*) = 1$. Since C is countable, the following is straightforward.

Lemma 2.1. *The map $\iota : (I, \lambda) \rightarrow (I^*, \mu)$ measurable.*

Denote by $D_0 = \{1 - 2^{-k} \mid k \geq 0\}$ the set of discontinuities of the von Neumann-Kakutani map \mathbf{a} , and let D^+ and D^- be the sets of forward and backward (whenever defined) orbits of points in D_0 . Since \mathbf{a} is continuous on the intervals $I_k = [1 - 2^{-(k-1)}, 1 - 2^{-k}]$, $k \geq 1$, and, moreover, the restriction $\mathbf{a}|_{I_k}$ for each $k \geq 1$ is a translation by $\pm p 2^{-s}$ for some $p, s \in \mathbb{N}$, the set $D_0 \cup D^+ \cup D^-$ of forward and backward orbits of the points of discontinuity of \mathbf{a} is contained in C .

We can extend $\mathbf{a} : I \rightarrow I$ to a continuous map $\mathbf{a}^* : I^* \rightarrow I^*$ by setting $\mathbf{a}^*(x) = \mathbf{a}(x)$ if $x \in I$, and

$$\mathbf{a}^*(x^-) = \lim_{y \nearrow x} \mathbf{a}(y), \quad \text{for all } x \in C \cup \{1\}.$$

Every point $x \in I$ except 0 has a two-sided orbit, and it follows that $\iota(x)$ has a two-sided orbit in I^* . For any sequence $y \nearrow 1$ the sequence of images $\mathbf{a}(y) \searrow 0$, so $\mathbf{a}^*(1) = 0$ and 0 has a two-sided orbit in I^* under \mathbf{a}^* . It follows that \mathbf{a}^* is a homeomorphism. It is immediate that $\mathbf{a}^* \circ \iota(x) = \iota \circ \mathbf{a}(x)$ for all $x \in I$.

Note that the finite IET $R_\pi : I \rightarrow I$ extends in a similar manner to a periodic homeomorphism $R_\pi^* : I^* \rightarrow I^*$, which satisfies $R_\pi^* \circ \iota(x) = \iota \circ R_\pi(x)$ for all $x \in I$. Then for the composition

$F_\pi^* = \mathbf{a}^* \circ R_\pi^*$ it follows that $F_\pi^* \circ \iota(x) = \iota \circ F_\pi(x)$ for all $x \in I$, and thus (I, F_π, λ) is measurably isomorphic to (I^*, F_π^*, μ) via the embedding ι .

2.2. Actions on trees. The binary tree and the grafted binary trees were defined in Definitions 1.1 and 1.2, and automorphisms of trees were defined in Definition 1.4. We now introduce a description of elements in $Aut(T_N)$ convenient for computations. This approach is a slight modification of the one routinely used in Geometric Group Theory to study actions on binary trees, see for instance [11]. The purpose of this modification is to take into account the fact that in the grafted binary tree the vertex set V_1 has more than 2 vertices.

Let $T = T_1$ be the binary tree with the labelling of vertices by finite words in \mathcal{A}_1 as defined in the Introduction. Let $w = w_1 \cdots w_k \in \prod_{i=1}^k \mathcal{A}_1$ and denote by $T(w)$ the subtree of T consisting of all paths starting with the finite word w . All such paths pass through the vertex in V_k labelled by w . Then there is an isomorphism of trees

$$(4) \quad \kappa_w : T(w) \rightarrow T, \quad w_1 \cdots w_k v_{k+1} \cdots \mapsto v_{k+1} \cdots, \quad v_i \in \{0, 1\} \text{ for } i > k.$$

For every $g \in Aut(T)$, the restriction $g|_{V_n}$ is a permutation of a set of 2^n elements.

