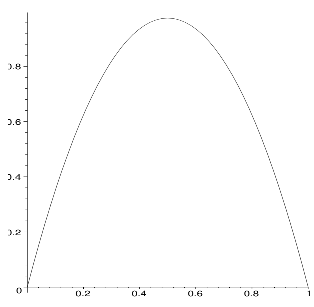
AN OVERVIEW OF UNIMODAL INVERSE LIMIT SPACES.

H. BRUIN

Abstract. An overview of unimodal inverse limit spaces inverse limit spaces, to support the mini-course “Interval dynamics and Inverse limit spaces”, at **IWCTA: International Workshop and Conference on Topology & Applications**, Rajagiri School of Engineering & Technology, Kochi, December 5-8 2018.

1. Introduction

Unimodal maps are maps of the interval with a single critical point, and increasing/decreasing at the left/right of the critical point. The best known examples are quadratic (logistic) maps and tent-maps, see Figure 1.



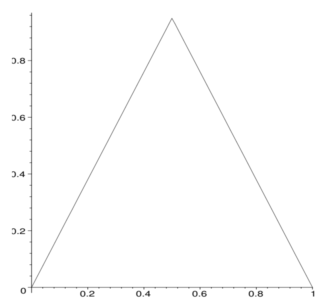


Figure 1. Unimodal maps: a quadratic map and a tent map.

They are among the simplest maps that, at least for some parameters, are chaotic in every sense that can be given to mathematical chaos. They are not invertible, however, and a simple way to make them invertible is by introducing a second coordinate, and **thicken** the map:

$$T_a : x \mapsto 1 - a|x|, \qquad L_{a,b} : (x, y) \mapsto 1 - a|x| + by, x),$$

$$Q_a : x \mapsto 1 - ax^2, \qquad H_{a,b} : (x, y) \mapsto (1 - ax^2 + by, x).$$

Date: December 7, 2022.

2000 *Mathematics Subject Classification.* 54H20, 37B45, 37E05.

Key words and phrases. inverse limit space, unimodal map, tent map, quadratic map, embeddings, endpoints, folding point, composant, Ingram conjecture.

In this way, the tent-map becomes a Lozi map and the quadratic map a Hénon map. Figure 2 gives a Lozi-attractor (resp. Hénon-attractor) obtained as $\cap_{n\geq 0} L^n_{a,b}(U)$ for some well-chosen, forward invariant open disk U. In order to understand the topology of such attractors, unimodal inverse limit spaces (UILs) are a first informative, but certainly not sufficient, step. In fact, all questions asked about UILs in these notes (and more!) can be asked about Lozi-attractors and Hénon-attractors.

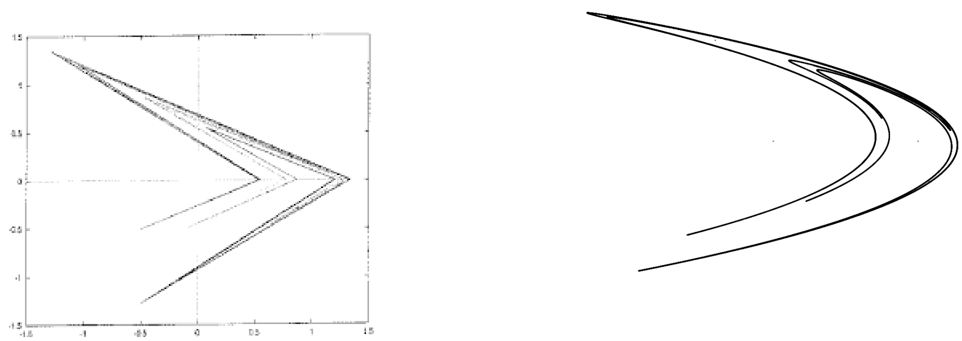


FIGURE 2. The Lozi and Hénon attractor

2. Definitions and Notation

Let $\mathbb{N} = \{1, 2, 3, \dots\}$ be the set of natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We consider two families of unimodal maps, the family of quadratic maps $Q_a : [0,1] \to [0,1]$, with $a \in [1,4]$, defined as $Q_a(x) = ax(1-x)$, and the family of tent maps $T_s : [0,1] \to [0,1]$ with slope $\pm s$, $s \in [1,2]$, defined as $T_s(x) = \min\{sx, s(1-x)\}$. Let f be a map from any of these two families. The **critical** or **turning** point is $c := 1/2$. Write $c_k := f^k(c)$. The closed f-invariant interval $[c_2, c_1]$ is called the **core**, and denoted as $\varprojlim([c_2, c_1], T)$.

