A renewal scheme for non-uniformly hyperbolic semiflows

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Abstract

We investigate a renewal scheme for non-uniformly hyperbolic semiflows that closely resembles the renewal scheme developed in the discrete time case, in order to obtain sharp estimates for the correlation function. The involved observables are supported on a flow-box of unbounded length and the present abstract setting does not require the use of Markov structure. However, the type of Dolgopyat inequality used here as an abstract hypothesis is at present only known for suspension flows over Markov maps.

1 Introduction

Mixing is a delicate phenomenon for flows. Exponential decay of correlations for Hölder observables has been established for Anosov flows with C^1 stable and unstable foliations in [8] and for contact Anosov flows in [14]. Building upon the techniques developed in these works, exponential decay of correlations has been later established for 'less smooth' or Markov systems (see, for instance, [5, 4, 3]).

The situation for superpolynomial decay of correlations (rapid mixing) is somewhat better. The work [9] established rapid mixing for (nontrivial) basic sets for typical Axiom A flows. This was extended in [17] to non-uniformly hyperbolic flows given by a suspension over a Young tower with exponential tails [24].

The recent work [21] develops an operator renewal theory framework for flows and applies this to the study of mixing properties of (non-uniformly hyperbolic flows that can be modeled as) suspension semiflows over Gibbs Markov maps. For this class of continuous time systems, [21] obtains: a) upper and lower bounds for polynomial decay of correlation in the finite measure preserving case and b) sharp mixing rates in the infinite measure preserving case.

The results obtained in [21] for infinite measure preserving suspension semiflows over Gibbs Markov maps (satisfying certain assumptions among which regular variation for the roof function

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is a must) are the direct analogue of the results in [20]. Polynomial *upper* bounds on the correlation for semiflows over Gibbs Markov maps (*e.g.* the class of flows that can be modeled as suspensions over maps with indifferent fixed points as in [15]) have previously been established in [18]. In [21], the authors show that for a large class of systems considered in [18], the established mixing rates are sharp; namely using operator renewal theory techniques they obtain lower and upper bounds. Although the method of proof is significantly different from the discrete time scenario, the results in [21] on decay of correlation for finite measure preserving suspension semiflows over Gibbs Markov maps (with polynomial roof function) are again the direct analogue of the results in the discrete time set-up [11, 22].

In the setting of Gibbs Markov semiflows the results of [21] are optimal, so this paper cannot improve on them. Instead, the aim of this paper is to investigate a different renewal scheme for semiflows that more closely resembles the renewal scheme developed in the discrete time case. As explained below, the main new elements are that we a) induce to a dynamical system with better mixing properties, and b) relate twisted transfer operators to inverse Laplace transforms, which allows us to show that the main techniques/computations used in the discrete time setting carry over to the continuous time case. Contrary to the results obtained in [21], this method allows us to study decay of correlation for observables that are supported on a flow-box of unbounded length. The abstract framework developed here does not require the use of Markov structure and allows us to obtain optimal results for observables supported on a flow-box of unbounded length. However, the type of Dolgopyat inequality used as abstract hypothesis is at present only known for suspension flows over Markov maps.

We provide an abstract framework similar in structure to the ones developed for discrete time systems. In Section 4 we list a set of hypotheses (H0)-(H6), with versions for the finite and infinite measure setting, under which the main theorems in Section 5, namely Theorems 5.1 and 5.2 for the finite and infinite measure setting respectively, give optimal bounds for the correlation function.

The main ingredients are:

(I) The type of renewal equation established in [21] (see Proposition 3.1), or more precisely, the argument used in establishing a renewal equation for flows in [21].

(II) A new inducing scheme which resembles the inducing scheme employed in the discrete time scenario, namely we induce to a hyperbolic map with exponential decay, see Section 2. The inducing scheme used in [21] involves observables supported on a flow-box of unit length, and the action of the inducing scheme in the flow direction is somewhat trivial. In contrast to inducing to a thickened Poincaré section as in [21], we induce to a flow-box $\tilde{Y}$ with in principle unbounded flow-time. We induce in such a way that the induced version of the semiflow is a uniformly hyperbolic map Φ, acting non-trivially in all dimensions, by forcing expansion in the flow direction.

The choice for the present inducing scheme creates certain technical complications that are overcome by introducing scaled versions of the measures and observables. Although at first this looks counter-intuitive and considerable complicates the formula for the induce time φ, this choice

ensures, given that R is the transfer operator associated with Φ, its twisted version $R(e^{-s\varphi})$, $\Re s \geq 0$, has the spectral properties required in (H2) and (H3).
(III) We notice that twisted transfer operators can be related to proper Laplace transforms of non delta functions. More precisely, the twisted version $R(e^{-s\varphi}v)$ of the transfer operator associated with Φ, can be related to $\int_0^\infty R_t v e^{-st}\, dt$, where $R_t v = R(1_{\{t<\varphi<t+1\}}v)$. For details we refer to Section 3. This makes it possible to show that many techniques/calculations from the discrete time scenario [22, 11, 20] carry over to the continuous case.

(I)-(III) above allow us to develop an abstract framework based on assumptions on

(H0,1) properties of the region $\tilde{Y}$ and tail estimates of the induce time φ,

(H2,3) functional analytic assumptions for the map Φ, in some appropriate Banach space.

(H4,5) the asymptotic behavior of the integral $\int_t^\infty \|R_\sigma\|\, d\sigma$ for the finite and infinite measure case respectively, and

(H6) a Dolgopyat-type inequality.

To ensure that (H6) holds, we further assume a Diophantine ratio condition (see Section 9.1 for details) for the return time φ , which is natural in this class; see [9, 17, 21])[3].

In Section 9, we provide a simple example with bounded length flow-box that, along with hypothesis (H0)-(H3) and (H6), illustrates the use of hypothesis (H5), which is a relaxation of (H4) for the infinite measure setting. For an example with unbounded length flow-box, we refer to [7].

Notation: We will write $a(t) \ll b(t)$ or $a(t) = O(b(t))$ if there is a constant $C > 0$ such that $a(t) \leq Cb(t)$ for all t. Similarly, $a(t) = o(b(t))$ means that $\lim_t a(t)/b(t) = 0$.

2 A general inducing scheme for flows

2.1 Inducing to a semiflow over an expanding base map

Let g_t be a C^2 semiflow on a manifold $\mathcal{M}$. Let $Y \times \{0\}$ be a section transversal to g_t, and $\tilde{Y} = \cup_{y\in Y}\{y\} \times [0, \tilde{h}(y))$ be a flow-box where the coordinates $\tilde{y} = (y, u)$ are chosen such that within $\tilde{Y}$ the flow becomes parallel and of unit speed:

$$g_t(y, u) = (y, u + t) \quad \text{for } 0 \leq u, u + t \leq \tilde{h}(y).$$

Let

$$\varphi_0 = \min\{t > 0 : g_t(y, 0) \in Y \times \{0\}\}$$

[3] Instead of a Diophantine condition, one could work with assumptions as in [3, 5] and as such obtain a better exponent α, namely $\alpha \in (0, 1)$, in assumption (H6). By working with this sort of assumptions one can establish optimal bounds for the correlation function $\int_{\tilde{Y}} vw \circ f_t d\tilde{\mu}$ for C^m-smooth w, where $m > \alpha$, but not arbitrarily large. We do not consider this sort of assumptions here because we do not exploit the advantage of a smaller m in the proofs of the present abstract results. For a future use of this type of assumption we refer to Remark 6.12.

be the first return time to the section. The function φ_0 can be in $L^1(\mu)$ (the finite case), or not (infinite case); in either case we will put some tail conditions on φ_0.

We assume that the Poincaré map $F = g_{\varphi_0}$ is a uniformly hyperbolic map with partition $\mathcal{P}$, and it preserves a probability measure μ. This means that the total mass of $\mathcal{M}$ is $\bar{\varphi}_0 := \int_Y \varphi_0 d\mu$. We also assume that F is uniformly expanding and has bounded distortion, see (9.1)

In the above, the height function $\tilde{h} : Y \to (0,\infty)$ is defined and $\tilde{h}(y) \leq \frac{1}{2}\varphi_0(y)$ μ-a.e. We will assume that $\tilde{h} \in L^p(\mu)$ for some $p > 1$, that $\inf_{y\in Y} \tilde{h}(y) \geq 1$, and that $\tilde{h}$ is C^2 smooth on each $Z \in \mathcal{P}$. The corresponding suspension semiflow on $\tilde{Y}$ is

$$\tilde{g}_t(y,u) = \begin{cases} (y, u+t) & \text{if } 0 \leq u, u+t < \tilde{h}(y), \\ (Fy, 0) & \text{if } t = \tilde{h}(y) - u, \end{cases}$$

and then continued for $t > \tilde{h}(y) - u$ by the usual group property of a flow.

Remark 2.1. *If $(\mathcal{M}, g_t)$ is itself a suspension flow over some base map $f : X \to X$ with roof function h, then we can take $Y \subset X$, $F = f^\tau : Y \to Y$ is the induced map with induce time τ, and $\varphi_0(y) = \sum_{i=0}^{\tau-1} h \circ f^i(y)$. For example, suspension flows over interval maps with a neutral fixed point (see Section 9 and [7]) fit in this framework.*

We define a return map to $\tilde{Y}$ which complements $F = g_{\varphi_0}$ with an artificial hyperbolic part (the doubling map) in the u-direction. Set

$$K(y) := \frac{2\tilde{h}(Fy)}{\tilde{h}(y)}. \tag{2.1}$$

Then taking $\Phi_1(y,u) := g_{\varphi(y,u)}(y,u)$ results in

$$\Phi_1(y,u) = \begin{cases} (Fy\ ,\ K(y)u), & \text{if } u < \tilde{h}(y)/2, \\ (Fy\ ,\ K(y)u - \tilde{h}(Fy)), & \text{if } u \geq \tilde{h}(y)/2, \end{cases} \tag{2.2}$$

for

$$\varphi_1(y,u) = \varphi_0(y) + \begin{cases} (K(y)-1)u, & \text{if } u < \tilde{h}(y)/2, \\ (K(y)-1)u - \tilde{h}(Fy), & \text{if } u \geq \tilde{h}(y)/2. \end{cases} \tag{2.3}$$

2.2 Remetrize to make Φ uniformly expanding

The idea behind Φ_1 is that it maps the flow-line $\{y\} \times [0, \tilde{h}(y))$ as a piecewise expanding map onto $\{Fy\} \times [0, \tilde{h}(Fy)]$. This is non-injective: for every $0 \leq u < \tilde{h}(y)/2$, there is another $u' := u + \tilde{h}(y)/2$ such that $\Phi_1(y,u) = \Phi(y,u')$. Yet, Φ_1 is invariant and ergodic with respect to $\frac{1}{\tilde{h}}\, d\mu\, du$ and mixing if (Y, F, μ) is mixing.

However, if $K(y) = \frac{2\tilde{h}(Fy)}{\tilde{h}(y)} \leq 1$, then Φ_1 is still not expanding in the vertical u-direction. This can be remedied by a change of coordinates:

$$\zeta : \tilde{Y} \to Y \times [0,1]. \quad (y,u) \mapsto (y, \frac{u}{\tilde{h}(y)}).$$

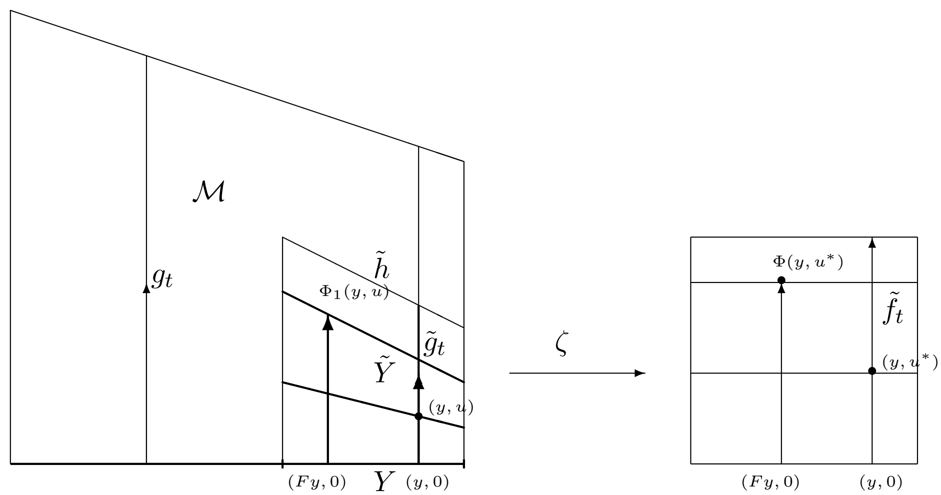

Figure 1: The flows $g_t : \mathcal{M} \to \mathcal{M}$, $\tilde{g}_t : \tilde{Y} \to \tilde{Y}$ and map $\Phi_1(y,u) = g_{\varphi(y,u)}(y,u)$ acting as doubling map on the vertical coordinate. On the right is the image under the change of coordinate $(y,u^*) = \zeta(y,u)$ make the map Φ uniformly expanding.

Then $\Phi := \zeta \circ \Phi_1 \circ \zeta^{-1}$ is precisely the doubling map in the vertical direction: $\Phi(y,u) = (Fy, 2u \bmod 1)$, and hence Φ is uniformly expanding. In these new coordinates the formulas are

$$\Phi(y,u) = f_{\varphi(y,u)}(y,u) = (Fy, 2u \bmod 1), \tag{2.4}$$

where

$$\varphi(y,u) = \varphi_0(y) + \begin{cases} (2\tilde{h}(Fy) - \tilde{h}(y))u & \text{if } 0 \le u < \frac{1}{2}; \\ (2\tilde{h}(Fy) - \tilde{h}(y))u - \tilde{h}(Fy) & \text{if } \frac{1}{2} \le u < 1. \end{cases} \tag{2.5}$$

Using the F-invariance of μ, it is straightforward to check that $\int_{\tilde{Y}} (\varphi - \varphi_0) d\mu_\Phi = 0$ for $d\mu_\Phi = d\mu\, du$. Hence,

$$\int_{\tilde{Y}} \varphi \, d\mu_\Phi = \int_Y \varphi_0 \, d\mu. \tag{2.6}$$

The change of coordinates ζ comes at the price that the semiflow $f_t = \zeta \circ g_t \circ \zeta^{-1}$, although parallel, is not of constant speed:

$$f_t(y,u) = (y, u + t/\tilde{h}(y)) \quad \text{for } 0 \le u < 1,\ 0 \le u + t/\tilde{h}(y) < 1.$$

That is, the speed is constant on each flow-line, but differs from flow-line to flow-line. Therefore f_t and Φ preserve the measures μ_Φ and $\tilde{\mu}$ respectively, and these measures are equivalent via the scaling

$$d\tilde{\mu} = \tilde{h}\, d\mu\, du = \tilde{h}\, d\mu_\Phi. \tag{2.7}$$

3 Operator renewal equation

Let $L_t : L^1(\mu_\Phi) \to L^1(\mu_\Phi)$ be the transfer operator for the flow f_t defined by $\int_Y L_t v^* \, w^* \, d\mu_\Phi = \int_Y v^* \, w^* \circ f_t \, d\mu_\Phi$ for all $w^* \in L^\infty(\mu_\Phi)$. Through-out, we write $v \in L^1(\tilde{\mu})$, $w \in L^\infty(\tilde{\mu})$ and $v^* \in L^1(\mu_\Phi)$, $w^* \in L^\infty(\mu_\Phi)$. In particular, we note that $v \in L^1(\tilde{\mu})$ can be written as $v = \frac{v^*}{\tilde{h}}$ where $v^* \in L^1(\mu_\Phi)$.

Define $T_t, U_t : L^1(\mu_\Phi) \to L^1(\mu_\Phi)$ by

$$T_t v^* = 1_{\tilde{Y}} L_t(1_{\tilde{Y}} v^*), \qquad U_t v^* = 1_{\tilde{Y}} L_t(1_{\{\varphi > t\}} v^*). \tag{3.1}$$

For $s \in \mathbb{C}$, we define the following Laplace transforms:

$$\hat{T}(s) := \int_0^\infty T_t e^{-st} dt, \qquad \hat{U}(s) := \int_0^\infty U_t e^{-st} dt.$$

Let $R : L^1(\mu_\Phi) \to L^1(\mu_\Phi)$ be the transfer operator for Φ defined by $\int_Y R v^* \, w^* \, d\mu_\Phi = \int_Y v^* \, w^* \circ \Phi \, d\mu_\Phi$ for all $w^* \in L^\infty(\mu_\Phi)$. For $s \in \mathbb{C}$, we define the following twisted/perturbed transfer operator

$$\hat{R}(s) v^* := R(e^{-s\varphi} v^*).$$

Clearly, $\hat{R}$, $\hat{T}$, $\hat{U}$ are analytic on $\mathbb{H} = \{\Re s > 0\}$ and $\hat{R}$ is well-defined on $\overline{\mathbb{H}} = \{\Re s \geq 0\}$.

Proposition 3.1. *The following holds μ_Φ-a.e. on $\tilde{Y}$ for all $s \in \mathbb{C}$:*

$$\hat{T}(s)(I - \hat{R}(s)) = \hat{U}(s).$$

Proof. The argument below goes exactly as the [21, Proof of Theorem 3.2]. By direct computation:

$$\begin{aligned}
\int_{\tilde{Y}} \hat{T}(s)\hat{R}(s) v^* \cdot w^* \, d\mu_\Phi &= \int_{\tilde{Y}} \int_0^\infty 1_{\tilde{Y}} L_t(1_{\tilde{Y}} R(e^{-s\varphi} v^*)) e^{-st} w^* \, dt \, d\mu_\Phi \\
&= \int_0^\infty \int_{\tilde{Y}} v^* \cdot w^* \circ f_{\varphi+t} e^{-s(\varphi+t)} \, d\mu_\Phi \, dt \\
&= \int_{\tilde{Y}} \int_\varphi^\infty v^* \cdot w^* \circ f_t e^{-st} \, dt \, d\mu_\Phi = \int_{\tilde{Y}} \int_\varphi^\infty L_t v^* \cdot w^* e^{-st} \, dt \, d\mu_\Phi \\
&= \int_{\tilde{Y}} \left(\int_0^\infty L_t v^* \cdot w^* e^{-st} \, dt - \int_0^\varphi L_t v^* \cdot w^* e^{-st} \, dt \right) d\mu_\Phi \\
&= \int_{\tilde{Y}} \hat{T}(s) v^* \cdot w^* \, d\mu_\Phi - \int_{\tilde{Y}} \hat{U}(s) v^* \cdot w^* \, d\mu_\Phi.
\end{aligned}$$

□

3.1 Relating the twisted transfer operator $\hat{R}(s)$ with a Laplace transform of non-delta functions

Although in the sequel we will not view $\hat{R}(s)$ as a Laplace transform (as noticed in [21]), any twisted transfer operator $\hat{R}(s)v^* := R(e^{-s\varphi} v^*)$ can be written as $\hat{R}(s) := \int_0^\infty R(\delta(\varphi - t) e^{-st} dt)$, we will sometimes make use of the following representation:

Lemma 3.2. *Set $R_{t,a}v^* = R(1_{\{t<\varphi<t+a\}}v^*)$ and define $\hat{L}_a(s) = \int_0^\infty R_{t,a}e^{-st}dt$. Then for $s \in \overline{\mathbb{H}}$, and $a \in \mathbb{R}_+$ such that $e^{sa} \neq 1$, we have*

$$\hat{R}(s) = \frac{s}{e^{sa}-1}\hat{L}_a(s), \quad \hat{R}(0) = \hat{L}_1(0) = R.$$

Proof. Compute that

$$\begin{aligned}
\int_{\tilde{Y}} \hat{L}_a(s)v^*w^*\,d\mu_\Phi &= \int_{\tilde{Y}}\int_0^\infty R(1_{\{t<\varphi<t+a\}}v^*)w^*e^{-st}dt\;d\mu_\Phi = \int_{\tilde{Y}}\int_{\varphi-a}^{\varphi} Rv^*w^*\,e^{-st}dt\,d\mu_\Phi \\
&= \int_{\tilde{Y}} R(e^{-s\varphi}v^*)\,w^*\cdot\int_{\varphi-a}^{\varphi} e^{-s(t-\varphi)}dt\,d\mu_\Phi = \int_{\tilde{Y}} R(e^{-s\varphi}v^*)\,w^*\,d\mu_\Phi\cdot\int_0^a e^{st}dt \\
&= \int_0^a e^{st}dt\cdot\int_{\tilde{Y}} R(e^{-s\varphi}v^*)w^*\,d\mu_\Phi = \frac{e^{sa}-1}{s}\int_{\tilde{Y}}\hat{R}(s)v^*w^*\,d\mu_\Phi.
\end{aligned}$$

Therefore, $\hat{R}(s) = s(e^{sa}-1)^{-1}\hat{L}_a(s)$, as required.

For the second equality, set $a = 1$ and note that $s(e^s-1)^{-1} = 1 + O(s)$, as $s \to 0$. □

4 Abstract set-up

We assume the setting and notation introduced in Section 2. Throughout, we assume that one of the two tail conditions holds:

(H0) **i)** Finite case: $\mu_\Phi((y,u) \in \tilde{Y} : \varphi(y,u) > t) = O(t^{-\beta})$, $\beta > 1$.

ii) Infinite case: $\mu_\Phi((y,u) \in \tilde{Y} : \varphi(y,u) > t) = \ell(t)t^{-\beta}$ where ℓ is slowly varying and $\beta \in (1/2, 1)$.

We require that

(H1) $\inf_{y\in Y}\tilde{h}(y) \geq 1$ and that $\tilde{h} = \varphi_0^\gamma$, where

i) Finite case. Under (H0) i), we assume that $\gamma \in (0,1)$.

ii) Infinite case. Under (H0) ii), we assume that $\gamma \in (0, \min\{\frac{2\beta-1}{1-\beta}, \frac{1-\beta}{2\beta-1}, \beta\})$.

Among others, as will be made clear by Lemma 4.4 below, assumption (H1) ensures that the tails $\mu(y \in Y : \varphi_0(y) > t)$ and $\mu_\Phi((y,u) \in \tilde{Y} : \varphi(y,u) > t)$ are of the same order. The assumption $\gamma < \beta$ ensures that in both cases of (H0), $\int_Y \tilde{h}^p\,d\mu < \infty$ for all $1 \leq p < \beta/\gamma$ and that $\tilde{\mu}(\tilde{Y}) < \infty$.

In this paper, we work with (H1) above, but simplifications for the case $\tilde{h}$ bounded from above and below will be pointed throughout the paper.

4.1 Functional analytic assumptions

We require that Φ satisfies the functional analytic assumptions listed below. We assume that there exists a Banach space $\mathcal{B}$, with norm $\|.\|_{\mathcal{B}}$ such that[4]

(H2) **i)** The space $\mathcal{B}$ contains constant functions and $\mathcal{B} \subset L^\infty(\mu_\Phi)$.

ii) 1 is a simple eigenvalue for R, isolated in the spectrum of R.

Recall that $\hat{R}(s)v^* = R(e^{-s\varphi}v^*)$ is the twisted transfer operator associated with the map Φ. By (H2) ii), 1 is an isolated eigenvalue in the spectrum of $\hat{R}(0)$. In addition to (H2) ii), we require

(H3) The spectral radius of $\hat{R}(s)$ is strictly less than 1 for $s \in \overline{\mathbb{H}} - \{0\}$ and is equal to 1 for $s = 0$.

