ON ISOTOPY AND UNIMODAL INVERSE LIMIT SPACES

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Abstract. We prove that every self-homeomorphism \( h : K_s \to K_s \) on the inverse limit space \( K_s \) of tent map \( T_s \) with slope \( s \in (\sqrt{2}, 2] \) is isotopic to a power of the shift-homeomorphism \( \sigma^R : K_s \to K_s \).

1. Introduction

The solution of Ingram’s Conjecture constitutes a major advancement in the classification of unimodal inverse limit spaces and the group of self-homeomorphisms on them. This conjecture was posed by Tom Ingram in 1992 for tent maps \( T_s : [0, 1] \to [0, 1] \) with slope \( \pm s, s \in [1, 2] \), defined as \( T_s(x) = \min\{sx, s(1-x)\} \). The turning point is \( c = \frac{1}{2} \) and we denote its iterates by \( c_n = T_s^n(c) \). The inverse limit space \( K_s = \lim\left([0, s/2], T_s\right) \) consists of the core \( \lim\left([c_2, c_1], T_s\right) \) and the 0-composant \( C_0 \), i.e., the composant of the point \( \bar{0} := (\ldots, 0, 0, 0) \), which compactifies on the core of the inverse limit space. Ingram’s Conjecture reads:

If \( 1 \leq s < s' \leq 2 \), then the corresponding inverse limit spaces \( \lim([0, s/2], T_s) \) and \( \lim([0, s'/2], T_{s'}) \) are non-homeomorphic.

The first results towards solving this conjecture were obtained for tent maps with a finite critical orbit [9, 12, 3]. Raines and Štimac [11] extended these results to tent maps with a possibly infinite, but non-recurrent critical orbit. Recently Ingram’s Conjecture was solved completely (in the affirmative) in [2], but we still know very little of the structure of inverse limit spaces (and their subcontinua) for the case that \( \text{orb}(c) \) is infinite and recurrent, see [1, 5, 8].

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Given a continuum $K$ and $x \in K$, the *composant* $A$ of $x$ is the union of the proper subcontinua of $K$ containing $x$. For slopes $s \in (\sqrt{2}, 2]$, the core is indecomposable (*i.e.*, it cannot be written as the union of two proper subcontinua), and in this case we also proved [2] that any self-homeomorphism $h : K_s \to K_s$ is pseudo-isotopic to a power $\sigma^R$ of the shift-homeomorphism $\sigma$ on the core. This means that $h$ permutes the composants of the core of $K_s$ in the same way as $\sigma^R$ does, and it is a priori a weaker property than isotopy. This is for instance illustrated by the sin $\frac{1}{x}$-continuum, defined as the graph $\{(x, \sin \frac{1}{x}) : x \in (0, 1]\}$ compactified with a bar $\{0\} \times [-1, 1]$. There are homeomorphisms that reverse the orientation of the bar, and these are always pseudo-isotopic, but never isotopic, to the identity. Since such sin $\frac{1}{x}$-continua are precisely the non-trivial subcontinua of Fibonacci-like inverse limit spaces [8], this example is very relevant to our paper.

In this paper we make the step from pseudo-isotopy to isotopy. To this end, we exploit so-called *folding points*, *i.e.*, points in the core of $K_s$ where the local structure of the core of $K_s$ is not that of a Cantor set cross an arc. In the next section we prove the following results:

**Theorem 1.1.** If $s \in (\sqrt{2}, 2]$ and $h : K_s \to K_s$ is a homeomorphism, then there is $R \in \mathbb{Z}$ such that $h(x) = \sigma^R(x)$ for every folding point $x$ in $K_s$.

Folding points $x = (\ldots, x_{-2}, x_{-1}, x_0)$ are characterized by the fact that each entry $x_{-k}$ belongs to the omega-limit set $\omega(c)$ of the turning point $c = \frac{1}{2}$, see [10]. This gives the immediate corollary for those slopes such that the critical orbit $\text{orb}(c)$ is dense in $[c_2, c_1]$, which according to [7] holds for Lebesgue a.e. $s \in [\sqrt{2}, 2]$.

**Corollary 1.2.** If $\text{orb}(c)$ is dense in $[c_2, c_1]$, then for every homeomorphism $h : K_s \to K_s$ there is $R \in \mathbb{Z}$ such that $h = \sigma^R$ on the core of $K_s$.

The more difficult case, however, is when $\text{orb}(c)$ is not dense in $[c_2, c_1]$. In this case, $h$ can be at best isotopic to a power of the shift, because at non-folding points, where the core of $K_s$ is a Cantor set cross an arc, $h$ can easily act as a local translation. It is shown in [4] that for tent maps with non-recurrent critical point (*or in fact, more generally long-branched tent maps*), every homeomorphism $h : K_s \to K_s$ is indeed isotopic to a power of the shift. The proof exploits the fact that in this case, so-called $p$-points (*indicating folds in the arc-components of $K_s$*) are separated from each other, at least in arc-length semi-metric. Here we prove the general result.

