FIBONACCI-LIKE UNIMODAL INVERSE LIMIT SPACES AND THE CORE INGRAM CONJECTURE

H. BRUIN AND S. ŠTIMAC

Abstract. We study the structure of inverse limit space of so-called Fibonacci-like tent maps. The combinatorial constraints implied by the Fibonacci-like assumption allow us to introduce certain chains that enable a more detailed analysis of symmetric arcs within this space than is possible in the general case. We show that link-symmetric arcs are always symmetric or a well-understood concatenation of quasi-symmetric arcs. This leads to the proof of the Ingram Conjecture for cores of Fibonacci-like unimodal inverse limits.

1. Introduction

A unimodal map is called Fibonacci-like if it satisfies certain combinatorial conditions implying an extreme recurrence behavior of the critical point. The Fibonacci unimodal map itself was first described by Hofbauer and Keller [15] as a candidate to have a so-called wild attractor. (The combinatorial property defining the Fibonacci unimodal map is that its so-called *cutting times* are exactly the Fibonacci numbers $1, 2, 3, 5, 8, \dots$) In [12] it was indeed shown that Fibonacci unimodal maps with sufficiently large critical order possess a wild attractor, whereas Lyubich [18] showed that such is not the case if the critical order is 2 (or $\leq 2 + \varepsilon$ as was shown in [17]). This answered a question in Milnor's well-known paper on the structure of metric attracts [20].

In [8] the strict Fibonacci combinatorics were relaxed to Fibonacci-like, see Definition 2.1. Intricate number-theoretic properties of Fibonacci-like critical omega-limit sets were revealed in [19] and [13], and [9, Theorem 2] shows that Fibonacci-like combinatorics are incompatible with the Collet-Eckmann condition of exponential derivative growth along the critical orbit. This makes Fibonacci-like maps an interesting class of maps in between the regular and the stochastic unimodal maps in the classification of [1].

2000 *Mathematics Subject Classification.* 54H20, 37B45, 37E05.

Key words and phrases. tent map, inverse limit space, Fibonacci unimodal map, structure of inverse limit spaces.

HB was supported by EPSRC grant EP/F037112/1. SŠ was supported in part by the MZOS Grant 037-0372791-2802 of the Republic of Croatia.

Our interest in the Fibonacci-like properties lies in the fact that they allow us to resolve the Ingram Conjecture for cores of Fibonacci-like inverse limit spaces. The original conjecture was posed by Tom Ingram in 1991 for tent maps $T_s : [0,1] \to [0,1]$ with slope $\pm s$, $s \in [1,2]$, defined as $T_s(x) = \min\{sx, s(1-x)\}$:

> If $1 \le s < s' \le 2$, then the corresponding inverse limit spaces $\varprojlim([0, s/2], T_s)$ and $\varprojlim([0, s'/2], T_{s'})$ are non-homeomorphic.

The first results towards solving this conjecture have been obtained for tent maps with a finite critical orbit [16, 23, 4]. Raines and Štimac [21] extended these results to tent maps with an infinite, but non-recurrent critical orbit. Recently Ingram's Conjecture was solved for all slopes $s \in [1,2]$ (in the affirmative) by Barge, Bruin and Štimac in [3], but we still know very little of the structure of inverse limit spaces (and their subcontinua) for the case that the orbit of a critical point is infinite and recurrent, see [2, 5, 10]. The inverse limit space $\varprojlim([0, s/2], T_s)$ is the union of the core $\varprojlim([c_2, c_1], T_s)$ and a ray $\mathfrak{C}$, containing the endpoint $\bar{0} := (\ldots, 0, 0, 0)$, converging onto the core. Since the arc-component $\mathfrak{C}$ is important in the proof of the Ingram Conjecture in [3], the "core" version of the Ingram Conjecture for tent maps with an infinite critical orbit stayed open. It is this version that we solve here for Fibonacci-like tent maps:

Main Theorem 1. If $1 \le s < s' \le 2$ are the parameters of Fibonacci-like tent-maps, then the corresponding cores of inverse limit spaces $\varprojlim([c_2, c_1], T_s)$ and $\varprojlim([c_2, c_1], T_{s'})$ are non-homeomorphic.

The set of all Fibonacci-like parameters $s \in [1,2]$ intersects every open subset of $[1,2]$ in an uncountable set. The inverse limit spaces of Fibonacci-like maps share the property that their only subcontinua are points, arcs and $\sin\frac{1}{x}$-continua, see [10], and they are not homeomorphic to the inverse limit spaces of tent maps with finite or non-recurrent critical orbits.

In this paper we develop tools to use the arc-component of the core which contains the point $(\ldots, r, r, r)$ fixed for the shift homeomorphism, where $r = \frac{s}{s+1}$ is a fixed point of T_s. One key observation of the paper is that this arc-component is fixed for every self-homeomorphism of the core, see Theorem 4.2. The other key observation is Proposition 3.2 which implies that every homeomorphism maps symmetric arcs to symmetric arcs, not just to quasi-symmetric arcs. (The difficulty that quasi-symmetric arcs pose was first observed and overcome in [21] in the setting of tent maps with non-recurrent critical

point.) To prove Proposition 3.2, the special structure of the Fibonacci-like maps is used. But assuming the result of Proposition 3.2, the proof of Theorem 4.2 and of the main theorem work for general tent maps. Therefore, this paper is an important step towards proving the core Ingram conjecture for all tent maps.

The paper is organized as follows. In Section 2 we set the notation, review some necessary tools, mostly from [3], and gradually generalize them to obtain new, more suitable tool for investigation of the core of Fibonacci-like inverse limits and its self-homeomorphisms. In Section 3 we show that any homeomorphism on the core of the Fibonacci-like inverse limit space maps symmetric arcs to symmetric arc, and using this we prove our main theorem in Section 4. Appendix A is devoted to the construction of the chains $\mathcal{C}$ having special properties that allow us to prove desired properties of folding structure in Appendix B. In Appendix C, we show that link-symmetric arcs are always symmetric or a well-understood concatenation of quasi-symmetric arcs.

2. Definitions

2.1. **Combinatorics of tent maps.** The tent map $T_s : [0,1] \to [0,1]$ with slope $\pm s$ is defined as $T_s(x) = \min\{sx, s(1-x)\}$. The critical or turning point is $c = 1/2$ and we write $c_k = T_s^k(c)$, so in particular $c_1 = s/2$ and $c_2 = s(1-s/2)$. We will restrict T_s to the interval $I = [0, s/2]$; this is larger than the *core* $[c_2, c_1] = [s - s^2/2, s/2]$, but it contains both fixed points 0 and $r = \frac{s}{s+1}$.

Combinatorics of unimodal maps can be described by cutting times, see *e.g.* [7]. The *cutting times* $\{S_k\}_{k\geq 0}$ are those iterates n (written in increasing order) for which the central branch of T_s^n covers c. More precisely, let $Z_n \subset [0,c]$ be the maximal interval with boundary point c on which T_s^n is monotone, and let $\mathfrak{D}_n = T_s^n(Z_n)$. Then n is a *cutting time* if $\mathfrak{D}_n \ni c$. Let $\mathbb{N} = \{1, 2, 3, \dots\}$ be the set of natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. It can be shown that the difference of consecutive cutting times is again a cutting time, so there is a function $Q : \mathbb{N} \to \mathbb{N}_0$ such that

$$S_k - S_{k-1} = S_{Q(k)} \tag{2.1}$$

for all k. This function is called the *kneading map*.

Definition 2.1. A unimodal map is called *Fibonacci map* if its kneading map $Q(k) = \max\{k-2, 0\}$ (and hence the cutting times $\{S_k\}_{k\geq 0} = \{1, 2, 3, 5, 8, \dots\}$ are the Fibonacci numbers). A unimodal map is called *Fibonacci-like* if its kneading map is eventually

non-decreasing, and satisfies Condition (2.2) as well:

$$Q(k+1) > Q(Q(k)+1) \qquad \text{for all } k \text{ sufficiently large.} \tag{2.2}$$

Remark 2.2. Condition (2.2) follows if the Q is eventually non-decreasing and $Q(k) \leq k-2$ for k sufficiently large. (In fact, since tent maps are not renormalizable of arbitrarily high period, $Q(k) \leq k-2$ for k sufficiently large follows from Q being eventually non-decreasing, see [7, Proposition 1].) Geometrically, it means that $|c - c_{S_k}| < |c - c_{S_{Q(k)}}|$, see Lemma A.4 and also [7].

2.2. **Inverse limit spaces.** The inverse limit space $K_s = \varprojlim([0, s/2], T_s)$ is the collection of all backward orbits

$$\{x = (\dots, x_{-2}, x_{-1}, x_0) : T_s(x_{i-1}) = x_i \in [0, s/2] \text{ for all } i \leq 0\},$$

equipped with metric $d(x,y) = \sum_{n\leqslant 0} 2^n |x_n - y_n|$ and *induced* (or *shift*) *homeomorphism*

$$\sigma(\dots, x_{-2}, x_{-1}, x_0) = (\dots, x_{-2}, x_{-1}, x_0, T_s(x_0)).$$

Let $\pi_k : \varprojlim([0, s/2], T_s) \to [0, s/2]$, $\pi_k(x) = x_{-k}$ be the k-th projection map. The *arc-component* of $x \in X$ is defined as the union of all arcs of X containing x. Since the fixed point $0 \in [0, s/2]$, the endpoint $\bar{0} = (\dots, 0, 0, 0)$ is contained in $\varprojlim([0, s/2], T_s)$, and the arc-component of $\varprojlim([0, s/2], T_s)$ of $\bar{0}$ will be denoted as $\mathfrak{C}$; it is a ray converging to, but disjoint from the core $\varprojlim([c_2, c_1], T_s)$ of the inverse limit space. The other fixed point $r \in [c_2, c_1]$, so the point $\rho = (\dots, r, r, r)$ is contained in $\varprojlim([c_2, c_1], T_s)$. The arc-component of ρ will be denoted as $\mathfrak{R}$; it is a continuous image of $\mathbb{R}$ and is dense in $\varprojlim([c_2, c_1], T_s)$ in both directions.

We fix $s \in (\sqrt{2}, 2]$; for these parameters T_s is not renormalizable and $\varprojlim([c_2, c_1], T_s)$ is indecomposable.

A point $x = (\dots, x_{-2}, x_{-1}, x_0) \in K_s$ is called a *p-point* if $x_{-p-l} = c$ for some $l \in \mathbb{N}_0$. The number $L_p(x) := l$ is called the *p-level* of x. In particular, $x_0 = T_s^{p+l}(c)$. By convention, the endpoint $\bar{0} = (\dots, 0, 0, 0)$ of $\mathfrak{C}$ and the point $\rho = (\dots, r, r, r)$ of $\mathfrak{R}$ are also p-points and $L_p(\bar{0}) = L_p(\rho) := \infty$, for every p.

The *folding pattern of the arc-component* $\mathfrak{C}$, denoted by $FP(\mathfrak{C})$, is the sequence

$$L_p(z^0), L_p(z^1), L_p(z^2), \dots, L_p(z^n), \dots,$$

where $E_p^{\mathfrak{C}} = \{z^0, z^1, z^2, \dots, z^n, \dots\}$ is the ordered set of all p-points of $\mathfrak{C}$ with $z^0 = \bar{0}$, and p is any nonnegative integer. Let $q \in \mathbb{N}$, $q > p$, and $E_q^{\mathfrak{C}} = \{y^0, y^1, y^2, \dots, y^n, \dots\}$. Since

σ^{q-p} is an order-preserving homeomorphism of $\mathfrak{C}$, it is easy to see that $\sigma^{q-p}(z^i) = y^i$ and $L_p(z^i) = L_q(y^i)$ for every $i \in \mathbb{N}$. Therefore the folding pattern of $\mathfrak{C}$ does not depend on p.

The *folding pattern of the arc-component* $\mathfrak{R}$, denoted by $FP(\mathfrak{R})$, is the sequence

$$\dots, L_p(z^{-n}), \dots, L_p(z^{-1}), L_p(z^0), L_p(z^1), \dots, L_p(z^n), \dots, \tag{2.3}$$

where $E_p^{\mathfrak{R}} = \{\dots, z^{-n}, \dots, z^{-1}, z^0, z^1, \dots, z^n, \dots\}$ is the ordered set (indexed by $\mathbb{Z}$) of all p-points of $\mathfrak{R}$ with $z^0 = \rho$, and p is any nonnegative integer. Since $r > 1/2$, we have $\pi_i(\rho) > 1/2$ for every $i \in \mathbb{N}_0$. It is easy to see that for every $i \in \mathbb{N}_0$, there exists an arc $A = A(i) \subset \mathfrak{R}$ containing ρ such that $\pi_i(A) = [c, c_1]$. Therefore two neighboring p-points of ρ have p-levels 0 and 1. From now on we assume, without loss of generality, that the ordering on $\mathfrak{R}$, *i.e.,* the parametrization of $\mathfrak{R}$, is such that $L_p(z^{-1}) = 0$ and $L_p(z^1) = 1$. Let $q \in \mathbb{N}$, $q > p$, and $E_q^{\mathfrak{R}} = \{\dots, y^{-n}, \dots, y^{-1}, y^0, y^1, \dots, y^n, \dots\}$ with $y^0 = \rho$. Since σ^{q-p} is an order-preserving (respectively, order-reversing) homeomorphism of $\mathfrak{R}$ if $q - p$ is even (respectively, odd), $\sigma^{q-p}(z^i) = y^i$ and $L_p(z^i) = L_q(y^i)$ for every $i \in \mathbb{Z}$. Therefore the folding pattern of $\mathfrak{R}$ does not depend on p.

Note that every arc of $\mathfrak{C}$ and of $\mathfrak{R}$ has only finitely many p-points, but an arc A of the core of K_s can have infinitely many p-points.

We will mostly be interested in the arc-component $\mathfrak{R}$, but also in some other arc-components 'topologically similar' to $\mathfrak{R}$. Therefore, unless stated otherwise, let $\mathfrak{A} \subset \varprojlim([c_2, c_1], T_s)$ denote an arc-component which does not contain any end-point, such that every arc $A \subset \mathfrak{A}$ contains finitely many p-points, and let $\mathfrak{A}$ be dense in the core of K_s in both directions. Let $E_p^{\mathfrak{A}} = (a^i)_{i\in\mathbb{Z}}$ denote the set of all p-points of $\mathfrak{A}$, where $a^0 = (\dots, a^0_{-2}, a^0_{-1}, a^0_0) \in \mathfrak{A}$ is the only p-point of $\mathfrak{A}$ with $a^0_{-j} \neq c$ for every $j \in \mathbb{N}_0$, and let by convention $L_p(a^0) = \infty$ for every p. Also, we abbreviate $E_p := E_p^{\mathfrak{A}}$. The *$p$-folding pattern of the arc-component* $\mathfrak{A}$, denoted by $FP_p(\mathfrak{A})$, is the sequence

$$\dots, L_p(a^{-n}), \dots, L_p(a^{-1}), L_p(a^0), L_p(a^1), \dots, L_p(a^n), \dots.$$

Given an arc $A \subset \mathfrak{A}$ with successive p-points $x^0, \dots, x^n$, the *p-folding pattern* of A is the sequence

$$FP_p(A) := L_p(x^0), \dots, L_p(x^n).$$

An arc A in $\varprojlim([0, s/2], T_s)$ is said to *p-turn at c_n* if there is a p-point $a \in A$ such that $a_{-(p+n)} = c$, so $L_p(a) = n$. This implies that $\pi_p : A \to [0, s/2]$ achieves c_n as a local extremum at a.

2.3. **Chainability and (quasi-)symmetry.** A space is *chainable* if there are finite open covers $\mathcal{C} = \{\ell_i\}_{i=1}^N$, called *chains*, of arbitrarily small *mesh* ($\operatorname{mesh}\mathcal{C} = \max_i \operatorname{diam}\ell_i$) with the property that the *links* ℓ_i satisfy $\ell_i \cap \ell_j \neq \emptyset$ if and only if $|i-j| \leqslant 1$. The combinatorial properties of Fibonacci-like maps allow us to construct chains $\mathcal{C}_p$ such that whenever an arc A p-turns in $\ell \in \mathcal{C}_p$, *i.e.,* enters and exits ℓ through the same neighboring link, then the projections $\pi_p(x) = \pi_p(y)$ of the first and last p-point x and y of $A \cap \ell$ depend only on ℓ and not on A, see Proposition A.6. We will work with the chains $\mathcal{C}_p$ which are the π_p^{-1} images of chains of the interval $[0, s/2]$.

Definition 2.3. An arc $A \subset \mathfrak{A}$ such that $\partial A = \{u, v\}$ and $A \cap E_p = \{x^0, \dots, x^n\}$ is called *p-symmetric* if $\pi_p(u) = \pi_p(v)$ and $L_p(x^i) = L_p(x^{n-i})$, for every $0 \leqslant i \leqslant n$.

If A is p-symmetric, then n has to be even and $L_p(x^{n/2}) = \max\{L_p(x^i) : x^i \in A \cap E_p\}$. The point $x^{n/2}$ is called the *midpoint* of A.

It frequently happens that $\pi_p(u) \neq \pi_p(v)$, but u and v belong to the same link $\ell \in \mathcal{C}_p$. Let us call the arc-components A_u, A_v of $\mathfrak{A} \cap \ell$ that contain u and v respectively the *link-tips* of A, see Figure 1. Sometimes we can make A p-symmetric by removing the link-tips. Let us denote this as $A \setminus \ell$-tips. Adding the closure of the link-tips can sometimes also produce a p-symmetric arc.

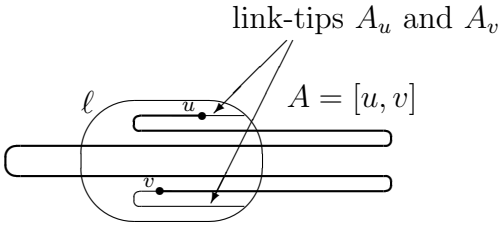

FIGURE 1. The arc A is neither p-symmetric, nor quasi-p-symmetric, but both arcs $A \setminus \ell$-tips and $A \cup \operatorname{Cl}(\ell\text{-tips})$ are p-symmetric.

Remark 2.4. (a) Let A be an arc and $m \in A$ be a p-point of maximal p-level, say $L_p(m) = L$. Then π_p is one-to-one on both components of $\sigma^{1-L}(A \setminus \{m\})$, so m is the only p-point of p-level L. It follows that between every two p-points of the same p-level, there is a p-point m of higher p-level.

(b) If $A \ni m$ is the maximal open arc such that m has the highest p-level on A, then we can write $\operatorname{Cl} A = [x, y]$ or $[y, x]$ with $L_p(x) > L_p(y) > L_p(m) =: L$, and π_p is one-to-one on $\sigma^{-L}(\operatorname{Cl} A)$. Here $L_p(x) = \infty$ is possible, but if $L_p(x) < \infty$, then $A' := \pi_p \circ \sigma^{-L}(A)$ is a

neighborhood of c with boundary points $c_{S_k} = \pi_p \circ \sigma^{-L}(x)$ and $c_{S_l} = \pi_p \circ \sigma^{-L}(y)$ for some $k, l \in \mathbb{N}$ such that $l = Q(k)$. By Lemma A.4 this means that the arc $[x, m]$ is shorter than $[m, y]$.