For $s \in \overline{\mathbb{H}}$, let $a \in \mathbb{R}_+$ such that $e^{sa} \neq 1$. Set $R_{t,a}v^* = R(1_{\{t<\varphi<t+a\}}v^*)$ and define $\hat{L}_a(s) = \int_0^\infty R_{t,a}e^{-st}dt$. By Lemma 3.2, $\hat{R}(s) = s(e^s - 1)^{-1}\hat{L}_a(s)$. Given $a > 0$, we make certain assumptions on $\|R_{t,a}\|$, which in the sequel will be used to obtain appropriate continuity properties for $\hat{R}$.

(H4) *Finite case*. Under (H0) i), we require that for any $\tau < \beta$, the following upper bound holds uniformly in $a \in [1, 2]$:

$$\int_0^\infty \sigma^\tau \|R_{\sigma,a}\|_{\mathcal{B}}\, d\sigma < \infty,$$

(H5) *Infinite case*. Under (H0) ii), we require that there exists a Banach space $\mathcal{B}_0$ such that $\mathcal{B} \subset \mathcal{B}_0 \subset L^\infty(\mu_\Phi)$ such that

i) There exists constants $C_1 > 0$, $C_2 < 1$ and some $\theta \in (0, 1)$ such that

$$\|\hat{R}^n(s)v\|_{\mathcal{B}} \le C_1\theta^n\|v\|_{\mathcal{B}} + C_2\|v\|_{\mathcal{B}_0}, \qquad \|\hat{R}(s)v\|_{\mathcal{B}_0} \le \|v\|_{\mathcal{B}_0}.$$

ii) The following upper bound holds uniformly in $a \in [1, 2]$,

$$\int_0^\infty \sigma^\tau \|R_{\sigma,a}\|_{\mathcal{B}\to\mathcal{B}_0}\, d\sigma < \infty,$$

for $\max\{1 - \beta, 2\beta - 1\} < \tau < \frac{\beta}{1+\gamma}$.

Remark 4.1. *Assumption (H4) is very strong: it does not hold in standard Banach spaces such as Hölder or BV, unless we make further very restrictive assumptions on the return time φ (such as piecewise constant on partition elements of the Φ-partition). However, as we show in [7], it can be verified for a Banach space of analytic functions, which also puts restrictions on the map Φ.*

Assumption (H5) ii) is rather mild. As we show in Section 9, under the assumption that $\tilde{h}$ is bounded from above, (H5) ii) holds for typical suspension flows over Markov maps with indifferent

[4] The assumption $\mathcal{B} \subset L^\infty(\mu_\Phi)$ can be relaxed to $\mathcal{B} \subset L^2(\mu_\Phi)$. Because the main results are cumbersome to state under the weaker assumption (and require more elaborated arguments), we do not pursue this issue here.

fixed points, for $\mathcal{B}_0 = L^\infty(\mu_\Phi)$. In this case (H5) is easy to check. Assumption (H5) ii) makes the proofs slightly more difficult. In particular, one has to estimate several operators in the $\|.\|_{\mathcal{B}\to\mathcal{B}_0}$ norm. We can pursue this issue in the proof of Theorem 5.2 along some arguments in [16], which deals with similar estimates in the context of discrete time hyperbolic infinite measure preserving systems; in the present set up the arguments in [16] are greatly simplified by the fact that $\mathcal{B} \subset \mathcal{B}_0 \subset L^\infty$.

Remark 4.2. *We believe that the argument we provide below for the proof of Theorem 5.1 under the strong (H4) can be adapted to work with an assumption of the type (H5) with appropriate τ; more precisely, we would assume that there exists a space $\mathcal{B}_0$ such that both $(\mathcal{B}, \mathcal{B}_0)$ and $(\mathcal{B}_0, L^\infty)$ satisfy the appropriate finite case version of (H5). However, because the involved argument is rather complicated, here we reduce the analysis to the case where (H4) holds. However, see Remarks 6.12 and 6.22 for an outline of future work using a weak form of (H4).*

Remark 4.3. *It is easy to see that (H4), (H5) ii) above implies that $\int_t^\infty \|R_{\sigma,a}\|\, d\sigma \ll t^{-\tau}$, for τ as in (H4) and (H5) ii), respectively.*

4.2 Assumptions (H0): analogy with the discrete time scenario

The first result below shows that assumption (H0) can be verified by estimating the tail $\mu(\varphi_0 > t)$, which is easier to verify. In a large class of examples the tail $\mu(\varphi_0 > t)$ can be estimated based on knowledge about $\mu(\tau > n)$: see [21].

Lemma 4.4. *Assume that $\tilde{h} = \varphi_0^\gamma$, $\gamma \in (0,1)$. Then for any $0 < \delta < 1$,*

$$\mu_\Phi(\varphi > t) = \mu(\varphi_0 > t(1 - t^{-\delta})) + O(\mu(\varphi_0 > t^{(1-\delta)/\gamma})).$$

Proof. Fix $0 < \delta < 1$. We argue considering each of two formulas for φ in (2.5). For $u \in [0, \frac{1}{2})$ we have $\varphi(y,u) = \varphi_0 + (2\tilde{h}(Fy) - \tilde{h}(y))u$. Since we also know $\tilde{h} = \varphi_0^\gamma$, $\gamma \in (0,1)$, $\varphi \geq t$ implies that $\varphi_0 > t(1-t^{-\delta})$ or $2\tilde{h}(Fy) > t^{1-\delta}$. Thus,

$$\begin{aligned} \mu(\varphi > t) &\leq \mu(\varphi_0 > t(1-t^{-\delta})) + \mu(2\tilde{h} \circ F > t^{1-\delta}) \\ &= \mu(\varphi_0 > t(1-t^{-\delta})) + \mu(2\tilde{h} > t^{1-\delta}), \end{aligned}$$

by F-invariance of μ. The conclusion for $u \in [0, \frac{1}{2})$ follows.

For $u \in [\frac{1}{2}, 1]$, the argument is similar since $\varphi(y,u) = \varphi_0 + (2\tilde{h}(Fy) - \tilde{h}(y))u - \tilde{h}(Fy)$. Again since $\tilde{h} = \varphi_0^\gamma$, $\gamma \in (0,1)$, $\varphi \geq t$ implies that $\varphi_0 > t(1-t^{-\delta})$ or $\tilde{h}(Fy) > t^{1-\delta}$. From here on the argument goes exactly same as in the case $u \in [0, \frac{1}{2})$. □

Also in analogy with the discrete time case, we note that

Proposition 4.5. *Assume that* $\varphi \in L^1(\mu_\Phi)$. *Let* $\frac{d}{d(-s)}\hat{R}(s)\Big|_{s=0}$ *be the derivative of* $\hat{R}(s)$ *in* $-s$ *evaluated at* 0. *Then*

$$\int_{\tilde{Y}} \frac{d}{d(-s)}\hat{R}(s)\Big|_{s=0} 1_{\tilde{Y}}\, d\mu_\Phi = \int_Y \varphi_0\, d\mu.$$

Proof. Using the pointwise formula for the twisted transfer operator, we write

$$\hat{R}(s)1_{\tilde{Y}} = R(e^{-s\varphi}1_{\tilde{Y}}) = \sum_{\Phi(y',u')=(y,u)} e^{p(y')}e^{-s\varphi(y',u')},$$

where $e^{p(y')}$ is the potential associated with the hyperbolic map Φ. Therefore,

$$\frac{d}{d(-s)}\hat{R}(s)1_{\tilde{Y}} = \sum_{\Phi(y',u')=(y,u)} e^{p(y')}e^{-s\varphi(y',u')}\varphi(y',u').$$

Evaluating at 0,

$$\frac{d}{d(-s)}\hat{R}(s)\Big|_{s=0} 1_{\tilde{Y}}(y,u) = \sum_{\Phi(y',u')=(y,u)} e^{p(y')}\varphi(y',u') = R\varphi(y,u).$$

Thus,

$$\int_{\tilde{Y}} \frac{d}{d(-s)}\hat{R}(s)\Big|_{s=0} 1_{\tilde{Y}}\, d\mu_\Phi = \int_{\tilde{Y}} R\varphi \cdot 1\, d\mu_\Phi = \int_{\tilde{Y}} \varphi\, d\mu_\Phi.$$

The conclusion follows from the above together with (2.6). □

4.3 Assumptions required in the continuous time case: Dolgopyat type inequality

We recall that $\tilde{\mu}$ is $f_t|_{\tilde{Y}}$ invariant. In the finite measure case we normalize the measure $\tilde{\mu}$ such that $d\hat{\mu} = d\tilde{\mu}/\bar{\varphi}_0$ with $\bar{\varphi}_0 = \int_Y \varphi_0\, d\mu$, is $f_t|_{\tilde{Y}}$ a probability measure. In the infinite measure case we let $\hat{\mu} = \tilde{\mu}$.

For appropriate v, w, we want to estimate the correlation function

$$\rho_t(v,w) = \int_{\tilde{Y}} v\, w\circ f_t\, d\hat{\mu} = \frac{1}{\bar{\varphi}_0}\int_{\tilde{Y}} \tilde{h}v\, w\circ f_t\, d\mu_\Phi = \frac{1}{\bar{\varphi}_0}\int_{\tilde{Y}} T_t(\tilde{h}v)w\, d\mu_\Phi.$$

Note that $v \in L^1(\tilde{\mu})$ if and only if $\tilde{h}v \in L^1(\mu_\Phi)$.

Let $\hat{\rho}(s)(v,w) = \int_0^\infty \rho_t(v,w)e^{-st}dt$ be its corresponding Laplace transform. By Proposition 3.1, hypotheses (H2) and (H3), for all $s \in \overline{\mathbb{H}} - \{0\}$

$$\hat{\rho}(s)(v,w) = \frac{1}{\bar{\varphi}_0}\int_{\tilde{Y}} \hat{T}(s)(\tilde{v})w\, d\mu_\Phi = \int_{\tilde{Y}} \hat{U}(s)(I-\hat{R}(s))^{-1}(\tilde{h}v)w\, d\mu_\Phi.$$

Hypothesis (H4) gives a good control of $(I-\hat{R}(a+ib))^{-1}$ for $a \geq 0$ and $|b| < 1$. To be able to estimate the inverse Laplace transform $\rho_t(v,w)$ of $\hat{\rho}(s)$, we need a good understanding of the asymptotics of $(I-\hat{R}(a+ib))^{-1}$, for $a \geq 0$ and large values of b. For this purpose we assume

(H6) *Dolgopyat type inequality*. There exist $C > 0$ and $\alpha > 0$ such that for all $|b| \geq 1$

$$\|(I-\hat{R}(ib))^{-1}\|_{\mathcal{B}} \leq C|b|^\alpha.$$

Sometimes in the sequel we will need the following form of (H6):

(H6') There exist $C_0 > 0$, $\theta \in (0,1)$ and some $\alpha_0 > 0$ such that for all $|b| \geq 1$ and $k > (1 + \log C_0|b|)/\log\theta$,

$$\|\hat{R}(a+ib)^k\|_{\mathcal{B}} \leq 1 - |b|^{-\alpha_0}.$$

Remark 4.6. *By a standard argument, one checks that (H6') implies (H6). More precisely, by (H2) and (H3), $\|\hat{R}^j(ib)\| \leq C_1$ for some constant C_1 independent of j. Together with (H6') this implies that*

$$\begin{aligned} \|(I - \hat{R}(ib))^{-1}\|_{\mathcal{B}} &= \|(I + \hat{R}(ib) + \cdots + \hat{R}^{k-1}(ib))(I - \hat{R}^k(ib))^{-1}\|_{\mathcal{B}} \\ &\leq kC_1 b^{\alpha_0} \leq C_1 \frac{1 + \log C_0 + \log|b|}{\log\theta} b^{\alpha_0} \leq C b^{\alpha}, \end{aligned}$$

for some $\alpha > \alpha_0$ and C depending only on C_0, C_1, θ and $\alpha - \alpha_0$.

4.4 Partitions of $\tilde{Y}$ and w observables

Let $\mathcal{P}$ be the partition of Y into domains of continuity of F, and for $n \geq 1$, let $\mathcal{P}_n = \mathcal{P} \vee F^{-1}\mathcal{P} \vee \cdots \vee F^{-(n-1)}\mathcal{P}$ be the n-th joint of this partition. On $\tilde{Y}$, in the vertical direction, let $\mathcal{Q}$ be defined as the partition of $\tilde{Y}$ into the complementary domains of the line $\{(y, 1/2) : y \in Y\}$, and the n-th joint $\mathcal{Q}_n$ as the partition of $\tilde{Y}$ into the complementary domains of the lines $\{(y, j2^{-n}) : y \in Y\}$ for the integers $0 < j < 2^n$. Then Φ is continuous on each element of the product partition $\tilde{\mathcal{P}}_n := \mathcal{P}_n \times \mathcal{Q}_n$.

Let $w^* : \tilde{Y} \to \mathbb{C}$, C^m smooth (for some $m \geq 0$ to be specified below) in the (vertical) u-direction and piecewise continuous (or smooth) in the (horizontal) y-direction. Assume also that

$$\frac{\partial^j w^*}{\partial u^j}(y, 0) = \frac{\partial^j w^*}{\partial u^j}(y, 1) \tag{4.1}$$

for all $y \in Y$ and $j = 0, \ldots, m$.

Note that for each n, $w^* \circ \Phi^n(y)$ is discontinuous at the lines $\{(y, j2^{-n}) : y \in Y\}$ for $0 < j < 2^n$, but the left and right limits of $\lim_{\varepsilon \to \pm 0} \Phi^n(y, u + \varepsilon)$ equal $(F^n y, 0)$ resp. $(F^n y, 1)$ due to the Markov property in the u-direction.

By our assumption, the function values and partial derivatives of w^* are identical at these points. Therefore, the partial derivatives $\frac{\partial^j}{\partial u^j} w^* \circ \Phi^n(y, u)$ in the u-direction exist at all points and they dependent smoothly on y within the domains of $\mathcal{P}_n$. Hence, we can assume that $\|\frac{\partial^j}{\partial u^j} w^* \circ \Phi^n(y, u)\|_{L^\infty(\mu_\Phi)} < \infty$, for all $j = 0, \ldots, m$.

In what follows, we let $C^m(\tilde{Y}, \mu_\Phi)$ be the class of functions w^* that satisfy (4.1) and such that $\|\frac{\partial^j}{\partial u^j} w^* \circ \Phi^n(y, u)\|_{L^\infty(\mu_\Phi)} < \infty$, for all $j = 0, \ldots, m$, and set

$$C^m(\tilde{Y}, \tilde{\mu}) = \{w : \tilde{Y} \to C,\ w = \frac{w^*}{\tilde{h}} \text{ with } w^* \in C^m(\tilde{Y}, \mu_\Phi)\}. \tag{4.2}$$

Proposition 4.7. *Let $m \geq 1$. Suppose that $v \in L^1(\mu)$ and $w \in C^m(\tilde{Y}, \tilde{\mu})$. Then*

$$\hat{\rho}(s)(v, w) = \sum_{j=1}^{m} \rho_{v,\partial_t^{j-1} w}(0) s^{-j} + s^{-m} \hat{\rho}_{v,\partial_t^m w}(s),$$

where $\partial_t^j w$ indicates the j-th partial derivative w.r.t. the second variable.

Proof. We recall the short argument for convenience (see for instance [21]). Note that $\rho_t(v, w)$ is m-times differentiable and ${\rho_{v,w}}^{(j)} = \rho_{v,\partial_t^j w}$ for $j = 0, \dots, m$. By Taylor's Theorem, $\rho_t(v, w) = P_m(t) + H_m(t)$, where

$$P_m(t) = \sum_{j=0}^{m-1} \frac{1}{j!} {\rho_{v,w}}^{(j)}(0) t^j, \quad H_m(t) = \int_0^t g(t - \tau) {\rho_{v,w}}^{(m)}(\tau) \, d\tau, \quad g(t) = \frac{t^{m-1}}{(m-1)!}.$$

Hence $\hat{\rho}(s)(v, w) = \sum_{j=0}^{m-1} \rho_{v,\partial_t^j w}(0) s^{-(j+1)} + \hat{H}_m(s)$, where $\hat{H}_m(s) = \hat{g}(s) \hat{\rho}_{v,\partial_t^m w}(s) = s^{-m} \hat{\rho}_{v,\partial_t^m w}(s)$. □

5 Main results in the abstract set-up

In contrast to the discrete time operator renewal theory which is concerned with estimating the operators T_t in the norm of some appropriate Banach space, here we follow the strategy in [21]. Namely, we adapt renewal theory techniques to estimate the correlation function

$$\rho_t(v, w) = \int_{\tilde{Y}} v w \circ f_t \, d\hat{\mu},$$

where $d\hat{\mu} = \frac{d\tilde{\mu}}{\bar{\varphi}_0}$ for $\bar{\varphi}_0 = \int_Y \varphi_0 \, d\mu$ in the finite case (under (H0) i)) and $d\hat{\mu} = d\tilde{\mu}$ in the infinite case (under (H0) ii)).

For the statement of the main results, we recall that $C^m(\tilde{Y}, \tilde{\mu})$ is the class of observables defined in (4.2). Recall that $\mathcal{B}$ is the Banach space defined by (H2) and (H3) and that the corresponding norm is denoted by $\|.\|_{\mathcal{B}}$.

5.1 Finite case

Under (H0) i), we let $\epsilon > 0$ and define

$$\eta(t) = \frac{1}{\bar{\varphi}_0} \int_t^\infty \mu_\Phi(\varphi > \tau) \, d\tau, \quad \xi_{\beta,\epsilon}(t) = \begin{cases} t^{-(\beta-\epsilon)}, & \beta \geq 2, \\ t^{-(2\beta-2)}, & 1 < \beta < 2. \end{cases} \tag{5.1}$$

With these specified we state:

Theorem 5.1 (Finite measure)**.** *Assume (H0) i), (H1) i), (H2), (H3), (H4) and (H6). Set α such that (H6) holds. Let $v = \frac{v^*}{\tilde{h}}$, with $v^* \in \mathcal{B}$. Let $w \in C^m(\tilde{Y}, \tilde{\mu})$. The following hold for all $m \in \mathbb{N}$ such that $m \geq 3 + \alpha(\beta + 1)$ and for any $\epsilon > 0$.*

(a) Let η and $\xi_{\beta-\epsilon}$ be as defined in (5.1). Then,

$$\rho_t(v,w) - \int_{\tilde{Y}} v\,d\hat{\mu} \int_{\tilde{Y}} w\,d\hat{\mu} = \eta(t) \int_{\tilde{Y}} v\,d\hat{\mu} \int_{\tilde{Y}} w\,d\hat{\mu} + O(\|v^*\|_{\mathcal{B}}\|w\|_{C^m(\tilde{Y},\tilde{\mu})}\,\xi_{\beta,\epsilon}(t)).$$

(b) Suppose further that $\int v d\hat{\mu} = 0$. Then,

$$\rho_t(v,w) = O\big(\|v^*\|_{\mathcal{B}}\|w\|_{C^m(\tilde{Y},\tilde{\mu})} t^{-(\beta-\epsilon)}\big).$$

5.2 Results in the infinite case

Set $d_\beta = \frac{1}{\pi}\sin\pi\beta$. With this specified we state:

Theorem 5.2 (Infinite measure)**.** *Assume (H0) ii), (H1) ii), (H2), (H3), (H5) and (H6). Set α such that (H6) holds. The following hold for all $m \in \mathbb{N}$ such that $m \geq 2(\alpha+1)$. Let $v = \frac{v^*}{\tilde{h}}$, with $v^* \in \mathcal{B}$. Let $w \in C^m(\tilde{Y},\tilde{\mu})$. Then*

$$\ell(t)t^{1-\beta}\rho_t(v,w) \to d_\beta \int_{\tilde{Y}} v\,d\tilde{\mu} \int_{\tilde{Y}} w\,d\tilde{\mu}.$$

Remark 5.3. *The results for the case $\beta = 1$ and higher order asymptotics of $\rho_t(v,w)$ obtained in [21] can be also obtained in this framework. To simplify the exposition we omit these issues here.*

6 Arguments for the finite case: proof of Theorem 5.1

Let $\hat{B}(s) = s(I - \hat{R}(s))^{-1}$, $s \in \overline{\mathbb{H}}$. Note that by Proposition 3.1, the Laplace transform of $\rho(t)$ is

$$\hat{\rho}(s)(v,w) = \frac{1}{\bar{\varphi}_0}\frac{1}{s}\int_{\tilde{Y}} \hat{U}(s)\hat{B}(s)(\tilde{h}v)w\,d\mu_\Phi. \tag{6.1}$$

The first result below on the asymptotic behavior of $\hat{B}$ will be essential in the proof of Theorem 5.1. Before its statement we establish the following

Notation: Because some of our result below have a direct analogue among the results in [21] we use the same notation here.

a) Let $\mathcal{A}$ be a general Banach space. Suppose that $S : [0,\infty) \to \mathcal{A}$ lies in L^1 with Laplace transform $\hat{S} : \overline{\mathbb{H}} \to \mathcal{A}$. In what follows we write $\hat{S} \in \mathcal{R}_{\mathcal{A}}(a(t))$ if $\|S(t)\| \leq Ca(t)$ for all $t \geq 0$. When $\mathcal{A} = L^1$ we simply write $\hat{S} \in \mathcal{R}(a(t))$.

b) Let $s \mapsto \hat{S}(s)$ be an analytic family of $\mathcal{A}$-valued operators, $s \in \mathbb{H}$ defined on some Banach space, such that the family extends continuously to $\overline{\mathbb{H}}$. If $p \geq 0$ is an integer, define

$$d_p\hat{S}(ib) = \max_{j=0,\ldots,p} \|\hat{S}^{(j)}(ib)\|_{\mathcal{A}}.$$

If $p > 0$ is not an integer, define

$$d_p\hat{S}(ib) = d_{[p]}\hat{S}(ib) + \sup_{h\neq 0} \|\hat{S}^{([p])}(i(b+h))) - \hat{S}^{([p])}(ib)\|/|h|^{p-[p]}.$$

We recall that $P : L^1(\mu_\Phi) \to L^1(\mu_\Phi)$ is the spectral projection associated with the eigenvalue 1 with $Pv^* = \int v^* d\mu_\Phi$. Let $P_{\bar\varphi_0} = (\bar\varphi_0)^{-1}P$. We also recall that by Lemma 3.2, $\hat{R}(s) = s(e^s - 1)^{-1}\hat{L}_1(s)$ for all $s \in \overline{\mathbb{H}}$ with $e^s \neq 1$, where $\hat{L}_1(s) = \int_0^\infty R_{t,1}e^{-st}\,dt$.

Proposition 6.1. *Assume (H0) i), (H1) i), (H2), (H3), (H4) and (H6). Then*

$$s^{-1}\hat{B}(s) = s^{-1}P_{\bar\varphi_0} + P_{\bar\varphi_0}\Big(\int_0^\infty \int_t^\infty \Big(\int_\sigma^\infty R_{\tau,1}d\tau\Big)\,d\sigma)e^{-st}\,dt\Big)P_{\bar\varphi_0} + \hat{E}(s),$$

where $\hat{E}(s)$ is as follows

a) There exists $0 < r < 1$ such that for any C^∞ function $\psi : \mathbb{R} \to [0,1]$ with supp $\psi \subset [-r,r]$,

$$\psi(b)\hat{E}(ib) \in \mathcal{R}_{\mathcal{B}}(\xi_{\beta,\epsilon}(t)).$$

b) Write $s = a + ib$, for $a \geq 0$ and $b \in \mathbb{R}$. Then for all $b \in \mathbb{R}$ with $|b| \geq 1$,

$$\|\hat{E}(s)\| \ll |b|^\alpha,$$

where α is as in (H6).