The inverse limit space $\varprojlim([0,1], f)$ is the collection of all backward orbits

$$\{x = (\dots, x_{-2}, x_{-1}, x_0) : f(x_{-i-1}) = x_{-i} \in [0, c_1] \text{ for all } i \in \mathbb{N}_0\},$$

equipped with metric $d(x,y) = \sum_{i\leq 0} 2^i |x_i - y_i|$. The map f is called the **bonding map** of $\varprojlim([0,1], f)$. We define the **induced**, or **shift homeomorphism**on $\varprojlim([0,1], f)$ as

$$\sigma(x) := \sigma_f(\dots, x_{-2}, x_{-1}, x_0) = (\dots, x_{-2}, x_{-1}, x_0, f(x_0)).$$

Let $\pi_i : \varprojlim([0,1], f) \to [0, c_1]$, $\pi_i(x) = x_{-i}$ be the i-th projection map.

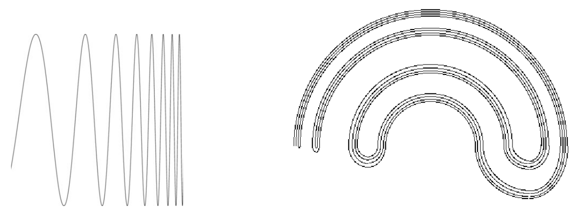


FIGURE 3. The $\sin\frac{1}{x}$ continuum and the Knaster continuum.

Simple examples of such unimodal inverse limit spaces are the $\sin\frac{1}{x}$-continuum and the Knaster continuum (bucket handle) shown in Figure 4.

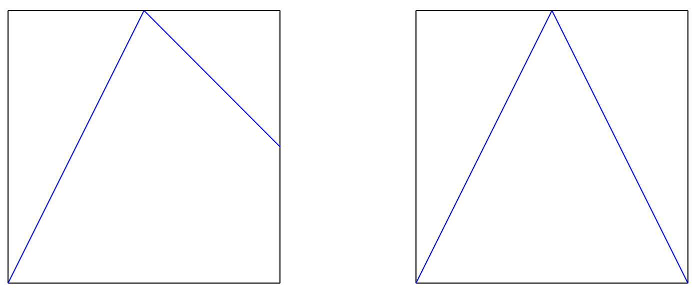


FIGURE 4. Maps with $\sin\frac{1}{x}$ continuum and the Knaster continuum as inverse limit spaces

The similarity between a Hénon attractors and the Knaster continuum may suggest that inverse limit spaces are homeomorphic to Hénon attractors in some generality, but in fact, the generality is very limited.

Theorem 2.1 (Barge & Holte [10])**.** *If a is such that 0 is a periodic for $Q_a(x) = 1 - ax^2$, then for $|b|$ sufficiently small, then the attractor of $H_{a,b}$ and the inverse limit space of Q_a are homeomorphic.*

Barge[5] on the other hand, showed that under fairly general assumptions, Hénon attractors (and homoclinic tangle emerging from a homoclinic bifurcations) are homeomorphic to unimodal inverse limit spaces, not even if you allow varying bonding maps.

Usually, the whole UIL is decomposable: For the case $c \leq c_1$ it follows from Bennett's Theorem in [12] that we can decompose $\varprojlim([0,1], T_s) = \varprojlim([c_2, c_1], T_s) \cup \mathfrak{C}$, where $\bar{0} := (\dots, 0, 0, 0) \in \mathfrak{C}$ is a continuous image of $[0, \infty)$ (called **zero-composant**) which compactifies on $\varprojlim([c_2, c_1], T_s)$. Inverse limit space of tent map $\varprojlim([c_2, c_1], T_s)$ obtained from the forward invariant interval $[c_2, c_1]$ is called the **core** of the UIL.

2.1. **Chainability.**

Definition 2.2. *Let X be a metric space. A* **chain in** X *is a set* $\mathcal{C} = \{\ell_1 \dots, \ell_n\}$ *of open subsets of X called* **links**, *such that* $\ell_i \cap \ell_j \neq \emptyset$ *if and only if* $|i - j| \leq 1$.

The **mesh of a chain** $\mathcal{C}$ *is defined as* $mesh(\mathcal{C}) = \max\{\operatorname{diam} \ell_i : i = 1, \dots, n\}$. *A space X is* **chainable** *if there exists chain covers of X of arbitrarily small mesh.*

A corollary of X being chainable is that X contains no **triods** (homeomorphic copies of the letter Y), or circles. All unimodal inverse limit spaces are chainable, and all chainable continua can be **embedded** in the plane, i.e., there is a continuous injection $h : X \to \mathbb{R}^2$ (called **embedding**) such that $h(X)$ and X are homeomorphic. They also possess the **fixed point property**: every continuous map $f : X \to X$ has a fixed point.