Definition 2.5. Let A be an arc of $\mathfrak{A}$. We say that the arc A is *quasi-p-symmetric with respect to $\mathcal{C}_p$* if

(i) A is not p-symmetric;
(ii) ∂A belongs to a single link ℓ;
(iii) $A \setminus \ell$-tips is p-symmetric;
(iv) $A \cup \ell$-tips is not p-symmetric. (So A cannot be extended to a symmetric arc within its boundary link ℓ.)

Definition 2.6. Let $\ell_0, \ell_1, \dots, \ell_k$ be the links in $\mathcal{C}_p$ that are successively visited by an arc $A \subset \mathfrak{A}$, and let $A_i \subset \mathrm{Cl}(\ell_i)$ be the corresponding maximal subarcs of A. (Hence $\ell_i \neq \ell_{i+1}$, $\ell_i \cap \ell_{i+1} \neq \emptyset$ but $\ell_i = \ell_{i+2}$ is possible if A turns in ℓ_{i+1}.) We call A *p-link-symmetric* if $\ell_i = \ell_{k-i}$ for $i = 0, \dots, k$. In this case, we say that A_i is p-link-symmetric to A_{k-i}.

Remark 2.7. Every p-symmetric and quasi-p-symmetric arc is p-link-symmetric by definition, but there are p-link-symmetric arcs which are not p-symmetric or quasi-p-symmetric. This occurs if A turns both at A_i and A_{k-i}, but the midpoint of A_i has a higher p-level than the midpoint of A_{k-i} and $i \notin \{0, k\}$. Note that for a p-link-symmetric arc A, if U and V are p-link-symmetric arc-components which do not contain any boundary point of A, then U contains at least one p-point if and only if V contains at least one p-point.

Appendix B is devoted to give a precise description of quasi-symmetric arcs and their concatenations. In Appendix C we use this structure to show that link-symmetric arcs are always symmetric or a well-understood concatenation of quasi-symmetric arcs.

2.4. **Salient Points.** In [3, Definition 2.7] we introduced salient p-points of the arc-component $\mathfrak{C}$. Let $(s_i)_{i \in \mathbb{N}}$ be the sequence of all p-points of the arc-component $\mathfrak{C}$ such that $0 \leq L_p(x) < L_p(s_i)$ for every p-point $x \in (\bar{0}, s_i)$. We call p-points satisfying this property *salient*.

For every slope $s > 1$ and $p \in \mathbb{N}_0$, the folding pattern of $\mathfrak{C}$ starts as $\infty\ 0\ 1\ 0\ 2\ 0\ 1\ \dots$, and since by definition $L_p(s_1) > 0$, we have $L_p(s_1) = 1$. Also, since $s_i = \sigma^{i-1}(s_1)$, $L_p(s_i) = i$, for every $i \in \mathbb{N}$. Note that the salient p-points depend on p: if $p \geq q$, then the salient p-point s_i equals the salient q-point s_{i+p-q}.

Definition 2.8. Recall that $\mathfrak{R}$ is the arc-component containing the point $\rho = (\dots, r, r, r)$ where $r = \frac{s}{s+1}$ is fixed by T_s. Let $(t^i)_{i\in\mathbb{Z}} \subset E_p^{\mathfrak{R}}$ be the bi-infinite sequence of all p-points of the arc-component $\mathfrak{R}$ such that for every $i \in \mathbb{N}$

$$\begin{cases} t^0 = \rho, \\ L_p(t^i) > L_p(x) & \text{for every } p\text{-point } x \in (\rho, t^i), \\ L_p(t^{-i}) > L_p(x) & \text{for every } p\text{-point } x \in (t^{-i}, \rho). \end{cases}$$

Note that p-points $(t^i)_{i\in\mathbb{Z}} \subset \mathfrak{R}$ are defined similarly as salient p-points $(s_i)_{i\in\mathbb{N}}$; we call them $\mathfrak{R}$-*salient* p-points, or simply *salient* p-points when it is clear which arc-component they belong to. There is an important difference between the sets $(s_i)_{i\in\mathbb{N}} \subset \mathfrak{C}$ and $(t^i)_{i\in\mathbb{Z}} \subset \mathfrak{R}$, namely $L_p(s_i) = i$ for every $i \in \mathbb{N}$, whereas $L_p(t^i) \neq |i|$ for all $i \in \mathbb{Z} \setminus \{1\}$.

Lemma 2.9. For $(t^i)_{i\in\mathbb{Z}} \subset \mathfrak{R}$ we have

$$L_p(t^i) = \begin{cases} 2i - 1 & \text{if } i > 0, \\ -2i & \text{if } i < 0. \end{cases}$$

Proof. Since r is the positive fixed point of T_s, the p-points closest to $\rho = (\dots, r, r, r)$ have p-levels 0 and 1. Also $\sigma(\rho) = \rho$ implies $\sigma(\mathfrak{R}) = \mathfrak{R}$. The parametrization of $\mathfrak{R}$, chosen in Section 2 below (2.3), is such that for $\rho \in [x^{-1}, x^1]$ we have $L_p(x^{-1}) = 0$ and $L_p(x^1) = 1$, thus $x^1 = t^1$. Since $\sigma(\rho) = \rho \in \sigma([x^{-1}, x^1]) \subset \mathfrak{R}$ and $\sigma|_{\mathfrak{R}}$ is order reversing, we have $\sigma(x^{-1}) = x^1$, $\sigma(x^1) \prec x^1$, *i.e.,* $\sigma([x^{-1}, x^1]) = [x^{-2}, x^1]$ with $L_p(x^{-2}) = 2$. Note that $x^{-2} = t^{-1}$. For the same reason, $\sigma([x^{-2}, x^1]) = [x^{-2}, x^j]$, where x^j is the first p-point to the right of x^1 such that $L_p(x^j) = 3$, *i.e.,* $x^j = t^2$. The claim of the lemma follows by induction. □

Analogously, we define $\mathfrak{A}$-salient p-points of an arc-component $\mathfrak{A}$ of the core of K_s.

Definition 2.10. Let $(u^i)_{i\in\mathbb{Z}} \subset E_p^{\mathfrak{A}} = (a^i)_{i\in\mathbb{Z}}$ be the bi-infinite sequence of all p-points of the arc-component $\mathfrak{A}$ such that for every $i \in \mathbb{N}$

$$\begin{cases} u^0 = a^0, \\ L_p(u^i) > L_p(x) & \text{for every } p\text{-point } x \in (u^0, u^i), \\ L_p(u^{-i}) > L_p(x) & \text{for every } p\text{-point } x \in (u^{-i}, u^0). \end{cases}$$

This fixes an orientation on $\mathfrak{A}$; the choice of orientation is immaterial, as long as we make one.

Lemma 2.11. If there exist $J, J', K \in \mathbb{N}_0$ such that for every $j \in \mathbb{N}$, $L_p(u^{J+j}) = 2(K + j) - 1$ and $L_p(u^{-(J'+j)}) = 2(K + j)$, then $\mathfrak{A} = \mathfrak{R}$.

In other words, the asymptotic shape of this folding pattern is unique to $\mathfrak{R}$.

Proof. Let $J, J', K \in \mathbb{N}_0$ be as in the statement of the lemma. Then for every $j \in \mathbb{N}$ we have:

(i) $L_p(u^{-(J'+j)}) - L_p(u^{J+j}) = L_p(u^{J+j+1}) - L_p(u^{-(J'+j)}) = 1$,
(ii) $L_p(x) < L_p(u^{J+j})$ for every p-point $x \in (u^{-(J'+j)}, u^{J+j})$, and
(iii) $L_p(x) < L_p(u^{-(J'+j)})$ for every p-point $x \in (u^{-(J'+j)}, u^{J+j+1})$.

Therefore, $\pi_{p+2(K+j)-1} : [u^{-(J'+j)}, u^{J+j}] \to [c, c_1]$ and $\pi_{p+2(K+j)} : [u^{-(J'+j)}, u^{J+j+1}] \to [c, c_1]$ are bijections, implying that for every $j \in \mathbb{N}$, $FP_p([u^{-(J'+j)}, u^{J+j}])$ and $FP_p([u^{-(J'+j)}, u^{J+j+1}])$ are uniquely determined by $T_s^{2(K+j)-1}$ and $T_s^{2(K+j)}$ respectively. Thus we have the following:

$$\begin{cases} FP_p([u^{-(J'+j)}, u^{J+j}]) = FP_p([t^{-(K+j)}, t^{K+j}]), \\ FP_p([u^{-(J'+j)}, u^{J+j+1}]) = FP_p([t^{-(K+j)}, t^{K+j+1}]), \end{cases}$$

whence $FP_p([u^{-(J'+j)}, u^{J+j+1}]) = FP_p(\sigma([u^{-(J'+j)}, u^{J+j}]))$ for every $j \in \mathbb{N}$. It follows that $FP_p(\sigma(\mathfrak{A})) = FP_p(\mathfrak{A}) = FP_p(\mathfrak{R})$ implying $\mathfrak{A} = \mathfrak{R}$. □

Note that in general J, J', K in the above lemma are not related since $u^0 = a^0$ can be any point, but there exists a point $a \in \mathfrak{A}$ such that for $u^0 = a$, we have $J = J' = K$.

3. Homeomorphisms and Symmetric Arcs

Note that in this section all proofs except the proof of Proposition 3.2 work in general, only the proof of Proposition 3.2 uses the special structure of the Fibonacci-like inverse limit spaces revealed in this paper.

Let $h : \varprojlim([c_2, c_1], T_s) \to \varprojlim([c_2, c_1], T_s)$ be a homeomorphism on the core of a (Fibonacci-like) inverse limit space. Let $q, p, g \in \mathbb{N}_0$ be such that $\mathcal{C}_q$, $\mathcal{C}_p$ and $\mathcal{C}_g$ are chains as in Proposition A.6, and such that

$$h(\mathcal{C}_q) \preceq \mathcal{C}_p \preceq h(\mathcal{C}_g).$$

It is straightforward that any q-link-symmetric arc $A \subset \varprojlim([c_2, c_1], T_s)$ maps to a p-link-symmetric arc $h(A) \subset \varprojlim([c_2, c_1], T_s)$.

In Appendix A, we construct special chains by which we are able to describe the structure of link-symmetric arcs (see Definition 2.6) precisely. The Fibonacci-like structure, and

the extra structure of these chains, allow us to conclude the stronger statement that q-symmetric arcs map to p-symmetric arcs. This is a rather technical undertaking, but let us paraphrase Remark C.6 so as to make this section understandable (although for the fine points we will still refer forward to the appendix). Link-symmetric arcs tend to be composed of smaller *(basic) quasi-symmetric arcs* A_k (see Definition B.1) that are ordered linearly such that A_k and A_{k+1} overlap, and the midpoint of A_{k+1} is the endpoint of A_k. An entire concatenation of such arcs is called *decreasing quasi-symmetric* (respectively *increasing quasi-symmetric*, see Definition C.1) if the levels of the successive midpoints (also called *nodes*) - all contained in, alternately, one of two given links - are decreasing (respectively increasing). The concatenation is called *maximal decreasing quasi-symmetric* (respectively *maximal increasing quasi-symmetric*, see Definition C.5) if it cannot be extended to a concatenation with more components. The last endpoint (respectively the first endpoint), namely, of the arc with midpoint of the lowest level, is then no longer a p-point.

For a point x, we denote a link of $\mathcal{C}_p$ which contains x by ℓ_p^x, and the arc-component of ℓ_p^x which contains x by A_x.

Definition 3.1. Let $x \in E_q^{\mathfrak{A}} \subset \mathfrak{A}$ be a q-point, and let $A_{h(x)} \subset \ell_p^{h(x)}$ be the arc-component of $\ell_p^{h(x)}$ which contains $h(A_x)$ (and therefore $h(x)$). Let $a, b \in \mathbb{N}$, $a \leqslant b$, be such that $h(\cup_{i=a}^b \ell_q^i) \subseteq \ell_p^{h(x)}$, $h(\ell_q^{a-1}) \not\subseteq \ell_p^{h(x)}$ and $h(\ell_q^{b+1}) \not\subseteq \ell_p^{h(x)}$. Let $\hat{A}_x$ be an arc-component of $\cup_{i=a}^b \ell_q^i$ such that $h(\hat{A}_x) \subseteq A_{h(x)} \subset \ell_p^{h(x)}$. We call $\hat{A}_x$ the *extended arc-component of the q-point x*. If a p-point u is the midpoint of $A_{h(x)}$, then we write $u \vdash h(x)$.

The extended arc-component $\hat{A}_x$ is obtained by extending A_x so much on both sides that $h(\hat{A}_x)$ fits almost exactly in the p-link containing $h(A_x)$. Note that the arc-component A_x of a q-point x depends on the chain $\mathcal{C}_q$, while the extended arc-component $\hat{A}_x$ of the q-point x also depends on the chain $\mathcal{C}_p$. But we still can define its midpoint as the q-point $z \in \hat{A}_x$ such that $L_q(z) \geqslant L_q(y)$ for every q-point $y \in \hat{A}_x = \hat{A}_z$. If a q-point x is the midpoint of its extended arc-component $\hat{A}_x$ we call it a *q_p-point*.

Proposition 3.2. Let $x, y \in E_q^{\mathfrak{A}} \subset \mathfrak{A}$ be q_p-points and let $u \vdash h(x)$ and $v \vdash h(y)$. Then $L_q(x) = L_q(y)$ implies $L_p(u) = L_p(v)$.

Since the endpoints of a symmetric arc have the same level, and q-link symmetric arcs are mapped to p-link-symmetric arcs by a homeomorphism h, Proposition 3.2 implies that h maps symmetric arcs to symmetric arcs.

Proof. Without loss of generality we suppose that between x and y, there are no q-points with q-level $L_q(x)$. Then the arc $A = [x, y]$ is q-symmetric. The midpoint m of A is a q_p-point. Let $w \vdash h(m)$.

Let us assume by contradiction that $L_p(u) \neq L_p(v)$. Then $D = [u, v]$ is not p-symmetric with midpoint w. Since A is q-symmetric, D is p-link symmetric. By Proposition C.8 and Remark C.6, D is contained either in an extended maximal decreasing/increasing (basic) quasi-p-symmetric arc, or in a p-symmetric arc which is concatenation of two arcs, one of which is a maximal increasing (basic) quasi-p-symmetric arc, and the other one is a maximal decreasing (basic) quasi-p-symmetric arc.

(1) Let us assume that D is contained in an extended maximal increasing (basic) quasi-p-symmetric arc G. Let B' and B be the link-tips of G, so $G = [B', B]$. Then, by Remark C.6, B' does not contain any p-point and hence $B' \neq A_u$.

(a) Suppose first that the p-point $z \in G$, such that $L_p(z) \geqslant L_p(d)$ for all p-points $d \in G$, does not belong to the open arc (u, v). Then $B \neq A_v$.

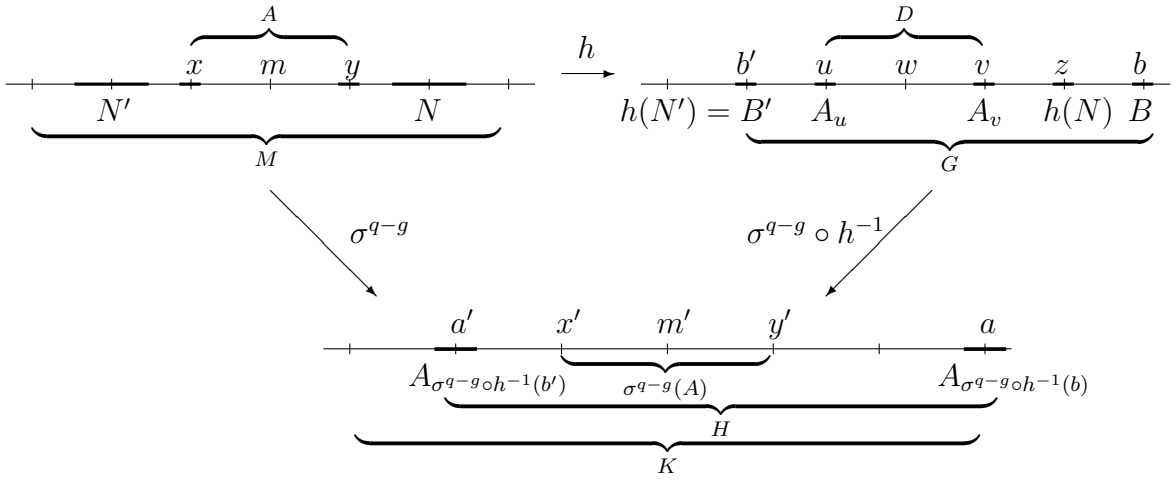

FIGURE 2. The relations between points and arcs in $\mathcal{C}_q$ (left), $\mathcal{C}_p$ (right), and $\mathcal{C}_g$ (bottom).

Let b' be any point of B' and let b be the midpoint of B. Then b' and b are nodes of G (see Remark C.6 for the definition of a node). Since $L_p(u) \neq L_p(v)$, u and v are also nodes of G, as well as w and z.

Let $a \vdash \sigma^{q-g} \circ h^{-1}(b)$ (note that b is a p_g-point, *i.e.,* b is the midpoint of the extended arc-component $\hat{A}_b$ such that $\sigma^{q-g} \circ h^{-1}(\hat{A}_b) \subseteq A_{\sigma^{q-g} \circ h^{-1}(b)} = A_a \subset \ell_g^a \in \mathcal{C}_g$). If the arc-component $A_{\sigma^{q-g} \circ h^{-1}(b')}$ contains a g-point, let a' be its midpoint; otherwise let a'

be any point of $A_{\sigma^{q-g}\circ h^{-1}(b')}$. Let us consider the arc $H = [a', a]$, see Figure 2. Let $x' \vdash \sigma^{q-g} \circ h^{-1}(u)$, $y' \vdash \sigma^{q-g} \circ h^{-1}(v)$, $z' \vdash \sigma^{q-g} \circ h^{-1}(z)$ and $m' \vdash \sigma^{q-g} \circ h^{-1}(w)$. Since $\mathcal{C}_p \prec h(\mathcal{C}_g)$, the arc H is g-link-symmetric and g-points a', x', m', y', a are some of its nodes. Note that $x' = \sigma^{q-g}(x)$ and $y' = \sigma^{q-g}(y)$, thus the arc $[x', y']$ is g-symmetric. Since there is at least one node in H on either side of $[x', y']$, Remark C.6 says that H is contained in the maximal g-symmetric arc K with midpoint m'. Therefore the arc $M = \sigma^{-q+g}(K) \supset A$ is q-symmetric with midpoint m.

