THE TOPOLOGICAL ENTROPY OF BANACH SPACES

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Abstract. We investigate some properties of (universal) Banach spaces of real functions in the context of topological entropy. Among other things, we show that any subspace of $C([0,1])$ which is isometrically isomorphic to $\ell_1$ contains a function with infinite topological entropy. Also, for any $t \in [0,\infty]$, we construct a (one-dimensional) Banach space in which any nonzero function has topological entropy equal to $t$.

1. Introduction

Let $C([0,1])$ denote the set of all continuous functions $f : [0,1] \to \mathbb{R}$ equipped with the supremum norm. A theorem of Banach and Mazur [2] states that the Banach space $C([0,1])$ is universal, i.e., every real, separable Banach space $X$ is isometrically isomorphic to a closed subspace of $C([0,1])$. It is known that one can require more properties of the functions of $C([0,1])$ in the image of $X$: a universal space containing only the zero function and nowhere differentiable functions [7], resp. consisting of the zero function and nowhere approximatively differentiable and nowhere Hölder functions [3] has been proved. On the other hand, no universal space can consist of functions of bounded variation [4] and every isometrically isomorphic copy of $\ell_1$ (i.e., the space of sequences with 1-norm) in $C([0,1])$ contains a function which is non-differentiable at every point of a perfect subset of $[0,1]$, see [6].

In this paper we are going to investigate some properties of (universal) Banach spaces of real functions of a real variable in the context of topological entropy. We show how to construct a universal Banach space using the zero function and functions with infinite topological entropy - Theorem 3, and as a supplement of the result from [6] we show that any subspace of $C([0,1])$ which is isometrically isomorphic to $\ell_1$ contains a function with infinite topological entropy - Theorem 2. Finally, for any $t \in [0,\infty]$ we construct a (one-dimensional) Banach space in which any nonzero function has its topological entropy equal to $t$ - Theorem 4.

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2. Preliminaries and auxiliary results

Let $C_b(X)$ denote the set of all \textit{bounded} continuous functions $f : X \to \mathbb{R}$ equipped with the supremum norm. Clearly, $C_b(\mathbb{R})$ is a non-separable Banach space. Let $[a, b]$ be a closed finite subinterval of $\mathbb{R}$. We identify $f : [a, b] \to \mathbb{R}$ with its extension

$$(1) \quad (\text{Ex } f)(x) = \begin{cases} f(x) & \text{if } x \in [a, b]; \\ f(b) & \text{if } x \geq b; \\ f(a) & \text{if } x \leq a. \end{cases}$$

Under this identification, $C([a, b]) \subset C_b(\mathbb{R})$. We will deal with the topological entropy of maps from $C_b(\mathbb{R})$ defined as $h_{\text{top}}(f) := h_{\text{top}}(f|_{\mathbb{R}})$ - see [1, Chapter 4].

The well known Banach - Mazur Theorem states that the Banach space $C([0, 1])$ is universal, \textit{i.e.}, every real, separable Banach space $D$ is isometrically isomorphic to a closed subspace of $C([0, 1])$. Since by our convention, $C([0, 1])$ is a closed subspace of $C_b(\mathbb{R})$, the non-separable space $C_b(\mathbb{R})$ is also universal. In our paper we will restrict ourselves to separable universal Banach spaces only.

Following [1] we recall the notion of horseshoe.

**Definition 1.** A function $f \in C_b(\mathbb{R})$ is said to have a $d$-horseshoe if there exist $d$ subintervals $I_1, I_2, \ldots, I_d$ of $\mathbb{R}$ with disjoint interiors such that $f(I_i) \supset I_j$ for all $1 \leq i, j \leq d$.

**Proposition 1.** [5] If $f \in C_b(\mathbb{R})$ has a $d$-horseshoe then $h_{\text{top}}(f) \geq \log d$.

In the next lemma we denote by $F(X)$ a linear space of functions $f : X \to \mathbb{R}$.