**Theorem 1.3.** If $s \in (\sqrt{2}, 2]$, and $h : K_s \to K_s$ is a homeomorphism, then there exists $R \in \mathbb{Z}$ such that $h$ is isotopic to $\sigma^R$. 
The paper is organized as follows. In Section 2 we give basic definitions and prove results on how homeomorphisms act on folding points, i.e., Theorem 1.1 and Corollary 1.2. These proofs depend largely on the results obtained in [2]. In Section 3 we present the additional arguments needed for the isotopy result and finally prove Theorem 1.3.

2. INVERSE LIMIT SPACES OF TENT MAPS AND FOLDING POINTS

Let \( N = \{1, 2, 3, \ldots\} \) be the set of natural numbers and \( N_0 = N \cup \{0\} \). The tent map \( T_s : [0, 1] \to [0, 1] \) with slope \( \pm s \) is defined as \( T_s(x) = \min\{sx, s(1-x)\} \). The critical or turning point is \( c = 1/2 \) and we write \( c_k = T_s^k(c) \), so in particular \( c_1 = s/2 \) and \( c_2 = s(1-s/2) \). Also let \( \text{orb}(c) \) and \( \omega(c) \) be the orbit and the omega-limit set of \( c \). We will restrict \( T_s \) to the interval \( I = [0, s/2] \); this is larger than the core \( [c_2, c_1] = [s - s^2/2, s/2] \), but it contains the fixed point 0 on which the 0-composant \( \mathcal{C}_0 \) is based.

The inverse limit space \( K_s = \lim_{\leftarrow}([0, s/2], T_s) \) is 
\[
\{x = (\ldots, x_{-2}, x_{-1}, x_0) : T_s(x_{i-1}) = x_i \in [0, s/2] \text{ for all } i \leq 0\},
\]
equipped with metric \( d(x, y) = \sum_{n \leq 0} 2^n|x_n - y_n| \) and induced (or shift) homeomorphism
\[
\sigma(\ldots, x_{-2}, x_{-1}, x_0) = (\ldots, x_{-2}, x_{-1}, x_0, T_s(x_0)).
\]
Let \( \pi_k : \lim_{\leftarrow}([0, s/2], T_s) \to I, \pi_k(x) = x_{-k} \) be the \( k \)-th projection map. Since 0 \( \in I \), the endpoint \( \bar{0} := (\ldots, 0, 0, 0) \) is contained in \( \lim_{\leftarrow}([0, s/2], T_s) \). The composant of \( \lim_{\leftarrow}([0, s/2], T_s) \) of \( \bar{0} \) will be denoted as \( \mathcal{C}_0 \); it is a ray converging to, but disjoint from the core \( \lim_{\leftarrow}([c_2, c_1], T_s) \) of the inverse limit space. We fix \( s \in (\sqrt{2}, 2] \); for these parameters \( T_s \) is not renormalizable and \( \lim_{\leftarrow}([c_2, c_1], T_s) \) is indecomposable. Moreover, the arc-component of \( \bar{0} \) coincides with the composant of \( \bar{0} \), but for points in the core of \( K_s \), we have to make the distinction between arc-component and composant more carefully.

A point \( x = (\ldots, x_{-2}, x_{-1}, x_0) \in K_s \) is called a p-point if \( x_{-p-l} = c \) for some \( l \in N_0 \). The number \( L_p(x) := l \) is the p-level of \( x \). In particular, \( x_0 = T_s^{p+l}(c) \). By convention, the endpoint \( \bar{0} \) of \( \mathcal{C}_0 \) is also a p-point and \( L_p(\bar{0}) := \infty \), for every \( p \). The ordered set of all p-points of the composant \( \mathcal{C}_0 \) is denoted by \( E_p \), and the ordered set of all p-points of p-level \( l \) by \( E_{p,l} \). Given an arc \( A \subset K_s \) with successive p-points \( x^0, \ldots, x^n \), the p-folding pattern of \( A \) is the sequence
\[
FP_p(A) := L_p(x^0), \ldots, L_p(x^n).
\]
Note that every arc of \( \mathcal{C}_0 \) has only finitely many p-points, but an arc \( A \) of the core of \( K_s \) can have infinitely many p-points. In this case, if \( (u^i)_{i \in \mathbb{Z}} \) is the sequence of successive p-points of \( A \), then \( FP_p(A) = (L_p(u^i))_{i \in \mathbb{Z}} \). The folding pattern of the composant \( \mathcal{C}_0 \), denoted by \( FP(\mathcal{C}_0) \),
Take \( \Rightarrow \) proves that Lemma 2.2.