Definition 2.2. *Given an automorphism $g \in Aut(T)$, and a finite word w , we define a section at w by*

$$(5) \quad g_w = \kappa_{g(w)} \circ g \circ \kappa_w^{-1} \in Aut(T).$$

Let $g|_{V_n} = \tau$. Then we can write g as a composition (we compose the maps on the left)

$$(6) \quad g = (g_{\tau^{-1}(0^n)}, g_{\tau^{-1}(0^{n-1}1)}, \dots, g_{\tau^{-1}(1^n)})\tau,$$

where $g_{\tau^{-1}(w)}$ are sections, for finite words w of n letters. Equation (6) means that to compute g , we first apply τ on V_n , and then we apply a section $g_{\tau^{-1}(w)}$ to the subtree $T(w)$, for all $w \in V_n$.

Example 2.3. Using sections, we can write automorphisms of T recursively. Recall that a generator of the adding machine action on a Cantor space $\{0, 1\}^{\mathbb{N}}$ is given by

$$(7) \quad a(w_1 w_2 \cdots) = \begin{cases} (w_1 + 1) w_2 \cdots & \text{if } w_1 = 0, \\ 0 0 \cdots 0 (w_{k+1} + 1) w_{k+2} \cdots & \text{if } w_i = 1 \text{ for } 1 \leq i \leq k, w_{k+1} = 0, \\ 0 0 \cdots & \text{if } w_k = 1 \text{ for all } k \geq 1. \end{cases}$$

Recall that for the binary tree T we have $\partial T \cong \{0, 1\}^{\mathbb{N}}$. Let σ be the non-trivial permutation of \mathcal{A}_1 . Then using (6) we can write

$$a = (a, 1)\sigma,$$

where 1 is the identity map in $Aut(T)$. Here σ performs the addition of 1 modulo two in the first entry of the sequence, interchanging 0 and 1, while $(a, 1)$ implements the recursive procedure of infinite carry to the right. For example, if $w = 10^\infty$, then applying σ to w interchanges 1 to 0 in the first component, so $\sigma(w) = 0^\infty$, and we must compute $(a, 1)(0^\infty)$ next. The sequence 0^∞ belongs to the subtree $T(0)$ which means that we must apply the section $a_0 = a$ to 0^∞ . That is, we apply a to 0^∞ starting from the second entry. Since $a|_{V_1} = \sigma$, we must interchange 0 and 1 in

the second entry, obtaining $010^\infty \in T(01)$. We have for the sections $a_1 = 1$, then also $a_{01} = 1$, and the computation stops with the result $a(10^\infty) = 010^\infty$.

Using (6) we can compute the compositions of elements in $\text{Aut}(T)$. The following statement is obtained by a straightforward computation.

Lemma 2.4. *Let $g, h \in \text{Aut}(T)$, and suppose $g = (g_0, \dots, g_{2^n-1})\tau$ and $h = (h_0, \dots, h_{2^n-1})\nu$, where τ, ν are permutations of 2^n symbols and $g_i, h_i \in \text{Aut}(T)$ for $0 \leq i < 2^n$. Then*

$$(8) \quad gh = (g_0, \dots, g_{2^n-1})\tau(h_0, \dots, h_{2^n-1})\nu = (g_0h_{\tau^{-1}(0)}, \dots, g_{2^n-1}h_{\tau^{-1}(2^n-1)})\tau\nu.$$

Now let T_N be the grafted binary tree. Similarly to (4), for any $w \in V_k$, $k \geq 1$ we define a map

$$(9) \quad \kappa_w : T_N(w) \rightarrow T_1, \quad w_1 \cdots w_k v_{k+1} \cdots \mapsto v_{k+1} \cdots .$$

The difference with (4) is that the range of κ_w is not the grafted tree T_N but the binary tree T . A section g_w of the grafted tree T_N at w is defined by (5) with κ_w given by (9). Again, the difference with the setting of the binary tree is that for the grafted binary tree T_N sections are elements of $\text{Aut}(T)$ and not of $\text{Aut}(T_N)$.

