ON "OBSERVABLE" LI-YORKE TUPLES FOR INTERVAL MAPS

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Abstract. In this paper we study the set of Li-Yorke d-tuples and its d-dimensional Lebesgue measure for interval maps $T\colon [0,1] \to [0,1]$. If a topologically mixing T preserves an absolutely continuous probability measure (with respect to Lebesgue), then the d-tuples have Lebesgue full measure, but if T preserves an infinite absolutely continuous measure, the situation becomes more interesting. Taking the family of Manneville-Pomeau maps as example, we show that for any $d \geq 2$, it is possible that the set of Li-Yorke d-tuples has full Lebesgue measure, but the set of Li-Yorke $(d+1)$-tuples has zero Lebesgue measure.

1. Introduction

Abundance of Li-Yorke pairs (see [25] and Definition 1.1 below) is a frequently used criterion for declaring that a dynamical system $T\colon X \to X$ on a compact metric space (X, ρ) is chaotic. A milestone was the result that positive topological entropy implies the existence of an uncountable scrambled set, *i.e.,* a set in which every pair of distinct points has the Li-Yorke property [6]. However, interval maps of type 2^∞ (*i.e.,* with periodic points of period 2^k for each $k \geq 0$, but no other periods) have zero entropy, but they can still have uncountable scrambled sets [38], so that Li-Yorke pairs give a slightly more refined view on mathematical chaos than the condition that $h_{top}(T) > 0$.

Multimodal interval maps never have closed invariant scrambled sets, and it can be proved that C^3 multimodal interval maps with nonflat critical points (see Section 2 for definitions) only have scrambled sets of Lebesgue measure zero, see [12]. In view of these results, when speaking about "observable chaos" it is more reasonable to consider the size of the set of Li-Yorke pairs, than scrambled sets themselves. This idea comes from Lasota, and was first employed by Piórek in [34].

In [4, Proposition 5 and Remark 3] it was shown that if (X, T) preserves a weakly mixing non-atomic measure μ, then the set of Li-Yorke pairs has positive $\mu \times \mu$-measure (the first results of this type for continuous transformations were obtained independently by Piórek [35] and Iwanik [20]). If (X, T) has an exact, non-singular but not necessarily invariant measure λ, then $\lambda \times \lambda$-a.e. pair is Li-Yorke [12, Lemma 18]. In the setting of smooth interval maps with Lebesgue measure, Barnes [3] showed that the weaker assumption of limsup full, see Definition 1.2, implies exactness and therefore that Lebesgue typical pairs are Li-Yorke, see also [12, Theorem D]. More precisely, [12, Theorem D] states that for every topologically

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mixing non-flat C^3 multimodal map without attracting Cantor set, the set of Li-Yorke pairs has full Lebesgue measure. The paper continues to discuss the intricate situation when attracting Cantor sets are present, but we will not go into that direction here. However, one question left open in [12] is whether there are smooth conservative maps for which λ_2-a.e. pair (x, y) is Li-Yorke and additionally its orbit under $T \times T$ is not dense in $[0,1]^2$. We will answer this question in Theorem B.

The topic of this paper is the d-fold measure of Li-Yorke d-tuples. We give the definitions first. Let (X, ρ) be a compact metric space and $T\colon X \to X$ a continuous map acting on it.

Definition 1.1. *A d-tuple $\underline{x} = (x_1, \dots, x_d)$ is called:*

(1) asymptotic *if* $\lim_n \max_{i,j} \rho(T^n(x_i), T^n(x_j)) = 0$*;*
(2) proximal *if* $\liminf_n \max_{i,j} \rho(T^n(x_i), T^n(x_j)) = 0$*;*
(3) δ-separated *if* $\limsup_n \min_{i\neq j} \rho(T^n(x_i), T^n(x_j)) \geq \delta$*; if $\underline{x}$ is δ-separated for some $\delta > 0$, we call it* separated*;*
(4) Li-Yorke (LY for short) *if it is proximal and separated, that is:*

$$\begin{cases} \liminf_n \max_{i,j} \rho(T^n(x_i), T^n(x_j)) = 0, \\ \limsup_n \min_{i\neq j} \rho(T^n(x_i), T^n(x_j)) > 0. \end{cases}$$

(5) δ-Li-Yorke (or simply, δ-LY) *for some $\delta > 0$, if $\underline{x}$ is LY and*

$$\limsup_n \min_{i\neq j} \rho(T^n(x_i), T^n(x_j)) \geq \delta.$$

Let LY_d and LY_d^δ be the sets of all LY and δ-LY d-tuples, respectively. Also we use T_d as abbreviation for the d-fold product map $T \times \cdots \times T$ on X^d.

It is known that a transitive system with a fixed point has a LY d-tuple for any $d \geq 2$ [44], and consequently, every totally transitive system with dense periodic points (hence topologically weakly mixing) must have such tuples.

The main motivation of the paper is the following question.

Question: *How large is the set of LY d-tuples?*

Obviously, there is no one good answer to this question without specifying what "large" means. In purely topological terms it would be a residual set. Each transitive map on the interval must have LY d-tuples, cf. Lemma 4.2, and every topologically mixing map on an infinite space has a dense Mycielski set M (*i.e.,* countable union of Cantor sets) such that any $d \geq 2$ distinct points in M form a LY d-tuple (see *e.g.* [24]). On the other hand, maps of the interval with zero topological entropy never have LY 3-tuples [23]. However, there are dynamical systems on the Cantor set such that each d-tuple of distinct points is LY, but no uncountable set with this property for $(d+1)$-tuples exists [24]. In fact, the system need not have LY $(d+1)$-tuples at all [15].

In many cases, *e.g.* on the interval, there is a natural reference measure such as Lebesgue measure, to express the size of the set of Li-Yorke tuples. Even with this natural tool at hand, the answers depend on the degree of smoothness of the map. The smoother the map is, the better this method of measurement is appropriate.

The following definitions come from [3] and [36] respectively:

Definition 1.2. *Let λ be a probability measure on X. Then λ is called:*

(1) limsup full *if* $\lambda(A) > 0$ *implies that* $\limsup_{n\to\infty} \lambda(T^n(A)) = 1$*;*
(2) full[1] *if* $\lambda(A) > 0$ *implies that* $\lim_{n\to\infty} \lambda(T^n(A)) = 1$.

Let $\lambda_d = \lambda \times \cdots \times \lambda$ denote the d-fold product measure on X^d. We show

Theorem A. *Let λ be a fully supported probability measure.*

(1) If λ is limsup full then $\lambda_2(\mathrm{LY}_2^\delta) = 1$ for any $\delta < diam(X)/2$,
(2) If λ is full then $\lambda_d(\mathrm{LY}_d^\delta) = 1$ for every $d \geq 2$ and some $\delta = \delta(d) > 0$. If X is additionally connected then $\lambda_d(\mathrm{LY}_d^\delta) = 1$ for every $\delta < diam(X)/2(d-1)$.

When $d \geq 3$, then λ being limsup full is no longer sufficient to guarantee that λ_d-a.e. d-tuple is Li-Yorke. Instead, if λ admits an equivalent weak mixing T-invariant probability measure μ, then this holds, see [4] and Lemma 3.2.

Remark 1.3. *Without the connectedness assumptions, a bound $\delta < diam(X)/2(d-1)$ cannot work. For instance, if X is a union of two small intervals but placed at long distance, then $diam(X)$ is large but two points in any triple have to be very close.*

For a smooth topologically mixing interval map T preserving a probability measure $\mu \ll \lambda$, the above theorem supplies an abundance of Li-Yorke d-tuples. If the T-invariant measure μ is only σ-finite, then the difficulty in showing the abundance of Li-Yorke d-tuples for $d \geq 3$ lies in the separation along a subsequence. Under mild conditions, any two points in a d-tuple separate infinitely often, but it is difficult to show that three or more points separate at the same time. The family of *Manneville-Pomeau maps* $T_\alpha : [0,1] \to [0,1]$ defined by

$$T_\alpha(x) = \begin{cases} x(1 + 2^\alpha x^\alpha) & \text{if } x \in [0, \frac{1}{2}), \\ 2x - 1 & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

illustrates this situation (although not continuous, but see Remark 5.5).

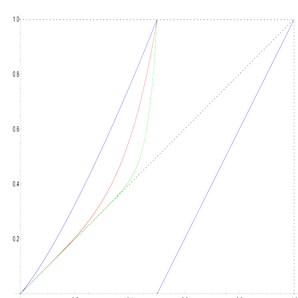

FIGURE 1. Graph of $T_{\frac{1}{2}}$, T_4 and T_{15}.

The point 0 is a neutral fixed point, weakly expanding, but the speed at which points move away from 0 is slower as α increases. The typical situation can thus be that one point in a d-tuple separates itself from the rest, while the other $d-1$ points linger in a neighborhood of 0. Using known techniques of renewal theory to a first return induced map we prove the following theorem.

[1]Compared to limsup full, lim full seems a more logical name for this property, but it is called *full* by Proppe & Boyarski in [36], hence we follow this terminology.