Since \( \forall i \in \mathbb{N}, \) and since by definition \( \ell_p(s_i) = i, \) for every \( i \in \mathbb{N}. \) Note that the salient \( p \)-points depend on \( p: \) if \( p \geq q, \) then the salient \( p \)-point \( s_i \) equals the salient \( q \)-point \( s_{i+p-q}. \)

A folding point is any point \( x \) in the core of \( K_s \) such that no neighborhood of \( x \) in core of \( K_s \) is homeomorphic to the product of a Cantor set and an arc. In [10] it was shown that \( x = (\ldots, x_{-2}, x_{-1}, x_0) \) is a folding point if and only if \( x_{-k} \in \omega(c) \) for all \( k \geq 0. \) We can characterize folding points in terms of \( p \)-points as follows:

Lemma 2.2. Let \( p \) be arbitrary. A point \( x \in K_s \) is a folding point if and only if there is a sequence of \( p \)-points \( (x^k)_{k \in \mathbb{N}} \) such that \( x^k \to x \) and \( L_p(x^k) \to \infty. \)

Proof. \( \Rightarrow \) Take \( m \geq p \) arbitrary. Since \( \pi_m(x) \in \omega(c) \) there is a sequence of post-critical points \( c_{n_i} \to \pi_m(x). \) This means that any point \( y^i = (\ldots, c_{n_i}, c_{n_{i+1}}, \ldots, c_{n_m}) \) is a \( p \)-point with \( p \)-level \( L_p(y^i) = n_i + m - p. \) Furthermore, for each \( 0 \leq j \leq m, \) \( |\pi_j(y^i) - \pi_j(x)| \to 0 \) as \( i \to \infty. \)

Since \( m \) is arbitrary, we can construct a diagonal sequence \( (x^k)_{k \in \mathbb{N}} \) of \( p \)-points, by taking a single element from \( (y^i)_{i \in \mathbb{N}} \) for each \( m, \) such that \( \sup_{j \leq k} |\pi_j(x^k) - \pi_j(x)| \to 0 \) as \( k \to \infty. \) This proves that \( x^k \to x \) and \( L_p(x^k) \to \infty. \)

\( \Leftarrow \) Take \( m \) arbitrary. Since \( x_k \to x, \) also \( |\pi_m(x^k) - \pi_m(x)| \to 0 \) and \( \pi_m(x^k) = c_n \) for \( n = L_p(x^k) + p - m. \) But \( L_p(x^k) \to \infty, \) so \( \pi_m(x) \in \omega(c). \)

A continuum is chainable if for every \( \varepsilon > 0, \) there is a cover \( \{\ell^1, \ldots, \ell^n\} \) of open sets (called links) of diameter \( \varepsilon \) such that \( \ell^i \cap \ell^j \neq \emptyset \) if and only if \( |i - j| \leq 1. \) Such a cover is called a chain. Clearly the interval \([0, s/2]\) is chainable.

Definition 2.3. We call \( C_p \) a natural chain of \( \lim([0, s/2], T_s) \) if

1. there is a chain \( \{I^1_p, I^2_p, \ldots, I^n_p\} \) of \([0, s/2]\) such that \( \ell^j_p := \pi_p^{-1}(I^j_p) \) are the links of \( C_p; \)
2. each point \( x \in \cup_{i=0}^{s} T_s^{-1}(c) \) is the boundary point of some link \( I^j_p; \)
3. for each \( i \) there is \( j \) such that \( T_s(I^j_{p+1}) \subset I^j_p. \)
If \( \max_j |I_j^p| < \varepsilon s^{-p}/2 \) then \( \text{mesh}(C_p) := \max\{\text{diam}(\ell) : \ell \in C_p\} < \varepsilon \), which shows that \( \lim([0, s/2], T_s) \) is indeed chainable. Condition (3) ensures that \( C_{p+1} \) refines \( C_p \) (written \( C_{p+1} \preceq C_p \)).

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let \( h : K_s \to K_s \) be a homeomorphism. Let \( x, y \in K_s \) be folding points with \( h(x) = y \). For \( i \in \mathbb{N}_0 \) let \( q_i, p_i \in \mathbb{N} \) be such that for sequences of chains \( (C_{q_i})_{i \in \mathbb{N}_0} \) and \( (C_{p_i})_{i \in \mathbb{N}_0} \) of \( K_s \) we have

\[
\cdots < h(C_{q_{i+1}}) \prec C_{p_{i+1}} \prec h(C_{q_i}) \prec C_{p_i} \prec \cdots < h(C_{q_1}) \prec C_{p_1} \prec h(C_q) \prec C_p,
\]