**Lemma 1.** Given $n$ linearly independent functions in $F(X)$, there exist $n$ points $x_1, \ldots, x_n \in X$ such that the vectors

$$
\begin{pmatrix}
  f_1(x_1) \\
  f_1(x_2) \\
  \vdots \\
  f_1(x_n)
\end{pmatrix}, \quad 
\begin{pmatrix}
  f_2(x_1) \\
  f_2(x_2) \\
  \vdots \\
  f_2(x_n)
\end{pmatrix}, \quad \ldots, \quad 
\begin{pmatrix}
  f_n(x_1) \\
  f_n(x_2) \\
  \vdots \\
  f_n(x_n)
\end{pmatrix}
$$

are linearly independent in $\mathbb{R}^n$.

**Proof.** This is clear if $n = 1$. Assume now by induction that the lemma holds for $k < n$ and points $x_1, \ldots, x_k$. For the unique linear combination such that $f_{k+1}(x_i) = a_1f_1(x_i) + \cdots + a_kf_k(x_i)$ for all $1 \leq i \leq k$. Now if $f_{k+1}(x) = a_1f_1(x) + \cdots + a_kf_k(x)$ for all $x \in X$, then $f_1, \ldots, f_{k+1}$ are linearly dependent, contrary to our assumption. So there must be some other point $x_{k+1}$ for which $f_{k+1}(x_{k+1}) \neq a_1f_1(x_{k+1}) + \cdots + a_kf_k(x_{k+1})$, which concludes the induction step. \qed
3. The Main Theorems

**Definition 2.** For a given set $B \subset C_b(\mathbb{R})$, let
\[
h_{top}^+(B) = \sup \{ h_{top}(f) : f \in B \},
\]
\[
h_{top}^-(B) = \inf \{ h_{top}(f) : f \in B, \text{ } f \text{ is non-zero} \}.
\]

**Theorem 1.** If a linear space $B \subset C_b(\mathbb{R})$ has dimension $n$, then
\[
h_{top}^+(B) \geq \log(n - 1).
\]
In particular, $h_{top}^+(B) = \infty$ if $\dim(B) = \infty$.

*Proof.* Take $f_1, \ldots, f_n \in B$ linearly independent and find points $x_1, \ldots, x_n$ as in Lemma 1. We can assume that $x_1 < x_2 < \cdots < x_n$. Form a linear combination $f = a_1 f_1 + \cdots + a_n f_n$ such that $f(x_i) = x_1$ if $i$ is odd, and $f(x_i) = x_n$ if $i$ is even. Then $f$ has an $(n - 1)$-horseshoe, so from Proposition 1 we get $h_{top}(f) \geq \log(n - 1)$ as required. \hfill \Box

**Example 1.** (i) Let $[a, b]$ be a closed subinterval of $\mathbb{R}$. Given a continuous function $f : \mathbb{R} \to \mathbb{R}$, let
\[
(Cr_{[a,b]} f)(x) = \begin{cases} 
  f(x) & \text{if } x \in [a, b]; \\
  f(b) & \text{if } x \geq b; \\
  f(a) & \text{if } x \leq a,
\end{cases}
\]
be the cropped version of $f$. Clearly $Cr_{[a,b]} f \in C([a,b]) \subset C_b(\mathbb{R})$. Let
\[
P^{n-1} = \{ Cr_{[a,b]} f : f \in C(\mathbb{R}) \text{ is a polynomial of degree } \leq n - 1 \}.
\]
Then $P^{n-1}$ has dimension $n$, each $f \in P^{n-1}$ is at most $n-2$-modal, so our definition of the entropy $h_{top}(f)$ and [1, Theorem 4.2.4] imply that $h_{top}(f) \leq \log(n - 1)$. This shows that the bound in Theorem 1 is sharp.