Let $j, k \in \mathbb{N}$, $j \leqslant k$, be such that $h(\cup_{i=j}^{k} \ell_q^i) \subseteq \ell_p^{b'}$, $h(\ell_q^{j-1}) \not\subseteq \ell_p^{b'}$ and $h(\ell_q^{k+1}) \not\subseteq \ell_p^{b'}$. Let N' be an arc-component of $\cup_{i=j}^{k} \ell_q^i$ such that $h(N') = B' \subset \ell_p^{b'}$. Obviously, $N' \subset M$. Since M is q-link symmetric, there exists an arc-component N of $\cup_{i=j}^{k} \ell_q^i$ such that the arc $[N', N] \subset M$ is q-symmetric with midpoint m. Then $h(N) \subset h(M)$ is an arc-component of $\ell_p^{b'}$. Since $[N', N]$ is q-symmetric, the arc-component $h(N')$ contains a p-point if and only if the arc-component $h(N)$ contains a p-point. Since $h(N') = B'$, the arc-components $h(N')$ and $h(N)$ do not contain any p-point, see Figure 2.

On the other hand, the arc $[h(N'), h(N)]$ is p-link-symmetric with midpoint w. Recall that w is also the midpoint of the arc $D \subset [h(N'), h(N)]$, D is not p-symmetric by assumption, and $D \subset G$, where G is an extended maximal increasing (basic) quasi-p-symmetric arc. The arc-component $h(N)$ can be contained in the arc $[A_v, B]$, as in Figure 2. In this case $h(N)$ does contain at least one p-point, a contradiction.

The other possibility is that $h(N)$ is not contained in $[A_v, B]$, *i.e.,* $h(N)$ is on the right hand side of B. Since $[h(N'), h(N)]$ is p-link symmetric and $h(N') = B'$ contains a node b' of G, we have that $h(N)$ also contains a node of G, say n. Hence, on the right hand side of z (which is the p-point with the highest p-level in G), there are at least two nodes, b and n. Therefore, by Remark C.6, G is contained in a p-symmetric arc with midpoint z and this arc conatins $h(N)$, implying that $h(N)$ does contain at least one p-point, a contradiction.

(b) Let us assume now that $B = A_v$. Then $z \in (u, v)$. Let a', x', m', z', y' and H be defined as in case (a). Since b', u, w, z, v are nodes of G, we have that a', x', m', z', y' are also nodes of H. Moreover, since $[x', y']$ is g-symmetric with midpoint m', there is $z'' \in [x', m']$ such that $[z'', z']$ is g-symmetric with midpoint m', and z'' is a node of H. Thus, the arc between nodes z'' and z' is g-symmetric, and on either side of $[z'', z']$ there is at least one additional node. By Remark C.6, H is contained in the maximal g-symmetric arc K with

midpoint m', and the arc $M = \sigma^{-q+g}(K) \supset A$ is q-symmetric with midpoint m. Now the proof follows in the same way as in case (a).

If D is contained in an extended maximal decreasing (basic) quasi-p-symmetric arc G, the proof is analogous.

(2) Let us assume that D is contained in a p-symmetric arc G which is concatenation of two arcs, one of which is a maximal increasing (basic) quasi-p-symmetric arc, and the other one is a maximal decreasing (basic) quasi-p-symmetric arc. Let B' and B be the link-tips of G, thus $G = [B', B]$. Then, by Remark C.6, B' and B do not contain any p-point and hence $B' \neq A_u$ and $B \neq A_v$. If for the midpoint z of G we have $z \notin (u, v)$, we are in case (1). If $z \in (u, v)$ (note $z \neq m$ since the arc D is not p-symmetric), then the proof is analogous to the proof of case (1a) (since $B \neq A_v$). □

Definition 3.3. Let $\kappa \in \mathbb{N}$, $\kappa > 2$, be the smallest integer with $c_\kappa < c$. It is easy to see that κ is odd. Set

$$\Lambda_\kappa := \mathbb{N} \setminus \{1, 3, 5, \dots, \kappa - 4\}.$$

Lemma 3.4. Let x, y be q-points of $\mathfrak{A}$. Then there exist q_p-points x', z' and y' such that the arc $A = [x', z']$ is q-symmetric with midpoint y', $L_q(x') = L_q(z') = L_q(x)$ and $L_q(y') = L_q(y)$ if and only if $L_q(y) - L_q(x) \in \Lambda_\kappa$.

This is proven in Lemma 46 of [16] and in Lemmas 3.13 and 3.14 of [22]. Although [16] deals with the periodic case and [22] with the finite orbit case, the proofs of the mentioned lemmas work in the general case, as stated above.

Proposition 3.5. Let $x, y \in E_q^{\mathfrak{A}} \subset \mathfrak{A}$ be q_p-points and let $u \vdash h(x)$ and $v \vdash h(y)$. Then $L_q(x) < L_q(y)$ implies $L_p(u) < L_p(v)$.

Proof. **(1)** Let us first assume that $L_q(y) - L_q(x) \in \Lambda_\kappa$. Then, by Lemma 3.4, there exist q_p-points x', z' and y' such that the arc $A = [x', z']$ is q-symmetric with midpoint y', $L_q(x') = L_q(z') = L_q(x)$, $L_q(y') = L_q(y)$ and between x' and z' there are no q_p-points with q-level $L_q(x')$. Let $u \vdash h(x)$, $v \vdash h(y)$, $u' \vdash h(x')$, $v' \vdash h(y')$, $w' \vdash h(z')$. By Proposition 3.2 we have $L_p(u) = L_p(u') = L_p(w')$, $L_p(v) = L_p(v')$ and between the points u' and w' there are no p-points with the p-level $L_p(u')$. Therefore, the arc $[u', w']$ is p-symmetric with midpoint v', implying $L_p(v) = L_p(v') > L_p(u') = L_p(u)$, which proves the proposition in this case. Note that also we have $L_p(v) - L_p(u) \in \Lambda_\kappa$.

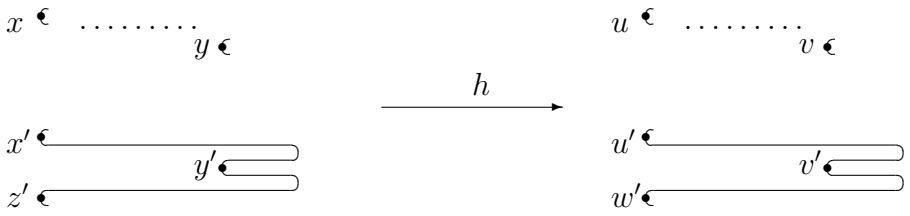

FIGURE 3. The points x and y, their companion arc $A = [x', z']$ and their images under h. Dots indicate some shape of the arc $[x, y]$ and $[u, v]$; the shape of $[x, y]$ can be very different from the shape of $[x', y']$ and similar for the shapes of $[u, v]$ and $[u', v']$.

(2) Let us now assume that $\Lambda_\kappa \neq \mathbb{N}$, $L_q(y) - L_q(x) \in \{1, 3, \dots, \kappa - 4\}$, and that for $u \vdash h(x)$ and $v \vdash h(y)$ we have, by contradiction, $L_p(u) > L_p(v)$.

Without loss of generality we suppose that x has the smallest q-level among all q_p-points which satisfy the above assumption and that, for this choice of x, the q_p-point y (which also satisfies the above assumption) is such that $L_q(y) - L_q(x) > 0$ is the smallest difference of q-levels.

Claim 1: $L_q(y) - L_q(x) = 1$.

Let us assume, by contradiction, that $L_q(y) - L_q(x) > 1$, and let z be a q_p-point such that $L_q(y) - L_q(z) = 2$. Note first that $L_q(z) \neq L_q(x)$ since $L_q(y) - L_q(x) \neq 2$ by assumption. Therefore, $L_q(z) > L_q(x)$.

Let $w \vdash h(z)$ and recall $u \vdash h(x)$ and $v \vdash h(y)$. By the choice of q_p-points x and y and since $L_q(z) - L_q(x) < L_q(y) - L_q(x)$, we have $L_p(w) > L_p(u)$ and $L_p(u) > L_p(v)$, implying $L_p(w) > L_p(v)$.

On the other hand, $L_q(y) - L_q(z) \in \Lambda_\kappa$ and by **(1)** we have $L_p(v) > L_p(w)$, a contradiction. This proves Claim 1.

Claim 2: $L_p(u) - L_p(v) = 1$.

Let us assume, by contradiction, that $L_p(u) - L_p(v) > 1$. For a q_p-point z let w denote the p-point with $w \vdash h(z)$. We will show that the above assumption implies that there is no q_p-point z such that $L_p(w) = L_p(v) + 1$. This contradicts assumption that both arc-components $\mathfrak{A}$ and $h(\mathfrak{A})$ are dense in $\varprojlim([c_2, c_1], T_s)$ in both directions.

By the choice of q_p-points x and y, for every q_p-point z such that $L_q(z) < L_q(x) < L_q(y) = L_q(x) + 1$ we have $L_p(w) < L_p(v)$ and hence $L_p(w) \neq L_p(v) + 1$.

Let $L_q(z) = L_q(x) + 2$. Since $L_q(z) - L_q(x) \in \Lambda_\kappa$, by **(1)** we have $L_p(w) > L_p(u) > L_p(v) + 1$.

Let $L_q(z) = L_q(x)+3$. Then $L_q(z)-L_q(y) \in \Lambda_\kappa$ (recall $L_q(y) = L_q(x)+1$ by Claim 1) and again by **(1)** we have $L_p(w) > L_p(v)$ and $L_p(w)-L_p(v) \in \Lambda_\kappa$. Note that $L_p(w)-L_p(v) \neq 1$ since $1 \notin \Lambda_\kappa$ (recall $\Lambda_\kappa \neq \mathbb{N}$ by assumption). Hence $L_p(w) > L_p(v) + 1$.

It follows now by induction that for every $i \in \mathbb{N}$, $L_q(z) = L_q(x) + 3 + i$ implies $L_p(w) > L_p(v) + 1$. To see this, for a q_p-point z' let w' denote the p-point with $w' \vdash h(z')$. Take $j \in \mathbb{N}$ such that $L_q(z) = L_q(x) + 3 + i$ implies $L_p(w) > L_p(v) + 1$ for every $i < j$. Let $L_q(z') = L_q(x) + 1 + j$ and $L_q(z) = L_q(x) + 3 + j$. Then $L_q(z) - L_q(z') \in \Lambda_\kappa$ and by **(1)** we have $L_p(w) > L_p(w')$. Since $L_p(w') > L_p(v) + 1$, we have $L_p(w) > L_p(v) + 1$. This proves Claim 2.

Claim 3: For a q_p-point z let w denote the p-point with $w \vdash h(z)$. For every $i \in \mathbb{N}$, $L_q(z) = L_q(x) + 2i$ implies $L_p(w) = L_p(u) + 2i$, and $L_q(z) = L_q(y) + 2i$ implies $L_p(w) = L_p(v) + 2i$.

Let $L_q(z) = L_q(x) + 2 = L_q(y) + 1$. Note first that $L_p(w) \neq L_p(u) + 1$, since by **(1)** $L_q(z) - L_q(x) \in \Lambda_\kappa$ implies $L_p(w) - L_p(u) \in \Lambda_\kappa$. Note also that $L_q(z) - L_q(y) \notin \Lambda_\kappa$. Therefore, $L_p(w) = L_p(v) + L = L_p(u) - 1 + L$, where $1 < L < \kappa - 2$ is odd.

For q_p-points z' and z'', let w' and w'' denote the p-points with $w' \vdash h(z')$ and $w'' \vdash h(z'')$ respectively.

Let us assume that $L_q(z') = L_q(y)+2$ and $L_p(w') \neq L_p(v)+2 = L_p(u)+1$. Then $L_p(w') > L_p(v) + 2$ and for every q_p-point z'' with $L_q(z'') > L_q(z')$ we have $L_p(w'') > L_p(v) + 2$. This implies that there is no q_p-point z'' such that $L_p(w'') = L_p(v) + 2 = L_p(u) + 1$, a contradiction. Therefore, $L_p(w') = L_p(v)+2$, and by Claims 1 and 2, $L_p(w) = L_p(v)+3 = L_p(u) + 2$. The proof of Claim 3 follows by induction in the same way.

Finally, to complete the proof of the proposition, let us consider q_p-point z such that $L_q(z)-L_q(x) = \kappa-2 \in \Lambda_\kappa$. Then, by Claim 3 (see Figure 4), $L_p(w)-L_p(u) = \kappa-4 \notin \Lambda_\kappa$, a contradiction.

Therefore, $L_q(x) < L_q(y)$ implies $L_p(u) < L_p(v)$, which proves the proposition. □

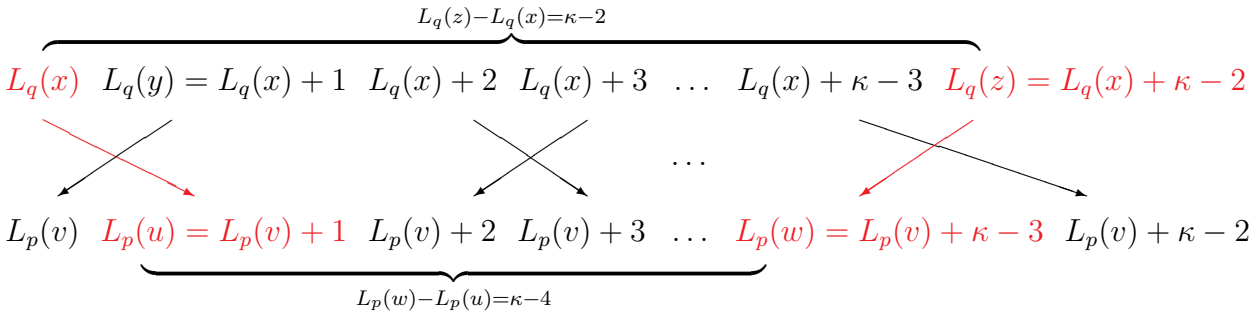

FIGURE 4. The configuration of levels that cannot exist.

4. Proof of the main theorems

Consider the arc-component $\mathfrak{A} := h(\mathfrak{R}) \subset \varprojlim([c_2, c_1], T_s)$, and let $E_p^{\mathfrak{A}} = (y^i)_{i\in\mathbb{Z}} \subset \mathfrak{A}$ be the set of all p-points of $\mathfrak{A}$ such that $y^0 = h(\rho)$. Let $(u^i)_{i\in\mathbb{Z}} \subset E_p^{\mathfrak{A}}$ be the set of all salient p-points of $\mathfrak{A}$, *i.e.*, the set of all $\mathfrak{A}$-salient p-points, with $u^0 = h(\rho)$. Recall that $\mathfrak{R}$ is dense in $\varprojlim([c_2, c_1], T_s)$ in both directions. Since h is a homeomorphism, $\mathfrak{A}$ and in fact $h^i(\mathfrak{R})$, $i \in \mathbb{Z}$, are also dense in the core $\varprojlim([c_2, c_1], T_s)$ in both directions.

We want to prove that $\mathfrak{A} = \mathfrak{R}$. For a p-point y we write $y \approx x$ if $y \in A_x$.

Lemma 4.1. There exist $M, M' \in \mathbb{Z}$ such that $h(t^i) \approx u^{i+M}$ and $h(t^{-j}) \approx u^{-j-M'}$, for every $i, j \in \mathbb{N}$ with $i + M > 0$, $j + M' > 0$, if h is order preserving, or $h(t^i) \approx u^{-i-M}$ and $h(t^{-j}) \approx u^{j+M'+1}$ if h is order reversing.

Proof. If $h : \mathfrak{R} \to \mathfrak{A}$ is order reversing, then $h \circ \sigma : \mathfrak{R} \to \mathfrak{A}$ is order preserving, and also if the proposition works for $h \circ \sigma$, it works for h. Therefore we can assume without loss of generality that h is order preserving.

Let $j \in \mathbb{N}$, and let B_j be the maximal q-symmetric arc with midpoint t^j. Since $s > \sqrt{2}$, $\rho \in B_j$. Therefore, for every q_p-point $x \in (\rho, t^j)$ there exists a q_p-point $y \in (t^j, t^{j+1})$, such that the arc $[x, y]$ is q-symmetric with midpoint t^j and $L_q(x) = L_q(y)$. Let u and v be p-points such that $u \vdash h(x)$ and $v \vdash h(y)$. By Proposition 3.2, we have $L_p(u) = L_p(v)$. Note that for the midpoint w of the arc $[u, v]$ we also have $w \vdash h(t^j)$. This implies, by Remark 2.4 (a), that $L_p(w) > L_p(z)$ for every $z \in (u^0, w)$. Therefore, w is a salient p-point, *i.e.*, $w \in (u^i)_{i\in\mathbb{N}}$.

Let $k, l \in \mathbb{N}$, $k < l$, be such that $u^k \vdash h(t^j)$ and $u^l \vdash h(t^{j+1})$. We want to prove that $l = k + 1$. Let us assume by contradiction that $l > k + 1$. Since $L_p(u^{k+1}) > L_p(u^k)$,

there exists a q_p-point $x \in (t^j, t^{j+1})$ such that $u^{k+1} \vdash h(x)$. But $x \in (t^j, t^{j+1})$ implies $L_q(x) < L_q(t^j)$, contradicting Proposition 3.5.

In this way we have proved that $h(t^i) \approx u^{i+M}$ for some $M \in \mathbb{Z}$ and every $i \in \mathbb{N}$ with $M + i > 0$. In an analogous way we can prove that $h(t^{-i}) \approx u^{-i-M'}$ for some $M' \in \mathbb{Z}$ and for every $i \in \mathbb{N}$ with $M' + i > 0$. □

Theorem 4.2. Every self-homeomorphism h of $\varprojlim([c_2, c_1], T_s)$ preserves $\mathfrak{R}$: $h(\mathfrak{R}) = \mathfrak{R}$.

Proof. Let $h : \mathfrak{R} \to \mathfrak{A}$, as before. We want to prove that $\mathfrak{A} = \mathfrak{R}$. Note that $h \circ \sigma^i : \mathfrak{R} \to \mathfrak{A}$ and $\sigma^i \circ h : \mathfrak{R} \to \sigma^i(\mathfrak{A})$ are homeomorphisms for every $i \in \mathbb{Z}$, and $\sigma^i(\mathfrak{A}) = \mathfrak{R}$ if and only if $\mathfrak{A} = \mathfrak{R}$. By using h^{-1} instead of h if necessary, we can assume that $M \geqslant 0$ (with M as in Lemma 4.1). Also, instead of studying h, we can study $\sigma^{1-a} \circ h : \mathfrak{R} \to \sigma^{1-a}(\mathfrak{A})$, where $a = L_p(u^{1+M})$ (recall that $h(t^1) \approx u^{1+M}$). Therefore, without loss of generality we can assume that $h(t^1) \approx u^1$ and $L_p(u^1) = 1$. Recall that $L_q(t^1) = 1$, $L_q(t^{-1}) = 2$ and for every $i \in \mathbb{N}$, $L_q(t^{-i}) - L_q(t^i) = L_q(t^{i+1}) - L_q(t^{-i}) = 1$. If $L_p(u^{-i}) - L_p(u^i) = L_p(u^{i+1}) - L_p(u^{-i}) = 1$, then $\mathfrak{A} = \mathfrak{R}$ by Lemma 2.11.