Theorem B. *For the Manneville-Pomeau map T_α, the following conditions hold:*

(1) *λ_2-a.e. pair is $\frac{1}{3}$-Li-Yorke,*
(2) *and for $d \geq 3$:*
  (i) *if $\alpha \leq \frac{d-1}{d-2}$ and $\delta < 1/2(d-2)$, then λ_d-a.e. d-tuple is δ-Li-Yorke;*
  (ii) *if $\alpha > \frac{d-1}{d-2}$ and $\varepsilon > 0$, then λ_d-a.e. d-tuple is not ε-separated; in particular λ_d-a.e. d-tuple is not Li-Yorke.*

Remark 1.4. *One can conclude from this theorem that for $\alpha > 2$, the product system is λ_2-dissipative, whence orbits of typical pairs are not dense in $[0,1]^2$, however still λ_2-a.e. pair is LY. This addresses a question posed in* [12, p. 527].

*If $\frac{d}{d-1} < \alpha \leq \frac{d-1}{d-2}$, then λ_d-a.e. $\underline{x}$ is LY in $[0,1]^d$, however λ_{d+1}-a.e $\underline{x}$ is not LY, nor asymptotic (*i.e., *$T^n_{\alpha,d+1}(\underline{x}) \not\to \Delta$, the diagonal).*

When measuring tuples using Lebesgue measure as reference, much may depend on the class of maps we are considering. In the class of continuous maps we can quite easily perturb the dynamics, and hence topological conjugacy can completely change qualitative description of the map in terms of reference measure. We prove the following:

Theorem C. *There exist pairwise topologically conjugate maps $P, S, T\colon [0,1] \to [0,1]$ and sets K, L, M such that:*

(1) *K, L, M have full Lebesgue measure;*
(2) *there are positive numbers δ_d such that every d-tuple of distinct points in L (resp. in M) is δ_d-LY for S (resp. T);*
(3) *$\omega(x, S) = [0,1]$ for every $x \in L$;*
(4) *there exists a Cantor minimal set A of positive measure such that $\omega(x,T) = A$ for every $x \in M$.*
(5) *none of the pairs $(x,y) \in K \times K$ is LY for P; in particular the set of LY pairs has zero Lebesgue measure.*

The maps P, S, T are the same from dynamical point of view, however the size of the set of LY tuples detected by Lebesgue measure is completely different in each case.

The paper is organized as follows. Section 2 gives preliminary information from ergodic theory. Section 3 shows the results on LY d-tuples that can be derived from the assumption that λ is full or limsup full. In Section 4 we present and prove our results for the C^0 setting. Finally, in Section 5 we discuss the situation of σ-finite measure (Manneville-Pomeau) which provides the most interesting examples where the theory of Section 3 doesn't apply.

2. Preliminaries from Measure Theory

Let X be a compact topological space with Borel σ-algebra $\mathcal{B}$ and let $T\colon X \to X$ be a Borel measurable map. We will only consider Borel measures λ, that is measures for which every $B \in \mathcal{B}$ is measurable.

Definition 2.1. *A Borel measure λ is (with respect to T):*

(1) non-singular *if $\lambda(T^{-1}(A)) = 0$ implies that $\lambda(A) = 0$,*
(2) fully supported *if $\lambda(U) > 0$ for every nonempty open set,*
(3) conservative *if for every set A with $\lambda(A) > 0$ there is $n > 0$ such that $\lambda(A \cap T^n(A)) > 0$; otherwise λ is called* dissipative.

(4) ergodic[2] *if* $A = T^{-1}(A) \pmod{\lambda}$ *implies that* $\lambda(A) = 0$ *or* $\lambda(A^c) = 0$,
(5) exact *if the* tail-field $\bigvee_n T^{-n}\mathcal{B}$ *is trivial, or equivalently* $\lambda(A) = 0$ *or* $\lambda(A^c) = 0$ *whenever* $T^{-n} \circ T^n(A) = A \pmod{\lambda}$ *for all* $n \in \mathbb{N}$.

Remark 2.2. *If λ is conservative, then $\lambda(A \cap T^n(A)) > 0$ for some $n > 0$ and so there is $m > 0$ such that*

$$\lambda(A \cap T^{n+m}(A)) \geq \lambda(A \cap T^n(A) \cap T^m(A \cap T^n(A))) > 0.$$

In other words (by induction), for every $n > 0$ there exists $k > n$ such that $\lambda(A \cap T^k(A)) > 0$, provided that $\lambda(A) > 0$, and hence it follows that λ-a.e. $x \in A$ returns to A infinitely often.

Lemma 2.3. *If a Borel measure λ is fully supported, then*

$$g(\varepsilon) := \inf_{x \in X} \lambda(B(x;\varepsilon)) > 0 \text{ for every } \varepsilon > 0. \tag{2.1}$$

Proof. If $g(\varepsilon) = 0$ for some $\varepsilon > 0$, then, due to compactness of X, we can find a convergent sequence $x_n \to x$ such that $\lambda(B(x_n;\varepsilon)) \to 0$. But then $B(x;\varepsilon/2) \subset B(x_n;\varepsilon)$ for n sufficiently large, so $\lambda(B(x;\varepsilon/2)) \leq \lim_n \lambda(B(x_n;\varepsilon)) = 0$, in contradiction to $\mathrm{supp}(\lambda) = X$. □

With any non-singular Borel measure λ w.r.t. T we can associate the Perron-Frobenius operator $\mathcal{L}\colon L^1(\lambda) \to L^1(\lambda)$, uniquely defined by the formula

$$\int_A \mathcal{L}f d\lambda = \int_{T^{-1}(A)} f d\lambda \quad \text{for all } f \in L^1(\lambda) \text{ and all } A \in \mathcal{B}.$$

Remark 2.4. *An equivalent property to exactness (assuming that λ is non-singular) is that $\int |\mathcal{L}^n f| d\lambda \to 0$ as $n \to \infty$ for any $f \in L^1(\lambda)$ with $\int f d\mu = 0$, where $\mathcal{L}$ is the Perron-Frobenius operator (see* [26] *or* [1, Theorem 1.3.3.]*).*

Lemma 2.5. *If λ is a non-singular, σ-finite and exact Borel measure, then d-dimensional direct product measure λ_d is ergodic for every integer $d \geq 1$.*

Proof. Similar to Lemma 18 of [12]. □

We repeat the following fact after Thaler [41].

Lemma 2.6. *Let λ be an exact non-singular Borel probability measure and $\mu \ll \lambda$ an infinite σ-finite T-invariant Borel measure. Then*

$$\int_A \mathcal{L}^n f d\lambda \to 0 \qquad \text{as } n \to \infty,$$

for all $A \in \mathcal{B}$, $\mu(A) < \infty$ and all $f \in L^1(\lambda)$.

Proof. Fix $B \in \mathcal{B}$, $0 < \mu(B) < \infty$ and denote

$$g = \frac{\int_X f d\lambda}{\mu(B)} 1_B\, h,$$

[2]If Borel sets A, B are such that $\lambda(A \setminus B) = 0$ then we write $A \subset B \pmod{\lambda}$. Similarly, when $A \subset B \pmod{\lambda}$ and $B \subset A \pmod{\lambda}$ then we denote this fact by $A = B \pmod{\lambda}$.

where $h = \frac{d\mu}{d\lambda}$ and observe that $\int_X (f-g)d\lambda = 0$. Then

$$\begin{aligned}\int_A \mathcal{L}^n f d\lambda &= \int_A \mathcal{L}^n (f-g) d\lambda + \int_{T^{-n}(A)} g d\lambda \\ &= \int_A \mathcal{L}^n (f-g) d\lambda + \frac{\int_X f d\lambda}{\mu(B)} \mu(T^{-n}(A) \cap B).\end{aligned}$$

Since λ is exact (see Remark 2.4), the first term tends to 0. By invariance of μ we obtain $\mu(T^{-n}(A) \cap B) \leq \mu(T^{-n}(A)) = \mu(A)$ and hence

$$\limsup_{n\to\infty} \int_A \mathcal{L}^n f d\lambda \leq \frac{\mu(A)}{\mu(B)} \int_X f d\lambda.$$

But μ is infinite and σ-finite, so we can start with arbitrarily large $\mu(B)$ which completes the proof. □

Lemma 2.7. *Let λ be an exact non-singular Borel probability measure and $\mu \ll \lambda$ an infinite σ-finite T-invariant Borel measure. Then λ is not full.*

Proof. Since μ is σ-finite, there is $A \in \mathcal{B}$ with $0 < \mu(A) < \infty$, in particular $\lambda(A) > 0$. While $\mu(X) = \infty$, there are countably many pairwise disjoint sets F_i such that $\mu(F_i) < \infty$ and $\bigcup_{i=1}^{\infty} F_i = X$. Fix any $k > 0$. Since $1 = \lambda(X) = \sum_{i=1}^{\infty} \lambda(F_i)$ there is $r > 0$ such that $\sum_{i=r+1}^{\infty} \lambda(F_i) < 2^{-k-1}$. Hence, for every $n > 0$ we have

$$\lambda(T^{-n}(A)) = \sum_{i=1}^{\infty} \lambda(T^{-n}(A) \cap F_i) \leq 2^{-k-1} + \sum_{i=1}^{r} \lambda(T^{-n}(A) \cap F_i).$$