Remark 6.2. *Item b) of the above result is not used as such in this work. It is an immediate consequence of item a) and (H6); we provide it here only for a complete description of $\hat{E}$.*

Proposition 6.3. *Assume the setting and notation of Proposition 6.1 a). Let $v^* \in \mathcal{B}$ and assume that $Pv^* = 0$. Then $\psi(b)b^{-1}\hat{B}(ib) \in \mathcal{R}_{\mathcal{B}}((t^{-(\beta-\epsilon)}))$.*

The proof of Proposition 6.1 is postponed to Section 6.2. Using equation (6.1) and Proposition 6.1 (which is new and required for the proof of Theorem 5.1 in our abstract setting) we can proceed to the proof of Theorem 5.1, following the main steps in [21]. First we notice that

$$\begin{aligned}\hat\rho(s)(v,w) &= \frac{1}{\bar\varphi_0}\frac{1}{s}\int_{\tilde Y} \hat{U}(s)P_{\bar\varphi_0}(\tilde{h}v)w\,d\mu_\Phi \\ &\quad + \frac{1}{\bar\varphi_0}\int_{\tilde Y} \hat{U}(s)P_{\bar\varphi_0}\int_0^\infty \Big(\int_t^\infty\int_\sigma^\infty R_{\tau,1}\,d\tau\,d\sigma\Big)e^{-st}\,dt P_{\bar\varphi_0}(\tilde{h}v)w\,d\mu_\Phi \\ &\quad + \frac{1}{\bar\varphi_0}\int_{\tilde Y} \hat{U}(s)\hat{E}(s)(\tilde{h}v)w\,d\mu_\Phi \\ &= \hat\rho_1(s)(v,w) + \hat\rho_2(s)(v,w) + \hat\rho_3(s)(v,w).\end{aligned}$$

Hence, it suffices to estimate the inverse Laplace transforms $\rho_1(t), \rho_2(t), \rho_3(t)$ of $\hat\rho_1(s), \hat\rho_2(s), \hat\rho_3(s)$. Following the strategy in [21], the inverse Laplace transforms $\rho_1(t), \rho_2(t), \rho_3(t)$ will be computed by moving the contour of integration to the imaginary axis (the functions in question are nonsingular on $\overline{\mathbb{H}}$). Hence we deal with inverse Fourier transforms. In this sense, we enlarge the definition of $\mathcal{R}(a(t))$ to include functions defined on the imaginary axis with inverse Fourier transform dominated by $a(t)$. For this purpose, we collect some technical estimates.

Lemma 6.4. *Assume the setting of Theorem 5.1. Let $\psi : \mathbb{R} \to [0,1]$ be a C^∞ function with supp $\psi \subset [-3,3]$. Then the inverse Laplace transform $\rho_1(t)$ of $\psi(b)\hat\rho_1(ib)$ is given by*

$$\rho_1(t)(v,w) = \int_{\tilde Y} v\, d\hat\mu \int_{\tilde Y} w\, d\hat\mu + E(t),$$

where $|E(t)| = O(t^{-\beta}\|v^*\|_{L^\infty(\mu_\Phi)}\|w^*\|_{L^\infty(\mu_\Phi)})$.

Proof. By Lemma 8.2 and the definition of $\hat\rho_1$,

$$\begin{aligned}\rho_1(t) =& \frac{1}{\bar\varphi_0}\int_{\tilde Y} \tilde h(y)\int_0^u \frac{1}{\tilde h(y)} P_{\bar\varphi_0}(\tilde h(y) v(y,\tau))\, w^*\, d\mu_\Phi \\ &+ \frac{1}{\bar\varphi_0}\int_{\tilde Y} \tilde h(y))\int_u^1 \frac{1}{\tilde h(y)} P_{\bar\varphi_0}(\tilde h(y) v(y,\tau))\, d\tau\, w^*\circ\Phi\, d\mu_\Phi + E(t),\end{aligned}$$

where $w = \frac{w^*}{\tilde h}$ with $w^* \in L^\infty(\mu_\Phi)$ and $|E(t)| = O(t^{-\beta}\|v^*\|_{L^1(\mu_\Phi)}\|w^*\|_{L^\infty(\mu_\Phi)})$. Now $P_{\bar\varphi_0}(\tilde h v) = \frac{1}{\bar\varphi_0}\int \tilde h v\, d\mu_\Phi = \frac{1}{\bar\varphi_0}\int v\, d\tilde\mu = \int v\, d\hat\mu$. This gives

$$\begin{aligned}\rho_1(t) &= \frac{1}{\bar\varphi_0}\int v\, d\hat\mu \int_{\tilde Y}\int_0^u d\tau\, w^* d\mu_\Phi + \frac{1}{\bar\varphi_0}\int v\, d\hat\mu \int_{\tilde Y}\int_u^1 d\tau\, w^*\circ\Phi d\mu_\Phi + E(t) \\ &= \int v\, d\hat\mu\, \frac{1}{\bar\varphi_0}\int_{\tilde Y}\int_0^1 d\tau\, w^* d\mu_\Phi + E(t) = \int v\, d\hat\mu \int w\, d\hat\mu + E(t),\end{aligned}$$

as required. □

Lemma 6.5. *Assume the setting of Theorem 5.1. Let $\psi : \mathbb{R} \to [0,1]$ be a C^∞ function with supp $\psi \subset [-3,3]$. Then the inverse Laplace transforms $\rho_2(t)$ of $\psi(b)\hat\rho_2(ib)$ is given by*

$$\rho_2(t)(v,w) = \eta(t)\int_{\tilde Y} v\, d\hat\mu \int_{\tilde Y} w\, d\hat\mu + E(t),$$

where $\eta(t)$ is as defined in (5.1) and $|E(t)| = O(t^{-\beta}\|v^*\|_{L^\infty(\mu_\Phi)}\|w^*\|_{L^\infty(\mu_\Phi)})$.

Proof. By definition

$$\begin{aligned}\hat\rho_2(s) &= \frac{1}{\bar\varphi_0}\int_{\tilde Y} \hat U(s) P_{\bar\varphi_0}\int_0^\infty \Big(\int_t^\infty\int_\sigma^\infty R_\tau\, d\tau\, d\sigma\Big) e^{-st}\, dt P_{\bar\varphi_0}(\tilde h v) w d\mu_\Phi \\ &= \frac{1}{\bar\varphi_0}\int_{\tilde Y} \hat U(0) P_{\bar\varphi_0}\int_0^\infty \Big(\int_t^\infty\int_\sigma^\infty R_\tau\, d\tau\, d\sigma\Big) e^{-st}\, dt P_{\bar\varphi_0}(\tilde h v) w d\mu_\Phi \\ &\quad + \frac{1}{\bar\varphi_0}\int_{\tilde Y} (\hat U(s) - \hat U(0)) P_{\bar\varphi_0}\int_0^\infty \Big(\int_t^\infty\int_\sigma^\infty R_\tau\, d\tau\, d\sigma\Big) e^{-st}\, dt P_{\bar\varphi_0}(\tilde h v) w d\mu_\Phi \\ &= \hat\rho_2^1(s) + \hat\rho_2^2(s).\end{aligned}$$

Recall $v = \frac{v^*}{\tilde h}$, $v^* \in \mathcal{B} \subset L^\infty(\mu_\Phi)$ and compute that

$$\begin{aligned}
P_{\bar\varphi_0}\Big(\int_t^\infty\int_\sigma^\infty R_\tau\,d\tau\,d\sigma\Big)P_{\bar\varphi_0}(\tilde h v) &= \frac{1}{\bar\varphi_0^2}\int_{\tilde Y}(\tilde h v)\,d\mu_\Phi\int_t^\infty\int_\sigma^\infty\int_{\tilde Y} R_\tau 1_{\tilde Y}\,d\mu_\Phi\,d\tau\,d\sigma \\
&= \frac{1}{\bar\varphi_0}\int_{\tilde Y}\tilde h v\,d\mu_\Phi\int_t^\infty\int_\sigma^\infty \mu_\Phi(\tau<\varphi<\tau+1)\,d\tau\,d\sigma \\
&= \int_{\tilde Y} v d\hat\mu\int_t^\infty\int_\sigma^{\sigma+1}\mu_\Phi(\varphi>\tau)\,d\tau\,d\sigma \\
&= \int_{\tilde Y} v d\hat\mu\Big(\int_t^\infty \mu_\Phi(\varphi>\sigma)\,d\sigma+\int_t^\infty\int_\sigma^{\sigma+1}\mu_\Phi(\varphi>\tau)-\mu_\Phi(\varphi>\sigma)\,d\tau\,d\sigma\Big) \\
&= (\eta(t)+K(t))\int_{\tilde Y} v d\hat\mu,
\end{aligned}$$

where

$$\begin{aligned}
|K(t)| &\le \int_t^\infty\int_\sigma^{\sigma+1}|\mu_\Phi(\varphi>\sigma+1)-\mu_\Phi(\varphi>\sigma)|\,d\tau\,d\sigma \\
&= \int_t^\infty \mu_\Phi(\varphi>\sigma)-\mu_\Phi(\varphi>\sigma+1)\,d\sigma = O(\mu_\Phi(\varphi>t)).
\end{aligned}$$

This together with Lemma 8.6 implies that the inverse Laplace transform $\rho_2^1(t)$ of $\hat\rho_2^1(s)$ is given by

$$\begin{aligned}
\rho_2^1(t)(v,w) &= \frac{1}{\bar\varphi_0}(\eta(t)+K(t))\int_{\tilde Y}\tilde h(y)\int_0^u\int_{\tilde Y} v\,d\hat\mu\,d\tau\,w^*\,d\mu_\Phi \\
&\quad+\frac{1}{\bar\varphi_0}(\eta(t)+K(t))\int_{\tilde Y}\tilde h(y)\int_u^1\int_{\tilde Y} v\,d\hat\mu\,d\tau\,w^*\circ\Phi\,d\mu_\Phi.
\end{aligned}$$

As in the proof of Lemma 6.4 we note that $\int_{\tilde Y} w^*\circ\Phi\,d\mu_\Phi = \int_{\tilde Y} w^*\,d\mu_\Phi$, for $w = w^*/\tilde h \in L^\infty(\tilde\mu)$ and that

$$\rho_2^1(t)(v,w) = \frac{1}{\bar\varphi_0}(\eta(t)+K(t))\int_{\tilde Y} v\,d\hat\mu\int_{\tilde Y}\int_0^1 d\tau\,\tilde h(y)w^*\,d\mu_\Phi = (\eta(t)+K(t))\int_{\tilde Y} v\,d\hat\mu\int_{\tilde Y} w\,d\hat\mu.$$

It remains to estimate the inverse Laplace transform $\rho_2^2(t)$ of $\hat\rho_2^2(s)$. Write

$$\hat\rho_2^2(s)(v,w) = \frac{1}{\bar\varphi_0}\int_{\tilde Y}\frac{\hat U(s)-\hat U(0)}{s}\hat M(s)(\tilde h v)w\,d\mu_\Phi, \tag{6.2}$$

where

$$\hat M(s)(\tilde h v) = sP_{\bar\varphi_0}\int_0^\infty\Big(\int_t^\infty\int_\sigma^\infty R_\tau\,d\tau\,d\sigma\Big)e^{-st}\,dtP_{\bar\varphi_0}(\tilde h v). \tag{6.3}$$

Note that

$$\begin{aligned}
-\frac{1}{\bar\varphi_0}\hat M(s)(\tilde h v) &= -\frac{1}{\bar\varphi_0}sP_{\bar\varphi_0}\int_0^\infty\Big(\int_t^\infty\int_\sigma^\infty R_{\tau,1}d\tau\,d\sigma\Big)e^{-st}\,dtP_{\bar\varphi_0}(\tilde h v) \\
&= \int_{\tilde Y}\int_0^\infty\int_t^\infty R_{\sigma,1}1_{\tilde Y}d\sigma\Big(\int_0^t -se^{-s\sigma}\,d\sigma\Big)\,dt\,d\mu_\Phi\int_{\tilde Y} v\,d\hat\mu \\
&= \int_{\tilde Y} v\,d\hat\mu\int_0^\infty\Big(\int_t^\infty\mu_\Phi(\sigma<\varphi<\sigma+1)d\sigma\Big)(e^{-st}-1)\,dt.
\end{aligned}$$

Together with Remark 8.7 this implies that

$$\hat{\rho}_2^2(s)(v,w) = \int_{\tilde{Y}} v\, d\hat{\mu} \int_{\tilde{Y}} \Big(\int_0^\infty \int_t^\infty \mu_\Phi(\sigma < \varphi < \sigma + 1) d\sigma (e^{-st} - 1)\, dt \Big) \Big(\int_0^\infty E(t) e^{-st} dt \Big),$$

where $|E(t)| \ll \|w^*\|_{L^\infty(\mu_\Phi)} t^{-\beta}$. Hence,

$$\rho_2^2(t) \ll \Big(\int_{\tilde{Y}} v\, d\hat{\mu} \Big) \Big(\int_t^\infty \mu_\Phi(\sigma < \varphi < \sigma + 1) d\sigma \Big) * E(t) \ll \Big(\int_{\tilde{Y}} v\, d\hat{\mu} \Big) \|w^*\|_{L^\infty(\mu_\Phi)} t^{-\beta},$$

as desired. □

Lemma 6.6. *Assume the setting and notation of Lemma 6.7. Then for all* $p > 0$*,*

$$(1 - \psi(b))(\hat{\rho}_1(ib) + \hat{\rho}_2(ib)) \in \mathcal{R}(\|v^*\|_{\mathcal{B}} \|w\|_{L^\infty(\tilde{\mu})} (1/t^p)).$$

Proof. By definition, $\hat{\rho}_1(s)(v,w) = \frac{1}{s} \int_{\tilde{Y}} \hat{U}(s) P_{\bar{\varphi}_0}(\tilde{h}v) w d\mu_\Phi$. Reasoning like in (6.2) we get

$$\hat{\rho}_2(s)(v,w) = \frac{1}{s} \int_{\tilde{Y}} \hat{U}(s) \hat{M}(s)(\tilde{h}v) w d\mu_\Phi,$$

where $\hat{M}(s)$ is defined in (6.3). By (H4), when viewed as an operator on $\mathcal{B}$, $\hat{M}(s)$ is bounded. Also by Lemma 8.4 we know that $\hat{U}(s) : \mathcal{B} \to L^1(\mu_\Phi)$ is bounded. Thus, for all $|b| > 1$, $|\frac{1}{ib} \int_{\tilde{Y}} \hat{U}(ib)(P_{\bar{\varphi}_0}(\tilde{h}v) + \hat{M}(s)(\tilde{h}v)) w d\mu_\Phi| \le C/|b|$, for some $C > 0$. Hence, $|(1-\psi(b))(\hat{\rho}_1(ib) + \hat{\rho}_2(ib))| \le C/|b|$. This together with Lemma 6.9(b) implies the desired conclusion. □

We state the result on the inverse Laplace transform of $(1 - \psi(b))\hat{\rho}_3(ib)$ below and postpone the proof to Section 6.1.

Lemma 6.7. *Assume the setting of Theorem 5.1. Let* $\psi : \mathbb{R} \to [0,1]$ *be a* C^∞ *function with supp* $\psi \subset [-3,3]$*. Then,* $(1 - \psi(b))\hat{\rho}_3(ib) \in \mathcal{R}(\|v^*\|_{\mathcal{B}} \|w\|_{C_w^m(\tilde{Y})} (1/t^{\beta - \epsilon}))$*, for any* $\epsilon > 0$*.*

As an immediate consequence of Lemma 6.7 and Lemma 6.6, we have

Corollary 6.8. *Assume the setting and notation of Lemma 6.7. Then the following holds for any* $\epsilon > 0$

$$(1 - \psi(b))\hat{\rho}(ib) \in \mathcal{R}(\|v^*\|_{\mathcal{B}} \|w\|_{L^\infty(\tilde{\mu})} (1/t^{\beta - \epsilon})).$$

We can now complete

Proof of Theorem 5.1. Item a) follows by Proposition 6.1 a), Lemma 6.4, Lemma 6.5 and Corollary 6.8. Item b) follows by Proposition 6.3 a) and and Corollary 6.8. □

6.1 Proof of Lemma 6.7

We start by collecting a few technical estimates.

Lemma 6.9. *[21, Proposition 14.1]*

(a) Suppose that the family $b \mapsto \hat{S}(ib)$ is C^p for some $p > 0$ and that there is a constant $C > 0$ such that $d_p\hat{S}(ib) \le C|b|^{-2}$ for $|b| > 1$. Then $\hat{S} \in \mathcal{R}(1/t^p)$.

(b) Suppose that $g : \mathbb{R} \to \mathbb{R}$ is C^∞, such that $g \equiv 0$ in a neighborhood of 0, and $g(b) \equiv 1$ for $|b|$ sufficiently large. Let $m \ge 1$. Then $g(b)/b^m \in \mathcal{R}(1/t^p)$ for all $p > 0$.

The next two results can be viewed as the analogue of [21, Propositions 2.1 and 12.2] in our abstract framework.

Lemma 6.10. *Let $\mathcal{B}$ be the Banach space defined by (H2) and (H3). Assume (H4). Then, for any $\epsilon > 0$, viewed as a family of operators on $\mathcal{B}$, $b \mapsto \hat{R}(ib)$ is $C^{\beta-\epsilon}$ and $d_{\beta-\epsilon}\hat{R}(ib) \le C(1 + |b|)$ for all $b \in \mathbb{R}$.*

Proof. Let $|b| \le 1$. By Lemma 3.2, $\hat{R}(ib) = -ib(e^{-ib}-1)^{-1}\hat{L}_1(ib)$, where $\hat{L}_1(ib) = \int_0^\infty R_{t,1}e^{ibt}\,dt$. It is easy to verify that for any $p > 0$,

$$d_p \frac{-ib}{e^{-ib} - 1} = 1 + O(b), \text{ as } b \to 0.$$

Also, by Lemma 3.2, $\hat{R}(ib) = -ib(e^{-iab} - 1)^{-1}\hat{L}_a(ib)$ for any $|b| > 1$ such that $e^{-iba} \neq 1$, where $\hat{L}(ib) = \int_0^\infty R_{t,a}e^{ibt}\,dt$. Given $b \in \mathbb{R}$, fix $a \in \mathbb{R}_+$ such that $|e^{-iba} - 1| > 1$. Then, there exists some constant $C > 0$ such that

$$d_p \frac{-ib}{e^{-iab} - 1} \le C|b|.$$

Hence, it suffices to show that for all $b \in \mathbb{R}$ and appropriate $a \in \mathbb{R}_+$, there exists $C > 0$ such that for any $\epsilon > 0$, $d_{\beta-\epsilon}\hat{L}_a(ib) \le C$.

Under (H4), let $\tau < \beta$. Put $\epsilon_0 = \tau - [\tau]$. By (H4),

$$\|i^{-[\tau]}d_{[\tau]}\hat{L}_a(ib)\| = \| \int_0^\infty t^{[\tau]}R_{t,a}e^{ibt}\,dt\| \le \int_0^\infty t^{[\tau]}\|R_{t,a}\|\,dt < \infty.$$

Moreover, there exists $C > 0$ such that

$$\begin{aligned}\|\hat{L}_a^{([\tau])}(i(b+h))) - \hat{L}_a^{([\tau])}(ib)\| &\le \| \int_0^\infty t^{[\tau]}R_{t,a}e^{ibt}(e^{iht} - 1)\,dt\| \\ &\le Ch^{\epsilon_0}\int_0^\infty t^{[\tau]+\epsilon_0}\|R_{t,a}\|\,dt \\ &= Ch^{\epsilon_0}\int_0^\infty t^\tau\|R_{t,a}\|\,dt \ll h^{\epsilon_0},\end{aligned}$$

where the last inequality follows since (H4) holds for any $\tau < \beta$. □

Corollary 6.11. *Assume the setting of Lemma 6.10. Suppose that (H6) holds and fix α accordingly. Then, viewed as a family of operators on $\mathcal{B}$, $b \mapsto (I - \hat{R}(ib))^{-1}$ is $C^{\beta-\epsilon}$ and there exists $C > 0$ such that $d_{\beta-\epsilon}(I - \hat{R}(ib))^{-1} \le C|b|^{\alpha(\beta-\epsilon+1)+1}$ for all $|b| > 1$.*

Proof. As in the proof of [21, Proposition 12.2], we give the details for $\beta - \epsilon$ not an integer. A straightforward induction argument shows that $\frac{d^j}{db^j}(I - \hat{R}(ib))^{-1}$ is a finite linear combination of factors

$$\hat{M}_k \in \{(I - \hat{R})^{-(k+1)}, \quad d_k\hat{R}\}, \;\; k = 1, \dots, j,$$

for each $j \in \mathbb{N}$, $j \le \beta-\epsilon$. Also, by (H6) and Lemma 6.10, $\max_{k=1,\dots,j} \|\hat{M}_k(ib)\| \ll |b|^{\alpha(\beta-\epsilon+1)+1}$, and $\max_{k=1,\dots,j}\, d_\epsilon \hat{M}_k(ib) \ll |b|^{\alpha(\beta-\epsilon+1)+1}$. The required estimate follows. □

Remark 6.12. *We believe that replacing (H4) with an assumption of the form (H5) (*i.e., *such that both $(\mathcal{B}, \mathcal{B}_0)$ and $(\mathcal{B}_0, L^\infty)$ satisfy (H5) with τ as appropriate) one can show that $\|\frac{d^k}{d^k b}(I - \hat{R}(ib))^{-1}\|_{\mathcal{B}\to\mathcal{B}_0} \ll C|b|^{\alpha(\beta-\epsilon+1)+1}$ for all $|b| > 1$ and $k < \beta$ by: a) obtaining $\alpha \in (0,1)$ as suggested in footnote 1; b) exploiting the type of arguments used in the proof of Lemma 7.7.*

Proof of Lemma 6.7. By Proposition 4.7, $\hat{\rho}(s)(v,w) = \hat{P}_m(s) + \hat{H}_m(s)$, where $\hat{P}_m(s)$ is a linear combination of s^{-j}, $j = 1, \dots, m$, and $\hat{H}_m(s) = s^{-m}\hat{\rho}_{v,\partial_t^m w}(s)$.

By the argument used in the proof of [21, Proposition 3.7], $(1 - \psi(b))\hat{P}_m(ib) \in \mathcal{R}(1/t^p)$ for all $p > 0$. Note that

$$\hat{\rho}(s)(v, \partial_t^m w) = \frac{1}{\bar{\varphi}_0} \int_{\tilde{Y}} \hat{U}(s)(I - \hat{R})^{-1}(s)(\tilde{h}v)\; \partial_t^m w \, d\mu_\Phi.$$

Recall $\tilde{h}v = v^* \in \mathcal{B}$. Therefore

$$\hat{H}_m(s) = s^{-m} \int_{\tilde{Y}} \hat{U}(s)(I - \hat{R})^{-1}(s)v^*\; \partial_t^m w \, d\mu_\Phi.$$

By Lemma 8.3, we know that $\hat{U}(s) : \mathcal{B} \to L^1(\mu_\Phi)$ lies in $\mathcal{R}_{\mathcal{B}\to L^1(\mu_\Phi)}(1/t^\beta)$. Since $\mathcal{B} \subset L^\infty(\mu_\Phi)$ it remains to show that $Q(ib) = b^{-m}(1 - \psi(b))(I - \hat{L}(ib))^{-1}(s)$ lies in $\mathcal{R}_\mathcal{B}(1/t^{\beta-\epsilon})$.