where \( q_0 = q \) and \( p_0 = p \). Let \( (\ell_{q_i}^s)_{i \in \mathbb{N}_0} \) be sequence of links such that \( x \in \ell_{q_i}^s \in C_{q_i} \), and similarly for \( (\ell_{p_i}^s)_{i \in \mathbb{N}_0} \). Then \( \ell_{q_{i+1}}^s \subset \ell_{q_i}^s \), \( \ell_{p_{i+1}}^s \subset \ell_{q_i}^s \), and \( h(\ell_{q_i}^s) \subset \ell_{p_i}^s \). Let \( (s_i^d_{d_i})_{i \in \mathbb{N}} \) be a sequence of salient \( q \)-points with \( s_i^d_{d_i} \to x \) as \( i \to \infty \). Then for every \( i \) there exist \( j_i \) such that \( s_i^d_{d_i} \in \ell_{q_i}^s \), \( h(s_i^d_{d_i}) \in \ell_{p_i}^s \), and \( h(s_i^d_{d_i}) \to y \) as \( i \to \infty \). By [2, Theorem 4.1] the midpoint of the arc component \( A_i \) of \( \ell_{p_i}^s \) which contains \( h(s_i^d_{d_i}) \) is a salient \( p \)-point \( s_i^m \). Since \( s_i^m, y \in \ell_{p_i}^s \), for every \( i \) and diam \( \ell_{p_i}^s \to 0 \) as \( i \to \infty \), we have \( s_i^m \to y \). Since \( s_i^d_{d_i} \) is a salient \( q \)-point and \( s_i^m \) can be also considered as a salient \( p \)-point and is also the midpoint of the arc component \( B_i \supset A_i \) of \( \ell_{p}^s \) which contains \( h(s_i^d_{d_i}) \). Therefore, \( s_i^m = s_i^d_{d_i} + M \), where \( M \) is as in [2, Theorem 4.1].

Let \( R = M - q + p \). By [2, Corollary 5.3], \( R \) does not depend on \( q, p \) and \( M \). Since \( \sigma^R : K_s \to K_s \) is a homeomorphism, and since \( s_i^d_{d_i} \to x \) as \( i \to \infty \), we have \( \sigma^R(s_i^d_{d_i}) \to \sigma^R(x) \) as \( i \to \infty \). Note that \( \sigma^R(s_i^d_{d_i}) = s_i^d_{d_i} + M \) and \( s_i^d_{d_i} + M \to y \). Therefore \( \sigma^R(x) = y \), i.e., \( \sigma^R(x) = h(x) \).

**Proof of Corollary 1.2.** If \( \text{orb}(c) \) is dense in \([c_2, c_1]\), every point \( x \) in the core of \( K_s \) satisfies \( \pi_k(x) \in \omega(c) \) for all \( k \in \mathbb{N} \). By [10], this means that every point is a folding point, and hence the previous theorem implies that \( h \equiv \sigma^R \) on the core of \( K_s \). \( \square \)

**Remark 2.4.** A point \( x \in K_s \) is an **endpoint** of an atriodic continuum, if for every pair of subcontinua \( A \) and \( B \) containing \( x \), either \( A \subset B \) or \( B \subset A \). The notion of folding point is more general than that of end-point. An example of a folding point that is not an endpoint is the midpoint \( x \) of a **double spiral** \( S \), i.e., a continuous image of \( \mathbb{R} \) containing a single folding point \( x \) and two sequences of \( p \)-points

\[
\cdots \, y^k < y^{k+1} < \cdots < x < \cdots < z^{k+1} < z^k \cdots
\]

converging to \( x \) such that the arc-length \( \bar{d}(y^k, y^{k+1}), \bar{d}(z^k, z^{k+1}) \to 0 \) as \( k \to \infty \). Here \( \prec \) denotes the induced order on \( S \).
It is natural to classify arc-components $\mathcal{A}$ according to the folding points they may contain. For arc-components $\mathcal{A}$, we have the following possibilities:

- $\mathcal{A}$ contains no folding point.
- $\mathcal{A}$ contains one folding point $x$, e.g. if $x$ is an end-point of $\mathcal{A}$ or $\mathcal{A}$ is a double spiral.
- $\mathcal{A}$ contains two folding points, e.g. if $\mathcal{A}$ is the bar of a $\frac{1}{x}$-continuum.
- $\mathcal{A}$ contains countably many folding points. One can construct tent maps such that the folding points of its inverse limit space belong to finitely many arc-components that are periodic under $\sigma$, but where there are still countably folding points.\(^1\)
- $\mathcal{A}$ contains uncountably many folding points, e.g. if $\omega(c) = [c_2, c_1]$, because then every point in the core is a folding point.

This is clearly only a first step towards a complete classification.

**Definition 2.5.** Let $\ell^0, \ell^1, \ldots, \ell^k$ be those links in $C_p$ that are successively visited by an arc $A \subset C_0$ (hence $\ell^i \neq \ell^{i+1}$, $\ell^i \cap \ell^{i+1} \neq \emptyset$ and $\ell^i = \ell^{i+2}$ is possible if $A$ turns in $\ell^{i+1}$). Let $A^i \subset \ell^i$ be the corresponding arc components such that $\text{Cl} A^i$ are subarcs of $A$. We call the arc $A$

- $p$-link symmetric if $\ell^i = \ell^{k-i}$ for $i = 0, \ldots, k$;
- maximal $p$-link symmetric if it is $p$-link symmetric and there is no $p$-link symmetric arc $B \supset A$ and passing through more links than $A$.