Lemma 2.5. *Given an automorphism $g \in \text{Aut}(T)$ of the binary tree T , there is always an automorphism $\hat{g} \in \text{Aut}(T_N)$ of the grafted tree T_N , such that the induced homeomorphisms on the boundaries of the corresponding trees are conjugate.*

Proof of Lemma 2.5. Vertices in the vertex level set V_N of T are labelled by words of length N in the alphabet \mathcal{A}_1 . Define the map

$$\kappa_N : \mathcal{A}_1^N \rightarrow \mathcal{A}_N, \quad w_1 \cdots w_N \mapsto \sum_{i=1}^N 2^{N-i} w_i.$$

Using the identification (2) of the path spaces ∂T and ∂T_N with products of finite sets we obtain a homeomorphism

$$\kappa_\infty : \partial T \rightarrow \partial T_N, \quad w_1 w_2 \cdots \mapsto \kappa_N(w_1 \cdots w_N) w_{N+1} \cdots .$$

It follows that the map $\tilde{g} = \kappa_\infty \circ g \circ \kappa_\infty^{-1} : \partial T_N \rightarrow \partial T_N$ is a homeomorphism. Moreover, by construction if two paths $(w_i), (v_i) \in \partial T$ coincide up to level $m \geq N$, then their images under κ_∞ coincide up to level $m - N$, so every subtree $T(w)$ for $w \in V_N$ is mapped isomorphically onto a subtree $T_N(\kappa_N(w))$. It follows that \tilde{g} defines an automorphism \hat{g} of T_N . \square

Given a recursive definition of $g \in \text{Aut}(T)$ as in (6), we can obtain a recursive definition of $\hat{g} \in \text{Aut}(T_N)$. Indeed, let $g|V_N = \tau$ be a permutation of V_N induced by g . Then $\tau_N = \kappa_N \circ \tau \circ \kappa_N^{-1}$ is a permutation of the level set V_1^N of T_N , and if $g = (g_0, \dots, g_{2^N-1})\tau$, then $\hat{g} = (g_0, \dots, g_{2^N-1})\tau_N$.

Example 2.6. Let $a = (a, 1)\sigma$ be the standard adding machine as in Example 2.3, and let $N \geq 2$. We can compute that

$$(10) \quad \tau_N = \kappa_N \circ (a|V_N) \circ \kappa_N^{-1} = (0, 2^{N-1}, 2^{N-2}, 2^{N-1} + 2^{N-2}, \dots, 2^N - 1),$$

and $\hat{a} = (a, 1, \dots, 1)\tau_N \in \text{Aut}(T_N)$.

More generally, given a finite set $S \subset V_N$, we can consider a subtree $T_S = \bigcup_{w \in S} T_N(w) \subset T_N$. Let η be a transitive permutation of S , and consider the map $a_S = (a, 1, \dots, 1)\eta$ on T_S . Then a_S is transitive on $V_n \cap T_S$, for any $n \geq 1$, so a_S is the adding machine on T_S .

We note that, given $h \in \text{Aut}(T_N)$, the composition $\kappa_\infty^{-1} \circ h \circ \kappa_\infty$ need not define an automorphism of T . Indeed, let $N = 2$, so T_N has 4 vertices at the first level, and let $h|V_2 = \tau_2 = (012)$, so the vertex 3 is fixed. We have $\kappa_2^{-1}(3) = 11 \in V_2$ and $\kappa_2^{-1}(2) = 10 \in V_2$. At the same time

$$\kappa_2^{-1} \circ \tau_2 \circ \kappa_2(10) = \kappa_2^{-1}(\tau_2(2)) = \kappa_2^{-1}(0) = 00.$$

Thus $\kappa_\infty^{-1} \circ h \circ \kappa_\infty$ maps paths starting with 1 in ∂T to paths starting with either 1 or 0 depending on the second symbol in the sequence. This means that $\kappa_2^{-1} \circ \tau_2 \circ \kappa_2$ is incompatible with the structure of the binary tree T , and so $\kappa_\infty^{-1} \circ h \circ \kappa_\infty$ does not define an automorphism of T .

2.3. Tree models for rotated odometers. In this section we prove Theorem 1.5.

Proof of Theorem 1.5. Recall that π is a permutation of 2^N symbols, and $\iota : (I, \lambda) \rightarrow (I^*, \mu)$ is a measurable embedding into a Cantor set. Write $x_{n,p} = p2^{-n}$ for points in C , and $x_{n,p}^-$ for the corresponding double points in I^* . For each $n \geq 1$, set $x_{n,2^n}^- = 1$.