By Lemma 6.10, $\hat{R}(ib)$ is $C^{\beta-\epsilon}$, for any $\epsilon > 0$. Hence, $(I - \hat{R}(ib))^{-1}$ is $C^{\beta-\epsilon}$ on $\mathbb{R} \setminus \{0\}$ and $Q(ib)$ is $C^{\beta-\epsilon}$ on $\mathbb{R}$. Moreover, by Corollary 6.11, for all $|b| > 1$ there exists $C > 0$ such that $d_{\beta-\epsilon}(I - \hat{R}(ib))^{-1} \le C|b|^{\alpha(\beta-\epsilon+1)+1}$. Hence for all $|b| > 1$ and all $m - \alpha(\beta - \epsilon + 1) > 3$, $d_{\beta-\epsilon}Q(ib) \ll |b|^{-2}$. Together with Lemma 6.9(a), we get that $Q \in \mathcal{R}_\mathcal{B}(1/t^{\beta-\epsilon})$, as required. □

6.2 Several technical results required in the proof of Proposition 6.1

As in [21], a main step in proving a result of the form of Theorem 5.1 is based on the following continuous time version of [11, 22, First Main Lemma]. In our abstract set-up, we state:

Lemma 6.13. *A version of [21, Lemma 13.1] Assume (H0) i), (H2), (H3) and (H4). Let $\psi : \mathbb{R} \to [0,1]$ be C^∞ with supp $\psi \subset [-r, r]$ where $r \in (0,1)$ is sufficiently small and such that $\psi \equiv 1$ in a neighborhood of 0. Then, for any $\epsilon > 0$, $\psi\hat{B} \in \mathcal{R}_\mathcal{B}(1/t^{\beta-\epsilon})$.*

The next result is required in the proof of Lemma 6.13.

Lemma 6.14. *Assume (H0) i), (H2), (H3) and (H4). For all* C^∞ *functions* $\psi : \mathbb{R} \to [0,1]$ *with supp* $\psi \subset [-3,3]$ *and for any* $\epsilon > 0$,

$$\psi(b)\frac{\hat{R}(ib) - \hat{R}(0)}{b} \in \mathcal{R}_{\mathcal{B}}(1/t^{\beta-\epsilon}).$$

Proof. We first prove a). Recall that $R_{t,1}v = R(1_{\{t<\varphi<t+1\}}v)$ and $\hat{L}_1(s) = \int_0^\infty R_{t,1}e^{-st}dt$, $s = a + ib =\in \overline{\mathbb{H}}$. By Lemma 3.2, for all $|b| < 1$,

$$\begin{aligned}\frac{\hat{R}(s) - \hat{R}(0)}{s} &= \frac{1}{e^s - 1}\hat{L}_1(s) - \frac{1}{s}\hat{L}_1(0) = \frac{1}{s}(\hat{L}_1(s) - \hat{L}_1(0)) + \frac{s - e^s + 1}{s(e^s - 1)}\hat{L}_1(s)\\ &= \frac{1}{s}(\hat{L}_1(s) - \hat{L}_1(0)) - \frac{1 - e^{-s} - se^{-s}}{s(1 - e^{-s})}\hat{L}_1(s), \qquad (6.4)\end{aligned}$$

and note that $\frac{1-e^{-s}-se^{-s}}{s(1-e^{-s})} = \frac{1}{2} + O(s)$. Applying this for $s = ib$, we know from the proof of Lemma 6.10 that $\hat{L}_1(ib)$ is $C^{\beta-\epsilon}$, for some small $\epsilon > 0$ and $d_{\beta-\epsilon}\hat{L}_1(ib) \le C$, for some constant $C > 0$. Also, it is easy to verify that for any $p > 0$, $\psi(b)\frac{1-e^{-ib}-ibe^{-ib}}{b(1-e^{-ib})}$ is C^p and $d_p\Big(\psi(b)\frac{1-e^{-ib}-ibe^{-ib}}{b(1-e^{-ib})}\Big) \le C$ for some constant $C > 0$. Hence $d_{\beta-\epsilon}\Big(\psi(b)\frac{1-e^{-ib}-ibe^{-ib}}{b(1-e^{-ib})}\hat{L}_1(ib)\Big) \le C$, for some constant $C > 0$. Note that the inverse Fourier transform $S(t)$ of $\psi(b)\frac{1-e^{-ib}-ibe^{-ib}}{b(1-e^{-ib})}\hat{L}_1(ib)$ is given by

$$S(t) = \int_{-3}^{3} \psi(b)\frac{1 - e^{-ib} - ibe^{-ib}}{b(1 - e^{-ib})}\hat{L}_1(ib)e^{ibt}db.$$

Integration by parts gives

$$\|S(t)\| \ll t^{-(\beta-\epsilon)}\int_{-3}^{3}\left|d_{\beta-\epsilon}\Big(\psi(b)\frac{1 - e^{-ib} - ibe^{-ib}}{b(1 - e^{-ib})}\hat{L}_1(ib)\Big)\right| db \ll t^{-(\beta-\epsilon)}.$$

Thus, the second term of (6.4) lies in $\mathcal{R}_{\mathcal{B}}(1/t^{\beta-\epsilon})$.

It remains to deal with the first term of (6.4). Compute that

$$\begin{aligned}\frac{\hat{L}_1(ib) - \hat{L}_1(0)}{-ib} &= \int_0^\infty R_{t,1}\frac{e^{-ibt} - 1}{-ib}dt = \int_0^\infty R_{t,1}(\int_0^t e^{-ib\sigma}d\sigma)\,dt \qquad (6.5)\\ &= \int_0^\infty (\int_t^\infty R_{\sigma,1}\,d\sigma)e^{-ibt}\,dt.\end{aligned}$$

The above equation together with (H4) (or more precisely, Remark 4.3) implies that $\psi(b)\frac{\hat{L}_1(ib)-\hat{L}_1(0)}{b} \in \mathcal{R}_{\mathcal{B}}(1/t^{\beta-\epsilon})$, which ends the proof. □

Remark 6.15. *For later use (in the proof of Proposition 6.1), we rewrite* (6.4) *to*

$$\frac{\hat{R}(s) - \hat{R}(0)}{s} = \frac{\hat{L}_1(s) - \hat{L}_1(0)}{s} - \frac{1}{2}\hat{L}_1(s) + \hat{F}(s),$$

where $\hat{F}(s) = g(s)\hat{L}_1(s)$ where $g(s) = O(s)$ belongs to C^p for any $p > 0$. By the argument used in the proof of Lemma 6.14, $\psi(b)\hat{F}(ib) \in \mathcal{R}_{\mathcal{B}}(1/t^{\beta-\epsilon})$. Also compute

$$\begin{aligned}\frac{d}{ds}\hat{R}(s) &= \frac{d}{ds}\frac{s}{e^s-1}\hat{L}_1(s) = \frac{e^s-1-se^s}{(e^s-1)^2}\hat{L}_1(s) - \frac{s}{e^s-1}\int_0^\infty tR_t e^{-st}\,dt \\ &\to -\frac{1}{2}\hat{L}_1(0) - \int_0^\infty tR_t\,dt = -\frac{1}{2}\hat{L}_1(0) - \int_0^\infty\int_t^\infty R_\sigma d\sigma\,dt \qquad \text{as } s \to 0.\end{aligned}$$

Together with (6.5) (with s instead of ib), the above equation implies that

$$\Big(\frac{\hat{R}(s)-\hat{R}(0)}{-s} - \frac{d}{d(-s)}\hat{R}(0)\Big) = \int_0^\infty \Big(\int_t^\infty R_{\sigma,1}d\sigma\Big)(e^{-st}-1)\,dt + \hat{Q}(s), \tag{6.6}$$

where $\|\hat{Q}(s)\| = O(s)$ as $s \to 0$.

The following result has been established in [21]. The corresponding proof in [21] builds upon the strategy in [11]. Roughly, it establishes the existence of an operator $\tilde{R}$ that is identical to $\hat{R}$ in a neighborhood of 0 and whose eigenvalue $\tilde{\lambda}$ is well defined on the imaginary axis. So, one can speak of the inverse Laplace transform of $(1-\tilde{\lambda}(b))/b$. Furthermore, the result below establishes that $(1-\tilde{\lambda}(b))/b$ is different from zero on a compact interval $[-r, r]$ for some $r > 0$, and one can speak of the inverse Laplace transform of $b(1-\tilde{\lambda}(b))^{-1}$.

Proposition 6.16. *[21, Proposition 13.4, Proposition 13.5] Assume (H0) i), (H2), (H3) and (H4). Let $\delta > 0$. For any $\epsilon > 0$ and for all $r > 0$ sufficiently small, there exists a $C^{\beta-\epsilon}$ family $b \mapsto \tilde{R}(ib)$ with a $C^{\beta-\epsilon}$ family of simple eigenvalues $\tilde{\lambda}(b) \in \{z \in \mathbb{C} : |z-1| < \delta\}$ such that*

(a) $\tilde{R}(ib) \equiv \hat{R}(ib)$ *for* $|b| \le r$.

(b) $\tilde{R}(ib) \equiv \hat{R}(0)$ *and* $\tilde{\lambda}^*(b) \equiv 1$ *for* $|b| \ge 2$.

(c) $\|\tilde{R}(ib) - \hat{R}(0)\|_{\mathcal{B}} < \delta$ *for all* $b \in \mathbb{R}$.

(d) For all $b \in \mathbb{R}$, the spectrum of $\tilde{R}(ib)$ consists of $\tilde{\lambda}(b)$ together with a subset of $\{z : |z-1| \ge 3\delta\}$.

(e) $(1-\tilde{\lambda}(b))/b$ *is bounded away from zero on* $[-r, r]$.

(f) $(1-\tilde{\lambda}(b))/b \in \mathcal{R}(1/t^{\beta-\epsilon})$.

(g) Let $\tilde{B}(ib) = b(I-\tilde{R}(ib))^{-1}$. Then, for $b \in [-r, r]$, $\tilde{B}(ib) = P + \tilde{D}(ib)$, where $\tilde{D}(ib) \in \mathcal{R}_{\mathcal{B}}(1/t^{\beta-\epsilon})$.

Remark 6.17. *The proof of Proposition 6.16 goes word by word as the proofs of [21, Proposition 13.4, Proposition 13.5] with Lemma 6.13 above replacing [21, Lemma 13.3].*

Proof of Lemma 6.13. The rest of the proof of Lemma 6.13 goes exactly as the proof of [21, Lemma 13.1] with the norm $\|.\|_{\mathcal{B}}$ of our function space $\mathcal{B}$ replacing the norm $\|.\|_\theta$ in [21]. This is possible due to Lemma 6.14. We provide the main steps for the reader's convenience.

By Proposition 6.16(a), $\psi\hat{B} = \psi\tilde{B}$ where $\tilde{B}(ib) = b(I - \tilde{R}(ib))^{-1}$. Let $\tilde{P}(b)$ be the spectral projection associated with $\tilde{\lambda}(b)$. By definition,

$$\tilde{B}(ib) = ((1 - \tilde{\lambda}(b))/b)^{-1}\tilde{P}(b) + b(I - \tilde{R}(ib))^{-1}(I - \tilde{P}(b)).$$

The second term is $C^{\beta-\epsilon}$, for any $\epsilon > 0$. Hence, it lies in $\mathcal{R}_{\mathcal{B}}(1/t^{\beta-\epsilon})$ when multiplied by ψ. By the argument used in the proof of [21, Lemma 13.1] (which applies to our setting because of Lemma 6.14), we have $\psi(b)((1 - \tilde{\lambda}(b))/b)^{-1}\tilde{P}(b) \in \mathcal{R}_{\mathcal{B}}(1/t^{\beta-\epsilon})$. The conclusion follows. □

6.3 A step in the proof of Proposition 6.1 analogous to the discrete time setting

In the present and next sections we show that Proposition 6.1 a), which can be viewed as the continuous time version of [11, Theorem 1] (a generalization of [22, Theorem 1]), can be proved by adapting the techniques developed in [11, 22]. A variant of Proposition 6.1 a) is implicit in [21], which restricts the analysis to suspension flows over Gibbs Markov maps. The argument used in the proof Proposition 6.1 a) is essentially different from the type of arguments used in [21]. This is, of course, required since this result is formulated in the abstract setting of Section 4 as opposed to the setting of Gibbs Markov maps in [21]. As we explain below, due to Lemma 3.2, we are able to adapt the main steps in [11, 22, Proof of Theorem 1] from discrete to continuous time systems and as such offer a transparent proof of Theorem 5.1.

We start by constructing a continuous time version of the polynomial operator valued function used in [11, 22]. Recall from Lemma 3.2 that $\hat{R}(s) = \frac{s}{e^s-1}\hat{L}_1(s)$ for all $s = -a + ib$, $a \geq 0$, $|b| < 1$, $\hat{L}_1(s) = \int_0^\infty R_{t,1}e^{-st}\,dt$. To simplify notation, throughout this section we let $\hat{L}(s) := \hat{L}_1(s) = \int_0^\infty R_{t,1}e^{-st}\,dt$.

For $N > 0$, define

$$\begin{aligned} \hat{L}_N(s) &= \int_0^N R_{t,1}e^{-st}\,dt + \int_N^\infty R_{t,1}\,dt - (e^{-s} - 1)\int_N^\infty tR_{t,1}\,dt \\ \hat{R}_N(s) &= \frac{s}{e^s - 1}\hat{L}_N(s), \quad \hat{B}_N(s) = s(I - \hat{R}_N(s))^{-1}. \end{aligned} \tag{6.7}$$

Throughout this section we assume (H0) i), (H2), (H3) and (H4).

The first result below can be viewed as the continuous time version of [22, Step 1, Proof of Lemma 5].

Lemma 6.18. *There exists $\delta > 0$ such that for all $s \in \overline{\mathbb{H}}$ with $b \in B_\delta(0)$*

$$\hat{B}_N(s) = P_{\bar{\varphi}_o} + (1 - e^{-s})\hat{D}_N(s),$$

where $\hat{D}_N(s)$ is analytic.

Proof. By definition, $\hat{L}_N(0) = \hat{L}_1(0) = \hat{R}(0)$. Hence, $\hat{R}_N(0) = \hat{L}_1(0)$. Denote by $\frac{d}{d(-s)}\hat{L}_N(s)$ the derivative in $-s$ and note that $\frac{d}{d(-s)}\hat{L}_N(s)\Big|_{s=0} = \frac{d}{d(-s)}\hat{L}_a(s)\Big|_{s=0}$. Since

$$\frac{d}{d(-s)}\hat{R}_N(s) = \frac{d}{d(-s)}\Big(\frac{s}{e^s-1}\Big)\hat{L}_N(s) + \frac{s}{e^s-1}\frac{d}{d(-s)}\hat{L}_N(s),$$

we have $\frac{d}{d(-s)}\hat{R}_N(s)\Big|_{s=0} = \frac{d}{d(-s)}\hat{R}(s)|_{s=0}$. Since $s \to \hat{R}_N(s)$ is differentiable in $\mathbb{H}$, there exists $\delta_0 > 0$ such that $\hat{R}_N(s)$ has an eigenvalue $\lambda_N(s)$ in $B_{\delta_0}(0)$ such that

1. Since $\hat{R}_N(0) = \hat{L}_1(0) = R$, $\lambda_N(0)$ is simple and isolated in the spectrum of $\hat{R}_N(0)$ and $\lambda_N(0) = 1$. (Recall that by (H2) ii), 1 is an isolated simple eigenvalue in the spectrum of R.)

2. The rest of the spectrum of $\hat{R}_N(s)$ is contained in $\{\lambda \in \mathbb{C} : |\lambda| < 1\}$.

Recall that P is the spectral projection associated with the eigenvalue 1 of $\hat{R}(0) = R$. Let $P_N(s)$ be the spectral projection associated with $\lambda_N(s)$ and note that $P_N(0) = P$.

To obtain the expansion of $\lambda_N(s)$ as $s \to 0$, we follow [22]. More precisely, starting from $\hat{R}_N(s)P_N(s) = \lambda_N(s)P_N(s)$, differentiating (in $-s$) and applying $P_N(s)$ to both sides,

$$P_N(s)\frac{d}{d(-s)}\hat{R}_N(s)P_N(s) = \lambda_N(s)P_N(s)\frac{d}{d(-s)}\hat{P}_N(s) + \frac{d}{d(-s)}\lambda_N(s)P_N(s).$$

Next, by Proposition 4.5, $P\frac{d}{d(-s)}\hat{R}(s)\Big|_{s=0}P = \bar{\varphi}_0 \neq 0$. Combined with the previous equation, we obtain that $s \to 0$,

$$\lambda_N(s) = 1 + s \cdot P\frac{d}{d(-s)}\hat{R}(s)\Big|_{s=0}P + o(s) = 1 + s\bar{\varphi}_0 + o(s). \tag{6.8}$$

Let $Q_N(s) = I - P_N(s)$ be the complementary spectral projection. Putting the above together, we have that there exists $0 < \delta < \delta_0$ such that for all $s \in B_\delta(0)$,

$$\hat{B}_N(s) = \Big(\frac{1-\lambda_N(s)}{s}\Big)^{-1}P_N(s) + (e^{-s}-1)(I - \hat{R}_N(s))^{-1}Q_N(s),$$

with $\|(I - \hat{R}_N(s))^{-1}Q_N(s)\| \le C$ for some constant $C > 0$. By (6.8), $s = 0$ is the only zero of $1 - \lambda_N(s)$. This together with the above equation and the analyticity of $\hat{R}_N$ ends the proof. □

The next result is the analogue of Lemma 6.14 for $\hat{R}_N$.

Lemma 6.19. *For all C^∞ functions $\psi : \mathbb{R} \to [0,1]$ with $\operatorname{supp}\psi \subset [-3,3]$.*

$$\psi(b)\frac{\hat{R}(ib) - \hat{R}_N(ib)}{b} \in \mathcal{R}_{\mathcal{B}}(1/t^{\beta-\epsilon}).$$

Proof. Write

$$\begin{aligned}\frac{\hat R(ib)-\hat R_N(ib)}{ib} &= \frac{\hat R(ib)-\hat R(0)}{ib} - \frac{\hat R_N(ib)-\hat R(0)}{ib}\\ &= \frac{\hat L(ib)-\hat L(0)}{ib} - \frac{1-e^{-ib}-ibe^{-ib}}{ib(1-e^{-ib})}\hat L(ib) - \frac{\hat L_N(ib)-\hat L(0)}{ib} + \frac{1-e^{-ib}-ibe^{-ib}}{ib(1-e^{-ib})}\hat L_N(ib)\\ &= \frac{\hat L(ib)-\hat L(0)}{ib} - \frac{\hat L_N(ib)-\hat L(0)}{ib} - \frac{1-e^{-ib}-ibe^{-ib}}{ib(1-e^{-ib})}(\hat L(ib)-\hat L_N(ib)). \qquad (6.9)\end{aligned}$$

By the argument used in the proof of Lemma 6.14 (which relies on Remark 4.3), when multiplied by ψ, each term in (6.9) lies in $\mathcal{R}_{\mathcal{B}}(1/t^{\beta-\epsilon})$. The conclusion follows. □

Based on the spectral properties of $\hat R_N$ mentioned in the proof of Lemma 6.18, we can obtain a continuous time version of [22, Second Main Lemma].

Lemma 6.20. *Choose ψ such that the conclusion of Lemma 6.13 holds. The following hold for any $p > 0$.*

a) Let $D_N(ib)$ be defined as in Lemma 6.18, so $\hat B_N(ib) = P_{\bar\varphi_0} + (1-e^{-ib})\hat D_N(ib)$. Then $\psi(b)\hat D_N(ib) \in \mathcal{R}_{\mathcal{B}}(t^{-p})$.

b) $\psi(b)b^{-1}\Big(\frac{\hat R_N(ib)-\hat R(0)}{-ib} - \frac{d}{d(-ib)}\hat R(ib)\Big|_{b=0}\Big) \in \mathcal{R}_{\mathcal{B}}(t^{-p})$.

Proof. First, we note that $\psi(b)\hat B_N(ib)$ is well defined. Let $B(t)$ be the inverse Fourier transform of $\psi\hat B$. By Lemma 6.13, $\psi\hat B \in \mathcal{R}_{\mathcal{B}}(1/t^{\beta-\epsilon})$. Hence, there exists C_0 such that

$$\int_0^\infty \|B(t)\|dt \le C_0. \qquad (6.10)$$

Recall $\operatorname{supp}\psi \subset [-r,r]$ where $r \in (0,1)$ is sufficiently small. For $|b| \le r$, write

$$\hat B_N(ib) = \hat B(ib)\Big(I - \frac{\hat R(ib)-\hat R_N(ib)}{b}\hat B(ib)\Big)^{-1}.$$

Continuing from equation (6.9),

$$\|\frac{\hat R(ib)-\hat R_N(ib)}{b}\| \ll \int_N^\infty t\|R_{t,1}\|dt + \Big|\frac{1-e^{-ib}-ibe^{-ib}}{b(1-e^{-ib})}\Big|\int_N^\infty \|R_{t,1}\|\,dt.$$

Clearly, $|\frac{1-e^{-ib}-ibe^{-ib}}{b(1-e^{-ib})}| = \frac{1}{2} + O(|b|)$, as $b \to 0$. Together with Remark 4.3, this implies that for any $\tau < \beta$ and for all $|b| \le 1$,

$$\|\frac{\hat R(ib)-\hat R_N(ib)}{b}\| \le CN^{-(\tau-1)}, \qquad (6.11)$$

for some positive finite constant C. Choosing $N \le C_0/2$ (with C_0 as in (6.10)), we have that $\psi(b)\hat B_N(ib)$ is well defined.

Reasoning as in [22, Step 1 of Proof of Lemma 6], for any $b_0 \in \operatorname{supp}\psi$, there exists $\delta_0 > 0$ such that $b \to \hat B_N(ib)$ is analytic in $B_{\delta_0}(b_0)$. Also, by Lemma 6.18, there exists $\delta > 0$ such for all $b \in B_\delta(0)$, $\hat{B}_N(ib) = P_{\bar{\varphi}_0} + b\hat{D}_N(ib)$, where $\hat{D}_N$ is analytic. It follows that for any $p > 0$, $\psi(b)\hat{D}_N(ib)$ is C^p with $d_p[\psi(b)\hat{D}_N(ib)] \le C$, for some constant $C > 0$. By the argument used in Lemma 6.14 (in estimating $S(t)$ there), we have $\psi(b)\hat{D}_N(ib) \in \mathcal{R}_\mathcal{B}(t^{-p})$, which ends the proof of a).

For the proof of b), we note that reasoning like in Remark 6.15, we have

$$\frac{1}{ib}\Big(\frac{\hat{R}_N(ib) - \hat{R}(0)}{ib} - \frac{d}{d(ib)}\hat{R}(ib)\Big|_{b=0}\Big) = \int_0^N \Big(\int_t^\infty R_{\sigma,1}\, d\sigma\Big)\frac{(e^{ibt}-1)}{ib}\, dt + \frac{1}{ib}\hat{Q}_N(ib), \tag{6.12}$$

where $Q_N(ib) \to 0$ as $b \to 0$ and $\psi(b)b^{-1}\hat{Q}_N(ib) \in \mathcal{R}_\mathcal{B}(1/t^p)$, for any $p > 0$. The conclusion follows since the first term has finitely many inverse Laplace transforms. □

6.4 Proofs of Proposition 6.1 and Proposition 6.3

At this point in the exposition we can summarize the rest of the argument and emphasize on the analogy with the discrete time situation.

Put $\hat{C}(s) = s^{-1}(\hat{R}_N(s) - \hat{R}(s))$. By equation (6.11), $\|\hat{C}(s)\| \le CN^{-(\tau-1)}$, for all $s \in \mathbb{H}$ with $|b| < 1$, for some positive finite constant C. By Lemma 6.20, $\|\hat{C}(s)\| \le C$, for some positive finite constant C. Hence, we can choose N such that $(I - \hat{C}(s)\hat{B}_N(s))^{-1}$ is well defined for all $s \in \mathbb{H}$ with $|b| < 1$.