The $p$-point of $A^{k/2}$ with the highest $p$-level is called the center of $A$, and the link $\ell^{k/2}$ is called the central link of $A$.

### 3. ISOTOPIC HOMEOMORPHISMS OF UNIMODAL INVERSE LIMITS

It is shown in [2] that every salient $p$-point $s_l \in C_0$ is the center of the maximal $p$-link symmetric arc $A_l$. We denote the central link that $s_l$ belongs to by $\ell^s_p$. For a better understanding of this section, let us mention that a key idea in [2] is that under a homeomorphism $h$ such that $h(C_q) \preceq C_p$, (maximal) $q$-link symmetric arcs have to map to (maximal) $p$-link symmetric arcs, and for this reason $h(s_m) \in \ell^s_p$ for some appropriate $m \in \mathbb{N}$ (see [2, Theorem 4.1]).

**Lemma 3.1.** Let $h : K_s \to K_s$ be a homeomorphism pseudo-isotopic to $\sigma^R$, and let $q, p \in \mathbb{N}_0$ be such that $h(C_q) \preceq C_p$. Let $x$ be a $q$-point in the core of $K_s$ and let $\ell^s_p \in C_p$ be the link $\ldots$ .

\(^1\)An example is the tent-map where $c_1$ has symbolic itinerary (kneading sequence) $\nu = 1001012013014015\ldots$. Then the two-sided itineraries of folding points are limits of $\{\sigma^j(\nu)\}_{j \geq 0}$. The only such two-sided limit sequences are $1^\infty, 1^\infty$ and $\{\sigma^j(1^\infty, 01^\infty) : j \in \mathbb{Z}\}$. Since they all have left tail $\ldots 1111$, these folding points belong to the arc-component of the point $\ldots, p, p, p$ for the fixed point $p = \frac{1}{1+2}$. This use of two-sided symbolic itineraries was introduced for inverse limit spaces in [6].
containing both $\sigma^p(x)$ and salient $p$-point $s_l$, where $l = L_p(\sigma^R(x))$. Suppose that the arc-component $W_x$ of $\ell^s_p$ containing $\sigma^R(x)$ does not contain any folding point. Then $h(x) \in W_x$.

**Proof.** Since $W_x$ does not contain any folding point, it contains finitely many $p$-points. Note that $W_x$ contains at least one $p$-point since $\sigma^R(x) \in W_x$ is a $p$-point. Since $C_0$ is dense in $K_s$, there exists a sequence $(W_i)_{i \in \mathbb{N}}$ of arc-components of $\ell^s_p$ such that $W_i \in C_0$, $FP_p(W_i) = FP_p(W_x)$ for every $i \in \mathbb{N}$, and $W_i \rightarrow W_x$ in the Hausdorff metric. Let $(x_i)_{i \in \mathbb{N}}$ be a sequence of $q$-points such that for every $i \in \mathbb{N}$, $L_q(x_i) = L_q(x)$, $x_i \rightarrow x$ and $\sigma^R(x_i) \in W_i$. Obviously $(x_i)_{i \in \mathbb{N}} \subseteq C_0$. Let $\sigma^R(x_i)$ be such that $i = L_p(\sigma^R(x))$. Let $W \subset \ell^s_p$ be arc-component containing $\sigma^R(x)$. Since $\sigma^R(x)$ does not contain any folding point, $h(x_i) \cong h(x)$. It follows by the construction in the proof of [2, Proposition 4.2] that $h(x_i) \in W_i$ for every $i \in \mathbb{N}$. Therefore $h(x) \in W_x$. □

**Corollary 3.2.** Let $h : K_s \rightarrow K_s$ be a homeomorphism pseudo-isotopic to $\sigma^R$. Then $h$ permutes arc-components of $K_s$ in the same way as $\sigma^R$.

**Proof.** Since $h$ is a homeomorphism, $h$ maps arc-components to arc-components. Let $\mathfrak{A}$ be an arc-component of $K_s$. Let us suppose that $\mathfrak{A}$ contains a folding point, say $x$. Then $h(x) = \sigma^R(x)$ implies $h(\mathfrak{A}) = \sigma^R(\mathfrak{A})$.