Definition 2.3. *A point $a \in X \subset \mathbb{R}^2$ is* **accessible** *if there exists an arc $A = [x, y] \subset \mathbb{R}^2$ such that $a = x$ and $A \cap X = \{a\}$.*

Unimodal inverse limit spaces can therefore be embedded in the plane, but in general there are many (in fact uncountably non-homotopic) ways to do so. There are two **standard ways** that yield an embedding very much like the Lozi-attractor (or Hénon-attractor) with $b > 0$ (orientation reversing, making the composant $\mathfrak{R}$ of the fixed point $p = (\dots, r, r, r)$ accessible, see [19]) and $b < 0$ (orientation preserving, making the zero-composant accessible, see [17]) respectively.

The result of Anušić et al. gives an idea how much variety there is in embeddings:

Theorem 2.4 ([1]). *For every point $a \in X$ there exists an embedding of X in the plane such that a is accessible.*

2.2. **Symbolic dynamics.** We can extend the Milnor-Thurston [28] **kneading theory** to UILs, as done originally in [17]. The symbolic itinerary of the critical value $c_1 \in [0, 1]$ under the action of T is called the **kneading sequence**, and we denote it as $\nu = \nu_1\nu_1\nu_3\dots$, where $\nu_i = 0$ if $c_i < c$ and $\nu_i = 1$ if $c_i > c$. Analogously, to each $x \in \varprojlim([0, 1], T)$, we can assign a symbolic sequence $\overleftarrow{x}\,.\,\overrightarrow{x} = \dots s_{-1} \in \{0, \frac{0}{1}, 1\}^{-\mathbb{N}}$ where

$$s_{-i} = \begin{cases} 0 & \pi_i(x) < c, \\ \frac{0}{1} & \pi_i(x) = c, \\ 1 & \pi_i(x) > c, \end{cases} \qquad i \geq 0.$$

Here $\frac{0}{1}$ means that both 0 and 1 are assigned to x. If c is non-periodic, this can happen only once, i.e., to every point we assign at most two symbolic itineraries. If c is periodic, say of period n, then we need to make a consistent choice, usually such hat $s_{i+1} \dots s_{i+n}$ contains an even number of 1s.

For a fixed left-infinite sequence $s = \dots s_{-2}s_{-1}s_0 \in \{0,1\}^{\mathbb{N}_0}$, the subset

$$A(s) := \overline{\{x \in X : \overleftarrow{s} \in \overleftarrow{x}\}}$$

of X is called a **basic arc**. It can be shown that $A(\overleftarrow{x})$ is the maximal closed arc A containing x such that $\pi_0 : A \to I$ is injective. In [19, Lemma 1] it was observed that $A(\overleftarrow{x})$ is indeed an arc (but it can be degenerate, i.e., a single point).

For every basic arc $A(\overleftarrow{x})$ we define

$$\begin{aligned} N_L(\overleftarrow{x}) &:= \{n > 1 : s_{-(n-1)} \dots s_{-1} = \nu_1\nu_2 \dots \nu_{n-1}, \#_1(\nu_1 \dots \nu_{n-1}) \text{ odd}\}, \\ N_R(\overleftarrow{x}) &:= \{n \geq 1 : s_{-(n-1)} \dots s_{-1} = \nu_1\nu_2 \dots \nu_{n-1}, \#_1(\nu_1 \dots \nu_{n-1}) \text{ even}\}. \end{aligned}$$

and

$$\tau_L(\overleftarrow{x}) := \sup N_L(\overleftarrow{x}) \quad \text{and} \quad \tau_R(\overleftarrow{x}) := \sup N_R(\overleftarrow{x}).$$

We can construct a model of the inverse limit space $\varprojlim([0,1], f)$ by gluing basic arcs $A(\overleftarrow{x})$ to $A(\overleftarrow{y})$ at their left (resp. right) endpoints if and only if $\overleftarrow{x}$ and $\overleftarrow{y}$ agree up to one index, and this index is exactly $\tau_L(\overleftarrow{x}) = \tau_L(\overleftarrow{y})$ (resp. $\tau_R(\overleftarrow{x}) = \tau_R(\overleftarrow{y})$).

3. Endpoints and Folding Points

Definition 3.1. *A point x in a chainable continuum is called* **endpoint** *if for every two subcontinua $A, B \subset X$, $A \subset B$ or $B \subset A$. We denote the set of endpoints by $\mathcal{E}$.*

As an example, $X = [0,1]$ has endpoints 0 and 1 according to this definition. But the triod would have four endpoints (the branch point too!), which speaks against our intuition. Therefore we required X to be chainable.

A geometric description of endpoints (using the notion of **crooked** graphs is due to Barge & Martin [11]. Here we give a symbolic classification of endpoints, following [19, Section 2].