(ii) Let
\[
P = \{ Cr_{[a,b]} f : f \in C(\mathbb{R}) \text{ is a polynomial } \}.
\]
Then $P$ is a normed linear subspace of $C([a,b])$ and $\dim(P) = \infty$, hence by Theorem 1, $h_{top}^+(P) = \infty$. By the same argument as above, the entropy of any $p \in P$ satisfies $h_{top}(p) \leq \log \deg(p)$, so it is finite.

Theorem 1 does not answer the question whether every infinite dimensional Banach space $A \subset C_b(\mathbb{R})$ contains a function with infinite entropy. Our next example shows that in general it is not the case.

**Example 2.** For $n \geq 1$ and $a \in \mathbb{R}$, let $f_{n,a} : \mathbb{R} \to \mathbb{R}$ be given by
\[
f_{n,a}(x) = \begin{cases} 
  a \cdot (x - 2 + \frac{1}{n}) \cdot (2 - \frac{1}{n+1} - x) & \text{if } x \in J_n := [2 - \frac{1}{n}, 2 - \frac{1}{n+1}]; \\
  0 & \text{otherwise}.
\end{cases}
\]
Clearly $f_{n,a}$ is unimodal, so its entropy $h_{top}(f_{n,a}) \leq \log 2$. Consider the smallest Banach space $Q$ (subspace of $C_b(\mathbb{R})$ with supremum norm) containing all finite sums
\( f_{1,a_1} + f_{2,a_2} + \cdots + f_{n,a_n} \). Then \( \dim(Q) = \infty \), \( \lim_{x \to 2-} f(x) = 0 \) for each \( f \in Q \) and if
\[
\max\{x \in \mathbb{R} : f(x) = 1\} \in J_n,
\]
then since the modality of \( f|_{1,1-\frac{1}{n+1}} \) is at most \( 2n \) and \( f^2(x) = 0 \) for \( x \notin [1,1-\frac{1}{n+1}] \), we conclude that \( h_{\text{top}}(f) \leq \log(2n + 1) \).

As a counterpart of the previous example we will prove the following theorem.

**Theorem 2.** Let \( A \subset C([0,1]) \) be isometrically isomorphic to \( \ell_1 \). Then \( A \) contains a function with infinite topological entropy.

**Proof.** Let \( \Phi \) be an isometrical isomorphism ensured by the statement, so \( \Phi(\ell_1) = A \). For \( i \in \mathbb{N} \) let \( e_i = (e_{ij})_{j=1}^{\infty} \in \ell_1 \) be defined by
\[
e_{ij} = \delta_{ij},
\]
where \( \delta_{ij} \) is the Kronecker delta. Then for every \( n \in \mathbb{N} \) and every choice of distinct positive integers \( i(1), \ldots, i(n) \)
\[
\| \pm e_{i(1)} \pm \cdots \pm e_{i(n)} \|_{\ell_1} = n.
\]
Denote \( f_i = \Phi(e_i) \in A \subset C([0,1]), \ i \in \mathbb{N} \). Clearly \( \|f_i\| = 1 \); in particular for every \( x \in [0,1], \)

\[
|f_i(x)| \leq 1.
\]

**Claim 1.** For every \( s = (s_i)_{i \in \mathbb{N}} \in \{1,-1\}^\mathbb{N} \) there exists a point \( x \in [0,1] \) such that the sequence \( (f_i(x))_{i \in \mathbb{N}} \) is equal to either \( s \) or \( -s \).