Let us assume now that $\mathfrak{A}$ does not contain any folding point. There exist $q, p \in \mathbb{N}_0$ such that $h(C_q) \preceq C_p$ and that $h(\mathfrak{A})$ is not contained in a single link of $C_p$. Then $\mathfrak{A}$ is not contained in a single link of $C_q$. Let $\ell_q \subset C_q$ and $V \subset \ell_q \cap \mathfrak{A}$ be an arc-component of $\ell_q$ such that $V$ contains at least one $q$-point, say $x$. Let $\ell^s_p \subset C_p$ be such that $l = L_p(\sigma^R(x))$. Let $W \subset \ell^s_p$ be arc-component containing $\sigma^R(x)$. Since $\mathfrak{A}$ does not contain any folding point, $h(\mathfrak{A})$ does not contain any folding point implying $W$ does not contain any folding point. Then, by Lemma 3.1, $h(x) \in W$ implying $h(\mathfrak{A}) = \sigma^R(\mathfrak{A})$. □

**Lemma 3.3.** Let $h : K_s \rightarrow K_s$ be a homeomorphism that is pseudo-isotopic to the identity. Then $h$ preserves orientation of every arc-component $\mathfrak{A}$, i.e., given a parametrization $\varphi : \mathbb{R} \rightarrow \mathfrak{A}$ (or $\varphi : [0, 1] \rightarrow \mathfrak{A}$ or $\varphi : [0, \infty) \rightarrow \mathfrak{A}$) that induces an order $\prec$ on $\mathfrak{A}$, then $x \prec y$ implies $h(x) \prec h(y)$.

**Proof.** Let us first suppose that $h : K_s \rightarrow K_s$ is any homeomorphism. Then, by [2, Theorem 1.2] there is an $R \in \mathbb{Z}$ such that $h$, restricted to the core, is pseudo-isotopic to $\sigma^R$, i.e., $h$ permutes the composants of the core of the inverse limit in the same way as $\sigma^R$. Therefore, by Corollary 3.2, it permutes the arc-components of the inverse limit in the same way as $\sigma^R$.

Let $\mathfrak{A}, \mathfrak{A}'$ be arc-components of the core such that $h, \sigma^R : \mathfrak{A} \rightarrow \mathfrak{A}'$, and let $x, y \in \mathfrak{A}$, $x \prec y$. We want to prove that $h(x) \prec h(y)$ if and only if $\sigma^R(x) \prec \sigma^R(y)$. Since $h$ and $\sigma^R$ are
homeomorphisms on arc-components, each of them could be either order preserving or order reversing. Therefore, to prove the claim we only need to pick two convenient points $u, v \in \mathfrak{A}$, $u < v$, and check if we have either $h(u) < h(v)$ and $\sigma^R(u) < \sigma^R(v)$, or $h(v) < h(u)$ and $\sigma^R(v) < \sigma^R(u)$. If $\mathfrak{A}$ contains at least two folding points, we can choose $u, v$ to be folding points. Then $h(u) = \sigma^R(u)$ and $h(v) = \sigma^R(v)$ and the claim follows.

Let us suppose now that $\mathfrak{A}$ contains at most one folding point. Then there exist $q, p \in \mathbb{N}_0$ such that $h(C_q) \preceq C_p$ and $q$-points $u, v \in \mathfrak{A}$, $u < v$ (on the same side of the folding point if there exists one) such that $\sigma^R(u)$ and $\sigma^R(v)$ are contained in disjoint links of $C_p$ each of which does not contain the folding point of $\mathfrak{A}$, if there exists one.

Let $\ell^q_p, \ell^k_p \in C_p$ with $j = L_p(\sigma^R(u))$ and $k = L_p(\sigma^R(v))$ be links containing $\sigma^R(u)$ and $\sigma^R(v)$ respectively. Let $W_u \subset \ell^q_p$ and $W_v \subset \ell^k_p$ be arc-components containing $\sigma^R(u)$ and $\sigma^R(v)$ respectively. Then $W_u$ and $W_v$ do not contain any folding point and by Lemma 3.1 $h(u) \in W_u$ and $h(v) \in W_v$. Therefore obviously $h(u) < h(v)$ if and only if $\sigma^R(u) < \sigma^R(v)$.

If $h$ is a homeomorphism that is pseudo-isotopic to the identity, then $R = 0$ and the claim of lemma follows. \(\Box\)

**Corollary 3.4.** If $h$ is pseudo-isotopic to the identity, then the arc $A$ connecting $x$ and $h(x)$ is a single point, or $A$ contains no folding point.

*Proof.* Since $h$ is pseudo-isotopic to the identity, $x$ and $h(x)$ belong to the same composant, and in fact the same arc-component. So let $A$ be the arc connecting $x$ and $h(x)$. If $x = h(x)$, then there is nothing to prove. If $h(x) \neq x$, say $x < h(x)$, and $A$ contains a folding point $y$, then $x < y = h(y) < h(x)$, contradicting Lemma 3.3. \(\Box\)

In particular, any homeomorphism $h$ that is pseudo-isotopic to the identity cannot reverse the bar of a sin $\frac{1}{x}$-continuum, or reverse a double spiral $S \subset K_s$, see Remark 2.4. The next lemma strengthens Lemma 3.1 to the case that $W_x$ is allowed to contain folding points.