Lemma 3.2. *([19], Lemmas 2 and 3) If $A(\overleftarrow{x}) \in \{0,1\}^{\mathbb{N}}$ is such that $\tau_L(\overleftarrow{x}), \tau_R(\overleftarrow{x}) < \infty$, then*

$$\pi_0(A(\overleftarrow{x})) = [T^{\tau_L(\overleftarrow{x})}(c), T^{\tau_R(\overleftarrow{x})}(c)].$$

Without the restriction that $\tau_L(\overleftarrow{x}), \tau_R(\overleftarrow{x}) < \infty$, we have

$$\begin{aligned} \sup \pi_0(A(\overleftarrow{x})) &= \inf\{c_n : n \in N_R(\overleftarrow{x})\}, \\ \inf \pi_0(A(\overleftarrow{x})) &= \sup\{c_n : n \in N_L(\overleftarrow{x})\}. \end{aligned}$$

This gives the following symbolic characterization of endpoints.

Proposition 3.3. [19, Proposition 2] *A point $x \in X$ such that $\pi_i(x) \neq c$ for every $i < 0$ is an endpoint of X if and only if $\tau_L(\overleftarrow{x}) = \infty$ and $\pi_0(x) = \inf \pi_0(A(\overleftarrow{x}))$ or $\tau_R(\overleftarrow{x}) = \infty$ and $\pi_0(x) = \sup \pi_0(A(\overleftarrow{x}))$.*

Definition 3.4. *A* **folding point** *in the core of a unimodal inverse limit is any point that does not have a neighborhood homeomorphic to a Cantor set of open arcs. We denote this set by $\mathcal{F}$.*

The omega-limit set of a point is defined as the set of adherence points of its forward orbit:

$$\omega(x) = \{y : \exists\, n_i \to \infty\ T^{n_i}(x) \to y\} = \cap_{j \in \mathbb{N}} \overline{\cup_{i>j} \{T^j(x)\}}.$$

The following characterization of folding points is due to Raines.

Proposition 3.5 (Theorem 2.2 in [30]). *A point $x \in \varprojlim([c_2, c_1], T)$ is a folding point if and only if $\pi_n(x)$ belongs to $\omega(c)$ for every $n \in \mathbb{N}$.*

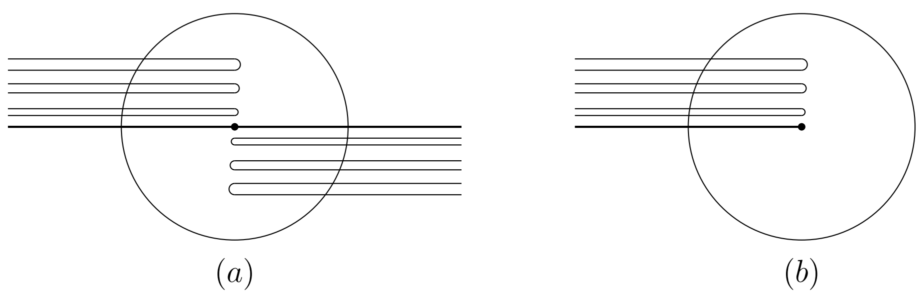


FIGURE 5. Neighborhoods of non-end folding point (a) and an endpoint (b).

Theorem 3.6. *The core $\varprojlim([c_2, c_1], T)$ contains exactly N endpoints if and only if c is periodic of period N.*

The core $\varprojlim([c_2, c_1], T)$ contains exactly N non-end folding points if and only if c is **pre***periodic of period N.*

Proof. This is a special case of the theory developed above (Proposition 3.3). □

Theorem 3.7. *If c is not recurrent, then the core $\varprojlim([c_2, c_1], T)$ contains no end-points, but folding points do exist.*

Proof. Since $\omega(c) \neq \emptyset$, there must be folding points, see Proposition 3.5. But there can't be any endpoints, because every backward itinerary $\overleftarrow{x}$ has $N_L(\overleftarrow{x}), N_L(\overleftarrow{x}) < \infty$, so Proposition 3.3 applies. $\square$

The following proposition follows implicitly from the proof of Corollary 2 in [19]. It shows that if c is recurrent, then $\#(\mathcal{E} \cap \varprojlim([c_2, c_1], T)) = N \in \mathbb{N}$ if and only if c is N-periodic, and otherwise $\mathcal{E} \cap \varprojlim([c_2, c_1], T)$ is uncountable. We prove it here for completeness.

Proposition 3.8. *If orb(c) is infinite and c is recurrent, then the core inverse limit space X' has uncountably many endpoints. Moreover, $\mathcal{E}$ has no isolated points and is dense in $\mathcal{F}$.*

Proof. Since c is recurrent, for every $k \in \mathbb{N}$ there exist infinitely many $n \in \mathbb{N}$ such that $\nu_1 \dots \nu_n = \nu_1 \dots \nu_{n-k} \nu_1 \dots \nu_k$.