**Proof.** Assume that for some \( n \),
\[
\forall x \in [0,1]: (f_i(x))_{i=1}^{n} \neq (s_i)_{i=1}^{n} \text{ and } (f_i(x))_{i=1}^{n} \neq (-s_i)_{i=1}^{n};
\]
then (2) implies \( |\sum_{i=1}^{n} s_i f_i(x)| < n \) for every \( x \in [0,1] \). This contradicts the equalities
\[
|\sum_{i=1}^{n} s_i e_i|_{\ell_1} = \| \sum_{i=1}^{n} s_i f_i \| = n.
\]
Thus, for each \( n \in \mathbb{N} \) one can find a point \( x_n \in [0,1] \) for which either \( (f_i(x_n))_{i=1}^{n} = (s_i)_{i=1}^{n} \) or \( (f_i(x_n))_{i=1}^{n} = (-s_i)_{i=1}^{n} \). Taking a limit point \( x \) of the sequence \( (x_n)_{n=1}^{\infty} \) from the continuity of the functions \( f_i \) we get either \( (f_i(x)) = s \) or \( (f_i(x)) = -s \). \( \square \)
Let us denote the $i$ for each $n > 1$. In particular, the particular matrix $A_8$ is
\[
\begin{pmatrix}
+1 & +1 & +1 & +1 & +1 & +1 & +1 \\
+1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & +1 & +1 & +1 & +1 & +1 \\
+1 & +1 & +1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & +1 & +1 & +1 & +1 \\
+1 & +1 & +1 & +1 & +1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & +1 & +1 \\
+1 & +1 & +1 & +1 & +1 & +1 & -1
\end{pmatrix}
\]

Passing to a subsequence if necessary, we can assume that $X_f(x)$, and we obtain that either
\[
(10) \quad e = x_0 e_1 + \sum_{n=2}^{\infty} \sum_{k=1}^{n} \alpha_k^n e_{2^n+k}, \quad \Phi(e) = f(x) = x_0 f_1(x) + \sum_{n=2}^{\infty} \sum_{k=1}^{n} \alpha_k^n f_2^{n+k}(x),
\]

Claim 2. For any $\beta \in \mathbb{R}^n$, the linear equation $A_n \alpha = \beta$ has a unique solution $\alpha$ given by the formulas
\[
\alpha_i = \frac{\beta_i + \beta_{i+1}}{(-1)^{i+1}2}, \quad i = 1, \ldots, n-1, \quad \alpha_n = \frac{\beta_1 + (-1)^n \beta_n}{2}.
\]

In particular, $\max |\alpha_i| \leq \max |\beta_i|$. 

Let us denote the $i$-th row of the matrix $A_n$ by $a_i^n = (a_{i1}^n, a_{i2}^n, \ldots, a_{in}^n)$. By Claim 1, for each $n > 1$ there are distinct points $x_1^n, \ldots, x_n^n \in [0,1]$ such that either
\[
(5) \quad f_1(x_i^n) = \cdots = f_{2^n}(x_i^n) = 1, \quad (f_{2^n+1}(x_i^n), f_{2^n+2}(x_i^n), \ldots, f_{2^n+n}(x_i^n)) = a_i^n,
\]
or
\[
(6) \quad f_1(x_i^n) = \cdots = f_{2^n}(x_i^n) = -1, \quad (f_{2^n+1}(x_i^n), f_{2^n+2}(x_i^n), \ldots, f_{2^n+n}(x_i^n)) = -a_i^n.
\]

Put $X_n = \{x_1^n, \ldots, x_n^n\}$. Since $n = \text{card}(X_n)$ is growing to infinity, one can consider subsets $X'_n \subset X_n$ satisfying
\[
(7) \quad \lim_{n \to \infty} \text{card}(X'_n) = \infty, \quad \lim_{n \to \infty} \text{diam}(X'_n) = 0.
\]

Passing to a subsequence if necessary, we can assume that $X'_n \to x_0 \in [0,1]$, i.e.,
\[
(8) \quad \forall \varepsilon > 0 \exists n_0 \forall n > n_0: X'_n \subset (x_0 - \varepsilon, x_0 + \varepsilon).
\]

Now, using (5) and (6), we obtain that either $(f_i(x_0))_i = (1)_i$ or $(f_i(x_0))_i = (-1)_i$. Without loss of generality assume the first possibility. Notice that then
\[
(9) \quad \forall n > 1: (f_{2^n+1}(x_0), f_{2^n+2}(x_0), \ldots, f_{2^n+n}(x_0)) = a_1^n.
\]