**Lemma 3.5.** Let $h : K_s \to K_s$ be a homeomorphism that is pseudo-isotopic to the identity.

Let $q, p \in \mathbb{N}_0$ be such that $h(C_q) \preceq C_p$. Let $x$ be a $q$-point in the core of $K_s$ and let $\ell^m_p \in C_p$ be such that $l = L_p(x)$. Let $W_x \subset \ell^m_p$ be an arc-component of $\ell^m_p$ containing $x$. Then $h(x) \in W_x$.

*Proof.* If $W_x$ does not contain any folding point the proof follows by Lemma 3.1 for $R = 0$.

Let $W_x$ contain at least one folding point. If $x$ is a folding point, then $h(x) = x \in W_x$ by Theorem 1.1. If $W_x$ contains at least two folding points, say $y$ and $z$, such that $x \in [y, z] \subset W_x$, then $h(x) \in [y, z] \subset W_x$ by Corollary 3.4.
The last possibility is that $x \in (y, z) \subset W_x$, where $z \in W_x$ is a folding point, $y \notin W_x$, i.e., $y$ is a boundary point of $W_x$, and $(y, z)$ does not contain any folding point. Since $C_0$ is dense in $K_s$, there exists a sequence $(W_i)_{i \in \mathbb{N}}$ of arc-components of $\ell^m$ such that $W_i \subset C_0$ and $W_i \to (y, z)$ in the Hausdorff metric. Note that for the sequence of $p$-points $(m_i)_{i \in \mathbb{N}}$, where $m_i$ is the midpoint of $W_i$, we have $m_i \to z$ and $L_p(m_i) \to \infty$. Also, for every $i$ large enough, every $W_i$ contains a $q$-point $x_i$ with $L_q(x_i) = L_q(x)$, and for the sequence of $q$-points $(x_i)_{i \in \mathbb{N}}$ we have $x_i \to x$. Obviously $(x_i)_{i \in \mathbb{N}} \subset C_0$ and $L_p(x_i) = L_p(x)$. By the proof of [2, Proposition 4.2] applied for $R = 0$ we have $h(x_i) \in W_i$ for every $i$. Since $h$ is a homeomorphisms, $h(x_i) \to h(x)$. Therefore, $h(x) \in (y, z) \subset W_x$.

**Proposition 3.6.** Let $h : K_s \to K_s$ be a homeomorphism. If $z^n \to z$ and $A^n = [z^n, h(z^n)]$, then $A^n \to A := [z, h(z)]$ in Hausdorff metric.

**Proof.** We know that $h$ is pseudo-isotopic to $\sigma^R$ for some $R \in \mathbb{Z}$; by composing $h$ with $\sigma^{-R}$ we can assume that $R = 0$. By Corollary 3.2, $h$ preserves the arc-components, and by Lemma 3.3, preserves the orientation of each arc-component as well.

Take a subsequence such that $A^n_k$ converges in Hausdorff metric, say to $B$. Since $x, h(x) \in B$, we have $B \supset A$. Assume by contradiction that $B \neq A$. Fix $q, p$ arbitrary such that $h(C_q)$ refines $C_p$, and such that $\pi_p(B) \neq \pi_p(A)$ and a fortiori, that there is a link $\ell \in C_p$ such that $\ell \cap A = \emptyset$ and $\pi_p(\ell)$ contains a boundary point of $\pi_p(B)$.

Let $d_n = \max\{L_p(y) : y \text{ is a $p$-point in } A^n\}$. If $D := \sup d_n < \infty$, then we can pass to the chain $C_{p+D}$ and find that all $A^n_k$'s go straight through $C_{p+D}$, hence the limit is a straight arc as well, stretching from $x$ to $h(x)$, so $B = A$. Therefore $D = \infty$, and we can assume without loss of generality that $d_n \to \infty$.

Since the link in $\ell$ is disjoint from $A$ but $\pi_p(\ell)$ contains a boundary point of $\pi_p(B)$, the arcs $A^n_k$ intersects $\ell$ for all $k$ sufficiently large. Therefore $A^n_k \cap \ell$ separates $x^{n_k}$ from $h(x^{n_k})$; let $W^{n_k}$ be a component of $A^n_k \cap \ell$ between $x^{n_k}$ and $h(x^{n_k})$. Since $\pi_p(\ell)$ contains a boundary point of $\pi_p(B)$, $W^{n_k}$ contains at least one $p$-point for each $k$. Lemma 3.5 states that there is $y^{n_k} \in W^{n_k}$ such that $h(y^{n_k}) \in W^{n_k}$ as well, and therefore $x^{n_k} \prec y^{n_k}, h(y^{n_k}) \prec h(x^{n_k})$ (or $y^{n_k} \prec x^{n_k}, h(x^{n_k}) \prec h(y^{n_k})$), contradicting that $h$ preserves orientation.