But, for any i we obtain by Lemma 2.6 that

$$\lambda(T^{-n}(A) \cap F_i) = \int_{T^{-n}(A)} 1_{F_i} d\lambda = \int_A \mathcal{L}^n(1_{F_i}) d\lambda \to 0.$$

Hence, there is $n_k > 0$ such that $\lambda(T^{-n_k}(A) \cap F_i) < 2^{-k-1}/r$ for each $1 \leq i \leq r$ and consequently $\lambda(T^{-n_k}(A)) < 2^{-k}$. In particular, if we set $B := \cup_{k\geq 1} T^{-n_k}(A)$ then $0 < \lambda(B) < 1$. The proof is finished by the fact, that $\lambda(B^c) > 0$, but $T^{n_k}(B^c) \cap A = \emptyset$ for all $k \geq 1$, which shows that λ is not full. □

Note that none of the properties in Definition 2.1 required that λ is *invariant i.e.,* $\lambda(A) = \lambda(T^{-1}(A))$ for every $A \in \mathcal{B}$. A definition that requires invariance is *weak mixing*: for every $A, B \in \mathcal{B}$, there is a sequence $n_k \to \infty$ such that $\mu(T^{-n_k}(A) \cap B) \to \mu(A)\mu(B)$ as $k \to \infty$. It is known that if μ is weak mixing, then for every $A, B \in \mathcal{B}$, there is a set $\mathcal{N} \subset \mathbb{N}$ of full density such that $\lim_{\mathcal{N} \ni n \to \infty} \mu(T^{-n}(A) \cap B) = \mu(A)\mu(B)$.

We write $C^3_{nf}([0,1])$ for the collection of C^3 multimodal interval maps $f\colon [0,1] \to [0,1]$ where all critical points c are *non-flat*, *i.e.,* there is $\ell_c \in (1,\infty)$ such that $|T(x) - T(c)|/|x-c|^{\ell_c}$ is bounded and bounded away from zero for all $x \neq c$ sufficiently close to c.

We call a closed invariant set $A \subset [0,1]$ an *attractor* (see [29]) if its basin $\{x \in [0,1] : \omega(x) \subset A\}$ has positive Lebesgue measure, and no proper subset of A has this property. Examples of attractors A which are also Cantor sets are the Feigenbaum attractor and the "wild" attractor of a Fibonacci unimodal map of sufficiently high critical order, see [13].

Let us recall two facts, which are Theorem 23 and Proposition 24 from [12], respectively.

Theorem 2.8. *Let $T \in C^3_{nf}([0,1])$ be a topologically mixing map having no Cantor attractors. If Lebesgue measure is conservative, then it is limsup full.*

Theorem 2.9. *Let $T \in C^3_{nf}([0,1])$ be topologically mixing and let λ be Lebesgue measure on $[0,1]$. Then the following are equivalent:*

1. *there exists an invariant probability Borel measure $\mu \ll \lambda$,*
2. $\liminf_{n\to\infty} \lambda(T^n(A)) > 0$ *for every measurable set $A \in \mathcal{B}$, $\lambda(A) > 0$,*
3. *λ is full for T .*

This shows that Lebesgue measure is limsup full but not full (for topologically mixing $T \in C^3_{nf}([0,1])$) precisely when it is conservative but admits no absolutely continuous probability Borel measure. The first such examples (within the quadratic family) were constructed by Johnson [21], and more detailed constructions can be found in [8, 19]. Whether there exists a quadratic map for which λ does not even admit a σ-finite invariant Borel measure is unknown. However, there are cases where a σ-finite Borel measure $\mu \ll \lambda$ exists such that $\mu(J) = \infty$ for all intervals $J \subset [0,1]$, see [2, 8, 11].

The following lemma shows that a version of $1 \Rightarrow 3$ in Theorem 2.9 holds not just in the setting of interval maps.

Lemma 2.10. *Let λ be a non-singular probability Borel measure which is limsup full. If λ admits an equivalent T-invariant probability Borel measure μ, then λ is full.*

Proof. Let $g^+(\varepsilon) = \sup\{\mu(A) : \lambda(A) \le \varepsilon\}$ and $g^-(\varepsilon) = \sup\{\lambda(A) : \mu(A) \le \varepsilon\}$. Since $\mu \ll \lambda$ and both μ, λ are probability Borel measures, Lusin's Theorem gives, for any $\delta > 0$, a uniformly continuous map h that coincides with $\frac{d\mu}{d\lambda}$ up to a set of measure $\delta/2$. Now choose $\varepsilon > 0$ so small that $\int_A h d\lambda < \delta/2$ for every set A with $\lambda(A) < \epsilon$. This gives $g^+(\varepsilon) \le \delta$, so we proved that $g^+(\varepsilon) \to 0$ as $\varepsilon \to 0$. Since $\lambda \ll \mu$ also $g^-(\varepsilon) \to 0$ as $\varepsilon \to 0$. Let B be an arbitrary measurable set with $\lambda(B) > 0$, and let (n_k) be a sequence such that $\lambda(T^{n_k}(B)) > 1 - 1/k$. Then $\mu(T^{n_k}(B)) \ge 1 - g^+(1/k)$ and by invariance, also $\mu(T^n(B)) \ge 1 - g^+(1/k)$ for all $n \ge n_k$. Using the definition of g^-, we find $\lambda(T^n(B)) \ge 1 - g^-(g^+(1/k))$ for all $n \ge n_k$. Since $\lim_{\varepsilon\to 0} g^+(\varepsilon) = \lim_{\varepsilon\to 0} g^-(\varepsilon) = 0$, it follows that λ is full. □

3. Li-Yorke tuples and d-fold product measures

Another way of expressing the set LY_d is

$$\mathrm{LY}_d = (\bigcap_{k=1}^{\infty} \bigcap_{r=1}^{\infty} \bigcup_{n>r} A_{k,n}) \cap (\bigcup_{k=1}^{\infty} \bigcap_{r=1}^{\infty} \bigcup_{n>r} D_{k,n}),$$

where

$$\begin{cases} A_{k,n} = \{\underline{x} : \max_{i,j} \rho(T^n(x_i), T^n(x_j)) < 1/k\}, \\ D_{k,n} = \{\underline{x} : \min_{i\neq j} \rho(T^n(x_i), T^n(x_j)) > 1/k\}. \end{cases}$$

Similarly, we can write

$$\mathrm{LY}^\delta_d = (\bigcap_{k=1}^{\infty} \bigcap_{r=1}^{\infty} \bigcup_{n>r} A_{k,n}) \cap (\bigcap_{r=1}^{\infty} \bigcup_{n>r} D^\delta_{r,n}),$$

where

$$D^\delta_{r,n} = \{\underline{x} : \min_{i\neq j} \rho(T^n(x_i), T^n(x_j)) > \delta - 1/r\}.$$

Since $A_{k,n}$ and $D_{k,n}$ and $D^{\delta}_{r,n}$ are open, LY_d and LY^{δ}_d are Borel sets (in fact G_δ-sets), hence their indicator function is measurable w.r.t. Borel measures on X^d, and we can use Fubini's theorem.

Proof of Theorem A. We argue by induction. The initiation step is for δ-LY-pairs with $\delta < \mathrm{diam}(X)/2$. Given $x \in X$, let $\mathrm{LY}(x) = \left\{ y \in X : (x,y) \in \mathrm{LY}^{\delta}_2 \right\}$. First we argue that λ-a.e. y is δ-separated in pair with x. If this was not the case, then

$$\lambda \left(\bigcup_{r=1}^{\infty} \bigcap_{n>r} \{ y : \rho(T^n(x), T^n(y)) \leq \delta - 1/r \} \right) > 0.$$

Therefore we can find $r \in \mathbb{N}$ such that $A := \cap_{n>r}\{y : \rho(T^n(x), T^n(y)) \leq \delta\}$ has positive measure. Fix $0 < \gamma < \mathrm{diam}(X)/4 - \delta/2$ and observe that $\mathrm{diam}(T^n(A)) \leq 2\delta < \mathrm{diam}(X) - 4\gamma$ for every $n > r$. Note that $X \neq B(T^n(A), \gamma)$ because for any two $p, q \in B(T^n(A), \gamma)$ we have

$$d(p,q) \leq 2\gamma + \mathrm{diam}(T^n(A)) \leq \mathrm{diam}(X) - 2\gamma < \mathrm{diam}(X).$$

Fix $x_n \in X \setminus B(T^n(A), \gamma)$ and observe that $T^n(A) \cap B(x_n; \gamma) = \emptyset$, hence $\lambda(T^n(A)) < 1 - g(\gamma) < 1$ (with g as in (2.1)) for all $n > r$. This contradicts the assumption that λ is limsup full.