Note that for any $n \geq 0$ we have

$$I^* = \bigcup \{[x_{n,p}, x_{n,p+1}^-] \mid 0 \leq p < 2^n - 1\}.$$

Consider the grafted tree T_N , and recall that $|V_1| = 2^N$. We are going to construct a homeomorphism $\tilde{\phi} : I^* \rightarrow \partial T_N$ inductively as follows.

Define $\tilde{\phi}_1 : I^* \rightarrow \mathcal{A}_N$ by setting

$$\tilde{\phi}_1(z) = p \quad \text{if and only if} \quad z \in [x_{N,p}, x_{N,p+1}^-].$$

For $n \geq 2$, there is a unique $0 \leq m < 2^n - 1$ such that $z \in [x_{n,m}, x_{n,m+1}^-]$. Set

$$w_n = \tilde{\phi}_n(z) = m \pmod{2}.$$

Then define

$$\tilde{\phi}_\infty : I^* \rightarrow \partial T_N, \quad z \mapsto (\tilde{\phi}_1(z), \tilde{\phi}_2(z), \dots).$$

This mapping is bijective, since every point in I^* has a system of clopen neighborhoods of the form $\{[x_{n,p}, x_{n,p+1}^-] \mid n \geq 0\}$, and every clopen neighborhood $[x_{n,p}, x_{n,p+1}^-]$ is non-empty. The mapping $\tilde{\phi}_\infty$ is clearly continuous and so it is a homeomorphism. Note that by construction the inclusions of clopen sets in I^* correspond to vertices in T_N joined by finite paths.

The measure μ assigns equal weight to each interval $\{[x_{n,p}, x_{n,p+1}^-] \mid 0 \leq p < 2^n\}$ in the partition of I^* , and $\mu(I^*) = 1$. By construction each $[x_{n,p}, x_{n,p+1}^-]$ is mapped onto a unique vertex in V_{n-N+1} . The Bernoulli measure μ_N assigns equal weight to every set $\partial T_N(w)$, where $w \in V_{n-N+1}$, and $\mu_N(\partial T) = 1$. It follows that $\tilde{\phi}_\infty$ is measure-preserving.

Every map of I^* which for all $n \geq 1$ induces a permutation of clopen sets $\{[x_{n,p}, x_{n,p+1}] \mid 0 \leq p < 2^n - 1\}$, induces a family of permutations of the vertex level sets V_{n-N+1} , $n \geq 1$ of T_N . Since paths in T_N correspond to inclusions of clopen sets in I^* , such permutations are compatible with the structure of the tree T_N and induce an automorphism of T_N . We note that the maps $\mathbf{a}^* : I^* \rightarrow I^*$ and $R_\pi^* : I^* \rightarrow I^*$ described in Section 2.1 satisfy this condition. Therefore, the composition $F_\pi^* = \mathbf{a}^* \circ R_\pi^* : I^* \rightarrow I^*$ induces an automorphism of T_N . The proof of Theorem 1.5 is completed by composing $\phi = \tilde{\phi}_\infty \circ \iota : I \rightarrow \partial T_N$ with the measurable isomorphism $\iota : (I, F_\pi, \lambda) \rightarrow (I^*, F_\pi^*, \mu)$. \square

Remark 2.7. Consider the set of added points $\{x_{m,p}^- \mid x_{m,p} \in C\}$. Suppose $x_{m,p} = p2^{-m}$ is an irreducible fraction, that is, p is odd. Then for $n > m$ we have that $\tilde{\phi}_n(x_{m,p}^-) = 1$ since in that case $x_{m,p}^-$ corresponds to a right endpoint of a clopen interval in the partition $\{[x_{n,r}, x_{n,r+1}^-] \mid 0 \leq r < 2^n\}$, and it is always contained in the second interval of the subdivision of $[x_{n,r}, x_{n,r+1}^-]$ into two intervals. Then the image of $x_{m,p}^-$ in ∂T_N is a sequence which is eventually constant with entries equal to 1.