Let us finally prove Theorem 1.3:
Proof of Theorem 1.3. Fix $R$ such that $h$ is pseudo-isotopic to $\sigma^R$. Then $\sigma^{-R} \circ h$ is pseudo-isotopic to the identity. So renaming $\sigma^{-R} \circ h$ to $h$ again, we need to show that $h$ is isotopic to the identity.

If $x$ is a folding point of $K_s$, then $h(x) = x$ by Theorem 1.1. In this case, and in fact for any point such that $h(x) = x$, we let $H(x, t) = x$ for all $t \in [0, 1]$. If $h(x) \neq x$, then $x$ and $h(x)$ belong to the same arc-component, and the arc $A = [x, h(x)]$ contains no folding point by Corollary 3.4. By Lemma 2.2, $A$ contains only finitely many $p$-points, so there is $m$ such that $\pi_m : A \to \pi_m(A)$ is one-to-one. In this case,

$$H(x, t) = \pi_m^{-1}[\pi_m(x) + t\pi_m(h(x))] = \pi_m^{-1}[\pi_m(x) + t\pi_m(x)] = \pi_m^{-1}([1 - t]\pi_m(x) + t\pi_m(h(x))].$$

Clearly $t \mapsto H(\cdot, t)$ is a family of maps connecting $h$ to the identity in a single path as $t \in [0, 1]$. We need to show that $H$ is continuous both in $x$ and $t$, and that $H(\cdot, t)$ is a bijection for all $t \in [0, 1]$.

Let $z \in K_s$ and $(z^n, t^n) \to (z, t)$. If $h(z) = z$, then $H(z, t) \equiv z$, and Proposition 3.6 implies that $H(z^n, t^n) \to z = H(z, t)$. So let us assume that $h(z) \neq z$. The arc $A = [z, h(z)]$ contains no folding point, so by Lemma 2.2, for all $x \in A$, there is $\varepsilon(x) > 0$ and $W(x) \in \mathbb{N}$ such that $B_{\varepsilon(x)}(x)$ contains no $p$-point of $p$-level $\geq W(x)$. By compactness of $A$, $\varepsilon := \inf_{x \in A} \varepsilon(x) > 0$ and $\sup_{x \in A} W(x) < \infty$, whence there is $m > p + W$ such that $V := \pi_m^{-1} \circ \pi_m(A)$ is contained in an $\varepsilon$-neighborhood of $A$ that contains no $p$-point.

By Proposition 3.6, there is $N$ such that $A^n \subset V$ for all $n \geq N$, and in fact $\pi_m(A^n) \to \pi_m(A)$. It follows that $H(z^n, t^n) \to H(z, t)$.

To see that $x \mapsto H(\cdot, t)$ is injective for all $t \in [0, 1]$, assume by contradiction that there is $t_0 \in [0, 1]$ and $x \neq y$ such that $H(x, t_0) = H(y, t_0)$. Then $x$ and $y$ belong to the same arc-component $\mathfrak{A}$, which is the same as the arc-component containing $h(x)$ and $h(y)$. The smallest arc $J$ containing all four point contains no folding point by Corollary 3.4. Therefore there is $m$ such that $\pi_m : J \to \pi_m(J)$ is injective, and we can choose an orientation on $\mathfrak{A}$ such that $x < y$ on $J$, and $\pi_m(x) < \pi_m(y)$. Since $t \mapsto \pi_m \circ H(\cdot, t)$ is monotone with constant speed depending only on $x$, we find

$$\pi_m(x) < \pi_m(y) < \pi_m \circ H(x, t_0) = \pi_m \circ H(y, t_0) < \pi_m \circ h(y) < \pi_m \circ h(x)$$

This contradicts that $h$ preserves orientation on arc-components, see Lemma 3.3.

To prove surjectivity, choose $x \in K_s$ arbitrary. If $h(x) = x$, then $H(x, t) = x$ for all $t \in [0, 1]$. Otherwise, say if $h(x) > x$, there is $y < x$ in the same arc-component as $x$ such that $h(y) = x$. The map $t \mapsto H(\cdot, t)$ moves the arc $[y, x]$ continuously and monotonically to $[h(y), h(x)] = [h(y), x]$. Since $\mathfrak{A}$ is connected, $H(\cdot, t)$ moves $[h(y), x]$ continuously and monotonically to $[h(x), x]$. Since $\mathfrak{A}$ is connected, $H(\cdot, t)$ moves $[y, x]$ continuously and monotonically to $[x, x]$. Therefore $H(\cdot, t)$ is a bijection for all $t \in [0, 1]$. Thus $H(\cdot, t)$ is the required isotopy. Therefore $h$ is isotopic to the identity.
Therefore, for every \( t \in [0, 1] \), there is \( y_t \in [y, x] \) such that \( H(y_t, t) = x \). This proves surjectivity.

We conclude that \( H(x, t) \) is the required isotopy between \( h \) and the identity. \(\square\)

References


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