Recall that $h(t^{-1}) \approx u^{-1-M'}$, where M' is as in Lemma 4.1. Since

$$L_q(t^1) < L_q(t^{-1}) < L_q(t^2) < L_q(t^{-2}) < \cdots,$$

by Proposition 3.5 we have

$$1 = L_p(u^1) < L_p(u^{-1-M'}) < L_p(u^2) < L_p(u^{-2-M'}) < \cdots < L_p(u^n) < L_p(u^{-n-M'}) < \cdots.$$

Let $L_p(u^n) = 1+a_1+b_1+\cdots+a_{n-1}+b_{n-1}$ and $L_p(u^{-n-M'}) = 1+a_1+b_1+\cdots+a_{n-1}+b_{n-1}+a_n$, for every $n \in \mathbb{N}$ and some $a_1, \ldots, a_n, b_1, \ldots, b_{n-1} \in \mathbb{N}$. We want to prove that $a_i = b_i = 1$ for every $i \in \mathbb{N}$.

Assume by contradiction that $k \in \mathbb{N}$ is the smallest integer with $a_i = b_i = 1$ for all $i < k$ and $a_k > 1$. Then, by Proposition 3.5, there is no salient p-point $u \in (u^i)_{i\in\mathbb{Z}}$ with $L_p(u) = L_p(u^k) + 1$. Thus, Proposition 3.2 implies that $\mathfrak{A}$ does not contain any p-point with p-level $L_p(u^k)+1$, contradicting that $\mathfrak{A}$ is dense in $\varprojlim([c_2, c_1], T_s)$ in both directions. If $k \in \mathbb{N}$ is the smallest integer with $a_i = b_i = 1$ for all $i < k$, $a_k = 1$ and $b_k > 1$, the proof follows in an analogous way. □

Remark 4.3. If h is order-preserving, then by proof of Theorem 4.2 we have $M' = M$, where M and M' are as in Lemma 4.1. Also, by Lemma 2.9, Lemma 4.1 and Theorem 4.2 we have $L_p(u^{i+M}) = 2(i + M) - 1 = (2i - 1) + 2M = L_q(t^i) + 2M$ for $i > 0$ and

$L_p(u^{i-M}) = 2(-i+M) = -2i+2M = L_q(t^i)+2M$ for $i<0$. Moreover, by Proposition 3.2, for every q_p-point x, and for the p-point u with $u \vdash h(x)$, we have $L_p(u) = L_q(x)+2M$.

We finish with the

Proof of Theorem 1. Let $1 \le s \leqslant \sqrt{2} < s' \le 2$. Then $\varprojlim([c_2,c_1],T_s)$ is decomposable, $\varprojlim([c_2,c_1],T_{s'})$ is indecomposable, and the proof follows.

Since Lemmas 2.1 and 2.2 of [3] show how to reduce the case $1 \le s < s' \le \sqrt{2}$ to the case $\sqrt{2} < s < s' \le 2$, it suffices to prove the latter case.

Let $\sqrt{2} < s < s' \le 2$. Suppose that there exists a homeomorphism $h : \varprojlim([c_2,c_1],T_{s'}) \to \varprojlim([c_2,c_1],T_s)$. Let $r' := \frac{s'}{s'+1}$ be the positive fixed point of $T_{s'}$ and $\rho' := (\dots,r',r',r') \in C_{s'} = \varprojlim([c_2,c_1],T_{s'})$. Let $\mathfrak{R}'$ denote the arc-component containing ρ'. Let r, ρ and $\mathfrak{R}$ be the analogous objects of $C_s = \varprojlim([c_2,c_1],T_s)$, as before. Take $q,p \in \mathbb{N}_0$ such that $h(\mathcal{C}_q) \prec \mathcal{C}_p$. Let $(t^i)_{i\in\mathbb{Z}}$ be the sequence of salient q-points of $\mathfrak{R}'$ with $t^0 = \rho'$. Let $(u^i)_{i\in\mathbb{Z}}$ be the sequence of salient p-points of $\mathfrak{R}$.

Let $f = h^{-1}\circ\sigma\circ h$, and assume by contradiction that $h(\mathfrak{R}') = \mathfrak{A} \neq \mathfrak{R}$. Since $\mathfrak{R}$ is the only arc-component in $\varprojlim([c_2,c_1],T_s)$ that is fixed by σ, we have $\sigma(\mathfrak{A}) \neq \mathfrak{A}$ implying $f(\mathfrak{R}') \neq \mathfrak{R}'$. But this contradicts Theorem 4.2. Therefore $h(\mathfrak{R}') = \mathfrak{R}$.

We want to prove that $FP(\mathfrak{R}') = FP(\mathfrak{R})$. Without loss of generality we suppose that h is order-preserving and that $M>0$ (with M as in Remark 4.3).

Claim 1: Let $l \in \mathbb{N}$ and let x be a q-point with $L_q(x) = l$. Then $u := h(x) \in \ell_p^{u^{l+2M}}$ and the arc component $A_u \subset \ell_p^{u^{l+2M}}$ containing u, also contains a p-point y such that $L_p(y) = l+2M$.

Note that Claim 1 is the same as Proposition 4.2 (1) of [3]. The proof is analogous:

By Remark 4.3, Claim 1 is true for all salient q-points and for all q_p-points. Note that there exists $j \in \mathbb{N}$ such that every q-point $x \in [t^{-j},t^j]$ is also a q_p-point. Therefore Claim 1 is true for all q-points $x \in [t^{-j},t^j]$, i.e., for every q-point $x \in [t^{-j},t^j]$ the arc-component $A_{h(x)}$ containing $h(x)$, also contains a p-point y such that $L_p(y) = L_q(x)+2M$. Also $h([t^{-j},t^j]) = [a_{-j},a_j]$, $u^{-j-2M} \in A_{a_{-j}}$ and $u^{j+2M} \in A_{a_j}$. Let q-point $x_1 \in [t^{-j},t^j]$ be such that the open arc (x_1,t^{j+1}) is q-symmetric with midpoint t^j. Such x_1 exists since $L_q(t^{j+1}) - L_q(t^j) = 2$ and $L_q(t^{-j}) - L_q(t^j) = 1$. Then $h((x_1,t^{j+1}))$ is p-link-symmetric with midpoint u^{j+2M}. Since there exists a unique p-point b_1 such that the open arc (b_1,u^{j+1+2M})

is p-symmetric with midpoint u^{j+2M}, for every q-point $x' \in (t^j, t^{j+1})$ the arc-component $A_{h(x')}$ containing $h(x')$, also contains a p-point y' such that $L_p(y') = L_p(y) = L_q(x)+2M = L_q(x') + 2M$, see Figure 5.

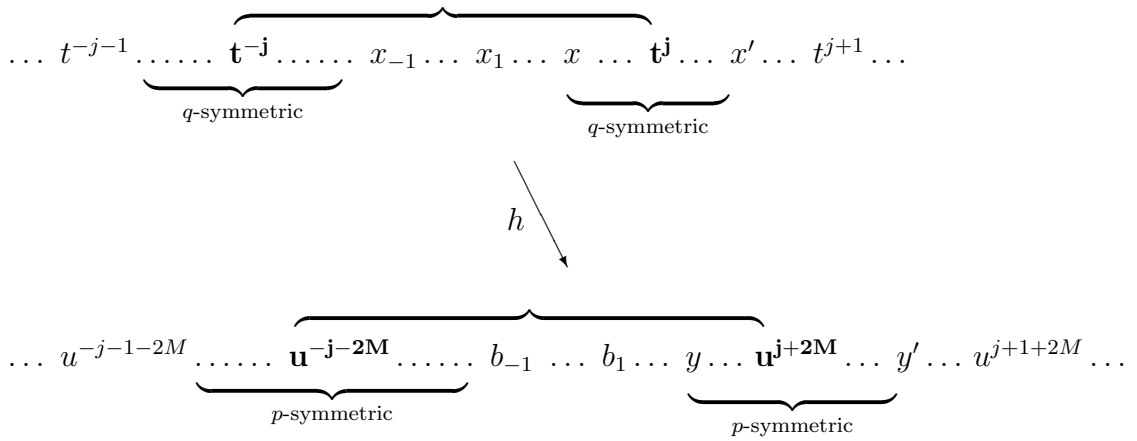

FIGURE 5. The configuration of symmetric arcs.

Let us consider now the arc $h([t^{-j-1}, t^{j+1}]) = [a_{-j-1}, a_{j+1}]$, $u^{-j-1-2M} \in A_{a_{-j-1}}$ and $u^{j+1+2M} \in A_{a_{j+1}}$. Let the q-point $x_{-1} \in [t^{-j}, t^{j+1}]$ be such that the open arc (t^{-j-1}, x_{-1}) is q-symmetric with midpoint t^{-j}. Such x_{-1} exists since $L_q(t^{-j-1}) - L_q(t^{-j}) = 2$ and $L_q(t^{j+1}) - L_q(t^{-j}) = 1$. Therefore $h((t^{-j-1}, x_{-1}))$ is p-link-symmetric with midpoint u^{-j-2M}. Since there exists a unique p-point b_{-1} such that the open arc $(u^{-j-1-2M}, b_{-1})$ is p-symmetric with midpoint u^{-j-2M}, for every q-point $x'' \in (t^{-j-1}, t^{-j})$ the arc-component $A_{h(x'')}$ containing $h(x'')$, also contains a p-point y'' such that $L_p(y'') = L_q(x'') + 2M$, as before. The proof of Claim 1 follows by induction.

Claim 2: For $l \in \mathbb{N}_0$ and $i \in \mathbb{N}$, the number of q-points in $[t^{-i}, t^i]$ with q-level l is the same as the number of p-points in $[u^{-i-2M}, u^{i+2M}]$ with p-level $l + 2M$.

Claim 2 is the same as Proposition 4.2 (2) of [3]. The proof is very similar and we omit it.

Claims 1 and 2 show that

$$FP_q([t^{-i}, t^i]) = FP_{p+2M}([u^{-i-2M}, u^{i+2M}]) = FP_p([u^{-i}, u^i]),$$

for every positive integer i, and therefore $FP(\mathfrak{R}') = FP(\mathfrak{R})$.

This proves the Ingram Conjecture for cores of the Fibonacci-like inverse limit spaces. □

Appendix A. The Construction of Chains

We turn now to the technical part, *i.e.,* the construction of special chains that will eventually allow us to show that symmetric arcs map to symmetric arcs (see Proposition 3.2).

Let $\beta(n) = n - \sup\{S_k < n\}$ for $n \geq 2$ and find recursively the images of the central branch of T_s^n (the levels in the Hofbauer tower, see *e.g.* [7, 6]) as

$$\mathfrak{D}_1 = [0, c_1] \text{ and } \mathfrak{D}_n = [c_n, c_{\beta(n)}].$$

It is not hard to see that $\mathfrak{D}_n \subset \mathfrak{D}_{\beta(n)}$ for each n, see [7], and that if $J \subset [0, s/2]$ is a maximal interval on which T_s^n is monotone, then $T_s^n(J) = \mathfrak{D}_m$ for some $m \leqslant n$.

If x and y are two adjacent p-points on the same arc-component, then $\pi_p([x, y]) = \mathfrak{D}_n$ for some n, so $\pi_p(x) = c_n$ and $\pi_p(y) = c_{\beta(n)}$ or vice versa. Let us call x and y (or $\pi_p(x)$ and $\pi_p(y)$) *β-neighbors* in this case. Notice, however, that there may be many post-critical points between $\pi_p(x)$ and $\pi_p(y)$. Obviously, every p-point of $\mathfrak{C}$ and $\mathfrak{R}$ has exactly two β-neighbors, except the endpoint $\bar{0}$ of $\mathfrak{C}$ whose β-neighbor (w.r.t. p) is by convention the first proper p-point in $\mathfrak{C}$, necessarily with p-level 1.

The condition that $Q(k) \to \infty$ has consequence on the structure of the critical orbit:

Lemma A.1. If $Q(k) \to \infty$, then $|\mathfrak{D}_n| \to 0$ as $n \to \infty$, c is recurrent and $\omega(c)$ is a minimal Cantor set.

Proof. See [11, Lemma 2.1.]. □

Remark A.2. As was shown by Hofbauer [14], a kneading map Q belongs to a unimodal map (with infinitely many cutting times) if and only if

$$\{Q(k+j)\}_{j\geq 1} \geq_{lex} \{Q(Q^2(k)+j)\}_{j\geq 1} \tag{A.1}$$

for all $k \geq 1$, where $\geq_{lex}$ indicates lexicographical order. Clearly, Condition (2.2) is compatible with (and for large k implies) Condition (A.1).

The condition $\{Q(k+j)\}_{j\geq 1} \geq_{lex} \{Q(l+j)\}_{j\geq 1}$ is equivalent to $|c - c_{S_k}| < |c - c_{S_l}|$. Therefore, because $c_{S_{k-1}} \in (\zeta_{Q(k)-1}, \zeta_{Q(k)})$, we find by taking the $T_s^{S_{Q(k)}}$-images, that $c_{S_k} \in [c_{S_{Q^2(k)}}, c]$ and (A.1) follows. The other direction, namely that (A.1) is sufficient for admissibility is much more involved, see [14, 7].

As mentioned before, we will work with the chains which are the π_p^{-1} images of chains of the interval $[0, s/2]$. More precisely, we will define a finite collection of points $G =$

$\{g_0, g_1, \dots, g_N\} \subset [0, s/2]$ such that $|g_m - g_{m+1}| \le s^{-p}\varepsilon/2$ for all $0 \le m < N$ and $|0 - g_0|$ and $|s/2 - g_N|$ positive but very small. ¿From this one can make a chain $\mathcal{C} = \{\ell_n\}_{n=0}^{2N}$ by setting

$$\text{(A.2)} \qquad \begin{cases} \ell_{2m+1} = \pi_p^{-1}((g_m, g_{m+1})) & 0 \le m < N, \\ \ell_{2m} = \pi_p^{-1}((g_m - \delta, g_m + \delta) \cap [0, s/2]) & 0 \le m \le N, \end{cases}$$

where $\min\{|0 - g_0|, |s/2 - g_N|\} < \delta \ll \min_m\{|g_m - g_{m+1}|\}$. Any chain of this type has links of diameter $< \varepsilon$.

Remark A.3. We could have included all the points $\cup_{j \le p} T_s^{-j}(c)$ in G to ensure that $T_s^p|_{(g_m, g_{m+1})}$ is monotone for each m, but that is not necessary. Naturally, there are chains of $\varprojlim([0, s/2], T_s)$ that are not of this form.

For a component A of $\mathfrak{C} \cap \ell$, we have the following two possibilities:
(i) $\mathfrak{C}$ goes straight through ℓ at A, *i.e.,* A contains no p-point and $\pi_p(\partial A) = \partial \pi_p(\ell)$; in this case A enters and exits ℓ from different sides.
(ii) $\mathfrak{C}$ turns in ℓ: A contains (an odd number of) p-points $x^0, \dots, x^{2n}$ of which the middle one x^n has the highest p-level, and $\pi_p(\partial A)$ is a single point in $\partial \pi_p(\ell)$, in this case A enters and exits ℓ from the same side.

Before giving the details of the p-chains we will use, we need two lemmas.

Lemma A.4. If the kneading map of T_s satisfies (2.2), i.e., $Q(k+1) > Q(Q(k)+1)$ for all k sufficiently large, then

$$\text{(A.3)} \qquad |c_{S_k} - c| < |c_{S_{Q(k)}} - c| \quad \text{and} \quad |c_{S_k} - c| < \frac{1}{2}|c_{S_{Q^2(k)}} - c|.$$

for all k sufficiently large.

Proof. For each cutting time S_k, let $\zeta_k \in Z_{S_k}$ be the point such that $T_s^{S_k}(\zeta_k) = c$. Then ζ_k together with its symmetric image $\hat{\zeta}_k := 1 - \zeta_k$ are closest precritical points in the sense that $T_s^j((\zeta_k, c)) \not\ni c$ for $0 \leqslant j \leqslant S_k$. Consider the points ζ_{k-1}, ζ_k and c, and their images under $T_s^{S_k}$, see Figure 6. Note that $Z_{S_k} = [\zeta_{k-1}, c]$ and $T_s^{S_k}([\zeta_{k-1}, c]) = \mathfrak{D}_{S_k} = [c_{S_{Q(k)}}, c_{S_k}]$. Since $S_{k+1} = S_k + S_{Q(k+1)}$ is the first cutting time after S_k, the precritical point of lowest order on $[c, c_{S_k}]$ is $\zeta_{Q(k+1)}$ or its symmetric image $\hat{\zeta}_{Q(k+1)}$. Applying this to c_{S_k} and $c_{Q(k)}$, and using (2.2), we find

$$c_{S_k} \subset (\zeta_{Q(k+1)-1}, \hat{\zeta}_{Q(k+1)-1}) \subset (\zeta_{Q(Q(k)+1)}, \hat{\zeta}_{Q(Q(k)+1)}) \subset (c_{S_{Q(k)}}, \hat{c}_{S_{Q(k)}}).$$

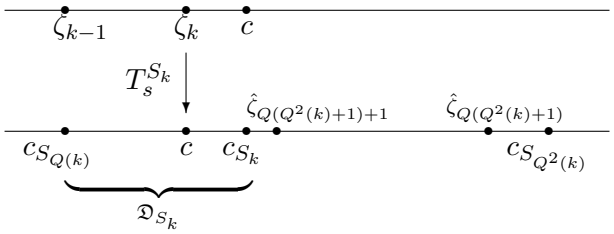

FIGURE 6. The points ζ_{k-1}, ζ_k and c, and their images under $T_s^{S_k}$.

Therefore $|c_{S_k} - c| < |c_{S_{Q(k)}} - c|$. Since $T_s^{S_k}|_{[\zeta_{k-1},c]}$ is affine, also the preimages ζ_{k-1} and ζ_k of $c_{S_{Q(k)}}$ and c satisfy $|\zeta_k - c| < |\zeta_{k-1} - \zeta_k|$. Applying (2.2) twice we obtain

$$Q(k+1) > Q(Q^2(k)+1)+1, \tag{A.4}$$

for all k sufficiently large. Therefore there are at least two closest precritical points ($\hat\zeta_{Q(Q^2(k)+1)}$ and $\hat\zeta_{Q(Q^2(k)+1)+1}$ in Figure 6) between c_{S_k} and $c_{S_{Q^2(k)}}$. Therefore

$$|c_{S_k} - c| < |\hat\zeta_{Q(Q^2(k)+1)+1} - c| < \frac{1}{2}|\hat\zeta_{Q(Q^2(k)+1)} - c| < \frac{1}{2}|c_{S_{Q^2(k)}} - c|, \tag{A.5}$$

proving the lemma. □

Lemma A.5. If the kneading map Q of T_s is eventually non-decreasing and satisfies Condition (A.4), then for all $n \in \mathbb{N}$ there are arbitrarily small numbers $\eta_n > 0$ with the following property: If $n' > n$ is such that $n \in \mathrm{orb}_\beta(n')$, then either $|c_{n'} - c_n| > \eta_n$ or $|c_{n''} - c_n| < \eta_n$ for all $n \le n'' \le n'$ with $n'' \in \mathrm{orb}_\beta(n')$.

To clarify what this lemma says, Figure 7 shows the configuration of levels $\mathfrak{D}_k$ that should be avoided, because then η_n cannot be found.