3. DYNAMICS OF ROTATED ODOMETERS

Using the tree model obtained in Theorem 1.5 we study the dynamics of rotated odometers and prove Theorems 1.6 and 1.8 and Corollary 1.7.

3.1. Periodic and non-periodic points. For the von Neumann-Kakutani map $\mathbf{a}^* : I^* \rightarrow I^*$ denote by $A = \tilde{\phi}_\infty \circ \mathbf{a}^* \circ \tilde{\phi}_\infty^{-1} : \partial T_N \rightarrow \partial T_N$ the induced map of the binary tree T . We want to describe A using the recursive formula (6).

Proof of Theorem 1.6 and Corollary 1.7. In what follows $n \geq N$. Let $L_n = [0, 2^{-n})$ and $M_n = [1 - 2^{-n}, 1)$, so that L_n is the first and M_n is the last set of the partition of I into 2^n sets of equal lengths. Then $\iota(L_n) \subset [x_{n,0}, x_{n,1}^-] \subset I^*$ and $\iota(M_n) \subset [x_{n,2^n-1}, 1] \subset I^*$. The definition of the von Neumann-Kakutani map in (1) implies that $\mathbf{a}(x) \in L_n$ if and only if $x \in M_n$, and for any $[x_{n,p}, x_{n,p+1})$ except M_n the restriction $\mathbf{a}|_{[x_{n,p}, x_{n,p+1})}$ is a translation. Thus it preserves the order \leq on the points in $[x_{n,p}, x_{n,p+1})$ induced from \mathbb{R} . The relation \leq is not preserved by the restriction $\mathbf{a} : M_n \rightarrow L_n$, where the order of two halves of M_n is interchanged, and the intervals inside the image of the second half of M_n are further interchanged. The second half of M_n is the set M_{n+1} , and we have $\mathbf{a}(M_{n+1}) = L_{n+1}$. Thus the restriction of \mathbf{a} to the set of intervals $\{[x_{n,p}, x_{n,p+1}) \mid 0 \leq p < 2^n\}$, and therefore of \mathbf{a}^* to the set of intervals $\{[x_{n,p}, x_{n,p+1}^-] \mid 0 \leq p < 2^n\}$, defines a permutation of 2^n symbols, which is transitive since \mathbf{a} is minimal on I .

It follows that $A|V_{n-N+1}$ is a transitive permutation of V_{n-N+1} . Since further permutations of subintervals, which do not respect the order $<$, only happen for the interval, mapped onto L_{n-N+1} , for any $w \neq 0^{n-N+1} \in V_{n-N+1}$, the section $A_w \in \text{Aut}(T)$ is the identity map. The restriction of the section $A_{0^{n-N+1}}$ to V_{n-N+2} is a non-trivial permutation of two symbols, since \mathbf{a} permutes two subintervals of L_{n-N+1} . For $n = N$, we have $A|V_{N-N+1} = A|V_1 = \tau_N$, for τ_N given by (10), and so $A = (a, 1, \dots, 1)\tau_N$, where $a = (a, 1)\sigma$ is described in Example 2.3.

Similarly, given a permutation π of 2^N symbols, and the corresponding finite IET $R_\pi : I \rightarrow I$, we deduce that the induced map $R = \tilde{\phi}_\infty \circ R_\pi^* \circ \tilde{\phi}_\infty^{-1}$ is given by $R = (1, 1, \dots, 1)\pi$, with $R|V_1 = \pi$.

Now using the law for composition of tree automorphisms (8) we can easily understand the dynamics of the system $(\partial T_N, A \circ R)$. In particular,

$$A \circ R = (a, 1, \dots, 1)\tau\pi,$$

which leads to the following conclusions:

- (i) Consider the decomposition of $\tau\pi$ into cycles, and suppose c is a cycle containing 0. Let $\mathcal{O} \subset \mathcal{A}_N$ be the set of symbols in c . Then

$$\partial T_N(\mathcal{O}) = \bigcup \{\partial T(s) \mid s \in \mathcal{O}\}$$

is a clopen subset of ∂T and the restriction of $A \circ R$ to this set satisfies

$$A \circ R|_{\partial T_N(\mathcal{O})} = (a, 1, \dots, 1)c,$$

which shows that this system is the addition of 1 in the first component with infinite carry to the right, and so it is minimal. Set $S = \{0, \dots, |c| - 1\}$, then $A \circ R|_{\partial T_N(\mathcal{O})}$ is isomorphic to the adding machine on $S \times \{0, 1\}^{\mathbb{N}}$ defined in Example 2.6, and Corollary 1.7 follows. Here $|c|$ denotes the length of the cycle c . In particular, the system $(\partial T_N, A \circ R)$ is minimal if and only if $\tau\pi$ is a transitive permutation.

- (ii) In the cycle decomposition of $\tau\pi$, let c' be a cycle not containing 0, and let $\mathcal{O}' \subset \mathcal{A}_N$ be the set of symbols in c' . Then $\partial T_N(\mathcal{O}') = \bigcup \{\partial T(s) \mid s \in \mathcal{O}'\}$ is a clopen subset of ∂T , and we have

$$A \circ R|_{\partial T_N(\mathcal{O}')} = (1, 1, \dots, 1)c'.$$

Thus every point in $T_N(\mathcal{O}')$ has period $|c'|$.

- (iii) Since $\tau\pi$ contains a finite number of cycles, the set of periods of periodic points in $(\partial T_N, A \circ R)$, and so in (I, F_π) , is finite. Also, it follows that a point $x \in \partial T_N$ is periodic if and only if $x \in \partial T_N(\mathcal{O}')$ for some cycle c' not containing 0. There is at most a finite number of such cycles c' in $\tau\pi$, and so there is a finite number of half-open intervals in I whose image under the inclusion map $\phi = \tilde{\phi}_\infty \circ \iota$ is contained in $\partial T_N \setminus \partial T_N(\mathcal{O})$. It follows that the set of periodic points in (I, F_π) is at most a finite union of half-open intervals.

These prove Theorem 1.6 and Corollary 1.7. □

Remark 3.1. We note that the periods of points in (I, F_π) need not be powers of 2. Let $N = 2$, then $A = (a, 1, 1, 1)(0213)$. Let $\pi = (03)$. Then

$$A \circ R = (a, 1, 1, 1)(0213)(03) = (a, 1, 1, 1)(0)(321),$$

so the orbit of every infinite sequence in ∂T_2 starting with 1, 2 or 3 is periodic with period 3.

It follows from this example that there exist rotated odometers whose action is not measurably isomorphic to the action of an automorphism of the binary tree T . Indeed, if $g \in \text{Aut}(T)$ is an automorphism and $x \in \partial T$ is a periodic point, then the period of x is a power of 2. To see this, consider $x = (x_0, x_1, \dots)$, and let r_k be the period of x_k in V_k . Then the period x_{k+1} in V_{k+1} is either r_k or $2r_k$. Since the period r_1 of x_1 in V_1 is either 1 or 2, the statement follows.

3.2. Applications. We prove Theorem 1.8.

Proof of Theorem 1.8. Suppose that $g \in \text{Aut}(T)$ is of finite order such that there is $m \geq 1$ such that for every $i > m$ the action of g leaves w_i unchanged. We need to show that there exists a rotated odometer (I, F_π) for some permutation π on 2^m intervals, such that (I, F_π, λ) is measurably isomorphic to $(\partial T, a \circ g, \mu_1)$, where λ is Lebesgue measure, μ_1 is the Bernoulli measure on the binary tree T and a is the adding machine described in Example 2.3.