We can formally put
\[
(10) \quad e = x_0 e_1 + \sum_{n=2}^{\infty} \sum_{k=1}^{n} \alpha_k^n e_{2^n+k}, \quad \Phi(e) = f(x) = x_0 f_1(x) + \sum_{n=2}^{\infty} \sum_{k=1}^{n} \alpha_k^n f_{2^n+k}(x),
\]
where coefficients $\alpha_n = (\alpha_1^n, \alpha_2^n, \ldots, \alpha_n^n)$ satisfy a linear equation $A_n\alpha_n = \beta_n^n$, $\beta_n^n = (\beta_1^n, \beta_2^n, \ldots, \beta_n^n) \in \mathbb{R}^n$. It can be easily seen that $f \in A$ if and only if
\[
\sum_{n=2}^{\infty} \sum_{k=1}^{n} |\alpha_k^n| < \infty.
\]
Moreover, if $\beta_1^n = 0$ for each $n$ and $f \in C([0,1])$ then $f(x_0) = x_0$ by the equation
\[
x_0 f_1(x_0) = x_0,
\]
and
\[
\text{Moreover, if } g \text{ gained as the unique solution of the linear equation } A_n g = \beta_n^n, \beta_n^n = (\beta_1^n, \beta_2^n, \ldots, \beta_n^n) \in \mathbb{R}^n. \text{ It can be easily seen that } f \in A \text{ if and only if }
\sum_{n=2}^{\infty} \sum_{k=1}^{n} |\alpha_k^n| < \infty.
\]

Using Claim 2 we will show in the sequel that there exists a sequence $(\beta_n^n = (0, \beta_2^n, \ldots, \beta_n^n))_n$ such that the corresponding function $f$ given by (10) satisfies $f \in A$ and $h_{top}(f) = \infty$. In what follows we denote
\[
g_1(x) = x_0 f_1(x), \ g_m(x) = g_1(x) + \sum_{n=2}^{m} \sum_{k=1}^{n} \alpha_k^n f_{2n+k}(x), \ m \geq 2.
\]

Let $\omega(f, X) = \sup_{x,y \in X} |f(x) - f(y)|$ denote the oscillation of a function $f$ on a set $X$. For a positive $\varepsilon(i)$ we use the notation $J(i) = [x_0 - \varepsilon(i), x_0 + \varepsilon(i)]$. The zero element in $\mathbb{R}^n$ is denoted by $0_n$. Let $(\gamma_m)_{m \in \mathbb{N}}$ be a sequence of positive numbers satisfying for each $m$
\[
\gamma_m > \sum_{i=m+1}^{\infty} 2(i + 3) \gamma_i.
\]

Step 0. $n(0) = 1$.

Step 1. We can find values $\varepsilon(1) > 0$ and $n(1) > n(0) + 1$ such that
\[
\varepsilon(1) + \omega(g_{n(0)}, J(1)) < \gamma_1 - \sum_{i=2}^{\infty} 2(i + 3) \gamma_i,
\]
\[
J(1) \cap X'_{n(1)} \supset \{ x_{i(1)}^{n(1)} < x_{i(2)}^{n(1)} < x_{i(3)}^{n(1)} < x_{i(4)}^{n(1)} \}.
\]

We put $\beta^n = 0_n$ for each $n(0) < n < n(1)$; the coefficients $\alpha_k^{n(1)}$, $k = 1, \ldots, n(1)$ are gained as the unique solution of the linear equation $A_{n(1)} \alpha^{n(1)} = \beta^{n(1)}$, where (as we already know) $\beta_1^{n(1)} = 0, \beta_i^{n(1)} = (-1)^j \gamma_1, j = 1, 2, 3, 4$ and $\beta_i^{n(1)} = 0$ otherwise.