Proof. We will show that the pattern in Figure 7 (namely with $c_{m_1} < c_{m_2} < c_{m_3} < \dots$ and $c_{m_{i-1}} < c_{k_i}$ for each i) does not continue indefinitely. To do this, we redraw the first few levels from Figure 7, and discuss four positions in $\mathfrak{D}_{m_1}$ where the precritical point $T_s^{-r}(c) \in \mathfrak{D}_{m_1}$ of lowest order r could be, indicated by points $a_1, \dots, a_4$, see Figure 8.

Case $a_1 \in (c_{m_1}, c_{m_2})$**:** Take the $r+1$-th iterate of the picture, which moves $\mathfrak{D}_{m_1}$ and $\mathfrak{D}_{k_1}$ to levels with lower endpoint c_1. then we can repeat the argument, until we arrive in one of the cases below.

Case $a_2 \in (c_{m_2}, c_{k_1})$**:** Take the r-th iterate of the picture, which moves $\mathfrak{D}_{m_1}$, $\mathfrak{D}_{k_1}$, $\mathfrak{D}_{m_2}$ and $\mathfrak{D}_{k_2}$ all to cutting levels and $c_{r+k_2} \in (c, c_{r+k_3})$. But $m_2 > m_1$, whence $k_2 > k_1$, and this contradicts that $|c_{S_{k_2}} - c| < |c_{S_{k_1}} - c|$. (If $a_2 \in (c_{m_3}, c_{k_2})$, then the same argument would give that $r + k_2 < r + k_3$ are both cutting times, but $|c - c_{r+k_2}| < |c - c_{r+k_3}|$.)

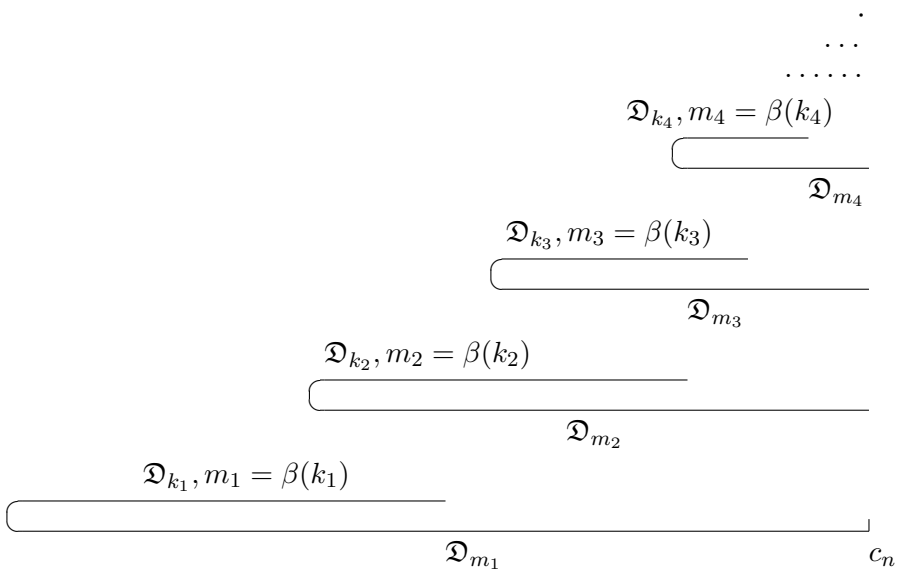

FIGURE 7. Linking of levels $\mathfrak{D}_{m_i}$ with $\beta(m_1) = \beta(m_2) = \beta(m_3) = \cdots = n$. The semi-circles indicates that two intervals have an endpoint in common.

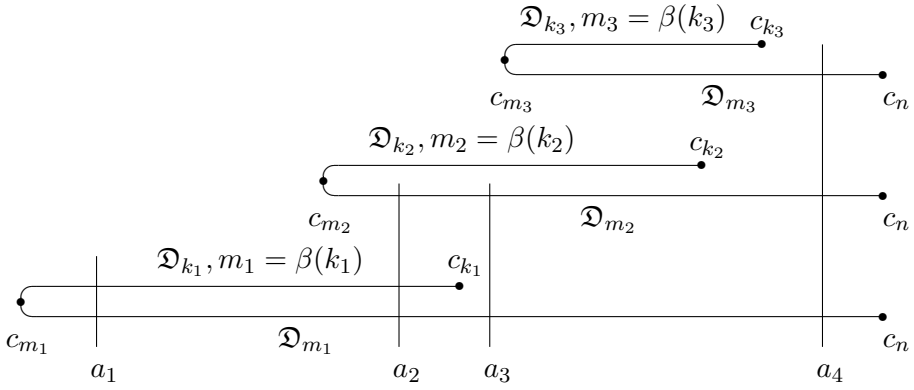

FIGURE 8. Linking of levels $\mathfrak{D}_{m_i}$, $i = 1, 2, 3$ and three possible positions of the precritical point $a_j = T_s^{-r}(c) \in \mathfrak{D}_{m_1}$ of lowest order r.

Case $a_3 \in (c_{k_1}, c_{m_3})$**:** Take the r-th iterate of the picture, which moves $\mathfrak{D}_{m_1}$, $\mathfrak{D}_{m_2}$ and $\mathfrak{D}_{k_2}$ to cutting levels, and $\mathfrak{D}_{m_3}$ to a non-cutting level $\mathfrak{D}_u$ with $u := m_3 + r$ such that

$$S_j := n + r = \beta(u) = \beta(m_2 + r) = \beta^2(k_2 + r).$$

The integer u such that c_u is closest to c is for $u = S_i + S_j$ where j is minimal such that $Q(i + 1) > i$, and in this case, the itineraries of $T_s(c)$ and $T_s(c_u)$ agree for at most $S_{Q^2(i+1)+1} - 1$ iterates (if $Q(i+1) = j+1$) or at most $S_{Q(j+1)} - 1$ iterates (if $Q(i+1) > j+1$). Call $S_h := k_2 + r$, then $j = Q^2(h)$ and the itineraries of $T_s(c_{S_h})$ and c agree up to $S_{Q(h+1)} - 1$

iterates. By assumption (A.4), we have

$$Q(j+1) \le Q^2(i+1)+1 = Q(j+1)+1 = Q(Q^2(h)+1)+1 < Q(h+1),$$

but this means that $\mathfrak{D}_u$ and $\mathfrak{D}_{S_h}$ cannot overlap, a contradiction.

Case $a_4 \in (c_{k_2}, c_n)$**:** Then take the $r+1$-st iterate of the picture, which has the same structure, with c_n replaced by $T_s^{r+1}(a_1) = c_1$. Repeating this argument, we will eventually arrive at Case a_2 or a_3 above.

Therefore we can find η_n such that $c_n - \eta_n$ separates c_n from all levels $\mathfrak{D}_{k_i}$, $\beta^2(k_i) = n$ that intersect $\mathfrak{D}_{m_1}$. Indeed, in Case a_2, we place $c_n - \eta_n$ just to the right of c_{k_1} and in Case a_3 (and hence $c_{k_1} \in \mathfrak{D}_{k_2}$), we place $c_n - \eta_n$ just to the right of c_{k_2}. □

Proposition A.6. Under the assumption of Lemma A.5, given $\varepsilon > 0$, there exists $p \in \mathbb{N}$ and a chain $\mathcal{C} = \mathcal{C}_p$ of $\varprojlim([0, s/2], T_s)$ with the following properties:

(1) The links of $\mathcal{C}$ have diameter $< \varepsilon$.
(2) For each $n \in \mathbb{N}$, there is exactly one link $\ell \in \mathcal{C}$ such that every $x \in \varprojlim([0, s/2], T_s)$ that p-turns at c_n belongs to ℓ.
(3) If $y \in \ell$ is a p-point not having the lowest p-level of p-points in ℓ, then both β-neighbors of y belong to ℓ.
(4) If $y \notin \ell$ is a β-neighbor of x above, then the other β-neighbor of y either lies outside ℓ, or has p-level n as well.

Proof. We will construct the chain $\mathcal{C}$ as outlined in the beginning of this section, see (A.2). So let us specify the collection G by starting with at least $\lceil 2s^p/\varepsilon \rceil$ approximately equidistant points $g_m \in [0, s/2]$ so that no g_m lies on the critical orbit, and then refining this collection inductively to satisfy parts 2.-4. of the proposition.

Start the induction with $n = 1$, *i.e.,* the point c_1. Note that $c_1 \notin G$, so there will be only one link $\ell \in \mathcal{C}$ with $c_1 \in \pi_p(\ell)$. Let $\eta_1 \in (0, s^{-p}\varepsilon/2)$ be as in Lemma A.5. Then, since each k contains 1 in its β-orbit, each $\mathfrak{D}_k$ intersecting $(c_1 - \eta_1, c_1]$ is either contained in $(c_1 - \eta_1, c_1]$ or has c_1 as lower endpoint (*i.e.,* $\beta(k) = 1$). In the latter case, also $\mathfrak{D}_l \cap (c_1 - \eta_1, c_1] = \emptyset$ for each l with $\beta(l) = k$. Hence by inserting $c_1 - \eta_1$ into G, we can refine the chain $\mathcal{C}$ so that properties 3. and 4. holds for the link ℓ with $\pi_p(\ell) \ni c_1$.

Suppose we have refined the chain to accommodate links ℓ such that $\pi_p(\ell) \ni c_i$ for each $i < n$. Then c_n does not belong to the set G created so far, so there will be only one link

$\ell \in \mathcal{C}$ with $\pi_p(\ell) \ni c_n$. Again, find $\eta_n \in (0, s^{-p}\varepsilon/2)$ as in Lemma A.5 and extend G with $c_n + \eta_n$ if c_n is a local minimum of T_s^n or with $c_n - \eta_n$ if c_n is a local minimum of T_s^n.

We skip the induction step if $\mathfrak{D}_n$ already belongs to complementary interval to G extended with all point $c_i \pm \eta_i$ created so far. Since $|\mathfrak{D}_n| \to 0$, the induction will eventually cease altogether, and then the required set G is found. □

Appendix B. Symmetric and Quasi-Symmetric Arcs

From now on all chains $\mathcal{C}_p$ are as in Proposition A.6. Also, we assume that the slope s is such that T_s is Fibonacci-like and we abbreviate $T := T_s$.

Suppose $A = [u, v] \subset \mathfrak{A}$ is a quasi-p-symmetric arc with $u, v \in \ell$, and let A_u and A_v be arc-components of ℓ that contain u and v respectively. We will sometimes say, for simplicity, that the arc $[A_u, A_v]$ between A_u and A_v, including A_u and A_v, is quasi-p-symmetric.

Definition B.1. A quasi-p-symmetric arc $A = [u, v]$ with midpoint m is called *basic* if there is no p-point $w \in (u, v)$ such that either $[u, w] \subset [u, m]$ or $[w, v] \subset [m, v]$ is a quasi-p-symmetric arc.

Example B.2. Let us consider the Fibonacci map and the corresponding inverse limit space. Then the arc-component $\mathfrak{C}$ (as well as an arc-component $\mathfrak{A}$) contains the arc $A = [x^0, x^{33}]$ such that the folding pattern of A is as follows (see Figure 9):

$$27\ 6\ \overbrace{\underbrace{\mathbf{1}_2\ \mathbf{14}_3\ \mathbf{1}\ 6\ \mathbf{1}_6}_{\text{basic}}\ 0\ 3\ 0\ 1\ 0\ 2\ 0\ 1\ 4\ 1\ \mathbf{9}\ 1\ 4\ 1\ 0\ 2\ 0\ 1\ 0\ 3\ 0\ \underbrace{1\ 6\ \mathbf{1}_{30}}_{\text{sym}}}^{\text{quasi-}p\text{-symmetric}}\ 0\ 3\ 0 \tag{B.1}$$

(for easier orientation we write sometimes for example 1_2 which means that the p-level 1 belongs to the p-point x^2). We can choose a chain $\mathcal{C}_p$ such that p-points with p-levels 1 and 14 belong to the same link. The arc $[x^2, x^6]$ with the folding pattern 1 14 1 6 1 is a basic quasi-p-symmetric arc; the arc $[x^2, x^{30}]$ with the folding pattern as in (B.1) under the wide brace is also a quasi-p-symmetric but not basic, because it contains $[x^2, x^6]$. Notice also that the arc $[x^3, x^{30}]$ is a quasi-p-symmetric arc for which Proposition B.11 and Proposition B.9 do not work (see the folding patterns to the left of $[x^3, x^{30}]$ and to the right of $[x^3, x^{30}]$).

Lemma B.3. Let $\mathcal{C}_p$ be a chain and $[x, y]$ a quasi-p-symmetric arc with respect to this chain (not contained in a single link) with midpoint m and such that $L_p(x) \geq L_p(m)$.

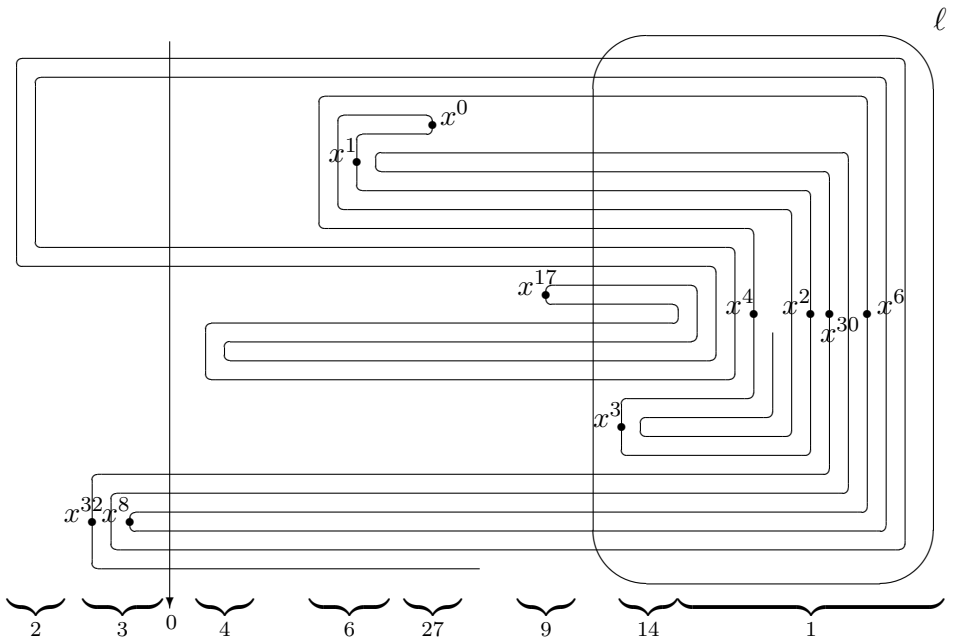

FIGURE 9. The arc A with folding pattern as in (B.1), with p-points of p-level 1 and 14 in a single link ℓ.

Let A_x be the link-tip of $[x, y]$ which contains x. Then $L_p(m) > L_p(z)$ for all p-points $z \in [x, y] \setminus (\{m\} \cup A_x)$.

Proof. Let $A = [a, b] \ni m$ be the smallest arc with p-points a, b of higher p-level than $L_p(m)$, say $m \in [a, b]$ and $L_p(m) \le L_p(a) \le L_p(b)$. By part (a) of Remark 2.4 we obtain $L := L_p(m) < L_p(a) < L_p(b)$. Since $L_p(x) \ge L_p(m)$, $[x, m]$ contains one endpoint of A. We can assume that $[x, m] \setminus A$ is contained in a single link, because otherwise $[x, y] \setminus \ell$-tips is not p-symmetric. If $[y, m]$ does not contain the other endpoint of A, then the statement is proved.

Let us now assume by contradiction that $A \subset [x, y]$. Again, we can assume that $[y, m] \setminus A$ is contained in a single link, because otherwise $[x, y] \setminus \ell$-tips is not p-symmetric. By part (a) of Remark 2.4 once more we have $\pi_{p+L}([a, b]) = [c_{S_l}, c_{S_k}] \ni c = \pi_{p+L}(m)$ for some k and $l = Q(k)$, and $|\pi_{p+L}(a) - c| > |\pi_{p+L}(b) - c|$, see the top line of Figure 10. It follows that $[a, b]$ contains a symmetric open arc (b', b) where $b' \in (a, b)$ is the unique point such that $T(\pi_{p+L}(b')) = T(\pi_{p+L}(b))$. Since $[x, y] \setminus \ell$-tips is p-symmetric, $L_p(b) > L_p(m)$ implies $b, b' \in \ell$-tips. Moreover, the arc $[a, b']$ is contained in the same link ℓ as b.

If k and l are relatively small, then $\pi_p^{-1}(c_{S_l})$ and $\pi_p^{-1}(c_{S_k})$ belong to different links of $\mathcal{C}_p$, so we can assume that they are so large that we can apply Condition (2.2). Let $r = Q(k+1)$ and $r' = Q(l+1)$ be the lowest indices such that the closest precritical points $\hat{\zeta}_{r'} \in [c_{S_l}, c]$

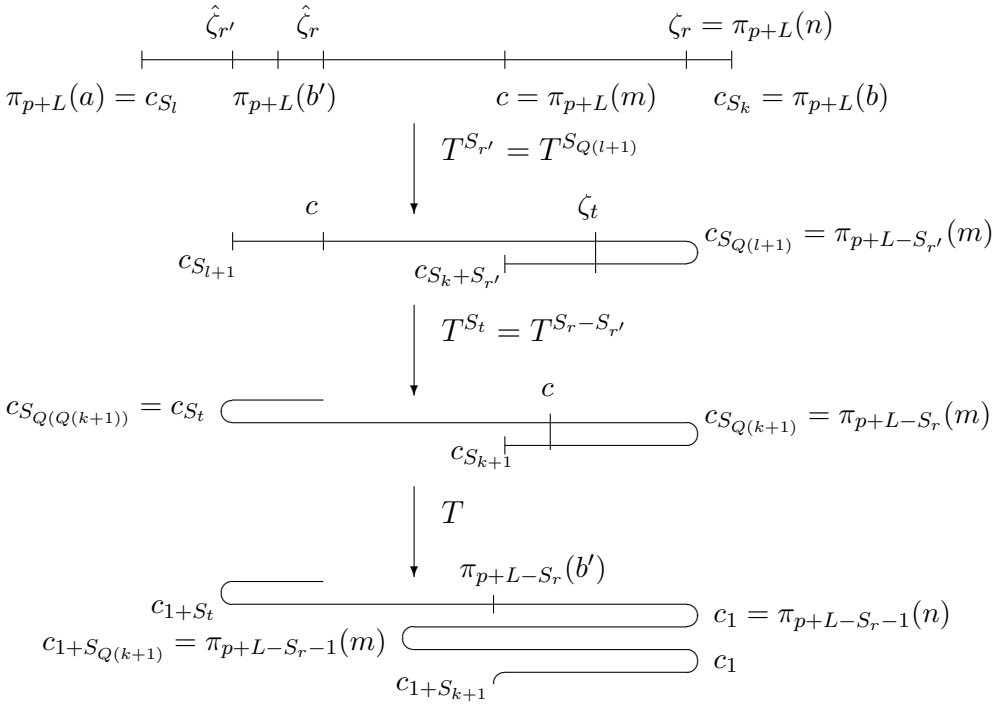

FIGURE 10. The image of $\pi_{p+L}([x,y]) \ni c = \pi_{p+L}(m)$ under appropriate iterates of T.

and $\zeta_r \in [c, c_{S_k}]$. By (2.2), $r' = Q(l+1) = Q(Q(k)+1) < Q(k+1) = r$. Consider the image of $[c_{S_l}, c_{S_k}]$ first under $T^{S_{r'}}$ and then under T^{S_r} (second and third level in Figure 10).