Take a sequence $(n_j)_{j\in\mathbb{N}}$ such that $\nu_1 \dots \nu_{n_{j+1}} = \nu_1 \dots \nu_{n_{j+1}-n_j} \nu_1 \dots \nu_{n_j}$ for every $j \in \mathbb{N}$. Then the basic arc given by the itinerary

$$\overleftarrow{x} := \lim_{j\to\infty} \nu_1 \dots \nu_{n_j},$$

is admissible and $\tau_L(\overleftarrow{x}) = \infty$ or $\tau_R(\overleftarrow{x}) = \infty$. Therefore, $A(\overleftarrow{x})$ contains an endpoint. Note that, since ν is not periodic, $\overleftarrow{x}$ is also not periodic and thus $\sigma^k(\overleftarrow{x}) \neq \overleftarrow{x}$ for every $k \in \mathbb{N}$.

To determine the cardinality of endpoints, we claim that for every fixed $n \in \mathbb{N}$ there are $m_2 > m_1 > n$ such that

$$\nu_1 \dots \nu_{m_2} = \nu_1 \dots \nu_{m_2-n} \nu_1 \dots \nu_n, \quad \nu_1 \dots \nu_{m_1} = \nu_1 \dots \nu_{m_1-n} \nu_1 \dots \nu_n,$$

but $\nu_1 \dots \nu_{m_1}$ is not a suffix of $\nu_1 \dots \nu_{m_2}$. Indeed, if m_2 didn't exist, then

$$\overleftarrow{x} = (\nu_1 \dots \nu_{m_1-n})^{-\infty} \nu_1 \dots \nu_n$$

would have a periodic tail. Since c is not periodic, no end-point can have such a tail.

We conclude that for every n_j there are at least two choices of n_{j+1} such that the corresponding tails $\overleftarrow{x}$ are different, and have $\#N_L(\overleftarrow{x}) \cup N_R(\overleftarrow{x}) = \infty$. It follows that there are uncountably many basic arcs containing at least one endpoint.

To show that $\mathcal{E}$ contains no isolated points and is in fact dense in $\mathcal{F}$, take any folding point x with **two-sided** itinerary $\dots s_{-2}s_{-1}s_0.s_1s_2\dots$ Then, for every $k \in \mathbb{N}$, there exists

$n \in \mathbb{N}$ such that $s_{-k} \dots s_k = \nu_n \dots \nu_{n+2k}$. Using the arguments as above, we can find a basic arc with itinerary $\overleftarrow{y} = \dots \nu_1 \dots \nu_{n-1}\nu_n \dots \nu_{n+2k}$ and such that $\tau_L(\overleftarrow{y}) = \infty$ or $\tau_R(\overleftarrow{y}) = \infty$. So $\sigma^{-k}(\overleftarrow{y})$ contains an endpoint with itinerary $\dots \nu_n \dots \nu_{n+k} . \nu_{n+k+1} \dots \nu_{n+2k} \dots$ Since $k \in \mathbb{N}$ was arbitrary, we conclude that there is some (in fact, uncountably many) endpoints arbitrarily close to x. □

The following result about comparing endpoints with folding points is due to [3]. We first need a definition, going back to Blokh & Lyubich [15]

Definition 3.9. *The critical point c is* reluctantly recurrent *if there is $\varepsilon > 0$ and an arbitrary long (but finite!) backward orbit $\bar{y} = (y_{-m}, \dots, y_{-1}, y_0)$ in $\omega(c)$ such that the ε-neighborhood of $y \in I$ has monotone pull-back along $\bar{y}$. Otherwise, c is* persistently recurrent.

Theorem 3.10. *In an UIL, $\mathcal{F} = \mathcal{E}$ if and only if c is persistently recurrent.*

4. Composants

Definition 4.1. *Let X be a continuum and $x \in X$. The* **arc-component** *$A(x)$ of x is the union of points y such that there is an arc in X connecting x and y. The* **composant** *$C(x)$ of a point x is the union of all proper subcontinua of X.*

For example, if $X = [0,1]$ then $A(0) = [0,1]$ but $C(0) = [0,1)$ (it doesn't contain 1 because $[0,1]$ is not a **proper** subcontinuum of X). Also $A(\frac{1}{2}) = C(\frac{1}{2}) = [0,1]$ because $[0,1] = [0, \frac{1}{2}] \cup [\frac{1}{2}, 1]$.

Two arc-components A and $\tilde{A}$ are asymptotic if there are parametrizations

$$\varphi, \tilde{\varphi} : \mathbb{R} \to A, \tilde{A} \qquad \text{such that} \qquad \lim_{t \to \infty} d(\varphi(t), \tilde{\varphi}(t)) = 0.$$

The trivial case when $A = \tilde{A}$ is excluded, but A is **self-asymptotic** if there is a parametrization φ such that

$$\lim_{t \to \infty} d(\varphi(t), \tilde{\varphi}(-t)) = 0.$$

Figure 6 give the UIL of a tent map with $T^3(c) = c$, for which the fixed composant $\mathfrak{R}$ is self-asymptotic. There is a single infinite Wada channel for which the entire shore is equal to $\mathfrak{R}$.