By the resolvent equality, $\hat{B}(s) = \hat{B}_N(s)(I - \hat{C}(s)\hat{B}_N(s))^{-1}$, for all $s \in \mathbb{H}$ with $|b| < 1$ and

$$\hat{B}(s) = \hat{B}_N(s) + \hat{B}_N(s)\hat{C}(s)\hat{B}_N(s) + [\hat{B}_N(s)\hat{C}(s)]^2\hat{B}(s).$$

Hence,

$$s^{-1}\hat{B}(s) = s^{-1}\hat{B}_N(s) + s^{-1}\hat{B}_N(s)\hat{C}(s)\hat{B}_N(s) + s^{-1}[\hat{B}_N(s)\hat{C}(s)]^2\hat{B}(s). \tag{6.13}$$

A discrete time version of the above identity has been used in [11, 22].

As already mentioned, in contrast to the discrete time scenario, following the strategy in [21] we only estimate the correlation function $\rho_t(v, w)$. For such a strategy it suffices to estimate the inverse Fourier transform (in the operator norm) of $\psi(b)B_N(ib)$, $\psi(b)\hat{B}_N(ib)\hat{C}(ib\hat{B}_N(ib)$ and $\psi(b)[\hat{B}_N(ib)\hat{C}(ib)]^2\hat{B}(ib)$, with ψ chosen as in Lemma 6.13.

The inverse Fourier transform of the first term is dealt with in Lemma 6.20. In what follows we estimate the inverse Fourier transform (in the operator norm) of $\psi(b)\hat{B}_N(ib)\hat{C}(ib\hat{B}_N(ib)$ and $\psi(b)[\hat{B}_N(ib)\hat{C}(ib)]^2\hat{B}(ib)$ by adapting the techniques in [11, 22] (including those steps in [11] needed to deal with the case $\beta > 1$) to the continuous time scenario. Again, the basic observation that makes this possible is Lemma 3.2.

Lemma 6.21. *Assume (H0) i), (H1) and (H2), (H3) and (H4). Choose ψ such that the conclusion of Lemma 6.13 holds. Then, the following hold for any $\epsilon > 0$, any $p > 0$ and all $s = a + ib$, $a \ge 0$, $|b| < 1$.*

a) $s^{-1}\hat{B}_N(s) = s^{-1}P_{\bar{\varphi}_0} + \hat{D}_N(s)$*, where* $\psi(b)\hat{D}_N(ib) \in \mathcal{R}_{\mathcal{B}}(t^{-p})$*.*

b) $s^{-1}\hat{B}_N(s)\hat{C}(s)\hat{B}_N(s) = P_{\bar{\varphi}_0}(\int_0^\infty \int_t^\infty (\int_\sigma^\infty R_{\tau,1} d\tau)\, d\sigma) e^{-st}\, dt) P_{\bar{\varphi}_0} + \hat{A}(s)$*, where* $\psi(b)\hat{A}(ib) \in \mathcal{R}_{\mathcal{B}}(t^{-(\beta-\epsilon)})$*.*

c) $\psi(b)b^{-1}[\hat{B}_N(ib)\hat{C}(ib)]^2\hat{B}(ib) \in \mathcal{R}_{\mathcal{B}}(a(t))$*, where*

$$a(t) \le \begin{cases} t^{-(\beta-\epsilon)} & \text{if } \beta > 2; \\ t^{-(2-\epsilon')} & \text{if } \beta = 2, \text{ for any } \epsilon' > \epsilon; \\ t^{-(2\beta-2)} & \text{if } \beta < 2. \end{cases}$$

Proof. Item a) follows immediately from Lemma 6.20 a). For the proof of b), write

$$\begin{aligned} s^{-1}\hat{C}(s) &= \Big(\frac{\hat{R}(s) - \hat{R}(0)}{-s} - \frac{d}{d(-s)}\hat{R}(s)\Big|_{s=0}\Big) - \Big(\frac{\hat{R}_N(s) - \hat{R}(0)}{-s} - \frac{d}{d(-s)}\hat{R}(s)\Big|_{s=0}\Big) \\ &:= \hat{C}_1(s) - \hat{C}_2(s). \end{aligned}$$

Continuing from (6.6),

$$\begin{aligned} \hat{C}_1(s) &= \int_0^\infty \Big(\int_t^\infty R_{\sigma,1} d\sigma\Big) \frac{e^{-st} - 1}{-s}\, dt + \hat{Q}(s) \\ &= \int_0^\infty \Big(\int_t^\infty R_{\sigma,1} d\sigma\Big)\Big(\int_0^t e^{-s\sigma}\, d\sigma\Big) dt + \hat{Q}(s) \\ &= \int_0^\infty \Big(\int_t^\infty \Big(\int_\sigma^\infty R_{\tau,1} d\tau\Big) d\sigma\Big) e^{-st}\, dt + \hat{Q}(s), \end{aligned}$$

where $\psi(b)b^{-1}\hat{Q}(ib) \in \mathcal{R}_{\mathcal{B}}(1/t^{\beta-\epsilon})$. By Lemma 6.20 b), for any $p > 0$, $\psi(b)\hat{C}_2(ib) \in \mathcal{R}_{\mathcal{B}}(t^{-p})$. Hence,

$$\hat{C}(s) = \int_0^\infty \Big(\int_t^\infty \Big(\int_\sigma^\infty R_{\tau,1} d\tau\Big) d\sigma\Big) e^{-st}\, dt + \hat{C}^*(s),$$

where $\hat{C}^*(s) = \hat{Q}(s) + \hat{C}_2(s)$ and thus, $\psi(b)b^{-1}\hat{C}^*(ib) \in \mathcal{R}_{\mathcal{B}}(1/t^{\beta-\epsilon})$.

Putting these together,

$$s^{-1}\hat{B}_N(s)\hat{C}(s)\hat{B}_N(s) = P_{\bar{\varphi}_0} \int_0^\infty \Big(\int_t^\infty \Big(\int_\sigma^\infty R_{\tau,1} d\tau\Big) d\sigma\Big) e^{-st}\, dt P_{\bar{\varphi}_0} + \hat{A}(s),$$

where $\hat{A}(s)$ is a sum of products, all of them including the factors $\hat{D}_N(s)$ and $\hat{C}^*(s)$. Hence, $\psi(b)\hat{A}(ib) \in \mathcal{R}_{\mathcal{B}}(t^{-(\beta-\epsilon)})$.

We continue with the proof of c). We first note that by (6.6) and (6.12) (with $s = ib$ and multiplied by s),

$$\hat{C}(s) = \int_N^\infty \Big(\int_t^\infty R_{\sigma,1} d\sigma\Big)(e^{-st} - 1)\, dt + (\hat{Q}(s) - \hat{Q}_N(s)),$$

where $\|\hat{Q}(s) - \hat{Q}_N(s)\| = O(s)$ as $s \to 0$. Also, for any $0 < \delta \le 1$ and s small,

$$\int_N^\infty \Big(\int_t^\infty \|R_{\sigma,1}\| d\sigma\Big)|e^{-st} - 1|\, dt \le |s|^\delta \int_N^\infty t^\delta \Big(\int_t^\infty \|R_{\sigma,1}\| d\sigma\Big) dt.$$

Since by Remark 4.3, $\int_t^\infty \|R_{\sigma,1}\| d\sigma \ll t^{-\tau}$, for any $\tau < \beta$ and $\beta > 1$ and δ is arbitrary small we have that $\int_N^\infty t^\delta \Big(\int_t^\infty \|R_{\sigma,1}\| d\sigma \Big) dt < \infty$. Hence,

$$\int_N^\infty \Big(\int_t^\infty R_{\sigma,1} d\sigma \Big)(e^{-st} - 1)\, dt \to 0, \text{ as } s \to 0$$

and as a consequence, $\hat{C}(s) \to 0$ as $s \to 0$.

Put $\hat{G}(ib) = \psi(b)\hat{B}_N(ib)\hat{C}(ib)$. By item a) and Lemma 6.14, $\hat{G}(ib) \in \mathcal{R}_{\mathcal{B}}(t^{-(\beta-\epsilon)})$. Clearly, $\hat{G}(s) \to 0$ as $s \to 0$ and $\hat{G}(0) = 0$. Writing $\hat{G}(s) = \int_0^\infty G(t)e^{-st}\, dt$, we have $\int_0^\infty G(t)\, dt = 0$ and $\|G(t)\| \ll t^{-(\beta-\epsilon)}$. Thus,

$$\frac{\hat{G}(ib)}{-ib} = \frac{\hat{G}(ib) - \hat{G}(0)}{-ib} = \int_0^\infty G(t) \frac{e^{-ibt} - 1}{-ib}\, dt = \int_0^\infty (\int_t^\infty G(\sigma)\, d\sigma) e^{-ibt}\, dt.$$

Since $\|G(t)\| \ll t^{-(\beta-\epsilon)}$, we have $\int_t^\infty \|G(\sigma)\| d\sigma \ll t^{-(\beta-1-\epsilon)}$ and hence $b^{-1}\hat{G}(ib) \in \mathcal{R}_{\mathcal{B}}(t^{-(\beta-1-\epsilon)})$.

Next, put $\hat{E}(ib) = b^{-1}\hat{G}(ib)^2\hat{B}(ib)$. We want to estimate the inverse Fourier transform of $\hat{E}(ib)$. Write $\hat{E}(ib)' := \frac{d}{db}\hat{E}(ib)$. To obtain the required estimates, we proceed as in the discrete time scenario [11, 22] by estimating the inverse Fourier transform of $\hat{E}(ib)'$ and then integrate. Let $\hat{B}'$ and $\hat{G}'$ denote the first derivative of $\hat{B}, \hat{G}$ in b. Compute that

$$\begin{aligned} \hat{E}(ib)' = &- \Big(\frac{\hat{G}(ib)}{b}\Big)^2 \hat{B}(ib) + i\Big[\Big(\frac{\hat{G}(ib)}{b}\hat{G}(ib)' + \hat{G}(ib)'\frac{\hat{G}(ib)}{b}\Big)\Big]\hat{B}(ib) \\ &+ i\hat{G}(ib)\Big(\frac{\hat{G}(ib)}{b}\Big)\hat{B}(ib)'. \end{aligned}$$

By Lemma 6.13, $\psi(b)\hat{B}(ib) \in \mathcal{R}_{\mathcal{B}}(1/t^{\beta-\epsilon})$. It follows that $\psi(b)\hat{B}' \in \mathcal{R}_{\mathcal{B}}(1/t^{\beta-1-\epsilon})$. Due to these estimates and the fact that $b^{-1}\hat{G}(ib) \in \mathcal{R}_{\mathcal{B}}(t^{-(\beta-1-\epsilon)})$, the rest of the argument goes exactly like in the discrete time case [11]. We recall the main elements. First, the statement and proof of [11, Lemma 4.3] on convolutions goes exactly the same as in the discrete time case (with of course, sums replaced by integrals). As a consequence we obtain, $b^{-2}\hat{G}(ib)^2\hat{B}(ib) \in \mathcal{R}_{\mathcal{B}}(b(t))$, where

$$b(t) \le \begin{cases} t^{-(\beta-1-\epsilon)} & \text{if } \beta > 2; \\ t^{-(1-\epsilon')} & \text{if } \beta = 2, \text{ for any } \epsilon' > \epsilon; \\ t^{-(2\beta-3)} & \text{if } \beta < 2. \end{cases}$$

Also, based on the continuous time version of [11, Lemma 4.3] and the fact that $\psi\hat{B}' \in \mathcal{R}_{\mathcal{B}}(1/t^{\beta-1-\epsilon})$, one obtains similar estimates for the other terms of $\hat{E}(ib)'$. Integrating, we obtain that $\hat{E}(ib) \in \mathcal{R}_{\mathcal{B}}(a(t))$, with $a(t)$ as in the statement of item c), as required. □

Proposition 6.1 follows immediately from equation (6.13) and Lemma 6.21. It remains to complete the

Proof of Proposition 6.3. Recall $Pv = 0$. Continuing from (6.13),

$$s^{-1}\hat{B}(s) = s^{-1}\hat{B}_N(s) + s^{-1}\hat{B}_N(s)\hat{C}(s)\hat{B}_N(s) + s^{-1}[\hat{B}_N(s)\hat{C}(s)]^2 + s^{-1}[\hat{B}_N(s)\hat{C}(s)]^3\hat{B}(s).$$

By Proposition 6.1 a), $s^{-1}\hat{B}_N(s)v = \hat{D}_N(s)v$, where $\psi(b)\hat{D}_N(ib) \in \mathcal{R}_{\mathcal{B}}(t^{-p})$, for any $p > 0$. By Proposition 6.1 b), $\hat{B}_N(s)\hat{C}(s)\hat{B}_N(s)v = \hat{A}(s)v$, where $\psi(b)\hat{A}(ib) \in \mathcal{R}_{\mathcal{B}}(t^{-(\beta-\epsilon)})$. Thus,

$$s^{-1}[\hat{B}_N(s)\hat{C}(s)]^2 v = \hat{A}(s)\hat{C}(s)\hat{D}_N(s)v$$

and

$$s^{-1}[\hat{B}_N(s)\hat{C}(s)]^3 \hat{B}(s)v = \hat{A}(s)\hat{C}(s)\hat{D}_N(s)\hat{C}(s)\hat{B}(s).$$

By Lemma 6.19, $\psi\hat{C}(ib) \in \mathcal{R}_{\mathcal{B}}(1/t^{\beta-\epsilon})$. By Lemma 6.13, $\psi\hat{B}(ib) \in \mathcal{R}_{\mathcal{B}}(1/t^{\beta-\epsilon})$. Hence, assuming that $Pv = 0$, $\psi b^{-1}[\hat{B}_N(ib)\hat{C}(ib)]^2 \in \mathcal{R}_{\mathcal{B}}(1/t^{\beta-\epsilon})$ and $\psi b^{-1}[\hat{B}_N(ib)\hat{C}(ib)]^3\hat{B}(ib) \in \mathcal{R}_{\mathcal{B}}(1/t^{\beta-\epsilon})$. The conclusion follows by putting the above estimates together. □

Remark 6.22. *Showing that the estimates provided by Lemma 6.21 hold with $\|.\|_{\mathcal{B}}$ replaced by $\|.\|_{\mathcal{B}\to\mathcal{B}_0}$ under a weakened (H4) (that is, such that both $(\mathcal{B}, \mathcal{B}_0)$ and $(\mathcal{B}_0, L^\infty)$ satisfy (H5) with τ as appropriate) brings up several complications. Among these, we note that a) one needs to work with an appropriate version of Lemma 6.13; b) the last term of equation (6.13) is a complicated product, so its inverse Laplace transform cannot be easily estimated under a weakened (H4).*

We believe that for a) one could exploit a decomposition of $\psi\hat{B}$ into the scalar part given by $\tilde{\lambda}$ (of which inverse Laplace transform can be estimated under a weaker (H4)) and the rest. Also, we believe that this route for dealing with a) can be further combined with repeated applications of the type of arguments and abstract results of [13] (or rather the improved version of the mentioned abstract result in [12]) to solve b). This type of argument for dealing with a) and b) above constitutes the subject of work in progress.

7 Arguments in the infinite case: proof of Theorem 5.2

As in Section 6, in this section we make transparent that the present abstract set-up allows us to show that main part of the techniques developed for the discrete time scenario (namely, [20] with some required generalizations in [16]) carry over to the continuous time case. Due to (H5) ii), in part of the arguments we follow the steps in [16], which exploits an analogue of (H5) ii) in the discrete time setting. Equally important, as in Section 6, some techniques/calculations required to deal with continuous time infinite measure preserving systems are directly borrowed from [21].

7.1 Estimates for $(I - \hat{R})^{-1}$

By (H2), (H3) there exist $\delta > 0$ and a continuous family $\lambda(ib)$ of simple eigenvalues of $\hat{R}(ib)$ such that $\lambda(ib)$ is well defined for all $|b| < \delta$ and $\lambda(0) = 1$. In what follows, we let $P(ib)$ be the associated spectral projection, $v(ib)$ be the associated eigenfunction and set $Q(ib) = I - P(ib)$.

The continuity properties of $\hat{R}(ib)$, $b \in \mathbb{R}$, are obtained via Lemma 3.2 and assumption (H4).

Lemma 7.1. *Assume (H2), (H3) and (H5) ii). Let τ be as defined by (H4). Then there exists $C > 0$ such that for any $h > 0$,*

$$\|\hat{R}(i(b+h)) - \hat{R}(ib)\|_{\mathcal{B}\to\mathcal{B}_0} \le C\max\{1,|b|\}\, h^{\tau}.$$

Proof. By Lemma 3.2, for all $b \in \mathbb{R}$ and $a > 0$ such that $e^{-iba} \neq 1$, $\hat{R}(ib) = g_a(ib)\hat{L}_a(ib)$ with $\hat{L}_a(ib) = \int_0^\infty R_{t,a} e^{-aibt}\, dt$ and $g_a(ib) = \frac{-ib}{e^{-iab}-1}$.

A standard calculation shows that $|g_1(i(b+h)) - g_1(ib)| \le h$, for all $|b| < 1$ and $|g_1(b)| \ll 1$. Next, given $b \in \mathbb{R}$, fix $a > 0$ such that $|e^{-iba} - 1| > 1$. Then, there exists some constant $C > 0$ such that $|g_a(i(b+h)) - g_a(ib)| \le Ch$ and $|g_a(b)| \le C|b|$.

It remains to show that $\|\hat{L}_a(i(b+h)) - \hat{L}_a(ib)\|_{\mathcal{B}\to\mathcal{B}_0} \ll h^{\tau}$, for all $b \in \mathbb{R}$ and appropriate $a > 0$. This is an immediate consequence of (H5) ii):

$$\begin{aligned}\|\hat{L}_a(ib_1) - \hat{L}_a(ib_2)\|_{\mathcal{B}\to\mathcal{B}_0} &= \|\int_0^\infty R_{t,a}(e^{-i(b+h)t} - e^{-ibt})\|_{\mathcal{B}\to\mathcal{B}_0} \\ &\ll h^{\tau}\int_0^\infty t^{\tau}\|R_{t,a}\|_{\mathcal{B}\to\mathcal{B}_0} dt \ll h^{\tau}.\end{aligned}$$

□

To estimate $\|P(ib) - P\|_{\mathcal{B}\to\mathcal{B}_0}$ for b close to zero (and as such $\|Q(ib) - Q\|_{\mathcal{B}\to\mathcal{B}_0}$, $\|v(ib) - v(0)\|_{\mathcal{B}\to\mathcal{B}_0}$) we recall the following abstract result of [13], which due to Lemma 7.1 applies to our setting with no modification of the involved proof (except adjusting the corresponding labeling of the quantities and parameters used there).

Lemma 7.2. *[13, Corollary 1] Assume (H2), (H3) and (H5). Then, there exists $\delta_0 > 0$ and some constant $C > 0$ such that for all $b \in spec(\hat{R}(ib)) \cap B_{\delta^*}(0)$, for all $h \le |b|$ and for any $\epsilon > 0$,*

$$\|P(ib) - P\|_{\mathcal{B}\to\mathcal{B}_0} \le C|b|^{\tau-\epsilon}, \quad \|P(i(b+h))) - P(ib)\|_{\mathcal{B}\to\mathcal{B}_0} \le Ch^{\tau-\epsilon}.$$

Moreover, the same estimates hold for the families $Q(z)$ and $v(z)$.

The result below is a consequence of (H0) ii), Lemma 7.1 and Lemma 7.2.

Lemma 7.3. *Assume (H0) ii), (H2), (H3), (H5) i) and (H5) ii) with $\max\{2\beta - 1, 1-\beta\} < \tau < \beta - \gamma$. Fix $\delta_0 > 0$ such that Lemma 7.2 holds.*

Let $c_\beta = i\int_0^\infty e^{-i\sigma}\sigma^{-\beta}\, d\sigma$. Then, for all $|b| < \delta_0$ and for any $\epsilon > 0$,

$$(1-\lambda(ib))^{-1} = c_\beta^{-1}\ell(1/|b|)^{-1}b^{-\beta} + O(|b|^{2\tau-\epsilon}), \quad (I-\hat{R}(ib))^{-1} = c_\beta^{-1}\ell(1/|b|)^{-1}b^{-\beta}(P+E(ib)),$$

where $E(ib)$ is a family of operators satisfying $\|E(ib)\|_{\mathcal{B}\to\mathcal{B}_0} = O(|b|^{\tau-\epsilon})$.

Proof. The argument is standard. For similar arguments we refer to, for instance, [2, 20, 19, 21]). Due to our assumption (H5) ii), we need to use Lemma 7.2 (for a similar use of Lemma 7.2 in a different set up we refer [16]).

Recall $\hat{R}(ib)v = R(e^{-ib\varphi}v)$. Following the formalism in [12] (a simplification of [2]), we write

$$\lambda(ib) = \int_{\tilde{Y}} \lambda(ib)v(ib)\, d\mu_\Phi = \int_{\tilde{Y}} R(e^{-ib\varphi}v(ib))\, d\mu_\Phi = \int_{\tilde{Y}} e^{-ib\varphi}\, d\mu_\Phi + V(ib), \qquad (7.1)$$

where $V(ib) = \int_{\tilde{Y}} (\hat{R}(ib) - \hat{R}(0))(v(ib) - v(0))\, d\mu_\Phi$.

By the argument in [10], $1 - \int_Y e^{-ib\varphi}\, d\mu \sim c_\beta \ell(1/b) b^\beta$ as $b \to 0^+$.

By Lemma 7.1 and Corollary 7.2, the families $\hat{R}(ib) : \mathcal{B} \to \mathcal{B}_0$ and $v(ib) : \mathcal{B} \to \mathcal{B}_0$ are C^τ and $C^{\tau-\epsilon_0}$, respectively, in $B_{\delta_0}(0)$. Since $\mathcal{B} \subset \mathcal{B}_0 \subset L^\infty(\mu_\Phi)$, $|V(ib)| = O(b^{2\tau-\epsilon})$. Thus,

$$1 - \lambda(ib) = c_\beta \ell(1/b) b^\beta + O(|b|^{2\tau-\epsilon}).$$

Next, for all $|b| < \delta_0$,

$$(I - R(ib))^{-1} = (1 - \lambda(ib))^{-1} P - (1 - \lambda(ib))^{-1}(P(ib) - P) + (I - R(ib))^{-1} Q(ib).$$

By (H3), $\|(I - R(ib))^{-1} Q(ib)\|_{\mathcal{B}} = O(1)$. By Corollary 7.2, $\|P(ib) - P(0)\|_{\mathcal{B} \to \mathcal{B}_0} \ll b^{\tau-\epsilon_0}$. Set

$$E(ib) = P(ib) - P(0) + (1 - \lambda(ib))(I - R(ib))^{-1} Q(ib)$$

and note that $\|E(ib)\|_{\mathcal{B} \to \mathcal{B}_0} = O(|b|^{\tau-\epsilon})$. Thus,

$$(I - \hat{R}(ib))^{-1} = (1 - \lambda(ib))^{-1}(P + E(ib)) = c_\beta^{-1} \ell(1/b)^{-1} b^{-\beta}(P + E(ib)),$$

as required. □

An immediate consequence of Lemma 7.3 is:

Corollary 7.4. *Assume the setting and notation of Lemma 7.3. Then*

i) $\|(I - \hat{R}(ib))^{-1}\|_{\mathcal{B} \to \mathcal{B}_0} \ll \ell(1/b)^{-1} |b|^{-\beta}$ *for all* $0 < |b| < \delta_0$.

ii) There exists $C > 0$ *such that* $\|(I - \hat{R}(ib))^{-1}\|_{\mathcal{B} \to \mathcal{B}_0} \le C$*, for all* $\delta_0 < |b| < 1$.

Using Lemma 7.1, (H5) i) and the type of arguments in [13] we obtain

Corollary 7.5. *Assume the setting and notation of Lemma 7.3. Then, there exists* $C > 0$ *such that* $\|(I - \hat{R}(ib))^{-1} - (I - \hat{R}(i(b+h))^{-1}\|_{\mathcal{B} \to \mathcal{B}_0} \le C\, h^\tau \log(1/h)$*, for all* $\delta_0 < |b| < 1$ *and* $h > 0$.