Step m. We can find values $\varepsilon(m) > 0$ and $n(m) > n(m - 1) + 1$ such that
\[
\varepsilon(m) + \omega(g_{n(m-1)}, J(m)) < \gamma_m - \sum_{i=m+1}^{\infty} 2(i + 3) \gamma_i,
\]
\[
J(m) \cap X'_{n(m)} \supset \{ x_{i(1)}^{n(m)} < x_{i(2)}^{n(m)} < \cdots < x_{i(m+2)}^{n(m)} < x_{i(m+3)}^{n(m)} \}.
\]
Theorem 3. There is a universal Banach space

\[ A \]

interval \( f \) has an \( (\cdot) \) horseshoe (created by the points \( x_{i(1)}; x_{i(2)}; \ldots; x_{i(m+3)} \)) on the interval \( J(\cdot) \). It means that \( h_{\text{top}}(f) \geq \log(m+2) \) and \( m \) can be arbitrarily large. \( \square \)

**Theorem 3.** There is a universal Banach space \( A \subset C_0(\mathbb{R}) \) such that \( h_{\text{top}}(f) = \infty \) for every non-zero \( f \) from \( A \).

**Proof.** Take \( p_n = 2^{-n} \) for \( n \geq 0 \) and \( \{q_n\}_{n \geq 0} \) a decreasing sequence such that \( q_0 = 1, q_n \geq p_n \) for all \( n, q_n/p_n \to \infty \), but \( q_n \to 0 \). Choose intervals \( I_n = [\frac{3}{2}p_n, \frac{5}{2}p_n] \) and \( J_n = (\frac{3}{2}p_n, \frac{5}{2}p_n) \subset I_n \), both ‘centered’ at \( p_n \). Notice also that the \( J_n \)'s are adjacent: \( \frac{2}{3}p_n \) is the common boundary point of \( J_n \) and \( J_{n+1} \). Now for a function \( f \in C([0,1]) \), construct \( g := \Psi(f) \in C_0(\mathbb{R}) \) as follows, see Figure 1:

\[
g(y) = \begin{cases} 
0 & \text{if } y = 0; \\
q_n \cdot f\left(\frac{2y}{p_n} - \frac{3}{2}\right) & \text{if } y \in I_n \text{ for some } n \geq 0; \\
0 & \text{if } y \in \cup_n \partial J_n; \\
0 & \text{if } y \geq \frac{4}{3}; \\
\text{by linear interpolation} & \text{if } y \in \cup_n (J_n \setminus I_n); \\
g(-y) & \text{if } y < 0;
\end{cases}
\]

Let \( \mathcal{A} = \mathcal{C}(C([0,1])) \subset C([-\frac{4}{3}, \frac{4}{3}]) \) equipped with the norm \( (q_0 = 1) \)

\[
\sup_{y \in \mathbb{R}} |g(y)| = \|g\| = \sup_{y \in I_0} |g(y)| = \|f\|,
\]
We define \( \Psi(f) = g \in C([-\frac{1}{3}, \frac{2}{3}]) \),

\[
p_n = \left(\frac{1}{2}\right)^n, \quad q_n = \left(\frac{2}{3}\right)^n, \quad n \geq 0.
\]

so \( \Psi \) is an isometrical isomorphism and \( \mathcal{A} \) is a separable Banach space.

If \( f \) is not constant zero, then \( g = \Psi(f) \) is not constant zero either and

\[
\sup_{y \in I_n} |g(y)| = q_n \|f\| > 0.
\]

Fix \( d \in \mathbb{N} \) arbitrary. Since \( q_n/p_n = q_n/2^{-n} \to \infty \), there is an \( n \in \mathbb{N} \) such that

\[
q_n\|f\| > 2^{-n+d} = p_{n-d}.
\]

Since \( \{q_i\}_i \) is decreasing and \( g(\pm\partial J_i) = 0 \) for all \( i \) (where \( -J_i = \{y : y \in J_i\} \)), it follows that \( g(I_i) = g(-I_i) \supset [0, \max J_{n-d+1}] \) or \([\max J_{n-d+1}, 0]\) for all \( n-d+1 \leq i \leq n \). Hence, within the intervals \( J_{n-d+1}, \ldots, J_n \), or within \( -J_{n-d+1}, \ldots, -J_n \), we can choose \( d \) intervals that form a \( d \)-horseshoe. This implies that \( h_{\text{top}}(g) \geq \log d \).