Theorem 4.2 (Barge, Diamond & Holton [9])**.** *Every UIL with periodic critical point has at least one asymptotic arc-component.*

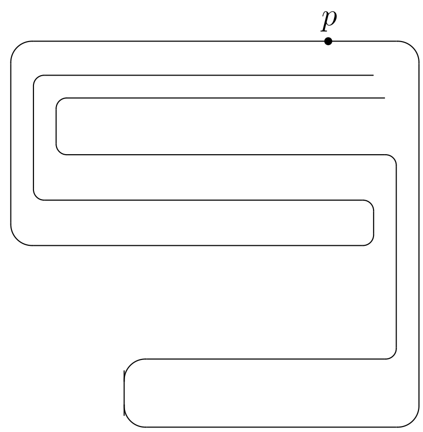


Figure 6. This representation has a single infinite Wada channel.

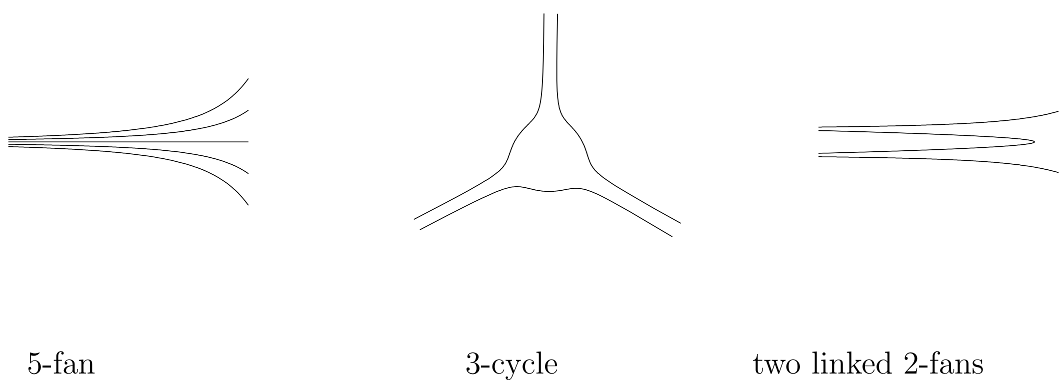


Figure 7. Configurations of asymptotic arc-components.

Proof. The proof relies on substitution tilings and the fact that these spaces act as 2-to-1 coverings of inverse limit spaces. In fact, if the period is N, then there are at least $N-1$ and at most $2(N-1)$ "halves" of arc-components asymptotic to some other "halves" of an arc-components. $\square$

Conjecture 4.3. *The upper bound is in fact* $2(N-2)$*. Given any two "halves" of arc-components* H *and* H'*,* H *is asymptotic to or coincides with* $\sigma^n(H')$ *for some* $n \in \mathbb{Z}$*.*

Question 4.4. *If c is non-recurrent, then there are no asymptotic ac-components, see* [20]*, but what is the situation of asymptotic arc-components when c is non-periodic but recurrent?*

In the Knaster continuum it was shown by Bandt [4] that every two arc-components without not containing the endpoint are homeomorphic. More generally, De Man [27] showed that every two arc-components inside any two one-dimensional solenoids are homeomorphic. (A solenoid is the inverse limit space of circles where the bonding maps are degree $n_i \geq 2$ covering maps of the circle as bonding maps f_i.)

Question 4.5. *Given two arc-components without endpoints, are they homeomorphic? In particular, can a self-asymptotic arc-component be homeomorphic to a non-self-asymptotic arc-component?*

In contrast, Fokkink (in his thesis and in [24]) showed that among all **matchbox manifolds** (i.e., continua that locally look like Cantor set of open arcs) there are uncountably many non-homeomorphic arc-components.

Question 4.6. *Are two lines with irrational slopes wrapping for ever around the torus be homeomorphic as spaces?*

This question is due to Aarts almost half a century ago, but beyond the fact that if the slopes θ and θ' have continued fraction expansion with the same tail then the lines are indeed homeomorphic, nothing is known.

Below, we gave a full list (taken from [20]) of what configurations asymptotic arc-components are possible for periodic kneading sequences