Similarly, we argue that λ-a.e. y is proximal w.r.t. x. Indeed, if not, then we can choose $k \in \mathbb{N}$ sufficiently large such that the set $D := \cap_{n \geq 0}\{y \in X : \rho(T^n(x), T^n(y)) > 1/k\}$ has positive measure. But then $T^n(D) \cap B(T^n(x); 1/k) = \emptyset$, and thus $\lambda(T^n(D)) < 1 - g(1/k) < 1$ for all n, so again λ is not limsup full.

The set of δ-LY-pairs can be written as $\mathrm{LY}^{\delta}_2 = \cup_x \mathrm{LY}(x)$, so by Fubini's theorem, it has full measure. This completes the first step of induction and proves also that if λ is limsup full then $\lambda_2(\mathrm{LY}^{\delta}_2) = 1$ for any $\delta < \mathrm{diam}(X)/2$. This proves part (1) of the theorem.

For part (2), we continue the induction, fixing $d \geq 2$ and assuming that $\mathrm{LY}^{\varepsilon}_d$ has full d-fold product measure for any $\varepsilon < \mathrm{diam}(X)/2(d-1)$ when X is connected, and for some $\varepsilon > 0$ otherwise (hence every ε sufficiently small). If X is connected, then we fix any $0 < \delta < \mathrm{diam}(X)/2d$. If X is not connected, then we fix any distinct points $a_1, \dots, a_{d+1}$ and put $\delta = \min\{\min_{s \neq t} \rho(a_s, a_t)/6, \varepsilon\}$.

Take $\underline{x} \in \mathrm{LY}^{\delta}_d$, let $(m_u)_{u \geq 1}$ and $(n_u)_{u \geq 1}$ be sequences along which $\underline{x}$ is asymptotic, resp. separated, that is

$$\begin{cases} \liminf_u \max_{i,j} \rho(T^{m_u}(x_i), T^{m_u}(x_j)) = 0, \\ \limsup_u \min_{i \neq j} \rho(T^{n_u}(x_i), T^{n_u}(x_j)) \geq \delta. \end{cases}$$

and let

$$\mathrm{LY}(\underline{x}) = \left\{ y \in X : \begin{array}{l} \liminf_u \max_i \rho(T^{m_u}(x_i), T^{m_u}(y)) = 0, \\ \limsup_u \min_i \rho(T^{n_u}(x_i), T^{n_u}(y)) \geq \delta. \end{array} \right\}$$

We show that λ-a.e. y is δ-separated in pair with each coordinate of $\underline{x}$ along the subsequence (m_u). If the set of non δ-separated points has positive measure, then there is $r \in \mathbb{N}$ such that $A := \bigcap_{u>r}\{y : \min_i \rho(T^{m_u}(x_i), T^{m_u}(y)) \leq \delta\}$ has positive measure. But then $T^{m_u}(A) \subset \bigcup_{i=1}^{d} B(T^{m_u}(x_i); \delta)$. If X is connected, then take any $\xi > 0$ such that $2d(\delta + 2\xi) < \mathrm{diam}(X)$ and in the other case put $\xi = \delta$.

We claim that $\bigcup_{i=1}^{d} B(T^{m_u}(x_i); \delta + \xi) \neq X$. First, we consider the case that X is connected. If the claim does not hold then, since X is connected, for every points $p, q \in X$ there are pairwise distinct numbers $i_1, \ldots, i_k$, where $k \leq d$, such that $p \in B(T^{m_u}(x_{i_1}); \delta + \xi)$, $q \in B(T^{m_u}(x_{i_k}); \delta + \xi)$ and $B(T^{m_u}(x_{i_j}); \delta + \xi) \cap B(T^{m_u}(x_{i_{j+1}}); \delta + \xi) \neq \emptyset$. But then

$$\begin{aligned} \rho(p, q) &\leq \delta + \xi + 2(k-1)(\delta + \xi) + \delta + \xi \leq 2d\delta + 2d\xi \\ &\leq 2d(\delta + 2\xi) - 2d\xi < \operatorname{diam}(X) - 2\xi. \end{aligned}$$

which is a contradiction, since p, q were arbitrary. Therefore in this case the claim holds. Similarly, if X is not connected, then by the definition of δ, each ball $B(T^{m_u}(x_{i_{j+1}}); \delta + \xi)$ can contain at most one point a_s, and hence also in this case $\bigcup_{i=1}^{d} B(T^{m_u}(x_i); \delta + \xi) \neq X$. Indeed, the claim holds.

By the above claim, for every u there is a point q_u such that

$$B(q_u; \xi) \cap \bigcup_{i=1}^{d} B(T^{m_u}(x_i); \delta) = \emptyset$$

which in particular implies that $\lambda(T^{m_u}(A)) < 1 - g(\xi)$ for all $u > r$. This contradicts that λ is full, which proves that λ-a.e. y is δ-separated with each coordinate of $\underline{x}$ along the subsequence (m_u).

Similarly, we argue that λ-a.e. y is proximal w.r.t. $\underline{x}$ along the subsequence (n_u). Indeed, otherwise we can choose $k \in \mathbb{N}$ sufficiently large such that the set $D := \bigcap_{u=0}^{\infty} \{y \in X : \max_i \rho(T^{n_u}(x_i), T^{n_u}(y)) \geq 1/k\}$ has positive measure. Passing to a subsequence of n_u if necessary we may assume that for every u there is j such that if $y \in D$ then $\rho(T^{n_u}(x_j), T^{n_u}(y)) \geq 1/k$. But then $T^{n_u}(D) \cap (B(T^{n_u}(x_j); 1/k)) = \emptyset$ and so $\lambda(T^{n_u}(D)) < 1 - g(1/k) < 1$ for all u, therefore again λ is not full.

The set of δ-LY $(d+1)$-tuples contains the set $\mathrm{LY}_{d+1}^{\delta} \supset \bigcup_{\underline{x} \in \mathrm{LY}_d^{\delta}} \mathrm{LY}(\underline{x})$, so again by Fubini's theorem, it has full measure. This completes the proof. □

Lemma 3.1. *Let λ be a Borel measure for a system (X, T). If the d-fold product measure λ_d is conservative for the product system (X^d, T_d) then the set of d-tuples that are not asymptotic for (X, T) has full λ_d-measure.*

Proof. First we deal with the case that λ is atomic, say $\lambda(\{x\}) > 0$. Conservativity then implies that x must be periodic. If $U = \{y \in X \setminus \{x\} : (x, y) \text{ is asymptotic}\}$ has positive measure, then there is $U' \subset U$ with $\lambda(U') > 0$ and $\varepsilon > 0$ such that $U' \cap B(\operatorname{orb}(x); \varepsilon) = \emptyset$, whereas $T^n(U') \subset B(\operatorname{orb}(x); \varepsilon)$ for all n sufficiently large. This contradicts that λ is conservative, and therefore we can assume for the rest of the proof that λ is non-atomic.

Assume by contradiction, that the set of asymptotic d-tuples has positive measure, *i.e.,*

$$\lambda_d\Big(\bigcap_{k>0} \bigcup_{r>0} \bigcap_{n>r} A_{k,n}\Big) = \alpha > 0. \tag{3.1}$$

where $A_{k,n} = \{\underline{x} : \max_{i,j} \rho(T^n(x_i), T^n(x_j)) < 1/k\}$. Since λ is non-atomic, we can find small $\varepsilon > 0$, such that if we denote by Δ_ε the ε-neighborhood of the diagonal $\Delta = \{\underline{x} : x_i = x_j \text{ for all } 1 \leq i, j \leq d\}$ then $\lambda_d(\Delta_\varepsilon) < \alpha/2$. Fix any $k \in \mathbb{N}$ such that $1/k < \varepsilon$. By (3.1) there is $m > 0$ such that

$$\lambda_d\Big(\bigcup_{r=1}^{m} \bigcap_{n>r} A_{k,n}\Big) > \alpha/2.$$

Denote $A = \bigcup_{r=1}^{m} \bigcap_{n>r} A_{k,n} \supset \bigcap_{n>m} A_{k,n}$. Note that $T_d^n(A) \subset \Delta_\varepsilon$ for all $n > m$ and $\lambda_d(A') > 0$ for $A' = A \setminus \Delta_\varepsilon$. By conservativity, $\lambda_d(T_d^n(A') \cap A') > 0$ for some $n > m$ which contradicts the definition of A. Therefore the set of asymptotic d-tuples has measure zero and the lemma follows. □

The following is a simple extension of a well-known fact for LY-pairs (see *e.g.* [4]):

Lemma 3.2. *If X has at least two points and μ is a weakly mixing, fully supported, T-invariant Borel measure, then for any integer $d > 1$, there is $\delta_d > 0$ such that the δ_d-LY d-tuples have full μ_d-measure.*

Proof. We start with proximality. Assume by contradiction that there is a set $A \subset X^d$ with $\mu_d(A) > 0$, and $m \in \mathbb{N}$, $\varepsilon > 0$ such that $T_d^n(A) \cap \Delta_\varepsilon = \emptyset$ for every $n \geq m$. Since μ is fully supported, $\mu_d(\Delta_\varepsilon) > 0$. But μ is weak mixing, hence the product measure μ_d is weak mixing too, and so there is a subsequence (n_k) such that $\mu_d(T_d^{-n_k}(\Delta_\varepsilon) \cap A) \to \mu_d(\Delta_\varepsilon)\mu_d(A) > 0$. This implies that there is $n > m$ such that $T_d^n(A) \cap \Delta_\varepsilon \neq \emptyset$, contradicting the definition of A. Therefore μ_d-a.e. $\underline{x}$ is proximal along a subsequence.