Proof. By the resolvent equality,

$$\begin{aligned}(I - \hat{R}(ib))^{-1} - (I - \hat{R}(i(b+h))^{-1} &= (I - \hat{R}(ib))^{-1}(\hat{R}(ib) - \hat{R}(i(b+h))(I - \hat{R}(i(b+h))^{-1} \\ &:= (I - \hat{R}(ib))^{-1} A(b,h).\end{aligned}$$

Next, by (H2), $\|(I - \hat{R}(ib))^{-1}\|_{\mathcal{B}} \le C$, for some $C > 0$ for all $\delta_0 < |b| < 1$. Together with (H5) i) and the type of arguments in [13], this implies that for any $v \in \mathcal{B}$,

$$\begin{aligned}\|(I - \hat{R}(ib))^{-1} A(b,h) v\|_{\mathcal{B}_0} &\le \sum_{j=1}^n \|\hat{R}(ib)^j A(b,h) v\|_{\mathcal{B}_0} + \|\hat{R}(ib)^n (I - \hat{R}(ib))^{-1} A(b,h)\|_{\mathcal{B}_0} \\ &\le n \|A(b,h) v\|_{\mathcal{B}_0} + C_1 \theta^n \|(I - \hat{R}(ib))^{-1} A(b,h) v\|_{\mathcal{B}} + C_2 \|(I - \hat{R}(ib))^{-1} A(b,h) v\|_{\mathcal{B}_0}.\end{aligned}$$

By Lemma 7.1, for all $\delta_0 < |b| < 1$, $\|A(b,h) v\|_{\mathcal{B}_0} \le h^\tau \|v\|_{\mathcal{B}}$. Recalling $C_2 < 1$ and using the last displayed inequality,

$$(1 - C_2) \|(I - \hat{R}(ib))^{-1} A(b,h) v\|_{\mathcal{B}_0} \le C_1 \theta^n \|v\|_{\mathcal{B}} + n h^\tau \|v\|_{\mathcal{B}}.$$

The conclusion follows by taking $n = [\tau \log(h) \log(\theta)^{-1}]$ (so, $\theta^n \ll h^\tau$ and $n \ll \log(1/h)$). □

The next result provides the required continuity estimates for $\lambda(ib)$ and $(I - \hat{R}(ib))^{-1}$ under (H5) ii).

Lemma 7.6. *[16, Proposition 3.7] Assume the setting and notation of Lemma 7.3. Then the following hold for any $0 < h \le |b| < \delta^*$ and any $\epsilon > 0$.*

i) $|\lambda(i(b+h)) - \lambda(ib)| \ll h^\beta \ell(1/h) + h^{\tau-\epsilon}|b|^\beta$.

ii) Also,

$$\|(I-\hat{R}(ib))^{-1}-(I-\hat{R}(i(b+h))^{-1}\|_{\mathcal{B}\to\mathcal{B}_0} \ll \ell(1/|b|)^{-2}h^{\tau-\epsilon}|b|^{-\beta}+\ell(1/|b|)^{-2}\ell(1/h)h^\beta|b|^{-2\beta}.$$

Proof. The proof goes word by word as the proof of [23, Proposition 4.2], [16, Proposition 3.8] (by setting $u = 0$, $\theta = b$ and relabeling the parameters used there), with the change that in the present set-up $\mathcal{B} \subset \mathcal{B}_0 \subset L^\infty(\mu_\Phi)$. Because the proof of item i) is short we provide below for completeness. As in [16, Proposition 3.8], item ii) follows from item i) together with Lemma 7.2.

Put $\Delta_\lambda = |\lambda(i(b+h)) - \lambda(ib)|$. Using eq. (7.1) we write

$$\begin{aligned}\Delta_\lambda &= \int_{\tilde{Y}} (e^{-i(b+h)\varphi} - e^{-ib\varphi})\, d\mu_\Phi + \int_{\tilde{Y}} (\hat{R}(i(b+h)) - \hat{R}(ib)(v(i(b+h)) - v(0))\, d\mu_\Phi \\ &\quad + \int_{\tilde{Y}} (\hat{R}(ib) - \hat{R}(0))(v(i(b+h)) - v(ib))\, d\mu_\Phi \\ &= \int_{\tilde{Y}} (e^{-i(b+h)\varphi} - e^{-ib\varphi})\, d\mu_\Phi + V_1(ib) + V_2(ib).\end{aligned}$$

Using the fact that $\mathcal{B} \subset \mathcal{B}_0 \subset L^\infty(\mu_\Phi)$, we get

$$\begin{aligned}|V_1(ib)| &\ll \|v(i(b+h)) - v(0)\|_{\mathcal{B}\to\mathcal{B}_0}\Big|\int_{\tilde{Y}} (\hat{R}(i(b+h)) - \hat{R}(ib))\, d\mu_\Phi\Big| \\ &= \|v(i(b+h)) - v(0)\|_{\mathcal{B}\to\mathcal{B}_0}\Big|\int_{\tilde{Y}} (e^{-i(b+h)\varphi} - e^{-ib\varphi})\, d\mu_\Phi\Big|\end{aligned}$$

and

$$\begin{aligned}|V_2(ib)| &\ll \|v(i(b+h)) - v(ib)\|_{\mathcal{B}\to\mathcal{B}_0}|\int_{\tilde{Y}} (\hat{R}(i(b+h)) - \hat{R}(0))\, d\mu_\Phi| \\ &= \|v(i(b+h)) - v(ib)\|_{\mathcal{B}\to\mathcal{B}_0}\Big|\int_{\tilde{Y}} (e^{-i(b+h)\varphi} - 1)\, d\mu_\Phi\Big|.\end{aligned}$$

By Corollary 7.2, for any $\epsilon > 0$, $\|(v(i(b+h)) - v(ib)\|_{\mathcal{B}\to\mathcal{B}_0} \ll h^{\tau-\epsilon}$. Similarly, $\|v(i(b+h)) - v(0)\|_{\mathcal{B}\to\mathcal{B}_0} \ll |b+h|^{\tau-\epsilon} \ll |b|^{\tau-\epsilon}$.

Next, let $G(x) = \mu_\Phi(\varphi \le x)$. It is easy to see that $\int_{\tilde{Y}}(e^{-i(b+h)\varphi} - e^{-ib\varphi}) d\mu_\Phi = \int_0^\infty e^{-ibx}(e^{-ihx} - 1)dG(x)$. The estimate $|\int_0^\infty e^{-ibx}(e^{-ihx} - 1)dG(x)| \ll h^\beta\ell(1/h)$ follows by the argument used in the proof of [10, Lemma 3.3.2]. Similarly, $|\int_{\tilde{Y}}(e^{-i(b+h)\varphi} - 1)\, d\mu_\Phi\Big| \ll |b+h|^\beta \ll b^\beta$.

Putting the above together and using that $0 < h \le |b|$, $|V_1(ib)| \ll |b+h|^{\tau-\epsilon}h^\beta \ll |b|^{\tau-\epsilon}h^\beta \ll h^{\tau-\epsilon}|b|^\beta$ and $|V_2(ib)| \ll h^{\tau-\epsilon}|b|^\beta$. Since $|\int_{\tilde{Y}}(e^{-i(b+h)\varphi} - e^{-ib\varphi})\, d\mu_\Phi| \ll h^\beta\ell(1/h)$, the conclusion follows. □

Lemma 7.7. *Assume (H0) ii) and (H5) ii). Assume (H6') and choose α such that (H6) holds. Assume that $|b| \geq 1$. Then the following holds as $h \to 0$,*

$$\|(I - \hat{R}(ib))^{-1} - (I - \hat{R}(i(b+h))^{-1}\|_{\mathcal{B}\to\mathcal{B}_0} \ll h^\tau \log(|b|)|b|^{2\alpha+1}.$$

Proof. Write $(I - \hat{R}(ib))^{-1} - (I - \hat{R}(i(b+h))^{-1} = (I - \hat{R}(ib))^{-1}A(b,h)$, with $A(b,h) = (\hat{R}(ib) - \hat{R}(i(b+h))(I - \hat{R}(i(b+h))^{-1}$. With the notation of Remark 4.6, let $k = \frac{1+\log C_0 + \log|b|}{\log\theta}$ and note that

$$\|(I - \hat{R}(ib))^{-1}A(b,h)v\|_{\mathcal{B}_0} \leq \sum_{j=1}^{k} \|\hat{R}(ib)^j A(b,h)v\|_{\mathcal{B}_0} + \|\hat{R}(ib)^k (I - \hat{R}(ib))^{-1}A(b,h)v\|_{\mathcal{B}_0}$$

Using (H6'), (H6) and the fact that $\|A(b,h)v\|_{\mathcal{B}_0} \leq |b|^{\alpha+1}h^\tau\|v\|_{\mathcal{B}}$ (by Lemma 7.1),

$$|b|^{-\alpha}\|(I - \hat{R}(ib))^{-1}A(b,h)v\|_{\mathcal{B}_0} \leq C\log(|b|)\|A(b,h)v\|_{\mathcal{B}_0} \leq C\log(|b|)|b|^{\alpha+1}h^\tau\|v\|_{\mathcal{B}},$$

for $C > 0$. The conclusion follows. □

7.2 Estimates for $\hat{T} = \hat{U}(I - \hat{R})^{-1}$

In this section, we combine our estimates for $(I - \hat{R})^{-1}$ obtained in Section 7.1 with the estimates for $\hat{U}$ obtained in Section 8.2. Recall that $c_\beta = i\int_0^\infty e^{-i\sigma}\sigma^{-\beta}\,d\sigma$.

Given that $\hat{T} = \hat{U}(I-\hat{R})^{-1}$ is a product, in the result below we speak of $\|\hat{U}(I-\hat{R})^{-1}\|_{(\mathcal{B}\to\mathcal{B}_0)\to L^1(\mu_\Phi)} \leq \|\hat{U}\|_{\mathcal{B}_0\to L^1(\mu_\Phi)}\,\|(I - \hat{R})^{-1}\|_{\mathcal{B}\to\mathcal{B}_0}$.

Lemma 7.8. *Assume (H0) ii), (H1) ii), (H2), (H3) and (H5). Assume (H6) and choose α accordingly. Let $v = \frac{v^*}{\tilde{h}}$ with $v^* \in \mathcal{B}$ and $w = \frac{w^*}{\tilde{h}}$ with $w^* \in L^\infty(\mu_\Phi)$. Let τ be given as in (H4). Then,*

(a) $\int_{\tilde{Y}} \hat{T}(ib)vw\,d\tilde{\mu} = c_\beta^{-1}\ell(1/|b|)^{-1}b^{-\beta}\int_{\tilde{Y}} v\,d\tilde{\mu}\,\int_{\tilde{Y}} w\,d\tilde{\mu}\,(1+o(1))$.

(b) $\|\hat{T}(ib)\|_{(\mathcal{B}\to\mathcal{B}_0)\to L^1(\mu_\Phi)} \ll \begin{cases} \ell(1/b)^{-1}|b|^{-\beta}, & 0 < |b| < 1, \\ |b|^\alpha, & |b| \geq 1. \end{cases}$

(c) For any $0 < h < |b| < 1$ and for any $\epsilon > 0$,

$$\|\hat{T}(i(b+h)) - \hat{T}(ib)\|_{(\mathcal{B}\to\mathcal{B}_0)\to L^1(\mu_\Phi)} \ll \ell(1/|b|)^{-2}h^{\tau-\epsilon}|b|^{-\beta} + \ell(1/|b|)^{-2}\ell(1/h)h^\beta|b|^{-2\beta}.$$

(d) For $|b| > 1$ and any $0 < h < 1$,

$$\|\hat{T}(i(b+h)) - \hat{T}(ib)\|_{(\mathcal{B}\to\mathcal{B}_0)\to L^1(\mu_\Phi)} \ll |b|^{2\alpha+1}h^\tau(\log|b|)|b|^{2\alpha+1}.$$

Proof. (a) By Lemma 8.6(b), $\hat{U}$ is continuous. Thus,

$$\begin{aligned}\lim_{b\to 0^+}\ell(1/b)b^\beta\int_{\tilde{Y}}\hat{T}(ib)v\,w\,d\tilde{\mu} &= \lim_{b\to 0^+}\int_{\tilde{Y}}\hat{U}(ib)\ell(1/b)b^\beta(I - \hat{R}(ib))^{-1}v\,w\,d\tilde{\mu} \\ &= \int_{\tilde{Y}}\hat{U}(0)\lim_{b\to 0^+}\ell(1/b)b^\beta(I - \hat{R}(ib))^{-1}v\,w\,d\tilde{\mu}.\end{aligned}$$

Combined with Lemma 8.6(a), the above equality yields

$$\begin{aligned}\lim_{b\to 0^+}\ell(1/b)b^\beta\int_{\tilde Y}\hat T(ib)v\,w\,d\tilde\mu &= \int_{\tilde Y}\tilde h\int_0^u \lim_{b\to 0^+}\ell(1/b)b^\beta(I-\hat R(ib))^{-1}v(y,\sigma)\,d\sigma\,w^*(y,u)\,d\mu_\Phi\\ &+\int_{\tilde Y}\int_u^1 R\Big(\tilde h\lim_{b\to 0^+}\ell(1/b)b^\beta(I-\hat R(ib))^{-1}v\Big)(y,\sigma)\,d\sigma\,w^*(y,u)\,d\mu_\Phi\\ &=\int_{\tilde Y}\tilde h\int_0^u \lim_{b\to 0^+}\ell(1/b)b^\beta(I-\hat R(ib))^{-1}v(y,\sigma)\,d\sigma\,w^*(y,u)\,d\mu_\Phi\\ &+\int_{\tilde Y}\tilde h\int_u^1\Big(\lim_{b\to 0^+}\ell(1/b)b^\beta(I-\hat R(ib))^{-1}v\Big)(y,\sigma)\,d\sigma\,w^*(y,u)\circ\Phi\,d\mu_\Phi.\end{aligned}$$

Recall $w=\frac{w^*}{\tilde h}$. Hence, by Lemma 7.3,

$$\begin{aligned}\lim_{b\to 0^+}\ell(1/b)b^\beta\int_{\tilde Y}\hat T(ib)v\,w\,d\tilde\mu &= c_\beta^{-1}\int_{\tilde Y}\tilde h\int_0^u v(y,\sigma))\,d\sigma d\mu_\Phi\int_{\tilde Y}w\,d\tilde\mu\\ &+c_\beta^{-1}\int_{\tilde Y}\tilde h\int_u^1 v(y,\sigma)\,d\sigma d\mu_\Phi\int_{\tilde Y}w^*\circ\Phi\,d\mu_\Phi.\end{aligned}$$

Since $\int_{\tilde Y}w^*\circ\Phi\,d\mu_\Phi=\int_{\tilde Y}w^*\,d\mu_\Phi=\int_{\tilde Y}w\,d\tilde\mu$,

$$\begin{aligned}\lim_{b\to 0^+}\ell(1/b)b^\beta\int_{\tilde Y}\hat T(ib)v\,w\,d\tilde\mu &= c_\beta^{-1}\int_{\tilde Y}\tilde h\Big(\int_0^u v(y,\sigma))+\int_u^1 v(y,\sigma)\,d\sigma\Big)d\mu_\Phi\int_{\tilde Y}w\,d\tilde\mu\\ &=c_\beta^{-1}\int_{\tilde Y}\tilde h v\,d\mu_\Phi\int_{\tilde Y}w\,d\tilde\mu=c_\beta^{-1}\int_{\tilde Y}v d\tilde\mu\int_{\tilde Y}w\,d\tilde\mu,\end{aligned}$$

which ends the proof of (a).
(b) Recall $\hat T(ib)=\hat U(ib)(I-\hat R(ib))^{-1}$ and note that

$$\|\hat T(ib)\|_{(\mathcal B\to\mathcal B_0)\to L^1(\mu_\Phi)}\ll\|\hat U(ib)\|_{\mathcal B_0\to L^1(\mu_\Phi)}\|(I-\hat R(ib))^{-1}\|_{\mathcal B\to\mathcal B_0}.$$

When $0<|b|<1$, the conclusion follows by Corollary 7.4 and Remark 8.5 (which ensures $\|\hat U(ib)\|_{\mathcal B_0\to L^1(\mu_\Phi)}\le C$, for some positive constant C). When $|b|>1$, the conclusion follows from (H6) and Lemma 8.3.
(c) Note that

$$\begin{aligned}\|\hat T(ib)-\hat T(i(b-h))\|_{(\mathcal B\to\mathcal B_0)\to L^1(\mu_\Phi)} &\ll\|\hat U(ib)-\hat U(i(b-h))\|_{\mathcal B_0\to L^1(\mu_\Phi)}\|(I-\hat R(ib))^{-1}\|_{\mathcal B\to\mathcal B_0}\\ &+\|\hat U(ib)\|_{\mathcal B\to L^1(\mu_\Phi)}\|(I-\hat R(ib)^{-1}-(I-\hat R(i(b-h)))^{-1}\|_{\mathcal B\to\mathcal B_0}\\ &=D(b,h)+E(b,h). \qquad (7.2)\end{aligned}$$

Recall $0<h<|b|<1$ and τ is such that (H5) ii) holds. By Lemma 8.6(b), $\|\hat U(ib)-\hat U(i(b-h))\|_{\mathcal B\to L^1(\mu_\Phi)}\ll h^\tau$. By Corollary 7.4, $\|(I-\hat R(ib))^{-1}\|_{\mathcal B\to\mathcal B_0}\ll\ell(1/|b|)^{-1}|b|^{-\beta}$. Hence,

$$D(b,h)\ll\ell(1/|b|)^{-1}|b|^{-\beta}h^\tau.$$

To deal with $E(b,h)$ we consider two cases: i) $0<h<|b|<\delta_0$ and ii) $\delta_0<|b|<1$ and $h>0$. In both cases, δ_0 is fixed such that the conclusion of Lemma 7.3 holds.

In case i), Lemma 7.6 ii) gives that for any $\epsilon > 0$,

$$\|(I-\hat{R}(i(b-h)))^{-1}-(I-\hat{R}(ib)^{-1}\|_{\mathcal{B}\to\mathcal{B}_0} \ll \ell(1/|b|)^{-2}h^{\tau-\epsilon}|b|^{-\beta}+\ell(1/|b|)^{-2}\ell(1/h)h^{\beta}|b|^{-2\beta}.$$

By Remark 8.5, $\|\hat{U}(ib)\|_{\mathcal{B}_0\to L^1(\mu_\Phi)} \le C$, for some positive constant $C > 0$. Hence,

$$E(b,h) \ll \ell(1/|b|)^{-2}h^{\tau-\epsilon}|b|^{-\beta} + \ell(1/|b|)^{-2}\ell(1/h)h^{\beta}|b|^{-2\beta}.$$

In case ii), by Corollary 7.5 we know that $\|(I-\hat{R}(i(b-h)))^{-1}-(I-\hat{R}(ib))^{-1}\|_{\mathcal{B}\to\mathcal{B}_0} \le C\,h^{\tau}$, for some constant $C > 0$. We already know that $\|\hat{U}(ib)\|_{\mathcal{B}_0\to L^1(\mu_\Phi)} \le C$, for some positive constant $C > 0$. Hence, $E(b,h) \le C\,h^{\tau}$.

The conclusion follows by putting together the estimates for $D(b,h)$ and $E(b,h)$.

(d) We continue from (7.2), recalling that we consider the case $|b| > 1$ and $0 < h < 1$. By Lemma 8.6(b), $\|\hat{U}(i(b+h)) - \hat{U}(ib)\|_{\mathcal{B}_0\to L^1(\mu_\Phi)} \ll h^{\tau}$. By (H6), $\|(I-\hat{R}(ib))^{-1}\|_{\mathcal{B}} \ll |b|^{\alpha}$. Hence,

$$D(b,h) \ll |b|^{\alpha}h^{\tau}.$$

By Lemma 8.3, $\|\hat{U}(ib)\|_{\mathcal{B}_0\to L^1(\mu_\Phi)} \le C$, for some positive constant C. By Lemma 7.7, $\|(I-\hat{R}(i(b-h)))^{-1}-(I-\hat{R}(ib))^{-1}\|_{\mathcal{B}\to\mathcal{B}_0} \ll h^{\tau}(\log|b|)|b|^{2\alpha+1}$. Hence,

$$E(b,h) \ll h^{\tau}(\log|b|)|b|^{2\alpha+1}.$$

The conclusion follows by putting together the estimates for $D(b,h)$ and $E(b,h)$. □

7.3 Proof of Theorem 5.2

Recall that $\rho_t(v,w) = \int_{\tilde{Y}} vw\circ f_t\,d\tilde{\mu}$ and for $s \in \bar{\mathbb{H}}$, set

$$\hat{\rho}(s)(v,w) = \int_0^{\infty} \rho_t(v,w)e^{-st}dt = \int_{\tilde{Y}} \hat{T}(s)v\,w\,d\tilde{\mu}. \tag{7.3}$$

The result below has been established in the set-up of [21]. It applies to the present set-up with no modification.

Proposition 7.9. *[21, Proposition 6.2] The analytic function $\hat{\rho}(s)(v,w)$, $\Re s > 0$, extends to a continuous function on $\{\Re s \ge 0\} - \{0\}$. Suppressing the dependence on v, w, the correlation function is given by*

$$\rho_t = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ibt}\hat{\rho}(ib)\,db = \frac{1}{\pi}\Re\Big(\int_0^{\infty} e^{ibt}\hat{\rho}(ib)\,db\Big).$$

The next three results provides an estimate for $\int_c^d e^{ibt}\hat{\rho}(ib)\,db$ for different regimes of $0 < c < d < \infty$, as specified below.

Lemma 7.10. *For any $a > 0$,*

$$\lim_{t\to\infty}\ell(t)t^{1-\beta}\int_0^{a/t} e^{ibt}\hat{\rho}(ib)(v,w)\,db = c_\beta^{-1}\int_0^a e^{i\sigma}\sigma^{-\beta}\,d\sigma\int_{\tilde{Y}} v\,d\tilde{\mu}\int_{\tilde{Y}} w\,d\tilde{\mu}.$$

Proof. The follows by the argument of [21, Proposition 6.2] with Lemma 7.8 (a) replacing [21, Corollary 5.10]. For the analogous argument in the discrete time scenario we refer to [20, Lemma 5.2]. □

Lemma 7.11. *Let $\beta' \in (\frac{1}{2}, \beta)$. Let τ be given as in (H4) and let $\tau' \in (1-\beta, \tau)$. Then for all $a \in (\pi, t)$,*

$$\int_{a/t}^{1} e^{ibt} \hat{\rho}_{v,w}(ib)\, db \ll \ell(t)^{-1} t^{-(1-\beta)} a^{-(2\beta'-1)} + \ell(t)^{-1} t^{-(1-\beta+\tau')} a^{1-\beta'}.$$

Proof. The argument below is similar to the argument used in the proofs of [21, Proposition 6.4], [20, Lemma 5.1], with obvious adaptation due to the estimates above.