	ν	type	periodic tail(s)	k	Case
1	101	1-cycle	1	2	I
2	1001	3-fan	101	3	I
3	10001	4-fan	1001	4	I
4	10010	3-cycle	101	3	II
5	10111	three 2-fans	101110	3	III
6	100001	5-fan	10001	5	I
7	100010	4-cycle	1001	4	II
8	100111	four 2-fans	10010011	4	III
9	101110	two linked 3-fans	$10, 1$	4	II, IV
10	1000001	6-fan	100001	6	I
11	1000010	5-cycle	10001	5	II
12	1000111	five 2-fans	1000100011	5	III
13	1000100	four 2-fans (l.i.p.)	$10, 1001$	4	II
14	1001101	four 2-fans	10011010	4	III
15	1001110	five 2-fans	$10010, 10111$	5	II
16	1001011	five 2-fans	1001011011	5	III
17	1011010	5-cycle	10111	5	II
18	1011111	five 2-fans	1011111110	5	III
19	10000001	7-fan	1000001	7	I
20	10000010	6-cycle	100001	6	II
21	10000111	six 2-fans	100001110000	6	III
22	10000100	five 2-fans	$10001, 10010$	5	II
23	10001101	five 2-fans	1000110100	5	III
24	10001110	six 2-fans	$100010, 100111$	6	II
25	10001011	six 2-fans	100010110011	6	III
26	10011010	six 2-fans (l.i.p.)	$101, 100111$	6	II
27	10011111	six 2-fans	100111110110	6	III
28	10011100	five 2-fans	$10010, 10111$	5	II
29	10010101	five 2-fans	1001010111	5	III
30	10010110	six 2-fans (l.i.p.)	$100, 101110$	6	II
31	10110111	three 3-cycles	101101110	3	III
32	10111110	two linked 4-fans	$101110, 1$	5	II, IV
					(l.i.p = linked in pairs.)

5. Ingram conjecture

In the early 90's a classification problem that became known as the **Ingram Conjecture** was posed:

> If $1 \leq s \leq \tilde{s} \leq 2$ then the inverse limit spaces $\varprojlim([0,1], T_s)$ and $\varprojlim([0,1], T_{\tilde{s}})$ are not homeomorphic.

In the "Continua with the Houston problem book" in 1995 [25, page 257], Ingram writes

> The [...] question was asked of the author by Stu Baldwin at the [...] summer meeting of the AMS at Orono, Maine, in 1991. ... There is a related question which the author has considered to be of interest for several years. He posed it at a problem session at the 1992 Spring Topology Conference in Charlotte for the special case (that the critical point has period) $n = 5$.

After partial results [7, 13, 18, 26, 31, 33, 34], the Ingram Conjecture was finally answered in affirmative by Barge, Bruin & Štimac in [6]. In addition (Bruin & Štimac [21]):

Proposition 5.1 (Rigidity)**.** *If $h : \varprojlim([0,1], T) \to \varprojlim([0,1], T)$ is a homeomorphism, then it is isotopic σ^n for some $n \in \mathbb{Z}$. In fact, if $\omega(c) = [c_1, c_2]$, then $h|_{\varprojlim([c_2,c_1],T)} = \sigma^n$.*

However, the proof presented in [6] crucially depends on using the zero-composant $\mathfrak{C}$, so the core version of the Ingram Conjecture still remains open. For Hénon maps, $\mathfrak{C}$ plays the role of the unstable manifold of the saddle point outside the Hénon attractor; it compactifies on the attractor, but it is somewhat unsatisfactory to have to use this (and the embedding in the plane that it presupposes) for the topological classification. It is not possible to derive the core version directly from the non-core version, because it is impossible to reconstruct $\mathfrak{C}$ from the core. This is for instance illustrated by the work of Minc [29] showing that in general there are many non-equivalent rays compactifying on the Knaster bucket handle.

Question 5.2. *Does the Core Ingram Conjecture hold? And the core rigidity proposition?*

Partial results here are by [26, 33] (because their proofs work without the zero-composant), and [2, 22, 23]. In short, the Core Ingram Conjecture holds if c is (pre)periodic or non-recurrent, or is persistently recurrent with so-called "Fibonacci-like" combinatorics, but all other cases remain unproved.

Question 5.3. *Does the Ingram Conjecture hold in the multimodal setting, e.g. for cubic maps?*