Since X has at least two points and μ is weakly mixing, X is infinite. In particular, there exists a nonempty open set $U \subset X^d$ such that $\overline{U} \cap \Delta_{i,j} = \emptyset$ for every $i \neq j$, where $\Delta_{i,j} := \{\underline{x} \in X^d : x_i = x_j\}$. Take any $0 < \delta < \inf_{\underline{x} \in U} \min_{i \neq j} \rho(x_i, x_j)$. We use the same argument as before, to show that a.e. $\underline{x}$ visits U infinitely often under action of T_d. Combining the two, we obtain that μ_d-a.e. d-tuple is δ-LY. □

Remark 3.3. *It is clear from the proof that the statement of Lemma 3.2 holds for any $\delta_2 < diam(X)$.*

4. Li-Yorke tuples in the topological setting

Definition 4.1. *Let T be a continuous map on compact metric space (X, ρ). We say that T is* topologically mixing *if for every pair of nonempty open sets $U, V \subset X$, $T^n(U) \cap V \neq \emptyset$ for all n sufficiently large. We say that T is* topologically weak mixing *if $T \times T$ is transitive on $X \times X$.*

The following fact is standard and its utility to Li-Yorke chaos dates back at least to works of Iwanik [20].

Lemma 4.2. *Let (X, ρ) be a compact metric space with at least two points and let $T \colon X \to X$ be topologically weakly mixing. Then for every $d > 1$ there is $\delta_d > 0$ such that the set of δ_d-LY d-tuples is residual.*

Proof. Since X is not a singleton, topological mixing implies that X is an infinite set without isolated points. Fix a sequence of pairwise distinct points $\{a_i\}_{i=1}^{\infty}$ and let $\delta_d = \inf_{1 \leq i < j \leq d} \rho(a_i, a_j)/3$. Now, any d-tuple with orbit dense in X^d is δ_d-LY and the set of such tuples is residual by transitivity of T_d. □

Let us recall an important fact which can be derived from works of Kuratowski and Mycielski (see *e.g.* [32]). We recall that $M \subset X$ is called a *Mycielski set* if it is a countable union of Cantor sets.

Theorem 4.3 (Kuratowski-Mycielski)**.** *Let X be a perfect complete metric space, and assume that R_k is a residual subset of X^{n_k}, where $n_k \geq 2$ for each $k \in \mathbb{N}$. Then there exists a Mycielski set M dense in X such that for each $k \in \mathbb{N}$ if points $x_1, \cdots, x_{n_k} \in M$ are pairwise distinct then $(x_1, \cdots, x_{n_k}) \in R_k$.*

The following fact seems to be well known (*e.g.* see [24, Example 4.6] and historical remarks therein), but since the proof is simple, we sketch it for completeness.

Lemma 4.4. *Let (X,ρ) be a compact metric space with at least two points and let $T\colon X\to X$ be topologically weakly mixing. Then there is a sequence $\{\delta_d\}_{d=2}^{\infty}\subset(0,1)$ and a Mycielski set $M\subset X$ such that for any $d\geq 2$, any d pairwise distinct points $x_1,\dots,x_d\in M$ and any integers $s_1,\dots,s_d\geq 0$ the d-tuple $(T^{s_1}(x_1),\dots,T^{s_d}(x_d))$ is δ_d-LY. Additionally, $\omega(x,T)=X$ for every $x\in M$.*

Proof. First observe, that if LY_d^{δ} is residual then by weak mixing also $T^{-s_1}\times\ldots\times T^{-s_d}(\mathrm{LY}_d^{\delta})$ is residual for any choice of integers $s_1,\dots,s_d\geq 0$. This produce for every d a countable family of residual subsets in X^d. Combining Lemma 4.2 with Theorem 4.3 and the fact that the set of points with dense orbit is residual in X, we obtain the desired Mycielski set. □

Let $\Sigma_2^+=\{0,1\}^{\mathbb{N}}$ be endowed with the standard prefix metric, and let σ be the left shift on Σ_2^+. The following fact was proved by Moothathu in [31] (which extends the existence of horseshoes argument from Misiurewicz & Szlenk [30] to the C^0 setting ensuring that the map π is really injective everywhere).

Lemma 4.5. *If a map $T:[0,1]\to[0,1]$ has positive topological entropy, then there exist $n>0$, a T^n-invariant closed set Λ and a homeomorphism $\pi\colon\Lambda\to\Sigma_2^+$ such that $\pi\circ(T^n|_{\Lambda})=\sigma\circ\pi$.*

Theorem 4.6. *Let $T\colon[0,1]\to[0,1]$ be topologically mixing. Then there exist dense Mycielski sets M_1,M_2,M_3 and numbers $\delta_n>0$ such that:*

1. *each n-tuple consisting of $n\geq 2$ distinct points in M_i is δ_n-LY, where $i=1,2$,*
2. *every point in M_1 has dense orbit in $[0,1]$,*
3. *there exists a minimal Cantor set A such that $\omega(x,T)=A$ for every $x\in M_2$ and $M_2\cap A$ contains a Cantor set,*
4. *$M_3\times M_3$ contains no Li-Yorke pairs.*

Proof. The set M_1 is obtained by a direct application of Theorem 4.4. Constructing the set M_2 requires a little more work.

It is known that every mixing map on the unit interval has positive topological entropy, so by Lemma 4.5 we can find $n>0$ and a T^n-invariant set Λ on which the dynamics is conjugate to the full one-sided shift Σ_2^+. Clearly, we may assume that $\Lambda\subset(0,1)$. Take any minimal weakly mixing subset of Σ_2^+ (*e.g.* one-sided version of the Chacón flow [5]) and let us denote it by A_n. Passing through the conjugating homeomorphism, we may assume that $A_n\subset\Lambda$. Clearly it is a Cantor set as a perfect subset of the Cantor set Σ_2^+ (up to conjugating homeomorphism) and also it is not hard to see that $A=\bigcup_{j=0}^{n-1}T^j(A_n)$ is a minimal subsystem of T. Let us apply Theorem 4.4 to the dynamical system $T^n|_{A_n}$, and let numbers $\delta_d>0$ and a Cantor set $C\subset A_n\subset A$ be such that for any $d>1$, any d pairwise distinct points $x_1,\dots,x_d\in C$ and any integers $s_1,\dots,s_d\in n\mathbb{N}\cup\{0\}$, the tuple $(T^{s_1}(x_1),\dots,T^{s_d}(x_d))$ is δ_d-LY for T^n (clearly, it is also δ_d-LY for T).

Since C is homeomorphic to C^2 we can find pairwise disjoint Cantor sets $\{C_i\}_{i=0}^{\infty}$ such that $\cup_i C_i\subset C$. There is also $\varepsilon>0$ such that $C\subset(\varepsilon,1-\varepsilon)$. Let $\{U_i\}_{i=1}^{\infty}$ be a sequence composed of all open subintervals of $[0,1]$ with rational endpoints. Since T is mixing, for every i there is k_i such that $T^{nk_i}(U_i)\cap[0,\varepsilon]\neq\emptyset$ and $T^{nk_i}(U_i)\cap$

$[1-\varepsilon, 1] \neq \emptyset$. It is known that if an image of a compact set via a continuous map is uncountable then there is a Cantor set on which this map is one-to-one (see *e.g.* [39, Remark 4.3.6]). Hence, for every i there is a Cantor set $D_i \subset U_i$ such that $T^{nk_i}|_{D_i}$ is one-to-one and $T^{nk_i}(D_i) \subset C_i$. Denote $M_2 = C_0 \cup \bigcup_{i=1}^{\infty} D_i$. Since for every $x \in M_2$ there is $j \geq 0$ such that $T^j(x) \in A$ and A is a minimal set, we immediately obtain that $\omega(x, T) = A$ for every $x \in M_2$.

Let us take any tuple $(x_1, \dots, x_d)$ of pairwise distinct points in M_2. There are numbers $i_1, \dots, i_d$ such that $x_j \in D_{i_j}$. Denote $z_j = T^{nk_{i_j}}(x_j) \in C_j$ and observe that points z_j are also pairwise distinct. Let $m = \max_j k_{i_j}$ and denote $s_j = m - k_{i_j}$. Now, it is enough to note that

$$\begin{aligned} \rho(T^{nm}(x_p), T^{nm}(x_q)) &= \rho(T^{n(s_p+k_{i_p})}(x_p), T^{n(s_q+k_{i_q})}(x_q)) \\ &= \rho(T^{ns_p}(z_p), T^{ns_q}(z_q)). \end{aligned}$$

The construction of M_2 completed by the definition of the set C.