By the choice of r, we obtain $\pi_{p+L-S_r}([m,b]) = [c_{S_{k+1}}, c_{S_{Q(k+1)}}]$, and $\pi_{p+L-S_r}([a,b']) \ni c_{S_t}$ for $t = Q(Q(k+1))$. As in (A.5), $|c_{S_t} - c| > |c_{S_{Q(k+1)}} - c| > |c_{S_{k+1}} - c|$, and taking one more iterate, we see that $[c_{1+S_{k+1}}, c_1] \subset [c_{1+S_{Q(k+1)}}, c_1] \subset [1 + c_{S_t}, c_1]$ (last level in Figure 10).

Let $n \in [m,b]$ be such that $\pi_{p+L}(n) = \zeta_r$, see the first level in Figure 10. Since $[a,b']$ belongs to a single link $\ell \in \mathcal{C}_p$, $m \in \ell$ as well. Suppose that $[a,m]$ is not contained in ℓ. Then there is a maximal symmetric arc $[d',d]$ with midpoint n such that the points $d, d' \notin \ell$. Then the arcs $[d',a]$ and $[d,m]$ both enter the same link ℓ but they have different 'first' turning levels in ℓ, contradicting the properties of $\mathcal{C}_p$ from Proposition A.6.

This shows that $[a,m] \subset \ell$. In the beginning of the proof we argued that the components of $[x,y] \setminus A$ belong to the same link, so that means that the entire arc $[x,y]$ is contained in a single link, contradicting the assumptions of the proposition. This concludes the proof. □

Remark B.4. In fact, this proof shows that the p-point $b \in \partial A$ of the highest p-level belongs to $[m, x]$. Indeed, if $a \in [m, x]$, then because $[m, b]$ has shorter arc-length than $[m, a]$, either a and b, and therefore x and y do not belong to the same link ℓ (whence $[x, y]$ is not quasi-p-symmetric), or the arc $[a, b]$ itself is quasi-p-symmetric and contradicts Lemma B.3.

Corollary B.5. Let $A = [x, y] \subset \mathfrak{A}$ be a quasi-p-symmetric arc with midpoint m. Let A_x, A_y be the link-tips of A containing x and y respectively. If x is the midpoint of A_x, and y is the midpoint of A_y, then either $L_p(x) > L_p(m) > L_p(y)$, or $L_p(x) < L_p(m) < L_p(y)$.

Remark B.6. Note that in general there are quasi-p-symmetric arcs $[x, y]$ with midpoint m such that $L_p(x) > L_p(y) > L_p(m)$. For example, if a tent map T_s has a preperiodic critical point, then for every quasi-p-symmetric arcs $[x, y]$ with midpoint m either $L_p(x) > L_p(y) > L_p(m)$, or $L_p(y) > L_p(x) > L_p(m)$.

Corollary B.7. Let $[x, y] \subset \mathfrak{A}$ be a quasi-p-symmetric arc with midpoint m, not contained in a single link, such that $L_p(x) > L_p(m) > L_p(y)$. If $[m, x]$ is longer than $[y, m]$ measured in arc-length, then there exists a p-point $y' \in A_x$ such that $[y, y']$ is p-symmetric.

Proof. As in the previous proof, $b \in [x, m]$ and $y \in [m, b']$ and take $y' \in [m, b]$ such that $\pi_{p+L}(y') = \pi_{p+L}(y)$. □

Remark B.8. If $A_x \ni x$ and $A_y \ni y$ are maximal arc-components of $\mathfrak{A} \cap \ell$ (with still $L_p(x) > L_p(m) > L_p(y)$), and m_y is the midpoint of A_y, then there is $y' \in A_x$ such that $[y', m_y]$ is p-symmetric.

In other words, when $\mathfrak{A}$ enters and turns in a link ℓ, then it folds in a symmetric pattern, say with levels $L_1, L_2, \ldots, L_{m-1}, L_m, L_{m-1}, \ldots, L_2, L_1$. The nature of the chain $\mathcal{C}_p$ is such that L_1 depends only on ℓ. The Corollary B.7 does not say that the rest of the pattern is the same also, but only that if $A \subset \mathfrak{A}$ is such that $A \setminus \ell$-tips is p-symmetric, then the folding pattern at the one link-tip is a subpattern (stopping at a lower center level) of the folding pattern at the other link-tip.

Proposition B.9 (Extending a quasi-p-symmetric arc at its higher level endpoint)**.** Let $A = [x, y] \subset \mathfrak{A}$ be a basic quasi-p-symmetric arc, not contained in a single link, such that the p-points $x, y \in \ell$ are the midpoints of the link-tips of A and $L_p(x) > L_p(y)$. Let m be the midpoint of A. Then there exists a p-point m' such that the arc $[m, m']$ is (quasi-)p-symmetric with x as its midpoint.

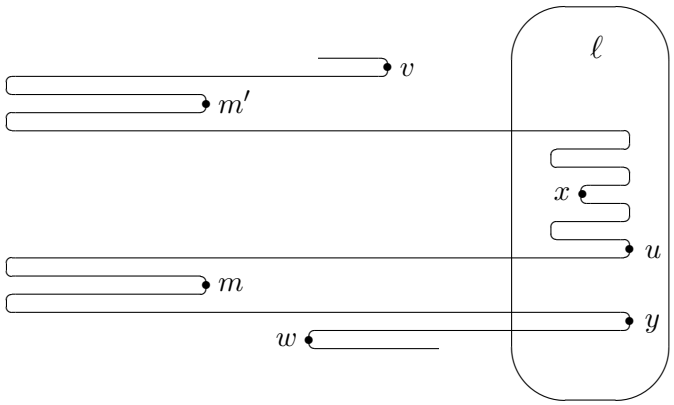

FIGURE 11. The configuration in Proposition B.9 where the existence of p-point m' is proved. v is the first p-point 'beyond' x such that $L_p(v) > L_p(x)$ and u is such that $[u, y]$ is p-symmetric with midpoint m.

Remark B.10. The conditions are all crucial in this lemma:

(*a*) It is important that y is a p-point. Otherwise, if $\mathfrak{A}$ goes straight through ℓ at y, then it is possible that x is the single p-point in A_x (where A_x is the arc-component of $\mathfrak{A} \cap \ell$ containing x) and $[v, x]$ is shorter than $[x, m]$, and the lemma would fail.

(*b*) Without the assumption that $[x, y]$ is basic the lemma can fail. If Figure 9 the quasi-p-symmetric arc $[x, y] = [x^3, x^{30}]$ is not basic, and indeed there is no p-point $m' \in [x, v] = [x^3, x^0]$ with $L_p(m') = L_p(m) = L_p(x^{17}) = 9$.

Proof. Since $[u, y]$ is p-symmetric, $L_p(u) = L_p(y) < L_p(m)$ and $x \neq u$. Let w be the first p-point 'beyond' y such that $L_p(w) > L_p(x)$. Take $L = L_p(x)$; Figure 12 shows the configuration of the relevant points on $[w, v]$ and their images under $\pi_p \circ \sigma^{-L}$ denoted by $\tilde{}$-accents. Clearly $\tilde{x} = c$.

Case I: $|\tilde{w} - c| < |\tilde{v} - c|$. Then by Remark 2.4 (b), $\tilde{w} = c_{S_l}$ and $\tilde{v} = c_{S_k}$ with $k = Q(l)$. The points $\tilde{y}, \tilde{m}, \tilde{u}$ have symmetric copies $\tilde{y}', \tilde{m}', \tilde{u}'$ (*i.e.,* $T(\tilde{y}) = T(\tilde{y}')$, etc.) in reverse order on $[c, \tilde{v}]$, and the pre-image under $\sigma^L \circ \pi_p^{-1}$ of the copy of $\tilde{m}'$ yields the required point m'.

Case II: $|\tilde{w} - c| > |\tilde{v} - c|$, so in this case, $l = Q(k)$. We can in fact assume that $|\tilde{m} - c| > |\tilde{v} - c|$ because otherwise we can find m' precisely as in Case I. Now take the p-point $a' \in (x, v)$ of maximal p-level, and let $a \in [m, x]$ be such that their $\pi_p \circ \sigma^{-L}$-images $\tilde{a}$ and $\tilde{a}'$ are each other symmetric copies. Let r be such that $T^r(\tilde{a}) = c$; the bottom part of Figure 12 shows the image of $[\tilde{m}, \tilde{v}]$ under T^r. The point $T^r(\tilde{x})$ and $T^r(v)$ are each

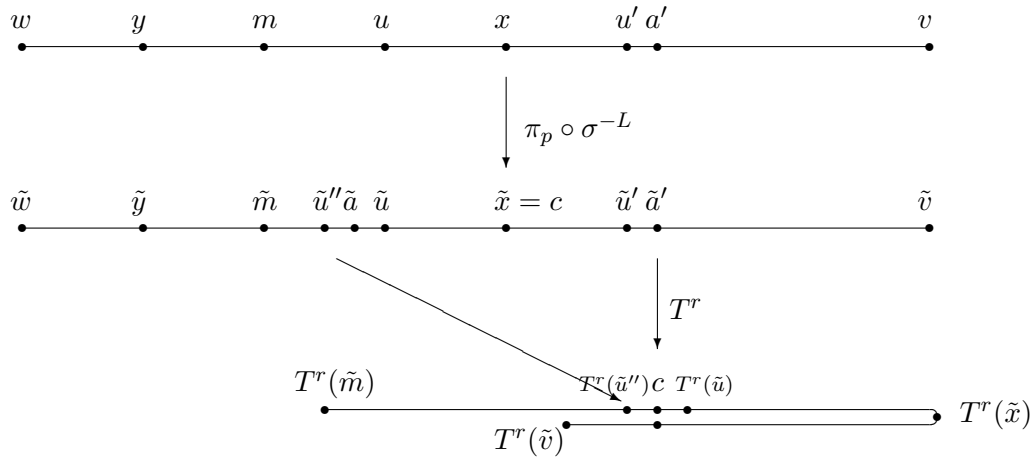

FIGURE 12. The configuration of points on $[w, v]$ and their images under $\pi_p \circ \sigma^{-L}$ and an additional T^r.

others β-neighbors, and since $L_p(v) > L_p(x)$, and by (2.2), $|T^r(\tilde{x}) - c| > |T^r(v) - c|$. Therefore $[T^{r+j}(\tilde{x}), T^{r+j}(\tilde{a}')] \supset [T^{r+j}(\tilde{v}), T^{r+j}(\tilde{a}')]$ for all $j \geq 1$.

If $a, a' \in \ell$, then since $[x, a] \subset \ell$, this would imply that $[a', v] \subset \ell$ as well, contrary to the fact that x is the midpoint of A_x.

If on the other hand $a, a' \notin \ell$, then there is a point $u'' \in [m, a]$ such that $T^r(\tilde{u}'')$ and $T^r(\tilde{u})$ are each other symmetric copies. It follows that $[u'', x]$ is a quasi-p-symmetric arc properly contained in $[x, y]$, contradicting that $[x, y]$ is basic. □

Proposition B.11 (Extending a quasi-p-symmetric arc at its lower level endpoint)**.** Let $A = [x, y] \subset \mathfrak{A}$ be a basic quasi-p-symmetric arc, not contained in a single link, such that x and y are the midpoints of the link-tips of A and $L_p(x) > L_p(y)$. Let m be the midpoint of A. Then there exists a point a such that $[m, a]$ is a quasi-p-symmetric arc with y as the midpoint.

Remark B.12. The assumption that $[x, y]$ is basic is essential. Without it, we would have a counter-example in $[x, y] = [x^3, x^{30}]$ in Figure 9. The quasi-p-symmetric arc $[x^3, x^{30}]$ is indeed not basic, because $[x^3, x^6]$ is a shorter quasi-p-symmetric arc in the figure. There is a point $n = x^{32}$ beyond y with $L_p(n) = L_p(x^{32}) = 3 > 1 = L_p(y)$, making it impossible that y is the midpoint of a quasi-p-symmetric arc stretching unto m.

Proof. A quasi-p-symmetric arc is not contained in a single link, so $[x, m] \not\subset \ell$. Let $H = [x, n] \supset A$ be the smallest arc containing a point n 'beyond' y with $L_p(n) > L_p(y)$.

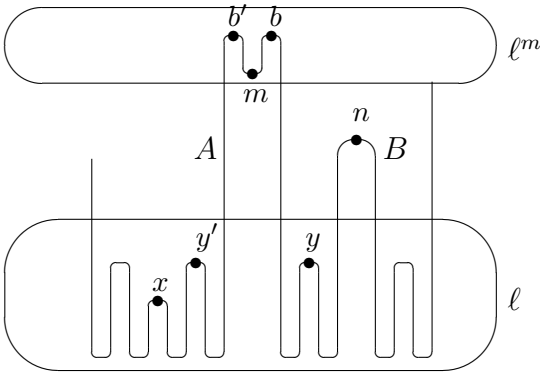

FIGURE 13. The arcs A and B and the relevant points for Proposition B.11, which is meant to show that the point n does not exist in B.

Corollary B.7 implies that the arc $[x, m]$ contains a p-point y' with $L_p(y') = L_p(y)$. Let b and b' be the p-points having the highest p-level on the arcs $[y, m)$ and $[y', m)$ respectively. By symmetry, $L_p(b) = L_p(b')$, and possibly $b = y$, $b' = y'$. Let $z \in [x, y']$ be the point closest to y' such that $L_p(z) > L_p(b)$; possibly $z = x$. Since $b' \in [y', m)$, we have

$$L_p(y) = L_p(y') \leqslant L_p(b) = L_p(b') < L_p(m).$$

Take $L := L_p(b)$ and let $\tilde{H} = \pi_p \circ \sigma^{-L}(H)$. Since y is the midpoint of its link-tip, $[y, n] \not\subset \ell$. Therefore $\pi_p^{-1}(c) \cap \sigma^{-L}(H) \supset \{\sigma^{-L}(b), \sigma^{-L}(b')\}$, and $\tilde{z} = \pi_p \circ \sigma^{-L}(z)$ and $\tilde{n} = \pi_p \circ \sigma^{-L}(n)$ have $\tilde{m} = \pi_p \circ \sigma^{-L}(m)$ as common β-neighbor, see Figure 14. Since $L_p(z) > L_p(b)$ there

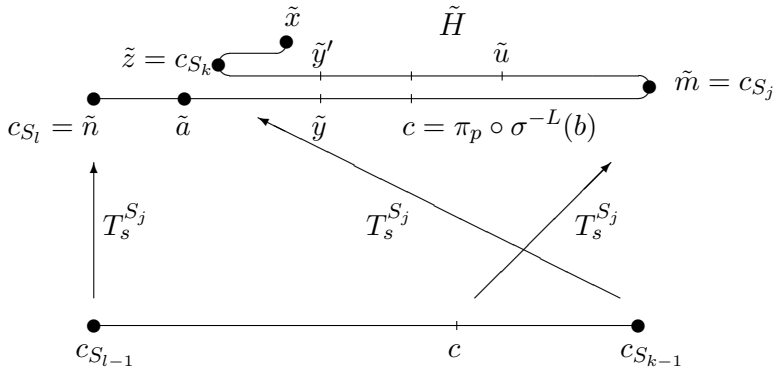

FIGURE 14. The arc $\tilde{H}$ drawn as multiple curve, its preimage under $T_s^{S_j}$ and the relevant points on them.

is k such that $\tilde{z} = c_{S_k}$. Also take l such that $\tilde{n} = c_{S_l}$ and j such that $\tilde{m} = c_{S_j}$. Let $\tilde{y} = \pi_p \circ \sigma^{-L}(y)$ and $\tilde{y}' = \pi_p \circ \sigma^{-L}(y')$.

We claim that there is a point $a \in [n, m]$ such that

$$\tilde{a} := \pi_p \circ \sigma^{-L}(a) \in [c_{S_l}, \tilde{y}] \quad \text{and} \quad T_s(\tilde{a}) = T_s(\tilde{m}).$$

Since c_{S_j} is β-neighbor to both c_{S_l} and c_{S_k}, we have three cases:

(1) $j = Q(k)$ and $l = Q(j)$, so $l = Q^2(k)$. In this case, Equation (2.2) and Remark 2.2 imply that $|c - c_{S_l}| > |c - c_{S_{Q(k)}}|$, so $[c_{S_l}, c]$ contains the required point $\tilde{a}$ with $T_s(\tilde{a}) = T_s(\tilde{m})$. By the same token, $|c_{S_k} - c| < |c_{S_j} - c| = \frac{1}{2}|\tilde{a} - \tilde{m}|$. Since $|\tilde{y} - c| = |\tilde{y}' - c| < |c_{S_k} - c|$, we indeed obtain that $\tilde{a} \in [c_{S_l}, \tilde{y}]$.

(2) $j = Q(l)$ and $k = Q(j)$, so $k = Q^2(l)$. Then Remark A.2 implies that $|c - c_{S_k}| > |c - c_{S_l}|$. But this would mean that the arc $[n, m]$ is shorter than $[z, m]$ and in particular that $[y, n] \subset \ell$, contradicting that y is the midpoint of its link-tip.

(3) $j = Q(k) = Q(l)$. In this case, we pull $\tilde{H}$ back for another S_j iterates, or more precisely, we look at the arc $\pi_p \circ \sigma^{-S_j - L}(H)$. The endpoints of this arc are $c_{S_{k-1}}$ and $c_{S_{l-1}}$ which are therefore β-neighbors. If $l - 1 = Q(k - 1)$, then we find

$$Q(k) = Q(l) = Q(Q(k - 1) + 1)$$

which contradicts Condition (2.2) with k replaced by $k - 1$. If $k - 1 = Q(l - 1)$, then we find

$$Q(l) = Q(k) = Q(Q(l - 1) + 1)$$

which contradicts Condition (2.2) with k replaced by $l - 1$.

This proves the claim.

Suppose now that $\tilde{y} \neq c$ (*i.e.*, $y \neq b$). Then $b, b' \notin \ell$ because y has the largest p-level in its link-tip. Since $|c_{S_k} - c| < |c - \tilde{m}|$, there is a point $u \in [z, m]$ such that $\tilde{u} = \pi_p \circ \sigma^{-L}(u) \in [c, \tilde{m}]$ and $T_s(\tilde{u}) = T_s(\tilde{y})$. This means that $[x, u]$ is a quasi-p-symmetric arc properly contained in $[x, m]$, contradicting the assumption that $[x, y]$ is a basic quasi-symmetric arc.