As \( d \) was arbitrary, \( h_{\text{top}}(g) = \infty \).

For a real, separable Banach space \( \mathcal{B} \) we will find an isometrical isomorphism \( \Phi : \mathcal{B} \to \mathcal{A} \). Since by the Banach–Mazur Theorem the space \( C([0,1]) \) is universal, there is an isometrical isomorphism \( \Phi : \mathcal{B} \to C([0,1]) \). Using the above constructed isometrical isomorphism \( \Psi : C([0,1]) \to \mathcal{A} \), the required \( \Phi \) is just \( \Psi \circ \Phi \).

**Remark 1.** Recall that \( f \in C^\alpha(\mathbb{R}) \) (\( f \) is \( \alpha \)-Hölder on \( \mathbb{R} \)) for some \( \alpha \in (0,1) \) if

\[
\sup \left\{ \frac{|f(x) - f(y)|}{|x-y|^\alpha} : x, y \in \mathbb{R}, \ 0 < |x-y| \leq 1 \right\} < \infty.
\]

For some fixed \( \alpha \in (0,1) \), if we choose \( q_n = p_n^\alpha \) and \( f \in C^\alpha([0,1]) \), then \( \Psi(f) \) is \( \alpha \)-Hölder on \( \mathbb{R} \). Therefore \( \mathcal{A}^\alpha := \Psi(C^\alpha([0,1])) \subset C^\alpha_b(\mathbb{R}) \) is a normed (infinite dimensional) linear space such that \( h_{\text{top}}(f) = \infty \) for every non-zero \( f \) from \( \mathcal{A}^\alpha \).
4. ENTROPY OF ONE-DIMENSIONAL BANACH SPACES

Even if \( \dim(B) = 1 \), it is still possible that \( h^+_{\text{top}}(B) = \infty \). As the following example shows, the upper bound for the entropy need not be attained.

Example 3. Let \( B \) be spanned by \( f(x) = \sin x \), then \( \lambda f \) admits a \( d \)-horseshoe whenever \( |\lambda| \geq 2\pi d \). Therefore \( h^+_{\text{top}}(B) = \infty \).

The above example also shows that there is no sensible upper bound for \( h^+_{\text{top}}(B) \) in terms of \( \dim(B) \) only. However, \( h^+_{\text{top}}(B) = 0 \) - see Definition 2.

In this section we will be investigating the equality \( h^+_{\text{top}}(B) = h^-_{\text{top}}(B) \) for one-dimensional subspaces \( B \) of \( C_b(\mathbb{R}) \); so far we know that for some \( B \),

- \( h^+_{\text{top}}(B) = h^-_{\text{top}}(B) = \infty \) (easy consequence of Theorem 3)
- \( h^+_{\text{top}}(B) = h^-_{\text{top}}(B) = 0 \) (\( B \) is spanned by a monotone map)

The following statement shows that the entropy can behave extremely rigidly on a one-dimensional subspace of \( C_b(\mathbb{R}) \).

Theorem 4. For any \( t \in [0, \infty) \), there exists a function \( f \in C_b(\mathbb{R}) \) such that for \( B = \{ \lambda f \}_{\lambda \in \mathbb{R}} \) satisfies \( h^+_{\text{top}}(B) = h^-_{\text{top}}(B) = t \).

Proof. The case \( t = 0 \) and \( t = \infty \) were covered previously, so let \( t \in (0, \infty) \) arbitrary and take an odd integer \( d > e^t \).