To construct M_3, let us first recall that every Sturmian minimal system does not contain Li-Yorke pairs [6, Example 3.15] (it is a so-called almost distal system) and Σ_2^+ contains an uncountable family of pairwise disjoint Sturmian systems [22, Proposition 4.44]. Repeating the conjugacy argument from the previous step we may view Sturmian minimal subshift M (which is a Cantor set) as a subset of $[0, 1]$. Note that since M is an infinite minimal system for T^n without Li-Yorke pairs, the set $\hat{M} = \bigcup_{j=0}^{n-1} T^j(M)$ is also minimal, and does not contain Li-Yorke pairs (sets $T^j(M), T^i(M)$ for $i \neq j$ are either disjoint of equal).

Proceed in the same way as in the construction of M_2 to obtain a dense Mycielski set M_3 such that every point $x \in M_3$ is eventually transformed into a point $z_x \in M$. Now, take any $x, y \in M_3$ and k sufficiently large, so that $T^k(x), T^k(y) \in \hat{M}$. If $T^k(x) \neq T^k(y)$ then (x, y) is not Li-Yorke pair by the definition of $\hat{M}$ and if $T^k(x) = T^k(y)$ then (x, y) is not Li-Yorke either. □

Proof of Theorem C. As a direct consequence of theorem by Oxtoby & Ulam (see [33, Theorem 9]) we obtain that if $B \subset [0, 1]$ is a dense Mycielski set then there exists a homeomorphism $\phi \colon [0, 1] \to [0, 1]$ such that $\phi(B)$ has full Lebesgue measure.

In fact, if $A, B \subset [0, 1]$ are dense in $[0, 1]$ and either of them is the union of pairwise disjoint Cantor sets, then there is an increasing homeomorphism $\varphi \colon [0, 1] \to [0, 1]$ such that $\varphi(A) \subseteq B$.

Note that the dense Mycielski sets M_1, M_2, M_3 in Theorem 4.6 are pairwise disjoint. In particular, if one of them has full Lebesgue measure, the others have measure zero. Hence Theorem C is a direct consequence of Theorem 4.6. □

The above theorem shows that in interval dynamics, Cantor sets that are attracting and not attracting always coexist, but their "physical" visibility depends on the special structure of the map. In fact, using the sets M_1, M_2, M_3 from Theorem 4.6, we can distribute Lebesgue measure in any proportion between Li-Yorke pairs made of points with dense orbits, Li-Yorke pairs with a Cantor attractor, or pairs which are not Li-Yorke.

Remark 4.7. *If $f \in C^3_{nf}(I)$ then by Theorem E in* [40]*, any minimal set must have zero Lebesgue measure. Hence, the situation described in Theorem C(4) can never occur for these maps.*

5. The Manneville-Pomeau map

5.1. **Inducing.** Suppose that every non-singular Borel measure $\mu \ll \lambda$ for T is σ-finite and infinite. Then Theorem 2.9 indicates that λ is not full, so the results of Section 3 are not applicable in this case. Another approach to address the question of Li-Yorke d-tuples is via inducing. The idea is to choose an appropriate subset $Y \subset X$ and consider the (first return) induced map (Y, F), such that Y can be decomposed in countably many intervals $\mathcal{Z} := \{Y_j\}_{j\in\mathbb{N}}$ with $\lambda(Y \setminus \cup_j Y_j) = 0$ and such that $F(Y_j) = T^{\tau_j}(Y_j) = Y$ where the first return time $\tau(x) = \min\{n \geq 1 : T^n(x) \in Y\}$ is constant τ_j on Y_j. Assume that the *distortion* of the Jacobian is bounded uniformly in the iterate n *i.e.,*

$$\sup_{n\in\mathbb{N}} \sup_{Z\in\bigvee_{i=0}^{n-1} F^{-i}\mathcal{Z}} \sup_{x,y\in Z} \frac{J_{F^n}(x)}{J_{F^n}(y)} < \infty. \tag{5.1}$$

We take the Jacobian w.r.t. Lebesgue measure λ: for any j, the push-forward measure $\lambda(F(A))$ for $A \subset Y_j$ is well-defined and absolutely continuous with respect to λ. Hence, $J_F = \frac{d\lambda\circ F}{d\lambda}$ is well-defined for λ-a.e. $x \in Y_j$. Similarly, we can define J_{F^n} on any set $Z \in \bigvee_{i=0}^{n-1} F^{-i}\mathcal{Z}$. By the definition $Y = \cup_j Y_j \pmod 0$ and sets Y_j are in practice intervals, hence we may view each Jacobian J_{F^n} as a function defined on Y.

In the case of bounded distortion (Y, F) preserves an absolutely continuous invariant probability Borel measure. Let us call this measure ν; its density $\frac{d\nu}{d\lambda}$ is bounded and bounded away from 0. It projects to a T-invariant measure

$$\mu(A) = \sum_n \sum_{i=0}^{n-1} \nu(T^{-i}(A) \cap \{y \in Y : \tau(y) = n\}), \tag{5.2}$$

which can be normalized if the normalizing constant $\int_Y \tau \, d\nu < \infty$. Otherwise, μ is σ-finite, see *e.g.* [7, Chapter 6]. The measure ν is ergodic and exact, and these properties carry over to (X, T, μ) (for exactness to carry over, we need the additional condition that greatest common divisor of all $\tau(x)$ over $x \in [0, 1]$ is 1). In fact, $\mu|_Y$ and $\nu|_Y$ differ by a fixed constant, regardless whether $\int_Y \tau \, d\nu < \infty$ or not, because $F : Y \to Y$ is the first return map, see (5.2).

If the tail $\nu(\{y : \tau(y) > n\})$ is sufficiently heavy (and regular), then the probability of two independently chosen initial points to return to Y at the same time infinitely often under iteration of T can be zero.

Let us denote

$$u_n = \lambda(\{y \in Y : T^n(y) \in Y\})$$

and observe that

$$\lambda_d(\{y \in Y^d : T_d^n(y) \in Y^d\}) = u_n^d.$$

Since $\frac{d\mu}{d\lambda}$ is bounded and bounded away from zero on Y, we can freely interchange μ and λ in these formulas. Therefore, if

$$\sum_n u_n^d < \infty, \tag{5.3}$$

then the Borel-Cantelli Lemma gives

$$\begin{aligned} 0 &= \mu_d(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}\{y \in Y^d : T_d^n(y) \in Y^d\}) \\ &= \mu_d(\{x \in Y^d : T_d^n(x) \in Y^d \text{ infinitely often}\}). \end{aligned}$$

The following recurrence lemma seems to be standard (see [1, Proposition 1.2.2]). However we could not find exact reference in the literature and hence decided to provide a proof for completeness.

Lemma 5.1. *Let μ be a σ-finite, non-singular Borel measure. If there exists $Y \in \mathcal{B}$ such that $0 < \mu(Y) < \infty$ and*

$$\sum_{n\geq 0} \mathcal{L}^n 1_Y = \infty \qquad \mu\text{-a.e. on } Y \tag{5.4}$$

(where $\mathcal{L}$ is the Perron-Frobenius operator), then Y is a recurrent set in the sense that

$$Y \subseteq \bigcup_{n\geq 1} T^{-n}(Y) \pmod{\mu}. \tag{5.5}$$

Moreover, if μ is ergodic and T-invariant, then μ is conservative.

Proof. First, we need to show that the set $A := Y \setminus \bigcup_{n\geq 1} T^{-n}(Y)$ of points which never return to Y has measure zero. We must have $A \cap T^{-j}(A) = \emptyset$ for every $j \geq 1$ because $A \subseteq Y$. This immediately implies that A is *wandering* for T, that is, the preimages $T^{-n}(A)$, $n \geq 0$, are pairwise disjoint. Directly from the definition of Perron-Frobenius operator $\mathcal{L}$ we have $\int_A \mathcal{L}^n 1_Y \, d\mu = \mu(Y \cap T^{-n}(A))$, hence

$$\int_A \sum_{n\geq 0} \mathcal{L}^n 1_Y \, d\mu = \sum_{n\geq 0} \mu(Y \cap T^{-n}(A)) \leq \mu(Y) < \infty,$$

By assumption (5.4) we obtain that $\mu(A) = 0$ which ends the proof of (5.5).

As a consequence of (5.5) we see that the set $E := \bigcup_{n\geq 0} T^{-n}(Y)$ satisfies $T^{-1}(E) = E \pmod{\mu}$. Since $Y \subset E$, we have $\mu(E) > 0$. If T is ergodic, then we can conclude that $X = \bigcup_{n\geq 0} T^{-n}(Y) \pmod{\mu}$. This, since T is measure preserving transformation, implies that assumptions of Maharam's Recurrence Theorem are satisfied (see *e.g.* [1, Theorem 1.1.7]) which thus ensures that T is conservative. □

Proposition 5.2. *Let λ be a non-singular exact Borel probability measure, and μ a T-invariant σ-finite measure equivalent to λ. If there exists a set $Y \subset X$ recurrent in the sense of* (5.5) *such that $\mu(Y) > 0$ and the first return map (Y, F) has only onto branches with bounded distortion (in the sense of* (5.1)*) then the following conditions are equivalent for every integer $d \geq 1$:*

(1) $\sum_n u_n^d = \infty$, where $u_n = \lambda(\{y \in Y : T^n(y) \in Y\})$,
(2) d-fold product measure λ_d is ergodic and conservative.