Therefore $y = b$, so there are no p-points between y and m of level higher than $L_p(y)$. Instead, the arc $[a, m]$ has midpoint y, and is the required quasi-p-symmetric arc, proving the lemma. □

Remark B.13. Let $A = [x, y]$ be a basic quasi-p-symmetric arc such that x and y are the midpoints of the link-tips of A and $L_p(x) > L_p(y)$. Let ℓ^m be the link which contains the midpoint m of A, and let A_m be the arc-component of ℓ^m containing m. Then, by the lemma above, $A \setminus (\ell\text{-tips} \cup A_m)$ does not contain any p-point z with $L_p(z) \geqslant L_p(y)$.

Appendix C. Link-Symmetric Arcs

Definition C.1. We say that an arc $[x, y]$ is *decreasing* (*basic*) *quasi-p-symmetric* if it is the concatenation of (basic) quasi-p-symmetric arcs where the p-levels of the midpoints decrease, *i.e.,* if there are p-points $x = x^0, x^1, x^2, \ldots, x^{n-1}$ and $x^n = y$ can be a p-point or not, such that the following hold:

(i) $[x^{i-1}, x^{i+1}]$ is a (basic) quasi-p-symmetric arc with midpoint x^i, for $i = 1, \ldots, n-1$. (By definition of a (basic) quasi-p-symmetric arc, the points x^{2i} all belong to a single link, and the points x^{2i-1} belong to a single link as well.)

(ii) $L_p(x^i) > L_p(x^{i+1})$, for $i = 1, \ldots, n-1$ (and if y is a p-point then also $L_p(x^{n-1}) > L_p(y)$).

Similarly, we say that the arc $[x, y]$ is *increasing* (*basic*) *quasi-p-symmetric* if it is the concatenation of (basic) quasi-p-symmetric arcs where the p-levels of the midpoints increase.

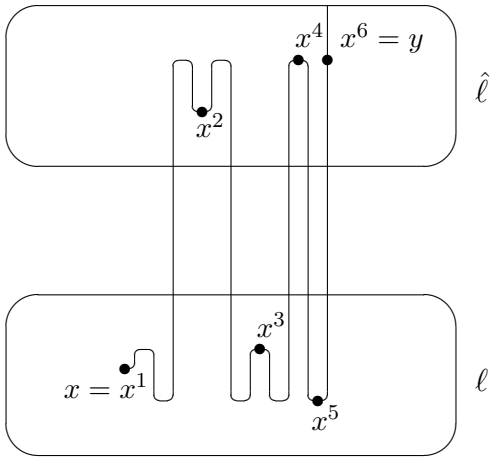

Figure 15. Illustration of a basic decreasing quasi-p-symmetric arc. The point y is not a p-point here; instead, the arc A goes straight through $\hat{\ell}$ at y.

Example C.2. Consider the Fibonacci inverse limit space, and let our chain $\mathcal{C}_p$ be such that p-points with p-levels 1 and 14 belong to the same link ℓ, but p-points with p-level 9 are not contained in ℓ. Since p-points with p-level 14 belong to the same link ℓ as p-points with p-level 1, also p-points with p-levels 22, 35, 56 and 77 belong to ℓ. Let p-points with p-level 43 belong to the same link as p-points with p-level 9.

(1) **Example of a basic decreasing quasi-p-symmetric arc.** Let $A = [y^0, y^{12}]$ be an arc with the following folding pattern (where the subscripts count important p-points):

$$\underbrace{\mathbf{1\ 22\ 77}_2\ \mathbf{22\ 1}\ 9\ 43_6\ 9\ \mathbf{1}}_{\text{basic}}\overbrace{\mathbf{22}_9\ \mathbf{1}\ 9_{11}\ \mathbf{1}_{12}}^{\text{basic}}$$

Let x^i be as in the above definition. Then $x^1 = y^2$, $x^2 = y^6$, $x^3 = y^9$, $x^4 = y^{11}$, and $x^5 = y^{12}$. So $[y^2, y^9]$ is basic quasi-p-symmetric with midpoint y^6, $[y^6, y^{11}]$ is basic quasi-p-symmetric with midpoint y^9, and $[y^9, y^{12}]$ is basic quasi-p-symmetric with the midpoint y^{11}. Also $L_p(y^2) = 77 > L_p(y^6) = 43 > L_p(y^9) = 22 > L_p(y^{11}) = 9 > L_p(y^{12}) = 1$.

(2) **Example of a non-basic decreasing quasi-p-symmetric arc.** Let $[y^0, y^{72}]$ be an arc with the following folding pattern:

$$\overbrace{\underbrace{\mathbf{1\ 22\ 1\ 56}_3\mathbf{\ 1\ 22\ 1}\ 9\ \mathbf{1}}_{\text{basic}}\ 4\ 1\ 0\ 2\ 0\ 1\ 0\ 3\ 0\ \mathbf{1}\ 6\ \mathbf{1\ 14\ 1\ 35}_{23}\mathbf{1\ 14\ 1}\ 6\ \mathbf{1}\ 0\ 3\ 0\ 1\ 0\ 2\ 0\ 1\ 4\ \underbrace{\mathbf{1}\ 9\ \mathbf{1}}_{\text{sym}}}^{\text{quasi-}p\text{-symmetric}}$$

$$\overbrace{\underbrace{\mathbf{22}_{41}\mathbf{1}\ 9\ \mathbf{1}}_{\text{basic}}\ 4\ 1\ 0\ 2\ 0\ 1\ 0\ 3\ 0\ \mathbf{1}\ 6\ \mathbf{1\ 14}_{57}\mathbf{1}\ 6\ \mathbf{1}\ 0\ 3\ 0\ 1\ 0\ 2\ 0\ 1\ 4\ \underbrace{\mathbf{1}\ 9\ \mathbf{1}_{72}}_{\text{sym}}}^{\text{quasi-}p\text{-symmetric}}$$

Let x^i be again as in the above definition. Then $x^1 = y^3$, $x^2 = y^{23}$, $x^3 = y^{41}$, $x^4 = y^{57}$, and $x^5 = y^{72}$. So, arcs $[y^3, y^{41}]$, $[y^{23}, y^{57}]$ and $[y^{41}, y^{72}]$ are quasi-p-symmetric, and $L_p(y^3) = 56 > L_p(y^{23}) = 35 > L_p(y^{41}) = 22 > L_p(y^{57}) = 14 > L_p(y^{72}) = 1$.

(3) **Example of an arc that is the concatenation of two quasi-p-symmetric arcs (one of them is basic), but is not decreasing quasi-p-symmetric.** Let $[y^0, y^{40}]$ be an arc with the following folding pattern:

$$\underbrace{\mathbf{1\ 22\ 77}_2\ \mathbf{22\ 1}\ 9\ 43_6\ 9\ \mathbf{1}}_{\text{basic}}\underbrace{\mathbf{22}_9\ \mathbf{1}\ 9_{11}\mathbf{1}_{12}4\ 1\ 0\ 2\ 0\ 1\ 0\ 3\ 0\ \mathbf{1}\ 6\ \mathbf{1\ 14}_{25}\mathbf{1}\ 6\ \mathbf{1}\ 0\ 3\ 0\ 1\ 0\ 2\ 0\ 1\ 4\ \mathbf{1}\ 9\ \mathbf{1}_{40}}_{\text{quasi-}p\text{-symmetric}}$$

Then $[y^2, y^9]$ is basic quasi-p-symmetric with midpoint y^6, $[y^6, y^{11}]$ is basic quasi-p-symmetric with midpoint y^9, and $[y^9, y^{12}]$ is basic quasi-p-symmetric with the midpoint y^{11}. However, $[y^9, y^{40}]$ is quasi-p-symmetric with midpoint y^{25} and $[y^6, y^{25}]$ is neither basic quasi-p-symmetric, nor quasi-p-symmetric. So $A = [y^0, y^{40}]$ is not a decreasingly quasi-p-symmetric arc. Note that $[y^0, y^{12}]$ is a decreasing quasi-p-symmetric arc.

Proposition C.3. Let A be a non-basic quasi-p-symmetric arc. Then there are $k, n, m, d \in \mathbb{N}$, $d < k$, such that

$$A \cap E_p = \{x^0, \dots, x^k, \dots, x^{k+n}, \dots, x^{k+n+m}\},$$

$[x^0, x^k]$ is a basic quasi-p-symmetric arc with midpoint x^{k-d} and $[x^k, x^{k+n}]$ is p-symmetric. Moreover,

(i) If $[x^{k+n}, x^{k+n+m}]$ is p-symmetric, then $[x^{-k+m/2}, x^{k+n+3m/2}]$ is not p-link-symmetric.
(ii) If $[x^{k+n}, x^{k+n+m}]$ is a basic quasi-p-symmetric arc, then A is contained in a decreasing quasi-p-symmetric arc consisting of at least two quasi-p-symmetric arcs. More precisely, $[x^{-k-n/2}, x^{k+n/2}]$ and $[x^{k+n/2}, x^{k+2m+3n/2}]$ are the quasi-p-symmetric arcs contained in the decreasing quasi-p-symmetric arc $[x^{-k-n/2}, x^{k+2m+3n/2}]$ containing A.

Proof. Since A is a non-basic quasi-p-symmetric arc, there is a basic quasi-p-symmetric arc which we can label $[x^0, x^k]$. The arc $[x^k, x^{k+n}]$ in the middle is p-symmetric by definition of quasi-p-symmetry, and it has the same midpoint $x^{k+n/2}$ as A. The arc $[x^{k+n}, x^{k+n+m}]$ could be either p-symmetric or basic quasi-p-symmetric.

(i) Assume that $[x^{k+n}, x^{k+n+m}]$ is p-symmetric. Without loss of generality we can suppose that x^0 and x^{k+n+m} are the midpoints of the link-tips of A, and also that x^k and x^{k+n} are the midpoints of their arc-components. Since the point $x^{k+n+m/2}$ is the midpoint of the p-symmetric arc $[x^{k+n}, x^{k+n+m}]$, and the symmetry of the arc $[x^k, x^{k+n}]$ can be extended to the midpoints of its neighboring (quasi-)symmetric arcs, we obtain that $d = m/2$ and the point $x^{k-m/2}$ is the midpoint of the basic quasi-p-symmetric arc $[x^0, x^k]$. Proposition B.9 implies that we can extend $[x^0, x^{k-m/2}]$ beyond x^0 to obtain the arc $[x^{-k+m/2}, x^{k-m/2}]$ which is either p-symmetric, or quasi-p-symmetric, and hence p-link-symmetric.

First, let us assume that $L_p(x^{k+n+m}) = 1$. Let us consider the arc $[x^{k+n+m/2}, x^{k+n+3m/2}]$. Its midpoint x^{k+n+m} has p-level 1. If $L_p(x^{k+n+m-1}) = L_p(x^{k+n+m+1})$, then $L_p(x^{k+n+m-1}) = 0$. Furthermore $x^{k+n+m-1} \neq x^{k+n+m/2}$ since a midpoint cannot have p-level zero. It follows that $x^{k+n+m-2}$ and $x^{k+n+m+2}$ have different p-levels, and are not in the same link, since by Lemma B.3 there is no quasi-p-symmetric arc whose both boundary points are p-points and whose midpoint has p-level 1.

If $L_p(x^{k+n+m-1}) \neq L_p(x^{k+n+m+1})$ then again $x^{k+n+m-1}$ and $x^{k+n+m+1}$ are not in the same link (by Lemma B.3 there is no quasi-p-symmetric arc whose both boundary points are

p-points and whose midpoint has p-level 1). In either case, $[x^{k+n+m/2}, x^{k+n+3m/2}]$ is not p-link-symmetric and hence $[x^{-m/2}, x^{k+n+3m/2}]$ is not p-link-symmetric. This proves statement (i) in the case that $L_p(x^{k+n+m}) = 1$.

Now for the general case, let $L := L_p(x^{k+n+m})$. The basic idea is to shift $[x^0, x^{k+n+m}]$ back by $L-1$ iterates, and use the above argument. Note that the arcs $[x^k, x^{k+n}]$ and $[x^{k+n}, x^{k+n+m}]$ are p-symmetric and hence $L_p(x^{k+n/2}) > L_p(x^{k+n}) = L_p(x^{k+n+m}) = L$. Then $\sigma^{-L+1}(A)$ is also a quasi-p-symmetric arc which is not basic, the arc $\sigma^{-L+1}([x^0, x^k])$ is a basic quasi-p-symmetric arc and $L_p(\sigma^{-L+1}(x^{k+n+m})) = 1$. Let

$$\sigma^{-L+1}(A) \cap E_p = \{u^0, \dots, u^{\hat{k}}, \dots, u^{\hat{k}+\hat{n}}, \dots, u^{\hat{k}+\hat{n}+\hat{m}}\},$$

where $u^{\hat{i}} = \sigma^{-L+1}(x^i)$. (Note that $\hat{k} \leqslant k$, $\hat{n} \leqslant n$ and $\hat{m} \leqslant m$, since not every $\sigma^{-L+1}(x^i)$ needs to be a p-point.) Then $G = [u^{-\hat{k}+\hat{m}/2}, u^{\hat{k}+\hat{n}+3\hat{m}/2}]$ is an arc with 'boundary arcs' $[u^{-\hat{k}+\hat{m}/2}, u^{\hat{k}-\hat{m}/2}]$ and $[u^{k+n+\hat{m}/2}, u^{k+n+3\hat{m}/2}]$ and the midpoint of the latter has p-level 1. The above argument shows that this arc cannot be p-link-symmetric, and therefore the whole arc G is not p-link-symmetric with midpoint $u = \sigma^{-L+1}(x^{k+n/2})$.

We want to prove that $\sigma^j(G)$ is also not p-link-symmetric with the midpoint $\sigma^j(u)$ for $j = L-1$.

Let us assume by contradiction that $\sigma^j(G)$ is p-link-symmetric. Since $[x^{-k+m/2}, x^{k-m/2}]$ is p-symmetric, also $\sigma^j([u^{\hat{k}+\hat{n}+\hat{m}/2}, u^{\hat{k}+\hat{n}+3\hat{m}/2}])$ is p-link-symmetric. But $[u^{\hat{k}+\hat{n}+\hat{m}/2}, u^{\hat{k}+\hat{n}+3\hat{m}/2}]$ has its midpoint at p-level 1, and hence is not p-link-symmetric. Therefore, there exists $l < j$ such that $\sigma^l([u^{\hat{k}+\hat{n}+\hat{m}/2}, u^{\hat{k}+\hat{n}+3\hat{m}/2}])$ is not p-link-symmetric and $\sigma^{l+1}([u^{\hat{k}+\hat{n}+\hat{m}/2}, u^{\hat{k}+\hat{n}+3\hat{m}/2}])$ is p-link-symmetric. By Proposition A.6, and since $L_p(\sigma^l(u^{\hat{k}+\hat{n}+\hat{m}})) = l+1 \neq 0$, there exist $v \in \sigma^l([u^{\hat{k}+\hat{n}+\hat{m}/2}, u^{\hat{k}+\hat{n}+\hat{m}}])$ and $w \in \sigma^l([u^{\hat{k}+\hat{n}+\hat{m}}, u^{\hat{k}+\hat{n}+3\hat{m}/2}])$ such that $L_p(v) = L_p(w) = 0$, see Figure 16.

Since $\sigma^{l+1}(u^{\hat{k}+\hat{n}+\hat{m}/2})$ and $\sigma^{l+1}(u^{\hat{k}+\hat{n}+3\hat{m}/2})$ belong to the same link and $L_p(\sigma^{l+1}(u^{\hat{k}+\hat{n}+\hat{m}/2})) \neq L_p(\sigma^{l+1}(u^{\hat{k}+\hat{n}+3\hat{m}/2}))$, Proposition A.6 implies that $\sigma^{l+1}(u^{\hat{k}+\hat{n}+\hat{m}/2})$ and $\sigma^{l+1}(u^{\hat{k}+\hat{n}+3\hat{m}/2})$ belong to the same link as $\sigma(v)$ and $\sigma(w)$. But then $\sigma^l(u^{\hat{k}+\hat{n}+\hat{m}/2})$ and $\sigma^l(u^{\hat{k}+\hat{n}+3\hat{m}/2})$ belong to the same link as v and w, contradicting the choice of l.

(*ii*) The rough idea of this proof is as follows: Whenever $[x^{k+n}, x^{k+n+m}]$ is not p-symmetric, there exists $N \in \mathbb{N}$ such that $\sigma^{-N}(A)$ is a basic quasi-p-symmetric arc and we can apply Propositions B.9 and B.11 to obtain the arc $B \supset \sigma^{-N}(A)$ which is decreasing basic quasi-p-symmetric. Then $\sigma^N(B) \supset A$ is the required decreasing quasi-p-symmetric arc.

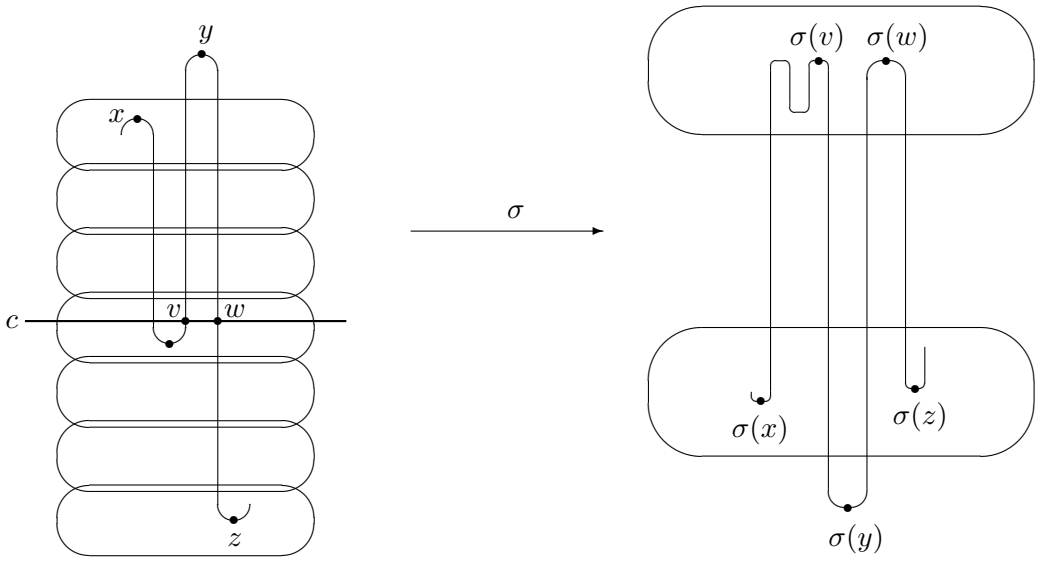

FIGURE 16. The configuration of p-levels that does not exist. Here $x = \sigma^l(u^{\hat{k}+\hat{n}+\hat{m}/2})$, $y = \sigma^l(u^{\hat{k}+\hat{n}+\hat{m}})$ and $z = \sigma^l(u^{\hat{k}+\hat{n}+3\hat{m}/2})$.