Let $0 < b < 1$. By Corollary 7.4,

$$|\hat{\rho}(ib)| \ll \ell(1/b)^{-1} b^{-\beta} \|v\|_{\mathcal{B}} \, \|w^*\|_{L^\infty(\mu_\Phi)}.$$

By Lemma 7.8(b,c) with $h = \pi/t$, we have that for any $\epsilon > 0$

$$\begin{aligned} |\hat{\rho}(ib) - \hat{\rho}(i(b-\pi/t))| \ll & \ell(1/b)^{-2}\, \ell(1/h)\, b^{-2\beta}\, t^{-\beta}\, \|v^*\|_{\mathcal{B}} \|w^*\|_{L^\infty(\mu_\Phi)} \\ & + \ell(1/b)^{-2}\, t^{-(\tau-\epsilon)}\, b^{-\beta}\, \|v^*\|_{\mathcal{B}} \|w^*\|_{L^\infty(\mu_\Phi)}. \end{aligned}$$

From here on we suppress the factor $\|v^*\|_{\mathcal{B}} \|w^*\|_{L^\infty(\mu_\Phi)}$. Write

$$I = \int_{a/t}^{1} e^{ibt} \hat{\rho}(ib)\, db = -\int_{(a+\pi)/t}^{1+\pi/t} e^{ibt} \hat{\rho}(i(b-\pi/t))\, db.$$

Then $2I = I_1 + I_2 + I_3$, where

$$\begin{aligned} I_1 &= -\int_{1}^{1+\pi/t} e^{ibt} \hat{\rho}(i(b-\pi/t))\, db, \qquad I_2 = \int_{a/t}^{(a+\pi)/t} e^{ibt} \hat{\rho}(ib)\, db, \\ I_3 &= \int_{(a+\pi)/t}^{1} e^{ibt} (\hat{\rho}(ib) - \hat{\rho}(i(b-\pi/t)))\, db. \end{aligned}$$

Clearly $I_1 = O(t^{-1})$, and by Potter's bounds (see, for instance, [6]),

$$\begin{aligned} |I_2| &\ll \int_{a/t}^{(a+\pi)/t} \ell(1/b)^{-1} b^{-\beta}\, db = \ell(t)^{-1} t^{-(1-\beta)} \int_{a}^{a+\pi} [\ell(t)/\ell(t/\sigma)] \sigma^{-\beta}\, d\sigma \\ &\ll \ell(t)^{-1} t^{-(1-\beta)} \int_{a}^{a+\pi} \sigma^{-\beta'}\, d\sigma \ll \ell(t)^{-1} t^{-(1-\beta)} a^{-\beta'}. \end{aligned}$$

Next,

$$\begin{aligned} |I_3| &\ll \ell(t) t^{-\beta} \int_{a/t}^{1} \ell(1/b)^{-2} b^{-2\beta}\, db + t^{-(\tau-\epsilon_0)} \int_{a/t}^{1} \ell(1/b)^{-2} b^{-\beta}\, db \\ &= I_{3,1} + I_{3,2}. \end{aligned}$$

By Potter's bounds,

$$I_{3,1} = \ell(t)^{-1} t^{\beta-1} \int_a^t [\ell(t)/\ell(t/\sigma)]^2 \sigma^{-2\beta} \, d\sigma \ll \ell(t)^{-1} t^{\beta-1} \int_a^\infty \sigma^{-2\beta'} \, d\sigma$$
$$\ll \ell(t)^{-1} t^{\beta-1} a^{-(2\beta'-1)}.$$

Also, $I_{3,2} = \ell(t)^{-2} t^{-(\tau-\epsilon_0)} \int_{a/t}^1 [\ell(t)/\ell(1/b)]^2 b^{-\beta} \, db$. By Potter's bounds, for any $\epsilon > 0$, $[\ell(t)/\ell(1/b)]^2 \ll t^\epsilon b^{-\epsilon}$. Hence,

$$I_{3,2} = \ell(t)^{-2} t^{-(\tau-2\epsilon)} \int_{a/t}^1 b^{-\beta-\epsilon} \, db.$$

Since ϵ can be taken arbitrarily small, $I_{3,2} \ll \ell(t)^{-2} t^{-(\tau-\epsilon')} a^{1-\beta-\epsilon}$, for any $\epsilon' > \epsilon$. The conclusion follows since $\tau' \in (1-\beta, \tau)$ and $\beta' \in (1/2, \beta)$. □

For the next result we recall that $C^m(\tilde{Y}, \tilde{\mu})$ is the class of observables defined in (4.2).

Lemma 7.12. *Assume (H4) (ii). Assume (H6) and choose α accordingly. Choose $m > 2\alpha + 2$. Let $w \in C^m(\tilde{Y}, \tilde{\mu})$. Assume (H4) with $\tau > 1 - \beta$ as given there. Then,*

$$\left| \int_1^\infty e^{ibt} \hat{\rho}(ib)(v, w) \, db \right| \ll t^{-\tau} \|v\|_{\mathcal{B}} \, \|w^*\|_{L^\infty(\mu_\Phi)}.$$

Proof. The argument below adapts the proof of [21, Proposition 6.5] to the present context.

By Proposition 4.7, $\hat{\rho}(s)(v, w) = \hat{P}_m(s) + \hat{H}_m(s)$, where $\hat{P}_m(s)$ is a linear combination of s^{-j}, $j = 1, \dots, m$, and $\hat{H}_m(s) = s^{-m} \hat{\rho}_{v, \partial_t^m w}(s)$.

By the argument used in the proof of [21, Proposition 6.5],

$$\left| \int_1^\infty e^{ibt} P_m(ib) \, db \right| \ll t^{-1}.$$

By Lemma 7.8(d) with $h = \pi/t$, there exists some constant $C > 0$ such that

$$|\hat{H}_m(i(b) - \hat{H}_m(i(b - \pi/t))| \le C b^{-(m-2\alpha-1)} (\log|b|) t^\tau \|v^*\|_{\mathcal{B}} |\partial_t^m w|_\infty.$$

Suppressing the term $\|v^*\|_{\mathcal{B}} |\partial_t^m w|_\infty$,

$$\left| 2 \int_1^\infty e^{ibt} \hat{H}_m(ib) \, db \right| \le \int_1^\infty |\hat{H}_m(ib) - \hat{H}_m(i(b - \pi/t))| \, db + \int_1^{1+\pi/t} |\hat{H}_m(i(b - \pi/t))| \, db$$
$$\ll t^{-\tau} \int_1^\infty (\log|b|) b^{-(m-2\alpha-1)} \, db + O(t^{-1}) = O(t^{-\tau}),$$

where in the last inequality we have used that $m > 2\alpha + 2$. □

We can now complete the

Theorem 5.2. By Lemma 7.10, Lemma 7.11 and Lemma 7.12, we obtain that for all $a \in (\pi, t)$,

$$\lim_{t\to\infty} t^{1-\beta}\ell(t) \int_0^\infty e^{ibt}\hat{\rho}(ib)(v,w)\, db = c_\beta^{-1} \int_0^a e^{i\sigma}\sigma^{-\beta}\, d\sigma \int_{\tilde{Y}} v\, d\tilde{\mu} \int_{\tilde{Y}} w\, d\tilde{\mu} + O(a^{-(2\beta'-1)}) + \lim_{t\to\infty} t^{-\tau'} a^{1-\beta'}.$$

Because $\beta' > 1/2$ and $\tau' > 1 - \beta'$,

$$\lim_{t\to\infty} t^{1-\beta} \int_0^\infty e^{ibt}\hat{\rho}(ib)(v,w)\, db = c_\beta^{-1} \int_0^\infty e^{i\sigma}\sigma^{-\beta}\, d\sigma \int_{\tilde{Y}} v\, d\tilde{\mu} \int_{\tilde{Y}} w\, d\tilde{\mu}.$$

Since $\Re(c_\beta^{-1} \int_0^\infty e^{i\sigma}\sigma^{-\beta}\, d\sigma) = \sin\beta\pi$, the result follows from Proposition 7.9. □

8 Several estimates related to the Laplace transform $\hat{U}$

In this section we study the asymptotic behavior of the Laplace transform $\hat{U}(s)$ defined before Proposition 3.1. To some extent, our calculations below follow the strategy of obtaining similar estimates for $\hat{U}$ as defined in [21]. Under the present assumption (H1) (so, $\tilde{h}$ unbounded) the calculations require more more work. Thus, we cannot simply cite the somewhat similar arguments in [21], but need to carry out the new calculations. However, we mention under the assumption that $\tilde{h}$ is bounded (limiting to the present (H2), that is $\mathcal{B} \in L^\infty(\mu_\Phi)$), all the arguments in [21] used to study the asymptotic behavior of $\hat{U}(s)$, simply go through.

We start by obtaining precise formula for the inverse Laplace transform U_t.

Proposition 8.1. *Assume that $\tilde{h}$ is bounded away from zero and $\varphi \geq \tilde{h}$. Let $v \in L^1(\tilde{\mu})$ and $w = w^*/\tilde{h} \in L^\infty(\tilde{\mu})$ with $w^* \in L^\infty(\mu_\Phi)$. If $t \leq \tilde{h}(y)u$ then*

$$\int_{\tilde{Y}} U_t(\tilde{h}v)w\, d\mu_\Phi = \int_{\tilde{Y}} 1_{[t/\tilde{h},1]}(u) v(y, u - \frac{t}{\tilde{h}(y)}) w^* d\mu_\Phi.$$

Also, if $t > \tilde{h}(y)u$, then

$$\int_{\tilde{Y}} U_t(\tilde{h}v)w\, d\mu_\Phi = \int_{\tilde{Y}} R(v_t) w^* d\mu_\Phi,$$

where $v_t(y,u) = 1_{\{t<\varphi<t+\tilde{h}(y)(1-u)\}}(y,u)\, v(y, u + \frac{\varphi - t}{\tilde{h}(y)})$.

Proof. First we deal with $t < \tilde{h}(y)u$. The assumption $\varphi > \tilde{h}$ implies $1_{\{\varphi>t\}} = 1_{\tilde{Y}}$ μ_Φ-a.e. on $\tilde{Y}$, and therefore, for $w \in L^\infty(\mu_\Phi)$,

$$\begin{aligned}
\int_{\tilde{Y}} U_t(\tilde{h}v)w\, d\mu_\Phi &= \int_{\tilde{Y}} 1_{\{\varphi>t\}} \tilde{h}vw \circ f_t\, d\mu_\Phi = \int_Y \int_0^{1-t/\tilde{h}(y)} v(y,u) w(y, u + t/\tilde{h}(y))\, \tilde{h}(y) du\, d\mu \\
&= \int_Y \int_{t/\tilde{h}(y)}^1 v(y, u - t/\tilde{h}(y)) w(y,u)\, \tilde{h}(y) du\, d\mu \\
&= \int_{\tilde{Y}} 1_{[t/\tilde{h},1]}(u) v(y, u - \frac{t}{\tilde{h}(y)}) w^* d\mu_\Phi,
\end{aligned}$$

where in the last equality we used that $w\tilde{h} = w^* \in L^\infty(\mu_\Phi)$.

For the case $t \geq \tilde{h}(y)u$, we compute

$$\begin{aligned}\int_{\tilde{Y}} U_t(\tilde{h}v)w \, d\mu_\Phi &= \int_{\tilde{Y}} 1_{\{\varphi > t\}} \tilde{h}v(w^*/\tilde{h}) \circ f_t \, d\mu_\Phi \\ &= \int_Y \int_0^1 \frac{\tilde{h}(y)}{\tilde{h}(Fy)} 1_{\{\varphi > t\}} v(y,u) w^*(Fy, \frac{t - \varphi_0(y) + \tilde{h}(y)u}{\tilde{h}(Fy)}) \, du \, d\mu,\end{aligned}$$

where we compute the argument of w^* by, starting from (y,u), first flowing to $(y,0)$ (which takes time $\tilde{h}(y)u$, then flowing to $(Fy,0)$ (which takes time φ_0) and then the remaining time $t - \varphi_0 + \tilde{h}(y)$ gives $(Fy, \frac{t-\varphi_0(y)+\tilde{h}(y)u}{\tilde{h}(Fy)})$.

We rewrite this integral twice, applying different changes of coordinates. First we take $2u' = \frac{t+\tilde{h}(y)u-\varphi_0}{\tilde{h}(Fy)}$ for $u' \in [0, \frac{1}{2})$, so $du = \frac{2\tilde{h}(Fy)}{\tilde{h}(y)} du'$. Also

$$u = \frac{2\tilde{h}(Fy)u' + \varphi_0 - t}{\tilde{h}(y)} = u' + \frac{\varphi(y,u') - t}{\tilde{h}(y)} \quad \text{and} \quad \frac{\varphi(y,u') - t}{\tilde{h}(y)} = \frac{\varphi(y,u) - t}{2\tilde{h}(Fy)},$$

hence the indicator function $1_{\{\varphi > t\}}$ keeps the same form in the coordinates (y,u'). This gives

$$I_1 := \frac{1}{2} \int_{\tilde{Y}} U_t(\tilde{h}v)w \, d\mu_\Phi = \int_Y \int_0^{\frac{1}{2}} 1_{\{\varphi(y,u') > t\}} v(y, u' + \frac{\varphi(y,u') - t}{\tilde{h}(y)})) w^* \circ \Phi_1(y,u') \; du' \, d\mu,$$

where $\Phi_1 : Y \times [0, \frac{1}{2}) \to \tilde{Y}$ is the first branch of the map Φ. Since v is supported on $\tilde{Y}$, the second argument $u' + \frac{\varphi(y,u')-t}{\tilde{h}(y)}$ needs to belong to $[0,1]$ to give a nonzero value. We emphasize this by the indicator function $1_{\{t < \varphi(y,u') + \tilde{h}(y)u' < t + \tilde{h}(y)\}}$, which combined with $1_{\{\varphi(y,u) > t\}}$ gives $1_{\{t < \varphi < \tilde{h}(y)(1-u')\}}$. Hence

$$I_1 = \int_Y \int_0^{\frac{1}{2}} 1_{\{t < \varphi < t + \tilde{h}(y)(1-u')\}} v(y, u' + \frac{\varphi - t}{\tilde{h}(y)}) w^* \circ \Phi_1(y,u') \; du' \, d\mu.$$

For I_2 we use the change of coordinates $2u' - 1 = \frac{t+\tilde{h}(y)u-\varphi_0}{\tilde{h}(Fy)}$, for $u' \in [\frac{1}{2}, 1)$ so $du = \frac{2\tilde{h}(Fy)}{\tilde{h}(y)} du'$. The definition of φ on this range gives

$$\begin{aligned}\varphi(y,u') + \tilde{h}(y)u' &= \varphi_0(y) + (2\tilde{h}(Fy) - \tilde{h}(y))u' - \tilde{h}(Fy) \\ &= \varphi_0(y) + (2\tilde{h}(Fy) - \tilde{h}(y))(u' - \frac{1}{2}),\end{aligned}$$

so the indicator function $1_{\{t < \varphi(y,u') + \tilde{h}(y)u' < t + \tilde{h}(y)\}}$ preserves the same form, as does $1_{\{\varphi(y,u) > t\}}$. Also $u = \frac{(2u'-1)\tilde{h}(Fy)+\varphi_0-t}{\tilde{h}(y)} = \frac{2u'\tilde{h}(y)+\varphi-t}{\tilde{h}(y)}$. Therefore, denoting the second branch of Φ by $\Phi_2 : Y \to [\frac{1}{2}, 1) \to \tilde{Y}$, we get

$$I_2 = \int_Y \int_{\frac{1}{2}}^1 1_{\{t < \varphi(y,u') < t + \tilde{h}(y)(1-u')\}} v(y, u' + \frac{\varphi - t}{\tilde{h}(y)}) w^* \circ \Phi_2(y,u') \; du' \, d\mu.$$

Adding I_1 and I_2, we obtain

$$\begin{aligned}\int_{\tilde{Y}} U_t(\tilde{h}v)w \, d\mu_\Phi &= \int_Y \int_0^1 1_{\{t < \varphi < t + \tilde{h}(y)(1-u')\}} v(y, u' + \frac{\varphi - t}{\tilde{h}(y)})) w^* \circ \Phi(y,u') \; du' \, d\mu \\ &= \int_Y \int_0^1 R(1_{\{t < \varphi < t + \tilde{h}(y)(1-u')\}} v(y, u' + \frac{\varphi - t}{\tilde{h}(y)})) w^* \, d\mu_\Phi,\end{aligned}$$

as required. □

8.1 Estimates for $\hat U$ required in the finite case

We first estimate the inverse Laplace transform of $s^{-1}\int_{\tilde Y}\hat U(s)(\tilde h v)w\,d\mu_\Phi$, as follows:

Lemma 8.2. *Assume (H0) i), and (H1) i). Let $\mathcal B$ the Banach space defined by (H2). Let $v=\frac{v^*}{\tilde h}$ with $v^*\in\mathcal B$ and $w=\frac{w^*}{\tilde h}$ with $w^*\in L^\infty(\mu_\Phi)$. Define $\hat r(s):=s^{-1}\int_{\tilde Y}\hat U(s)(\tilde h v)\,w\,d\mu_\Phi$. Let $r(t)$ be the inverse Laplace transform of $\hat r(s)$. Then*

$$r(t)(v,w=\int_{\tilde Y}\tilde h(y)\int_0^u v(y,\tau)\,d\tau\,w^*\,d\mu_\Phi+\int_{\tilde Y}\tilde h(y)\Big(\int_u^1 v(y,\tau)\,d\tau\Big)\,w^*\circ\Phi\,d\mu_\Phi+E(t),$$

where $\|E(t)\|\le\|v^*\|_{L^\infty(\mu_\Phi)}\|w^*\|_{L^\infty(\mu_\Phi)}t^{-\beta}$.

Proof. Write $\hat r(s)=s^{-1}\int_0^\infty\int_{\tilde Y}U_t(\tilde h v)\,w\,d\mu_\Phi\,e^{-st}\,dt$. Using Proposition 8.1,

$$\begin{aligned}\int_{\tilde Y}s^{-1}\hat U(s)(\tilde h v)(y,u)w(y,u)\,d\mu_\Phi&=\int_{\tilde Y}s^{-1}\int_0^{\tilde h(y)u}e^{-s\tau}v(y,u-\frac{\tau}{\tilde h})1_{[\frac{\tau}{\tilde h},1]}(u)\,d\tau w^*(y,u)\,d\mu_\Phi\\&\quad+\int_{\tilde Y}s^{-1}\int_{\tilde h(y)u}^\infty e^{-s\tau}(Rv_\tau)(y,u)\,d\tau\,w^*d\mu_\Phi\end{aligned}$$

with inverse Laplace transform (using that $1_{[\frac{\tau}{\tilde h},1]}\equiv1$ for $0\le\tau\le\tilde h(y)u$)

$$\int_{\tilde Y}\int_0^{\tilde h(y)u}1_{[\tau,\infty)}(t)v(y,u-\tau)\,d\tau\,w\,d\mu_\Phi+\int_{\tilde Y}\int_{\tilde h(y)u}^\infty 1_{[\tau,\infty)}(t)(Rv_\tau)(y,u)\,d\tau\,w^*d\mu_\Phi.$$

Hence $r(t)=r_1(t)+r_2(t)$ where

$$\begin{cases}r_1(t)=\int_{\tilde Y}\int_0^{\tilde h(y)u}1_{[\tau,\infty)}(t)v(y,u-\frac{\tau}{\tilde h})\,d\tau\,w^*\,d\mu_\Phi,\\ r_2(t)=\int_{\tilde Y}\int_{\tilde h(y)u}^\infty 1_{[\tau,\infty)}(t)(Rv_\tau)(y,u)\,d\tau\,w^*d\mu_\Phi.\end{cases}$$

Changing coordinates $u-\tau/\tilde h\to\tau$ gives

$$r_1(t)=\int_{\tilde Y}\tilde h\int_0^u 1_{[(u-\tau)\tilde h,\infty)}(t)v(y,\tau)\,d\tau\,w^*\,d\mu_\Phi=\int_{\tilde Y}\tilde h\int_0^u v(y,\tau)\,d\tau\,w^*\,d\mu_\Phi+E_1(t),$$

where

$$|E_1(t)|\le\int_{\tilde Y}\tilde h\int_0^u 1_{[0,\tilde h(u-\tau)}(t)|v(y,\tau)|\,d\tau\,|w^*(y,u)|\,d\mu_\Phi\le\|v^*\|_{L^\infty(\mu_\Phi)}\|w^*\|_{L^\infty(\mu_\Phi)}\mu(t<\tilde h).$$

Because $\tilde h=\varphi_0^\gamma$, using (H0) i) and Lemma 4.4, we get $\mu(t<\tilde h)\ll t^{-\beta/\gamma}$.

For $r_2(t)$, recall that $v_t(y,u)=1_{\{t<\varphi<t+\tilde h(y)(1-u)\}}\,v(y,u+\frac{\varphi-t}{\tilde h(y)})$. For $t>\tilde h(y)u$,

$$\begin{aligned}r_2(t)&=\int_{\tilde Y}\Big(\int_{\varphi+\tilde hu-\tilde h}^{\varphi}1_{[\tau,\infty)}(t)Rv(y,u+\frac{\varphi-\tau}{\tilde h(y)})d\tau\Big)w^*\,d\mu_\Phi\\&=\int_{\tilde Y}\Big(\int_{\varphi+\tilde hu-\tilde h}^{\varphi}1_{[\tau,\infty)}(t)v(y,u+\frac{\varphi-\tau}{\tilde h(y)})d\tau\Big)w^*\circ\Phi\,d\mu_\Phi\\&=\int_{\tilde Y}\tilde h\Big(\int_u^1 1_{[u+\frac{\varphi-\tau}{\tilde h},\infty)}(t)v(y,\tau)d\tau\Big)w^*\circ\Phi\,d\mu_\Phi\\&=\int_{\tilde Y}\tilde h\Big(\int_u^1 v(y,\tau)d\tau\Big)w^*\circ\Phi\,d\mu_\Phi-E_2(t),\end{aligned}$$

where

$$|E_2(t)| = \int_{\tilde{Y}} \tilde{h}\Big(\int_u^1 1_{\{\varphi > \tilde{h}t+\tau-u\tilde{h}\}}(t)|v(y,\tau)|\,d\tau\Big)|w^*\circ\Phi|\,d\mu_\Phi$$

$$\le \|w^*\|_{L^\infty(\mu_\Phi)} \int_{\tilde{Y}} \tilde{h}\Big(\int_u^1 1_{\{\varphi > \tilde{h}t+\tau-u\tilde{h}\}}|v(y,\tau)|\,d\tau\Big)\,d\mu_\Phi. \tag{8.1}$$

The set of the indicator function can be rewritten as

$$\{\varphi > \tilde{h}t+\tau-u\tilde{h}\} = \begin{cases} \{\varphi_0(y) > \tilde{h}(y)t+\tau-2\tilde{h}F(y)u\} & \text{if } u < \frac12, \\ \{\varphi_0(y) > \tilde{h}(y)t+\tau-2\tilde{h}F(y)u - \tilde{h}F(y)\} & \text{if } u \ge \frac12. \end{cases}$$

In either case, we get the inclusion

$$\{\varphi > \tilde{h}t+\tau-u\tilde{h}\} \subset \{\varphi_0 > \tilde{h}t - \tilde{h}\circ F\} \subset \{\varphi_0 > \tilde{h}t/2\} \cup \{\tilde{h}\circ F > t/2\}. \tag{8.2}$$

Recall $v = \frac{v^*}{\tilde{h}}$ with $v^* \in \mathcal{B}$, $\tilde{h} = \varphi_0^\gamma$, $\gamma < 1$ and compute that

$$|E_2(t)| \le \|v^*\|_{L^\infty(\mu_\Phi)}\|w^*\|_{L^\infty(\mu_\Phi)}\Big(\mu(\varphi_0^{1-\gamma} > t) + \mu(\varphi_0^\gamma\circ F > t)\Big).$$

Using the μ invariance under F, $\mu(\varphi_0^\gamma \circ F > t) = \mu(\varphi_0^\gamma > t)$. This together with (H0) i) and Lemma 4.4 implies that $\mu(\varphi_0^\gamma \circ F > t) \ll t^{-\frac{\beta}{\gamma}} < t^{-\beta}$, since by (H1) i), $\gamma < 1$. Also, $\mu(\varphi_0^{1-\gamma} > t) \ll t^{-\frac{\beta}{1-\gamma}} < t^{-\beta}$. The conclusion follows by combining E_1 and E_2. □