Proof. To prove (2) ⇒ (1) let us first observe, that if λ_d is conservative then

$$\begin{aligned} 0 < \lambda_d(Y^d) &= \lambda_d(\{\underline{y} \in Y^d : T_d^n(\underline{y}) \in Y^d \text{ infinitely often}\}) \\ &= \lambda_d(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}\{\underline{y} \in Y^d : T_d^n(\underline{y}) \in Y^d\}) \end{aligned}$$

and hence by the Borel-Cantelli Lemma (see (5.3)) we must have $\sum_n u_n^d = \infty$.

To prove (1) $\Rightarrow$ (2), denote by $\mathcal{L}$ and $\mathcal{P}$ the Perron-Frobenius operator for T and F, respectively, both w.r.t. λ. Then for every $n > 0$ and a.e. $x \in Y$ we have

$$\mathcal{P}^n f(x) = \sum_{y \in F^{-n}(x)} \frac{f(y)}{J_{F^n}(y)}, \tag{5.6}$$

where J_{F^n} is the Jacobian w.r.t. λ. The bounded distortion of J_{F^n} applied to (5.6) allows us to verify that there is a constant $\kappa > 0$ such that for every measurable $A \subset X$ and $n > 0$ we have

$$\inf_{x \in A} \sum_{k=0}^{n} \mathcal{P}^k 1_Y(x) \geq \kappa \cdot \sup_{x \in A} \sum_{k=0}^{n} \mathcal{P}^k 1_Y(x).$$

A similar estimate for the d-fold product system with Perron-Frobenius operator $\mathcal{P}_d$,

$$\inf_{\underline{x} \in A} \sum_{k=0}^{n} \mathcal{P}_d^k 1_{Y^d}(\underline{x}) \geq \kappa^d \cdot \sup_{\underline{x} \in A} \sum_{k=0}^{n} \mathcal{P}_d^k 1_{Y^d}(\underline{x}).$$

Hence, if there is a set $A \subset Y$ of positive measure such that $\sum_{k=0}^{n} \mathcal{P}_d^k 1_{Y^d}(\underline{x})$ is uniformly bounded for all $n \geq 0$ and $\underline{x} \in A$, then $\sum_{k=0}^{n} \mathcal{P}_d^k 1_{Y^d}(\underline{x})$ is uniformly bounded for all $n \geq 0$ and λ-a.e. $\underline{y} \in Y$.

Observe that for any Borel set $A \subset Y^d$ we have

$$\sum_{k=0}^{n} \int_A \mathcal{L}_d^k 1_{Y^d} \, d\lambda_d \leq \sum_{k=0}^{n} \int_A \mathcal{P}_d^k 1_{Y^d} \, d\lambda_d \leq \sum_{k=0}^{\infty} \int_A \mathcal{L}_d^k 1_{Y^d} \, d\lambda_d.$$

If there exists a set $A \subset Y^d$ with $\lambda_d(A) > 0$ and $M > 0$ such that $\sum_{k \geq 0} \mathcal{L}_d^k 1_{Y^d}(\underline{x}) < M$ for every $\underline{x} \in A$ then there is $c > 0$ such that

$$M \geq \sum_{k \geq 0} \int_A \mathcal{P}_d^k 1_{Y^d} \, d\lambda_d \geq c \sum_{k \geq 0} \int_{Y^d} \mathcal{P}_d^k 1_{Y^d} \, d\lambda_d \geq c \sum_{k \geq 0} \int_{Y^d} \mathcal{L}_d^k 1_{Y^d} \, d\lambda_d.$$

Consequently, divergence of $\sum_{k \geq 0} u_k^d = \sum_{k \geq 0} \int_{Y^d} \mathcal{L}_d^k 1_{Y^d} \, d\lambda_d$ implies that

$$\sum_{k \geq 0} \mathcal{L}_d^k 1_{Y^d} = \infty \qquad \lambda_d\text{-a.e. on } Y^d.$$

By Lemma 2.5 (and here we need the assumption that λ is exact), λ_d is ergodic with respect to T_d for every $d \geq 1$. Recall that μ_d is a T_d-invariant σ-finite measure. But λ_d is ergodic and μ_d is equivalent to λ_d, hence μ_d is ergodic as well. Applying Lemma 5.1 to μ_d and T_d, we obtain that μ_d is conservative, and using once again equivalence of μ_d and λ_d we conclude that λ_d is conservative as well. This completes the proof. □

5.2. **Manneville-Pomeau maps.** The classical example from interval dynamics where the u_n can be computed is the Manneville-Pomeau family, where Lebesgue measure λ will be our reference measure for μ. These are interval maps with a neutral fixed point and the inducing is with respect to a set Y bounded away from this fixed point. For us, it is convenient to use the family $T_\alpha : [0,1] \to [0,1]$ defined by

$$T_\alpha(x) = \begin{cases} x(1 + 2^\alpha x^\alpha) & \text{if } x \in [0, \frac{1}{2}), \\ 2x - 1 & \text{if } x \in [\frac{1}{2}, 1] =: Y. \end{cases}$$

It has a neutral fixed point at 0, and the first return map $F : Y \to Y$ is uniformly expanding with bounded distortion (uniformly in all iterates, in the sense of (5.1), see *e.g.* [28]), $F\colon Y \to Y$ preserves an ergodic Borel measure which pull back to an ergodic and conservative T_α-invariant measure μ, see Theorem 1 and Corollary 2 in [42].

If $y_0 = \frac{1}{2}$ and y_{n+1} is the unique point in $T_\alpha^{-1}(y_n) \cap [0, y_n]$, then $y_n \searrow 0$ such that

$$y_n \sim n^{-\beta} \qquad \text{for } \beta = \frac{1}{\alpha},$$

where $a_n \sim b_n$ stands for $\lim_n a_n/b_n \in (0,\infty)$, see [14]. If $y'_{n+1} \in [\frac{1}{2}, 1]$ is the other preimage of y_n, then $[\frac{1}{2}, y'_{n+1}) = \{y \in Y : \tau(y) \geq n+2\}$. By Remark 5.3, $\mu(\{y \in Y : \tau(y) \geq n\}) \sim |y'_{n-2} - \frac{1}{2}| = \frac{1}{2} y_{n-1} \sim n^{-\beta}$ and therefore

$$\begin{aligned} \int \tau d\mu &= \sum_n n\mu(\{y \in Y : \tau(y) = n\}) \\ &= \sum_n \mu(\{y \in Y : \tau(y) \geq n\}) \begin{cases} < \infty & \text{if } \alpha \in (0,1); \\ = \infty & \text{if } \alpha \geq 1. \end{cases} \end{aligned} \tag{5.7}$$

Remark 5.3. *More precise estimates on the invariant density h were given by Thaler* [42, Corollary 1]*, who showed that $h(x)x^\alpha$ is bounded and bounded away from* 0*. In addition (see* [27, Lemma 2.3]*) for $\alpha \in (0,1)$ the density $h = \frac{d\mu}{d\lambda}$ is Lipschitz outside a neighborhood of* 0 *and μ is mixing (see* e.g. [28, Theorem 7]*).*

Theorem 5.4. *Assume that $\alpha > 1$ and write $\beta = 1/\alpha \in (0,1)$. Then $u_n \sim n^{\beta-1}$. In particular, $\sum_n u_n^d = \infty$ if and only if $\alpha \leq \frac{d}{d-1}$.*

Proof. From the above considerations, we have for the Manneville-Pomeau map that

$$\mu(x \in Y : \tau(x) = n) \sim \lambda([y_n, y_{n-1}]) \sim (n-1)^{-\beta} - n^{-\beta} \sim n^{-(1+\beta)}.$$

Then the estimate on u_n is a special case of the results of Gouëzel [18], partly correcting results from Doney [16] (see [18, Section 1.3]). More precisely, [18, Proposition 1.7] applied to $u = v = 1_Y$ gives $u_n = \int_Y u \cdot v \circ T^n d\mu \sim n^{\beta-1}$. Now it follows immediately that $\sum_n u_n^d = \infty$ if and only if $\alpha \leq \frac{d}{d-1}$. □

Proof of Theorem B. Recall that by Remark 5.3 for $\alpha < 1$ the map T_α preserves a mixing absolutely continuous probability Borel measure, and hence the result follows by Lemma 3.2 (see also Remark 3.3).