Consider the partition of ∂T into clopen sets $\partial T(w)$, where $w = w_1 \cdots w_n$. Also, consider a partition of I^* into subintervals $\{[x_{n,p}, x_{n,p+1}) \mid 0 \leq p < 2^n\}$. By construction every such subinterval is mapped under $\phi = \tilde{\phi}_\infty \circ \iota$ into a distinct clopen set $\partial T(w)$, and ϕ is injective on I . Define

$$\tilde{g} : I \rightarrow I, \quad x \mapsto (\tilde{\phi}_\infty \circ \iota)^{-1} \circ g \circ (\tilde{\phi}_\infty \circ \iota)(x).$$

The map \tilde{g} is well-defined. Indeed, by Remark 2.7 the points in I^* which do not have preimages in I under ι are mapped into sequences which are eventually constant with entries equal to 1. Since g does not change w_i for $i \geq m$, $w \in \partial T$ is eventually a sequence of 1's if and only if $g(w)$ is eventually a sequence of 1's. Therefore, the map ϕ is invertible at $g \circ \phi(x)$. Since g does not change w_i for $i \geq m$, \tilde{g} preserves the order of points in the sets $\{[x_{n,p}, x_{n,p+1}) \mid 0 \leq p < 2^n\}$, and it follows that the restriction of \tilde{g} to every interval $\{[x_{m,p}, x_{m,p+1}) \mid 0 \leq p < 2^m\}$ is a translation. We conclude that $\tilde{g} : I \rightarrow I$ is a finite IET.

It is proved in Section 3.1 that the von Neumann-Kakutani map (I, \mathbf{a}, λ) is measurably isomorphic to $(\partial T, a, \mu_1)$, where $a = (a, 1)\sigma$ is the standard adding machine. Set $F_\pi = \mathbf{a} \circ \tilde{g}$, then (I, F_π, λ) is measurably isomorphic to $(\partial T, a \circ g, \mu_1)$. The second statement of Theorem 1.8 follows from Theorem 1.6. \square

Remark 3.2. In the literature a transformation g such that the cyclic group $\langle h \rangle$ acts transitively on every level V_n , $n \geq 1$, of the tree T , is sometimes called an *odometer*. Every odometer is conjugate to the adding machine in Example 2.3 by a tree automorphism [12]. We note that Theorem 1.8 only holds for the adding machine, but need not hold for its conjugates.

Indeed, define $a_1 = \sigma$ and $a_2 = (a_1, a_2)$ using the recursive notation, then $\langle a_1, a_2 \rangle$ is the dihedral group. Both a_1 and a_2 have order two, and $h = a_1 a_2$ generates an infinite cyclic group whose action on every V_n , $n \geq 1$, is transitive. The element h is conjugate to $a = (a, 1)\sigma$ but it is not equal to a . Recall that σ acts on $(w_1, w_2, \dots) \in \partial T$ by interchanging 0 and 1 in the first entry, and keeps the remaining entries fixed. We compute that $h\sigma = \sigma(a_1, a_2)\sigma = (a_2, a_1)$ has order two.

Remark 3.3. We have seen in the proof of Theorem 1.8 that, under its hypotheses, a finite order element $g \in \text{Aut}(T)$ respects the embedding of I into ∂T . More precisely, by Remark 2.7 points which do not have preimages under this embedding correspond to sequences which are eventually constant with entries equal to 1, and if $g \in \text{Aut}(T)$ satisfies the hypotheses of Theorem 1.8, then it preserves the set of such sequences. Suppose $g \in \text{Aut}(T)$ does not satisfy the hypotheses of Theorem 1.8, that is, for any $n \geq 1$ there is $(w_i) \in \partial T$ and $m_n \geq n$ such that $g(w_{m_n}) \neq w_{m_n}$. Then the action of g on ∂T may or may not respect the embedding of I . If such $g \in \text{Aut}(T)$ respects the embedding of I into ∂T (h in Remark 3.2 is an example), then g induces an IET of *infinite* number of intervals. At the moment we do not have a unified way of describing the dynamics of a composition of such an IET with the von Neumann-Kakutani map, and we pose this as an open question in Problem 1.10. An example of an element which does not respect the embedding is, for instance, g_{even} given for any $(w_i) \in \partial T$ by

$$g_{\text{even}}(w_{2n}) = w_{2n} + 1 \pmod{2}, \quad g_{\text{even}}(w_{2n+1}) = w_{2n+1}, \quad n \geq 1.$$

It is not clear whether such elements induce IETs on the interval I , even if one discards the measure 0 set of orbits in ∂T which do not have preimages under the embedding, and/or allows reflections of subintervals.

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