References

[1] A. Anušić, H. Bruin, Činč, *Uncountably many planar embeddings of unimodal inverse limit spaces,* Discrete Contin. Dyn. Syst. **37** (2017), 2285–2300.
[2] A. Anušić, H. Bruin, Činč, *The core Ingram conjecture for non-recurrent critical points,* Fund. Math. **241** (2018), no. 3, 209–235.
[3] L. Alvin, A. Anušić, H. Bruin, Činč, *Folding points of unimodal inverse limit spaces,* Nonlinearity **33** (2020), 224–248.
[4] C. Bandt, *Composants of the horseshoe,* Fund. Math. **144** (1994), 231–241. and Erratum to the paper: "Composants of the horseshoe". Fund. Math. **146** (1995), 313.
[5] M. Barge, *Homoclinic intersections and indecomposability,* Proc. Amer. Math. Soc. **101** (1987), 541-544.
[6] M. Barge, H. Bruin, S. Štimac, *The Ingram Conjecture,* Geom. Topol. **16** (2012), 2481-2516.
[7] M. Barge, B. Diamond, *Inverse limit spaces of infinitely renormalizable maps*, Topology Appl. **83** (1998), 103–108.
[8] M. Barge, B. Diamond, *Inverse limit spaces of infinitely renormalizable maps*, Topology Appl. **83** (1998), 103–108.
[9] M. Barge, B. Diamond, C. Holton, *Asymptotic orbits of primitive substitutions,* Theoret. Comput. Sci. **301** (2003), 439–450.
[10] M. Barge, S. Holte, *Nearly one-dimensional Hénon attractors and inverse limits*, Nonlinearity **8** (1995), 29-42.
[11] M. Barge, J. Martin, *Endpoints of inverse limit spaces and dynamics*, Continua (Cincinnati, OH, 1994), volume **170** of Lecture Notes in Pure and Appl. Math., 165–182. Dekker, New York, 1995.
[12] R. Bennett, *On Inverse Limit Sequences*, Master's Thesis, University of Tennessee, 1962.
[13] L. Block, S. Jakimovik, J. Keesling, L. Kailhofer, *On the classification of inverse limits of tent maps*, Fund. Math. **187** (2005), no. 2, 171–192.
[14] L. Block, J. Keesling, B. E. Raines, S. Štimac, *Homeomorphisms of unimodal inverse limit spaces with a non-recurrent critical point*, Topology Appl. **156** (2009), 2417–2425.
[15] A. Blokh, L. Lyubich, *Measurable dynamics of S-unimodal maps of the interval*, Ann. Sci. Ec. Norm. Sup. 24 (1991), 545–573.
[16] K. Brucks, H. Bruin, *Subcontinua of inverse limit spaces of unimodal maps,* Fund. Math. **160** (1999) 219–246.
[17] K. Brucks, B. Diamond, *A symbolic representation of inverse limit spaces for a class of unimodal maps*, Continuum Theory and Dynamical Systems, Lecture Notes in Pure Appl. Math. **149** (1995), 207–226.
[18] H. Bruin, *Inverse limit spaces of post-critically finite tent maps,* Fund. Math. **165** (2000) 125–138.
[19] H. Bruin, *Planar embeddings of inverse limit spaces of unimodal maps*, Topology Appl. **96** (1999), 191–208.
[20] H. Bruin, *Asymptotic arc-components of unimodal inverse limit spaces,* Topology Appl. **152** (2005) 182–200.
[21] H. Bruin, S. Štimac, *On isotopy and unimodal inverse limit spaces*, Discrete and Continuous Dynamical Systems - Series A **32** (2012), 1245–1253.
[22] H. Bruin, S. Štimac, *Fibonacci-like unimodal inverse limit spaces and the core Ingram conjecture*, Topol. Methods Nonlinear Anal. **47** (2016), 147–185.
[23] C. Good, B. Raines, *Continuum many tent maps inverse limits with homeomorphic postcritical ω-limit sets*, Fund. Math. **191** (2006), 1–21.
[24] R. Fokkink, *There are uncountably many homeomorphism types of orbits in flows,* Fund. Math. **136** (1990), 147–156.

[25] W. T. Ingram, *Inverse limits on* $[0,1]$ *using tent maps and certain other piecewise linear bounding maps,* Continua with the Houston problem book (Cincinnati, OH, 1994), in: H. Cook et al. (Eds.), Lecture Notes in Pure and Appl. Math.170, Dekker (1995), 253–258.
[26] L. Kailhofer, *A classification of inverse limit spaces of tent maps with periodic critical points*, Fund. Math. **177** (2003), 95–120.
[27] R. de Man, *On composants of solenoids,* Fund. Math. **147** (1995), 181–188.
[28] J. Milnor, W. Thurston, *On iterated maps of the interval: I, II*, Preprint 1977. Published in Lect. Notes in Math. **1342**, Springer, Berlin New York (1988) 465–563.
[29] P. Minc, $2^{\aleph_0}$ *ways of approaching a continuum with* $[1,\infty)$, Topology Appl. **202** (2016), 47–54.
[30] B. Raines, *Inhomogeneities in non-hyperbolic one-dimensional invariant sets*, Fund. Math. **182** (2004), 241–268.
[31] B. Raines, S. Štimac, *A classification of inverse limit spaces of tent maps with non-recurrent critical point*, Algebraic and Geometric Topology **9** (2009), 1049–1088.
[32] S. Štimac, *Structure of inverse limit spaces of tent maps with finite critical orbit*, Fund. Math. **191** (2006), 125–150.
[33] S. Štimac, *A classification of inverse limit spaces of tent maps with finite critical orbit*, Topology Appl. **154** (2007), 2265–2281.
[34] R. Swanson, H. Volkmer, *Invariants of weak equivalence in primitive matrices,* Ergodic Theory Dynam. Systems **20** (2000) 611–626.

Faculty of Mathematics, University of Vienna

Oskar Morgensternplatz 1, 1090 Wien, Austria

henk.bruin@univie.ac.at

FACULTY OF MATHEMATICS, UNIVERSITY OF VIENNA, OSKAR MORGENSTERNPLATZ 1, 1090 VIENNA, AUSTRIA