Let \( \theta_a : [0, \infty) \to [0, \infty) \) be a one-parameter family (with \( a \in [0, 1] \)) of at most \( d \)-modal continuous maps such that for each \( a \in [0, 1] \), \( \theta_a([9, 10]) \subset [9, 10] \) and \( \theta_0(x) = \theta_1(x) \) whenever \( x \not\in (9, 10) \), \( \theta_0 \) is the identity, and \( \theta_1 \) has a full \( d \)-horseshoe on \([9, 10] \). In the \( C^1 \) topology for maps of fixed modality, topological entropy depends continuously on the map, see [1, Cor. 4.5.5], so there is no loss in generality in assuming that \( h_{\text{top}}(\lambda \cdot \theta_a) \) is continuous in both \( a \in [0, 1] \) and \( \lambda \in [\frac{9}{10}, \frac{10}{9}] \). (Note that \( h_{\text{top}}(\lambda \cdot \theta_a) \equiv 0 \) for \( \lambda \geq 0 \) outside this interval.) Therefore \( r_0 = \sup_{\lambda \geq 0} h_{\text{top}}(\lambda \cdot \theta_a) \) is continuous in \( a \) as well, and \( r_0 = 0 \), \( r_1 = \log d > t \). Therefore there is \( a^* \) such that \( r_{a^*} = t \). Fix \( \Theta = \theta_{a^*} \).

Next let \( \{ \lambda_i \}_{i \geq 0} \) be a denumeration of the positive rationals such that \( \lambda_1 = 1 \) and

\[
\lambda_{n+1} \leq 2\lambda_n \quad \text{for all } n \geq 0.
\]

Let \( x_n = 4^{-n} \) and \( I_n = [0.9x_n, x_n] \) for \( n \geq 0 \). Now we set

\[
f(x) = \begin{cases} 
\lambda_n \cdot \frac{x}{x_n} \cdot \Theta(\frac{10}{x_n} \cdot x) & \text{if } x \in I_n; \\
0 & \text{if } x = 0; \\
10 & \text{if } x \geq 10; \\
\text{by linear interpolation} & \text{if } x \in (0, 10) \setminus \cup_n I_n; \\
f(-x) & \text{if } x < 0.
\end{cases}
\]
Fix $\lambda > 0$. By assumption (14) we have that $\lambda f(x) \leq \lambda f(y)$ for all $x \in I_{n+1}$, $y \in I_n$ and $n \geq 0$. It is not hard to see that every orbit with respect to $\lambda f$ can visit only finitely many intervals $I_n$, and at most one of them infinitely often. Therefore, if we choose some $x > 0$, then $\omega(x)$ can only belong to a single $I_n$, and only if the diagonal intersects the box $I_n \times \lambda f(I_n)$. By our choice of $a^*$ (and hence $\Theta$), $h_{\text{top}}(\lambda f|_{I_n}) \leq t$. Since $x \geq 0$ is arbitrary, $h_{\text{top}}(\lambda f) \leq t$.

For $\varepsilon > 0$ let $\lambda^*$ satisfy $h_{\text{top}}(\lambda^*\Theta) \geq t - \varepsilon$. Since $\{\lambda_n\}_{n \geq 0}$ is dense in $[0, \infty)$ there is some interval $I_m$ such that $\lambda_m\lambda$ is sufficiently close to $\lambda^*$ hence $h_{\text{top}}(\lambda_m\lambda\Theta) \geq t - 2\varepsilon$ and also $h_{\text{top}}(\lambda f|_{I_m}) \geq t - 2\varepsilon$. This shows that $h_{\text{top}}(\lambda f) \geq t$, and so we have $h_{\text{top}}(\lambda f) = t$.

Finally, the dynamics of $-\lambda f$ on $(-\infty, 0]$ is conjugate to the dynamics of $\lambda f$ on $[0, \infty)$, so also $h_{\text{top}}(-\lambda f) = t$. \hfill \square

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