If $\alpha \geq 1$, then there is an infinite σ-finite Borel measure $\mu \sim \lambda$ (by Remark 5.3, the density of μ is bounded and bounded away from 0). More precisely, using Remark 5.3 again, $\mu([y_k, 1]) < \infty$ for fixed k, but $\mu([y_l, y_k]) \to \infty$ as $l \to \infty$. This means that given a neighborhood U of the fixed point 0, Lebesgue-a.e. point x spends almost every iterate in U (*i.e.*, $N(x, U) := \{n \geq 0 : T_\alpha^n(x) \in U\}$ has density 1 for ν-a.e. x). Indeed, for $U = [0, y_k)$, we have

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \#\{0 \leq i < n : T_\alpha^i(x) \in [y_k, 1]\} &\leq \lim_{n\to\infty} \frac{\#\{0 \leq i < n : T_\alpha^i(x) \in [y_k, 1]\}}{\#\{0 \leq i < n : T_\alpha^i(x) \in [y_l, 1]\}} \\ &= \frac{\mu([y_k, 1])}{\mu([y_l, 1])} \to 0 \text{ as } l \to \infty, \end{aligned}$$

by the Ratio Ergodic Theorem (see *e.g.* [1, Theorem 2.2.5.]). It follows that λ_d-a.e. d-tuple is proximal along a subsequence.

Next, we claim that λ_2-a.e. every pair is LY. This follows from the fact that T_α is expanding away from 0. More precisely, for every $x, y \in [0,1]$ which are not eventually mapped to one another, $\rho(T_\alpha^i(x), T_\alpha^i(y)) \le \frac{1}{3}$ implies that $\rho(T_\alpha^{i+1}(x), T_\alpha^{i+1}(y)) > \rho(T_\alpha^i(x), T_\alpha^i(y))$. Therefore (x,y) is $\frac{1}{3}$-separated.

Now let $d \ge 3$, $1 \le \alpha < \frac{d-1}{d-2}$ and choose $\delta < 1/2(d-2)$. By Theorem 5.4 applied to $d-1$ coordinates, $\sum_n u_n^{d-1} = \infty$ and hence, $([0,1]^{d-1}, \lambda_{d-1}, (T_\alpha)_{d-1})$ is conservative by Proposition 5.2. (Since the induce times of the right-most two branches of the induced map F are 1 and 2, *i.e.,* with greatest common divisor 1, the exactness requirement in Proposition 5.2 is fulfilled.) Therefore, taking $A \subset Y^{d-1}$ with $\mu_{d-1}(A) > 0$, for μ_{d-1}-a.e. $\underline{x}$, there is a sequence (t_n) such that $((T_\alpha)_{d-1})^{t_n}(\underline{x}) \in A$ for all $n \in \mathbb{N}$. The set

$$A = [\frac{1}{2}, \frac{1}{2} + \eta] \times [\frac{1}{2} + \frac{1}{2(d-2)}, \frac{1}{2} + \frac{1}{2(d-2)} + \eta] \times \cdots \times [1-\eta, 1]$$

has positive λ_{d-1}-measure, and clearly points $\underline{y} \in A$ are $\frac{1}{2(d-2)} - 2\eta$-separated $(d-1)$-tuples. Choosing η so small that $\frac{1}{2(d-2)} - 2\eta > \delta$ then shows that the $d-1$-tuple $\underline{x}$ is δ-separated along a subsequence (t_n).

We claim that the remaining coordinate x_d is close to 0, or more precisely: for λ-a.e. x_d we have $T_\alpha^{t_n}(x_d) < \frac{1}{2} - \delta$ infinitely often. Indeed, take $U_N := \{x_d \in [0,1] : T_\alpha^{t_n}(x_d) \ge \frac{1}{2} - \delta$ for all $n \ge N\}$ and let $x_d^* \in U_N$ be arbitrary. Let $H_n \ni x_d^*$ be the maximal interval such that $T_\alpha^{t_n}(H_n) = [\frac{1}{2}(\frac{1}{2} - \delta), 1]$ and $H_n' \subset H_n$ is such that $T_\alpha^{t_n}(H_n') = [\frac{1}{2}(\frac{1}{2} - \delta), \frac{1}{2} - \delta)$. Since t_n is increasing, we clearly have $\lim_{n\to\infty} |H_n| = 0$. The maps $T_\alpha^{t_n} : H_n \to [\frac{1}{2}(\frac{1}{2} - \delta), 1]$ have uniformly bounded distortion (this is proved in virtually the same way as the distortion bound for the branches of F^k is proven, cf. [28]). Therefore there is $K > 0$ such that $|H_n'| > |H_n|/K$ for all $n \in \mathbb{N}$. Since $U_N \cap H_n' = \emptyset$ for $n \ge N$, it follows that x_d^* cannot be a Lebesgue density point of U_N. This means that $\lambda(U_N) = 0$ and hence $\lambda(\cup_N U_N) = 0$ as well, proving the claim. Using Fubini's Theorem, we conclude that λ_d-a.e. d-tuple is indeed δ-separated along a subsequence.

Finally, if $\alpha > \frac{d-1}{d-2}$, typical $(d-1)$-tuples $\underline{x} = (x_1, \dots x_{d-1})$ visit Y simultaneously only finitely often. Now let $\varepsilon = y_k$ and assume by contradiction that $\underline{x}$ visits $[\varepsilon, 1]^{d-1}$ infinitely often for a set of $\underline{x}$ of positive λ_{d-1}-measure. Hence, there are integers $a_1, \dots a_{d-1} \in \{0, 1, \dots, k\}$ such that $T_\alpha^{n+a_1}(x_1), \dots, T_\alpha^{n+a_{d-1}}(x_{d-1}) \in Y$ for infinitely many n, still for a set U of positive λ_{d-1}-measure. Take $a = \max_i a_i$ and for each $\underline{x} \in U$ take a $d-1$-tuple $\underline{y}$ with coordinates $y_i \in T_\alpha^{a_i - a}(x_i) \cap Y$. Since T_α is non-singular (and the Cartesian product $\prod_{i=1}^{d-1} T_\alpha^{a-a_i}$ is non-singular too), we have $\lambda_{d-1}(\{\underline{y} : \underline{x} \in U\}) > 0$. Now for every $\underline{y}$, there is an infinite sequence $(n_k)_{k\in\mathbb{N}}$ (depending on $\underline{y}$ but not on the index $i = 1, \dots, d-1$), such that $T_\alpha^{n_k}(y_i) = T_\alpha^{n_k + a_i - a}(x_i) \in Y$ for each i and k. This contradicts the first statement of this paragraph.

Coming back to a typical d-tuple $(x_1, \dots, x_d)$, the above argument shows that no matter how we select a $d-1$-tuple $\underline{x}$ from it, for all sufficiently large n, at least one coordinate $T_\alpha^n(x_i) \in [0, \varepsilon)$. Therefore at least two coordinates of the d-tuple belong to $[0, \varepsilon)$. Since $\varepsilon > 0$ can be taken arbitrary small, the proof is complete. □

Remark 5.5. *We can replace the right branch of the Manneville-Pomeau map by* $2(1-x)$ *if a continuous map is preferred. The dynamical properties that we are concerned with remain the same. The same is true, if we view Manneville-Pomeau map as a continuous map on the unit circle.*

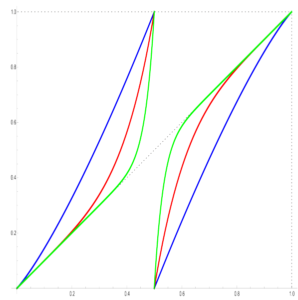

Figure 2. Graph of $T_{\alpha,\beta}$ for $\alpha = \beta = \frac{1}{2}, 4, 15$.

Remark 5.6. *One can increase the number of neutral fixed points,* e.g. *define the map*

$$T_{\alpha,\beta}(x) = \begin{cases} x + 2^\alpha x^{1+\alpha} & \text{if } x \in [0, \frac{1}{2}), \\ x - 2^\beta (1-x)^{1+\beta} & \text{if } x \in [\frac{1}{2}, 1], \end{cases}$$

see Figure 2, and consider the Li-Yorke behavior of tuples for this map. If $\alpha \geq 1$ *and* $\alpha > \beta$*, then neutral fixed point* 0 *dominates, and we expect the same behavior as in Theorem B. For the case* $\alpha = \beta \geq 1$*, we expect that* λ_3*-a.e.* 3*-tuple is Li-Yorke (where for typical triples* (x_1, x_2, x_3)*, there are infinitely many* n *with* $T^n(x_1) \approx 0$*,* $T^n(x_2) \approx 1$ *and* $T^n(x_3) \in [\frac{1}{3}, \frac{2}{3}]$*, as well as there are infinitely many* m *with* $T^m(x_1) \approx T^m(x_2) \approx T^m(x_3) \approx 0$*. Conjecturally, for* $d \geq 4$*, typical* d*-tuples are Li-Yorke if and only if* $\alpha \leq \frac{d-2}{d-3}$*.*

The above analysis can possibly be continued if the number of neutral fixed points increases even more. Only the neutral fixed points with largest order of contact matter; if there are N *such points with parameters* $\alpha_1 = \alpha_2 = \cdots = \alpha_N \geq 1$ *and these are larger than the parameter of any other neutral fixed point, then we conjecture* T *to have typical* d*-fold Li-Yorke tuples if* $\alpha_i \leq \frac{d-N}{d-N-1}$*. Less clear is whether all* $N \geq 3$ *neutral fixed points can be treated equally the way it is possible to do so for the two of* $T_{\alpha,\beta}$ *(using the symmetry* $x \mapsto 1-x$ *that holds there when* $\alpha = \beta$*).*

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