Let us assume now that $[x^{k+n}, x^{k+n+m}]$ is basic quasi-p-symmetric. Let us denote by ℓ the link which contains x^0. Then $x^k, x^{k+n}, x^{k+n+m} \in \ell$. We can assume without loss of generality that x^k and x^{k+n} are the p-points in the link-tips of $[x^k, x^{k+n}]$ furthest away from the midpoint $x^{k+n/2}$ and, similarly, x^0 and x^{k+n+m} are the p-points in the link-tips of $[x^0, x^{k+n+m}]$ furthest away from the midpoint $x^{k+n/2}$. Then from the properties of the chain in Proposition A.6 we conclude that $L_p(x^0) = L_p(x^k) = L_p(x^{k+n}) = L_p(x^{k+n+m})$. Let us denote by x^a and x^b the midpoints of arc-components which contains x^0 and x^{k+n+m} respectively. Then $x^a, x^b \in \ell$ and $x^b \neq x^{k+n+m}$. Without loss of generality we can assume that $L_p(x^a) > L_p(x^b)$.

Since x^{k-d} is the midpoint of $[x^0, x^k]$ and A is quasi-p-symmetric, x^{k+n+d} is the midpoint of $[x^{k+n}, x^{k+n+m}]$.

By Proposition B.9, $L_p(x^{-d}) = L_p(x^{k-d})$ and $L_p(x^{k+n+d}) = L_p(x^{k+n+m+d})$, see Figure 17.

Let us denote by ℓ^d the link which contains x^{-d}, and by A_d the arc-component of ℓ^d which contains x^{-d}.

Claim x^{-d} is the midpoint of its arc-component A_d.

Consider the arc $\sigma^{-L+1}(A)$, where $L := L_p(x^b)$. Since $L_p(x^a) > L_p(x^{k+n/2}) > L_p(x^b) = L$,

the preimage $\sigma^{-L+1}(A)$ contains the points $\sigma^{-L+1}(x^b)$ with $L_p(\sigma^{-L+1}(x^b)) = 1$, $\sigma^{-L+1}(x^a)$ and $\sigma^{-L+1}(x^{k+n/2})$ is the midpoint of $\sigma^{-L+1}(A)$.

By Corollary B.7 the arc-component containing x^a also contains p-points x' and x'' with the property that $[x', x'']$ is p-symmetric with midpoint x^a and $L_p(x') = L_p(x'') = L_p(x^b)$, Assume also that x' and x'' are furthest away from x^a with these properties. Therefore, $\sigma^{-L+1}(A) \cap E_p \supseteq \{u^0, u^{\hat{a}}, u^{2\hat{a}}, u^{2\hat{a}+\hat{n}}, u^{2\hat{a}+2\hat{n}}\}$, where $u^{\hat{a}} = \sigma^{-L+1}(x^a)$, $u^{2\hat{a}+\hat{n}} = \sigma^{-L+1}(x^{k+n/2})$, $u^{2\hat{a}+2\hat{n}} = \sigma^{-L+1}(x^b)$, $u^0 = \sigma^{-L+1}(x')$, $u^{2\hat{a}} = \sigma^{-L+1}(x'')$ and $L_p(u^0) = L_p(u^{2\hat{a}}) = 1$.

Let us suppose that $\sigma^{-L+1}(A)$ is not contained in a single link. Since $\sigma^{-L+1}(x^a)$ and $\sigma^{-L+1}(x^b)$ are contained in the same link, $\sigma^{-L+1}(A)$ is a basic quasi-p-symmetric arc. Let ℓ^n be the link containing $u^{2\hat{a}+\hat{n}}$, and let A_{2a+n} be the arc component of ℓ^n containing $u^{2\hat{a}+\hat{n}}$. Since $L_p(u^{2\hat{a}+2\hat{n}}) = 1$, by Remark B.13, $(u^{2\hat{a}+\hat{n}}, u^{2\hat{a}+2\hat{n}}) \setminus A_{2a+n}$ can contain at most one p-point and its p-level is 0. Therefore $(u^{2\hat{a}}, u^{2\hat{a}+\hat{n}}) \setminus A_{2a+n}$ can also contain at most one p-point and its p-level is 0. By Proposition B.9, $[u^{-\hat{n}}, u^{2\hat{a}+\hat{n}}]$ is either a p-symmetric arc, or a basic quasi-p-symmetric arc, see Figure 17. Let us denote by A_n the arc-component of ℓ^n containing $u^{-\hat{n}}$. Then $(u^{-\hat{n}}, u^0) \setminus A_n$ also does not contain any p-point with non-zero p-level.

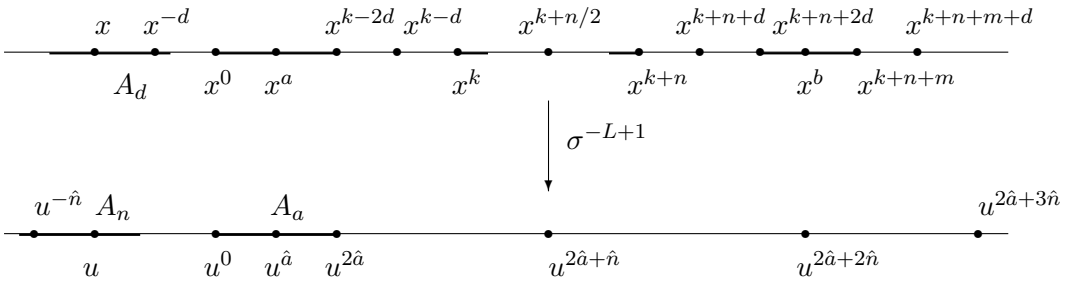

FIGURE 17. The configuration of points on $[x^{-d}, x^{k+n+m+2d}]$ and their images under σ^{-L+1} as in (ii).

Assume by contradiction that x^{-d} is not the midpoint of its arc-component A_d. Let us denote the midpoint of A_d by x, and let $u := \sigma^{-L+1}(x)$. Since $L_p(x) > L_p(x^a)$, also $L_p(u) > L_p(u^{\hat{a}})$. Let ℓ^a be the link which contains $u^{\hat{a}}$, and let A_a be the arc-component of ℓ^a containing $u^{\hat{a}}$. Then $u \in A_n$ and $[u^{-\hat{n}}, u^{2\hat{a}+\hat{n}}]$ is basic quasi-p-symmetric. But, since $u^{2\hat{a}+\hat{n}} \in \ell^n$ and $\sigma^{L-1}(u^{2\hat{a}+\hat{n}}) = x^{k+n/2}$, $x^{k+n/2} \in \ell^d$. Since the arc $[x, x^{k-d}]$ is

quasi-p-symmetric, $[x^{k-d}, x^{k+n/2}]$ is also quasi-p-symmetric and $L_p(x^a) > L_p(x^{k-d})$ implies $L_p(x^{k-d}) > L_p(x^{k+n/2})$, a contradiction.

Let us assume now that $\sigma^{-L+1}(A)$ is contained in a single link. Since $L_p(u) > L_p(u^{\hat{a}})$ and $L_p(u^0) = 1$, we have $\pi_p([u, u^0]) \subset \pi_p([u^{\hat{a}}, u^0])$. Then $\sigma^{L-1}([u^{\hat{a}}, u^0]) \subset \ell$ implies $\sigma^{L-1}([u, u^{\hat{a}}]) \subset \ell$ and hence $[x^{-d}, x^{k-d}] \subset \ell$, a contradiction.

These two contradictions prove the claim.

In the same way we can prove that $x^{k+n+m+d}$ is the midpoint of its arc-component, and by Proposition B.11 the arc $[u^{2\hat{a}+\hat{n}}, u^{2\hat{a}+3\hat{n}}]$ is either p-symmetric, or quasi-p-symmetric.

So we have proved that the arcs $[u^{-\hat{n}}, u^{2\hat{a}+\hat{n}}]$ and $[u^{2\hat{a}+\hat{n}}, u^{2\hat{a}+3\hat{n}}]$ are both either p-symmetric, or quasi-p-symmetric. Since $[x^a, x^b] = \sigma^{L-1}([u^{\hat{a}}, u^{2\hat{a}+2\hat{n}}])$ is quasi-p-symmetric, the arcs $\sigma^{L-1}([u^{-\hat{n}}, u^{2\hat{a}+\hat{n}}])$ and $\sigma^{L-1}([u^{2\hat{a}+\hat{n}}, u^{2\hat{a}+3\hat{n}}])$ are both either p-symmetric, or quasi-p-symmetric. This implies that $[x^{-2d-n/2}, x^{k+n/2}]$ and $[x^{k+n/2}, x^{k+n+m+2d+n/2}]$ are contained in the decreasing quasi-p-symmetric arc $[x^{-2d-n/2}, x^{k+n+m+2d+n/2}]$ containing A. □

Example C.4. (Example for (ii) of Proposition C.3.) Let us consider the Fibonacci map and the corresponding inverse limit space. The arc-component $\mathfrak{C}$ contains an arc $A = [x^0, x^{77}]$ with the following folding pattern:

$$1\ 9_1\ \underbrace{\mathbf{1}_2\ \mathbf{22\ 1\ 56\ 1\ 22\ 1}\ 9\ \mathbf{1}}_{\text{basic}}\ 4\ 1\ 0\ 2\ 0\ 1\ 0\ 3\ 0\ 1\ 6$$

$$\overbrace{\underbrace{\mathbf{1}_{22}\mathbf{14\ 1\ 35\ 1\ 14\ 1}\ 6\ \mathbf{1}}_{\text{basic}}\ 0\ 3\ 0\ 1\ 0\ 2\ 0\ 1\ 4\ \mathbf{1}\ 9\ \mathbf{1\ 22\ 1}\ 9\ \mathbf{1}\ 4\ 1\ 0\ 2\ 0\ 1\ 0\ 3\ 0\ \underbrace{\mathbf{1}\ 6\ \mathbf{1\ 14\ 1}_{60}}_{\text{basic}}}^{\text{quasi-}p\text{-symmetric}}$$

$$6\ 1\ 0\ 3\ 0\ 1\ 0\ 2\ 0\ 1\ 4\ \underbrace{1\ 9\ \mathbf{1}_{74}}_{\text{sym}}\ 4_{75}\ 1\ 0$$

We can choose a chain $\mathcal{C}_p$ such that p-points with p-levels 1, 14, 22, 35 and 56 belong to the same link. Then the arc $[x^{22}, x^{60}]$ is quasi-p-symmetric, and it is not basic. The arc $\sigma^{-13}([x^{22}, x^{60}])$ is basic quasi-p-symmetric with the folding pattern 1 22 1 9 1. So we can apply Propositions B.9 and B.11 as in the above proof. The arc $[x^2, x^{74}]$ is decreasing quasi-p-symmetric. Note that the arc $[x^1, x^{75}]$ is not p-link-symmetric.

Definition C.5. An arc $A = [x, y]$ is called *maximal decreasing (basic) quasi-p-symmetric* if it is decreasing (basic) quasi-p-symmetric and there is no decreasing (basic) quasi-p-symmetric arc $B \supset A$ that consists of more (basic) quasi-p-symmetric arcs than A.

Similarly we define a *maximal increasing (basic) quasi-p-symmetric* arc.

Remark C.6. (a) Propositions B.9 and B.11 imply that $A = [x, y]$ is a maximal decreasing basic quasi-p-symmetric arc if and only if A is a decreasing basic quasi-p-symmetric and for $x = x^0, x^1, \dots, x^{n-1}, x^n = y$ which satisfy (i) of Definition C.1, there exists a point x^{-1} such that $[x^{-1}, x^1]$ is p-symmetric with midpoint x^0 and x^n is not a p-point. The arc $[x^{-1}, x^n]$ we call the *extended* maximal decreasing basic quasi-p-symmetric arc. The points x^{-1}, $x = x^0, x^1, \dots, x^{n-1}, x^n = y$ we call the *nodes* of $[x^{-1}, x^n]$.

The analogous statement holds if A is a maximal increasing basic quasi-p-symmetric arc: If $A = [x^0, x^{n+1}]$ is an extended maximal increasing basic quasi-p symmetric arc, then x^0 is not a p-point, $L_p(x^n) > L_p(z)$ for every p-point $z \in A$, $z \neq x^n$, and $L_p(x^{n-1}) = L_p(x^{n+1})$.

(b) Let $A = [x^0, x^{n+1}]$ be an extended maximal increasing basic quasi-p symmetric arc. If there exists an additional p-point x^{n+2} such that the arc $[x^n, x^{n+2}]$ is quasi-p symmetric with midpoint x^{n+1}, Propositions B.9 and B.11 imply that A is contained in an p-symmetric arc $B = [x^0, x^{2n}]$ where the arc $[x^{n-1}, x^{2n}]$ is an extended maximal decreasing basic quasi-p-symmetric arc.

The analogous statement holds if A is a maximal decreasing basic quasi-p-symmetric arc.

Lemma C.7. Every (basic) quasi-p-symmetric arc A can be extended to a maximal decreasing/increasing (basic) quasi-p-symmetric arc $B \supset A$.

Proof. We take the largest decreasing (basic) quasi-p-symmetric arc B containing A. The only thing to prove is that there really is a largest B. If this were not the case, then there would be an infinite sequence $(x_i)_{i \geqslant 0}$ with $x_0 \in \partial A$, $L_p(x_i) < L_p(x_{i+1})$ and $[x_i, x_{i+2}]$ is a (basic) quasi-p-symmetric arc for all $i \geqslant 0$. By the definition of (basic) quasi-p-symmetric arc, there are two links ℓ and $\hat{\ell}$ containing x_i for all even i and odd i respectively. (Note that $\ell = \hat{\ell}$ is possible.) By Lemma B.3 for the basic case, the p-points in $\bigcup_{i \geqslant 0}[x_0, x_i] \setminus (\ell \cup \hat{\ell})$ can only have finitely many different p-levels. By the construction in the proof of Proposition C.3 (ii), the same conclusion is true for the non-basic case as well. But $\bigcup_{i \geqslant 0}[x_0, x_i]$ is a ray, and contains p-points of all (sufficiently high) p-levels. Since the closure of $\pi_p(\{x : L_p(x) \geqslant N\})$ contains $\omega(c)$ for all N, this set is not contained in the π_p-images of the two links ℓ and $\hat{\ell}$ only. So we have a contradiction. □

Proposition C.8. Let A be a p-link-symmetric arc with midpoint m and $\partial A = \{x, y\} \subset E_p$. Then A is p-symmetric, or is contained in an extended maximal decreasing/increasing

(basic) quasi-p-symmetric arc, or is contained in a p-symmetric arc which is the concatenation of two arcs, one of which is a maximal increasing (basic) quasi-p-symmetric arc, and the other one is a maximal decreasing (basic) quasi-p-symmetric arc.

Proof. Let $A \cap E_p = \{x^{-k'}, \dots, x^{-1}, x^0, x^1, \dots x^k\}$ and $x^0 = m$. Without loss of generality we assume that $x^{-k'}$ and x^k are the midpoints of the link-tips of A. If $L_p(x^{-i}) = L_p(x^i)$, for $i = 1, \dots, \min\{k', k\}$, then the arc A is either p-symmetric, or (basic) quasi-p-symmetric. Hence in this case the theorem is true.

Let us assume that there exists $j < \min\{k', k\}$ such that $L_p(x^{-i}) = L_p(x^i)$, for $i = 1, \dots, j-1$, and $L_p(x^{-j}) \neq L_p(x^j)$. The arc $[x^{-j}, x^j]$ is (basic) quasi-p-symmetric and by Lemma C.7 and Remark C.6, there exists an extended maximal decreasing/increasing (basic) quasi-p-symmetric arc which contains $[x^{-j}, x^j]$. Hence in this case the theorem is also true. □

References

[1] A. Avila, M. Lyubich, W. de Melo, *Regular or stochastic dynamics in real analytic families of unimodal maps*, Invent. Math. **154** (2003) 451–550.

[2] M. Barge, K. Brucks, B. Diamond, *Self-similarity in inverse limit spaces of the tent family*, Proc. Amer. Math. Soc. **124** (1996) 3563–3570.

[3] M. Barge, H. Bruin, S. Štimac, *The Ingram Conjecture*, Geometry and Topology **16** (2012) 2481–2516.

[4] L. Block, S. Jakimovik, J. Keesling, L. Kailhofer, *On the classification of inverse limits of tent maps*, Fund. Math. **187** (2005) 171–192.

[5] K. Brucks, H. Bruin, *Subcontinua of inverse limit spaces of unimodal maps*, Fund. Math. **160** (1999) 219–246.

[6] K. Brucks, H. Bruin, *Topics in one-dimensional dynamics*, London Math. Soc. Student texts **62** Cambridge University Press 2004.

[7] H. Bruin, *Combinatorics of the kneading map*, Int. Jour. of Bifur. and Chaos **5** (1995) 1339–1349.

[8] H. Bruin, *Topological conditions for the existence of absorbing Cantor sets*, Trans. Amer. Math. Soc. **350** (1998) 2229–2263.

[9] H. Bruin, *Quasi-symmetry of conjugacies between interval maps*, Nonlinearity **9** (1996) 1191–1207.

[10] H. Bruin, *Subcontinua of Fibonacci-like unimodal inverse limit spaces*, Topology Proceedings **31** (2007) 37–50.

[11] H. Bruin, *(Non)invertibility of Fibonacci-like unimodal maps restricted to their critical omega-limit sets*, Topology Proceedings **37** (2011) 459–480.

[12] H. Bruin, G. Keller, T. Nowicki, S. van Strien, *Wild Cantor Attractors exist*, Ann. of Math. (2) **143** (1996) 97–130.

[13] H. Bruin, G. Keller, M. St.-Pierre, *Adding machines and wild attractors*, Ergod. Th. & Dynam. Sys. **17** (1997) 1267–1287.

[14] F. Hofbauer, *The topological entropy of a transformation $x \mapsto ax(1-x)$*, Monath. Math. **90** (1980) 117–141.

[15] F. Hofbauer, G. Keller, *Some remarks on recent results about S-unimodal maps,* Ann. Inst. Henri Poincaré, Physique théorique, **53** (1990) 413–425.

[16] L. Kailhofer, *A classification of inverse limit spaces of tent maps with periodic critical points*, Fund. Math. **177** (2003) 95–120.
[17] G. Keller, T. Nowicki, *Fibonacci maps re(al)-visited*, Ergod. Th. & Dynam. Sys. **15** (1995) 99–120.
[18] M. Lyubich, *Combinatorics, geometry and attractors of quasi-quadratic maps*, Ann. of Math. (2) **140** (1994) 347–404.
[19] M. Lyubich, J. Milnor, *The Fibonacci unimodal map*, J. Amer. Math. Soc. **6** (1993) 425–457.
[20] J. Milnor, *On the concept of attractor*, Commun. Math. Phys. **99** (1985) 177–195.
[21] B. Raines, S. Štimac, *A classification of inverse limit spacers of tent maps with non-recurrent critical point*, Algebraic and Geometric Topology **9** (2009) 1049–1088.
[22] S. Štimac, *Structure of inverse limit spaces of tent maps with finite critical orbit*, Fund. Math. 191 (2006) 125–150.
[23] S. Štimac, *A classification of inverse limit spaces of tent maps with finite critical orbit*, Topology Appl. **154** (2007) 2265–2281.

Faculty of Mathematics, University of Vienna,
Oskar Morgensternplatz 1, A-1090 Vienna, Austria
henk.bruin@univie.ac.at
http://www.mat.univie.ac.at/∼bruin

Department of Mathematics, University of Zagreb.
Bijenička 30, 10 000 Zagreb, Croatia
sonja@math.hr
http://www.math.hr/∼sonja