Lemma 8.3. *Assume either case i) or ii) of (H0). Let $\mathcal{B}$ be the Banach space defined by (H2). Then, the function $\hat{U}(s) : \mathcal{B} \to L^1(\mu_\Phi)$ lies in $\mathcal{R}_{\mathcal{B}\to L^1(\mu_\Phi)}(1/t^\beta)$*

Proof. Let $v^* = \tilde{h}v \in \mathcal{B}$. Then $\int_{\tilde{Y}} U_t v^* w d\mu_\Phi = \int_{\tilde{Y}} 1_{\{\varphi>t\}} v^*\cdot w\circ f_t\,d\mu_\Phi$. Hence, $\|U_t v^*\|_{L^1(\mu_\Phi)} \le \|v^*\|_{L^\infty(\mu_\Phi)}\mu_\Phi(\varphi > t)$. The conclusion follows since $\mathcal{B} \subset L^\infty(\mu_\Phi)$. □

Lemma 8.4. *Assume either form of (H0) and (H1). Let $\mathcal{B}$ the Banach space defined by (H2). Then, the family of linear operators $\hat{U}(s) : \mathcal{B} \to L^1(\mu_\Phi)$, $s \in \bar{\mathbb{H}}$, $b \in \mathbb{R}$, is uniformly bounded; that is, there exists $C > 0$ such that $\|\hat{U}(s)\|_{\mathcal{B}\to L^1(\mu_\Phi)} \le C$ for all $s \in \bar{\mathbb{H}}$.*

Proof. By Proposition 8.1

$$\|\hat{U}(ib)v\|_{L^1(\mu_\Phi)} = \int_{\tilde{Y}} |\hat{U}(s)v|\,d\mu_\Phi \le \int_{\tilde{Y}}\int_0^{\tilde{h}(y)u} 1_{[t/\tilde{h}(y),1]}(u)|v(y,u-\frac{t}{\tilde{h}(y)})|dt\,d\mu_\Phi$$
$$+ \int_{\tilde{Y}}\int_{\tilde{h}(y)u}^\infty 1_{\{t<\varphi<t+\tilde{h}(y)(1-u)\}}|v(y,u+\frac{\varphi-t}{\tilde{h}(y)})|\,dt\,d\mu_\Phi$$
$$= I_1 + I_2.$$

Using the change of coordinates $\tau = u - t/\tilde{h}(y)$ gives

$$I_1 = \int_Y\int_0^1\int_0^u |\tilde{h}(y)v(y,\tau)|d\tau du\,d\mu.$$

Similarly,

$$I_2 = \int_Y\int_0^1\int_{\varphi+\tilde{h}(y)(1-u)}^{\varphi} \tilde{h}(y)|v(y,u+\frac{\varphi-t}{\tilde{h}(y)})|dt\,du\,d\mu = \int_Y\int_0^1\int_u^1 \tilde{h}(y)|v(y,\tau)|d\tau du\,d\mu$$

Thus, $I_1 + I_2 \le \int_{\tilde{Y}} \tilde{h}(y)|v(y,u)|\,d\mu_\Phi \le \|v\|_{L^\infty(\mu_\Phi)}\tilde{\mu}(\tilde{Y})$, as required. □

Remark 8.5. *Let $\mathcal{B}_0 \subset L^\infty(\mu_\Phi)$ as defined in (H4). By the argument of Lemma 8.4, we have $\|\hat{U}(s)\|_{\mathcal{B}_0 \to L^1(\mu_\Phi)} \leq C$ for all $s \in \bar{\mathbb{H}}$.*

8.2 Estimates for $\hat{U}$ required in the infinite case

Lemma 8.6. *(a) Assume either form of (H0) and (H1). Let $v \in L^1(\tilde{\mu})$ and $w = \frac{w^*}{\tilde{h}}$ with $w^* \in L^\infty(\mu_\Phi)$. Then*

$$\int_{\tilde{Y}} \hat{U}(0)v(y,u)w(y,u)\,d\tilde{\mu} = \int_{\tilde{Y}} \int_0^u \tilde{h}(y)v(y,\tau)\,d\tau\, w^*(y,u)\,d\mu_\Phi + \int_{\tilde{Y}} \int_u^1 R(\tilde{h}(y)v(y,\tau))\,d\tau w^*(y,u)\,d\mu_\Phi.$$

(b) Assume (H0) ii) and (H1) ii). Let $\mathcal{B}$ the Banach space defined by (H2). Let $\mathcal{B}_0$ be a Banach space with $\mathcal{B} \subset \mathcal{B}_0 \subset L^\infty(\mu_\Phi)$. Let τ be as defined in (H4). Then,

$$\|\hat{U}(ib_2) - \hat{U}(ib_1)\|_{\mathcal{B}_0 \to L^1(\mu_\Phi)} \ll |b_1 - b_2|^\tau.$$

Remark 8.7. *Lemma 8.2 and Lemma 8.6(a) imply that*

$$\int_{\tilde{Y}} \frac{\hat{U}(s)}{s}(\tilde{h}v)wd\mu_\phi = \int_0^\infty \Big(\int_{\tilde{Y}} \hat{U}(0)(\tilde{h}v)wd\mu_\Phi\Big)e^{-st}dt + \int_0^\infty E(t)e^{-st}dt,$$

where $|E(t)| \leq \|v^*\|_{L^\infty(\mu_\Phi)}\|w^*\|_{L^\infty(\mu_\Phi)}t^{-\beta}$. Also, note that $\int_{\tilde{Y}} \frac{\hat{U}(0)}{s}(\tilde{h}v)wd\mu_\phi = \int_0^\infty \Big(\int_{\tilde{Y}} \hat{U}(0)(\tilde{h}v)wd\mu_\Phi\Big)e^{-st}dt$. Thus,

$$\int_{\tilde{Y}} \frac{\hat{U}(s) - \hat{U}(0)}{s}(\tilde{h}v)wd\mu_\phi = \int_0^\infty E(t)e^{-st}dt,$$

with $|E(t)| \leq \|v^*\|_{L^\infty(\mu_\Phi)}\|w^*\|_{L^\infty(\mu_\Phi)}t^{-\beta}$.

Proof. (a) By Proposition 8.1,

$$\begin{aligned}
\int_{\tilde{Y}} \hat{U}(0)vwd\tilde{\mu} &= \int_{\tilde{Y}} \int_0^{\tilde{h}(y)u} 1_{[\frac{t}{\tilde{h}(y)},1]}(u)v(y,u+\frac{t}{\tilde{h}(y)})w^*(y,t)\,dt\,d\mu_\Phi \\
&\quad + \int_{\tilde{Y}} \int_{\tilde{h}(y)u}^\infty R(1_{\{t<\varphi<t+\tilde{h}(y)(1-u)\}}v(y,u+\frac{\varphi-t}{\tilde{h}(y)}))w^*(y,t)\,dt\,d\mu_\Phi \\
&= I_1 + I_2.
\end{aligned}$$

We compute the first integral with the change of coordinates $\tau = u - \frac{t}{\tilde{h}(y)}$:

$$\begin{aligned}
I_1 &= \int_{\tilde{Y}} \int_0^{\tilde{h}(y)u} v(y,u-\frac{t}{\tilde{h}(y)})w^*(y,t)\,dt\,d\mu_\Phi \\
&= \int_{\tilde{Y}} \int_0^u \tilde{h}(y)v(y,\tau)\,d\tau\, w^*(y,t)\,dt\,d\mu_\Phi.
\end{aligned}$$

We continue with the estimate for I_2. Under (H1), $\varphi > \tilde{h}$. Thus, the bounds of the t-integral become $\varphi - \tilde{h}(y)(1-u) < t < \varphi$:

$$\begin{aligned} I_2 &= \int_{\tilde{Y}} \int_{\tilde{h}(y)u}^{\infty} R(1_{\{t<\varphi<t+\tilde{h}(y)(1-u)\}} v(y, u + \frac{\varphi - t}{\tilde{h}(y)}) w^*(y,u)\, dt\, d\mu_\Phi \\ &= \int_{\tilde{Y}} \int_{\varphi+\tilde{h}(y)(1-u)}^{\varphi} R(v(y, u + \frac{\varphi - t}{\tilde{h}(y)}) w^*(y,u)\, dt\, d\mu_\Phi \\ &= \int_{\tilde{Y}} \int_{\varphi+\tilde{h}(y)(1-u)}^{\varphi} v(y, u + \frac{\varphi - t}{\tilde{h}(y)}) w^*(y,u) \circ \Phi\, dt\, d\mu_\Phi \\ &= \int_{\tilde{Y}} \int_{u}^{1} v(y,\tau)\, d\tau\, w^*(y,u) \circ \Phi\, d\mu_\Phi = \int_{\tilde{Y}} R\Big(\int_u^1 v(y,\tau)\, d\tau \Big)\, w^*(y,u)\, d\mu_\Phi \\ &= \int_{\tilde{Y}} \int_u^1 R(v(y,\tau))\, d\tau\, w^*(y,u)\, d\mu_\Phi. \end{aligned}$$

For (b), we write $\psi(y,u,t) = 1_{\{t<\varphi<t+\tilde{h}(y)(1-u)\}}$, and $B(b_1, b_2, t) = |e^{ib_2 t} - e^{ib_1 t}|$. By Proposition 8.1,

$$\begin{aligned} |\hat{U}(ib_2)v - \hat{U}(ib_1)v|_{L^1(\mu_\Phi)} &= \int_{\tilde{Y}} \tilde{h}(y) |\hat{U}(ib_2)v - \hat{U}(ib_1)v|\, d\mu_\Phi \\ &\le \int_{\tilde{Y}} \tilde{h}(y) \int_0^{\tilde{h}(y)u} B(b_1,b_2,t) 1_{[t/\tilde{h}(y),1]}(u) |v(y, u - \frac{t}{\tilde{h}(y)})|\, dt\, d\mu_\Phi \\ &\quad + \int_{\tilde{Y}} \tilde{h}(y) \int_{\tilde{h}(y)u}^{\infty} \psi(y,u,t) B(b_1,b_2,t) |v(y, u + \frac{\varphi - t}{\tilde{h}(y)})|\, dt\, d\mu_\Phi \\ &= I_1 + I_2. \end{aligned}$$

For I_1 we use the change of coordinates $\tau = u - \frac{t}{\tilde{h}(y)}$. This gives

$$\begin{aligned} I_1 &\le |b_1 - b_2| \int_{\tilde{Y}} \tilde{h}^2(y) \int_0^u (u-\tau) |v(y,\tau)|\, d\tau\, d\mu_\Phi \\ &\le |b_1 - b_2| \int_Y \tilde{h}^2(y) \int_0^1 u \int_0^u |v(y,\tau)| d\tau\, du\, d\mu \\ &\le |b_1 - b_2| \int_{\tilde{Y}} \tilde{h}^2(y) |v(y,u)|\, d\mu_\Phi. \end{aligned} \tag{8.3}$$

Recall that $v \in L^1(\tilde{\mu})$. So, $v = \frac{v^*}{\tilde{h}}$ with $v^* \in L^1(\mu_\Phi)$. Let $v^* \in \mathcal{B}_0 \subset L^\infty(\mu)$ and recall that $\tilde{h} \in L^1(\mu_\Phi)$. Hence,

$$|I_1| \le |b_1 - b_2| \int_{\tilde{Y}} \tilde{h} \cdot v^*(y,u)\, d\mu_\Phi \le |b_1 - b_2| \|v^*\|_{L^\infty(\mu_\Phi)} \int_Y \tilde{h}\, d\mu,$$

as required.

For I_2, we first mimic the argument used in the proof of Lemma 8.2 in estimating $r_2(t)$ there. Namely, we use the change $t \to \sigma \tilde{h}$ and compute that

$$\begin{aligned}
I_2 &= \int_{\tilde{Y}} \tilde{h} \int_{\tilde{h}(y)u}^{\infty} \psi(y,u,t)B(b_1,b_2,t)|v(y,u+\frac{\varphi-t}{\tilde{h}(y)})|\,dt\,d\mu_\Phi \\
&= \int_{\tilde{Y}} \tilde{h}^2 \int_{u}^{\infty} \psi(y,u,\sigma\tilde{h})B(b_1,b_2,\sigma\tilde{h})|v(y,u+\frac{\varphi}{\tilde{h}}-\sigma)|\,d\sigma\,d\mu_\Phi \\
&= \int_{\tilde{Y}} \tilde{h}^2 \int_{\frac{\varphi}{\tilde{h}}+u-1}^{\frac{\varphi}{\tilde{h}}} B(b_1,b_2,\sigma\tilde{h})|v(y,u+\frac{\varphi}{\tilde{h}(y)}-\sigma)|\,d\sigma\,d\mu_\Phi.
\end{aligned}$$

Using the change of coordinates $u + \frac{\varphi}{\tilde{h}} - \sigma \to \sigma$ and the fact that $B(b_1, b_2, \sigma) \le |b_1 - b_2|^\tau \sigma^\tau$, with τ as in (H4), we have

$$\begin{aligned}
I_2 &= \int_{\tilde{Y}} \tilde{h}^2 \int_{u}^{1} B(b_1,b_2,u\tilde{h}+\varphi-\sigma\tilde{h})|v(y,\sigma)|\,d\sigma\,d\mu_\Phi \\
&\le |b_1-b_2|^\tau \int_{\tilde{Y}} \tilde{h}^2 \int_{u}^{1} (\varphi+\tilde{h}(u-\sigma))^\tau |v(y,\sigma)|\,d\sigma\,d\mu_\Phi \\
&\le |b_1-b_2|^\tau \int_{\tilde{Y}} \tilde{h}^2 \int_{u}^{1} \varphi^\tau |v(y,\sigma)|\,d\sigma\,d\mu_\Phi. \qquad (8.4)
\end{aligned}$$

Since $v = \frac{v^*}{\tilde{h}}$ with $v^* \in \mathcal{B}_0 \subset L^\infty(\mu)$ and $\tilde{h} \in L^1(\mu_\Phi)$,

$$|I_2| \le |b_1 - b_2|^\tau \|v^*\|_{L^\infty(\mu_\Phi)} \int_Y \tilde{h} \cdot \varphi^\tau\,d\mu.$$

By definition, $\varphi(y) \le \varphi_0(y) + \tilde{h}(Fy)$ and by (H1), $\tilde{h} = \varphi_0^\gamma$. Hence,

$$\tilde{h}\varphi^\tau \le \varphi_0^{\tau+\gamma} + \varphi_0^\gamma (h \circ F)^\tau.$$

Using the μ invariance under F,

$$\int_Y \tilde{h} \cdot \varphi^\tau\,d\mu \le \int_Y \varphi_0^{\tau+\gamma}\,d\mu + \int_Y \varphi_0^{\tau(1+\gamma)}\,d\mu < 2\int_Y \varphi_0^{\tau(1+\gamma)}\,d\mu.$$

By (H0) ii) and Lemma 4.4, $\mu(\varphi_0 > t) \ll t^{-(\beta-\epsilon)}$, for any small $\epsilon > 0$. By (H4), $\tau(1+\gamma) < \beta$. Thus, $\int_Y \tilde{h} \cdot \varphi^\tau\,d\mu < \infty$. As a consequence, $|I_2| \le |b_1 - b_2|^\tau \|v^*\|_{L^\infty(\mu_\Phi)}$. The conclusion follows. □

9 Semiflows over Markov maps with bounded $\tilde{h}$

Markov maps represent a class of examples where the conditions of the abstract setting are likely to hold, except that condition (H4) is problematic for standard norms. However, in the infinite measure case, we are allowed pass to a weaker space $\mathcal{B}_0$ (see Theorem 5.2) in which to check (H5). Here we shall take $\mathcal{B}_0 = L^\infty(\mu_\Phi)$ and $\mathcal{B}$ the space of θ-Hölder functions. Contrary to (H1), we stipulate that $\tilde{h}$ is bounded (we can assume without loss of generality that $\tilde{h} \equiv 1$).

9.1 The set-up and results

Recall the partitions $\mathcal{P}, \mathcal{P}_n$ and $\tilde{\mathcal{P}}, \tilde{\mathcal{P}}_n$ from Section 4.4. For $y_1, y_2 \in Y$, define the *separation time* $s(y_1, y_2)$ as the smallest integer $n \geq 0$ such that $F^n y_1$ and $F^n y_2$ lie in different elements of $\mathcal{P}$. Similarly for $\tilde{y}_1, \tilde{y}_2 \in \tilde{Y}$, let $\tilde{s}(\tilde{y}_1, \tilde{y}_2)$ be the smallest integer $n \geq 0$ such that $\Phi^n \tilde{y}_1$ and $\Phi^n \tilde{y}_2$ lie in different elements of $\tilde{\mathcal{P}}$.

For given $\theta \in (0,1)$, let $\mathcal{B}(\tilde{Y})$ be the Banach space of function v supported on $\tilde{Y}$, with norm $\|v\|_\theta = |v|_\theta + \|v\|_\infty$, where $\|v\|_\infty = \|v\|_{L^\infty(\mu_\Phi)}$ and the seminorm $|v|_\theta$ is defined as

$$|v|_\theta = \sup_{\tilde{y}_1 \neq \tilde{y}_2 \in \tilde{Y}} \theta^{-\tilde{s}(\tilde{y}_1, \tilde{y}_2)} |v(\tilde{y}_1) - v(\tilde{y}_2)|.$$

Let $f : X \to X$ be a non-uniformly expanding map with a single indifferent fixed point, say at $0 \in X$. Consider a suspension flow over f with continuous roof function h and assume that h is bounded and bounded away from zero. Assume that $F : Y \to Y$ is an induced map over f, with the following properties:

(1) F is full-branched, *i.e.*, $F(Z) = Y$ for every Z in the Markov partition $\mathcal{P}$, and the induced time $\tau_F : Y \to \mathbb{N}$ such that $F = f^{\tau_F}$ is constant on each $Z \in \mathcal{P}$.

(2) F is expanding and there is a distortion constant C_{dis} such that

$$\frac{|DF^k(y_1)|}{|DF^k(y_2)|} \leq C_{dis}, \tag{9.1}$$

for all $k \geq 0$, $Z \in \mathcal{P}_k$ and $y_1, y_2 \in Z$. This condition implies that F preserves a measure μ, absolutely continuous w.r.t. Lebesgue, such that $\frac{1}{C_\mu} \leq \frac{d\mu}{dx} \leq C_\mu$ for some $C_\mu > 0$.

Define the potential $p : Y \to \mathbb{R}$, $p = \log \frac{d\mu}{d\mu \circ F}$ and $p_n = \sum_{j=0}^{n-1} p \circ F^j$; we assume that there is a constant C_p such that

$$e^{p_n(y)} \leq C_p \mu(Z) \quad \text{and} \quad |e^{p_n(y_1)} - e^{p_n(y_2)}| \leq C_p \mu(Z) \theta^{s(F^n(y_1), F^n(y_2))}, \tag{9.2}$$

for every $y, y_1, y_2 \in Z$, $Z \in \mathcal{P}_n$.

(3) The roof function of the induced system $F : Y \to Y$ is $\varphi_0 = \sum_{i=0}^{\tau_F - 1} h \circ f^i \leq \tau_F \sup h$. We assume that there exists $C_{\varphi_0} > 2$

$$|\varphi_0(y_1) - \varphi_0(y_2)| \leq C_{\varphi_0} \theta^{s(y_1, y_2)}, \tag{9.3}$$

for all $y_1, y_2 \in Z$, $Z \in \mathcal{P}$.

If there is only one $Z \in \mathcal{P}$ with $\tau_F(Z) = n$, the following is immediate: There is $h_n = h(0)n + o(n)$ such that

$$|\varphi_0(y) - h_n| \leq C_{\varphi_0} \tag{9.4}$$

for all $y \in Z, Z \in \mathcal{P}$ with $\tau_F(Z) = n$. Let us write $\eta : \mathbb{R} \to \mathbb{N}$ for the asymptotic inverse of h_n in the sense that $\eta(t)$ is minimal such that $h_{\eta(t)} \geq t$. For example, for the case $\beta \in (0,1)$, if the roof function h is differentiable near 0, and the branch of f with the indifferent fixed point is $x \mapsto x + x^{1+1/\beta}$, then $h_n = h(0)n + \frac{h'(0)}{1-\beta} n^{1-\beta} + o(n^{1-\beta})$, and $\eta(t) = t/h(0) + O(t^{1-\beta})$.

(4) The induce time τ_F satisfies the tail condition

$$\mu(y \in Y : \tau_F(y) \geq n) = O(n^{-\beta}). \tag{9.5}$$

By the argument [21, Proposition 2.6], the same tail condition holds for $\mu(y \in Y : \varphi_0(y) \geq h_n)$.

The following Diophantine condition below plays the role (A2) in [21] (namely that there exists periodic points $y_1, y_2 \in Y$ such that the ratio $\varphi_0(y_1)/\varphi_0(y_2)$ is Diophantine):

(♣) There exist two periodic points $\tilde{y}_1, \tilde{y}_2 \in \tilde{Y}$ such that the ratio $\varphi(\tilde{y}_1)/\varphi(\tilde{y}_2)$ is Diophantine.

Proposition 9.1. *Every system satisfying conditions (♣) and (1)-(4) in Section 9.1 for the above spaces* $(\mathcal{B}, \| \|_{\mathcal{B}})$ *and* $(\mathcal{B}_0, \| \|_{\mathcal{B}_0})$ *satisfies the conclusion of Theorem 5.2.*

9.2 Verifying (H0)-(H3) and (H6) with bounded $\tilde{h}$

The proof consists of verifying the conditions required for Theorem 5.2, that is, verifying that assumptions (H0)–(H6) are satisfied. Condition (H0) is supplied by (9.5) and Lemma 4.4, and it does not rely on the Markov structure. Condition (H1) can freely be assumed since h is bounded away from zero. Since Φ is Gibbs Markov, (H2) is satisfied with the space $\mathcal{B}$ as described above (see, for instance, [21, Lemma 4.1]). Because of the Diophantine assumption (♣), condition (H3) follows as in [21, Proposition 3.5 (a)]. Also, given that Φ is Gibbs Markov and (♣), (H6) can be verified as in [17, Lemma 3.13].

9.3 Verifying (H5) with bounded $\tilde{h}$

We start with the Lasota-Yorke inequality for the spaces $\mathcal{B}$ and $L^\infty(\mu_\Phi)$.

Lemma 9.2 (LY($\| \|_{\mathcal{B}}, \| \|_\infty$)). *Assume that $\Re s \geq 0$ and that $\epsilon > 0$ is so small that $\varphi^\epsilon \in L^1(\mu_\Phi)$. There exist constants $K_1, K_2, K_3 > 0$ such that*

$$|\hat{R}^n(s)v|_\theta \leq K_1 \theta^n |v|_\theta^* + K_2 (1 + |s|^\epsilon) \|v\|_\infty \quad \textit{and} \quad \|\hat{R}^n(s)v\|_{L^\infty(\mu_\Phi)} \leq K_3 \|v\|_\infty, \tag{9.6}$$

for all $v \in \mathcal{B}$ and $n \in \mathbb{N}$.

Proof. We have the pointwise formula for the transfer operator R:

$$(Rv)(y,u) = \sum_{\Phi(y',u')=(y,u)} \frac{1}{2} e^{p(y')} v(y', u'), \tag{9.7}$$

with potential $p = \log \frac{d\mu}{d\mu\circ F}$, and obviously the factor $\frac{1}{2}$ comes from the constant expansion 2 in the u-direction. Since $\tilde{h} \equiv 1$, formula (2.5) gives

$$\varphi(y,u) = \varphi_0(y) + \psi(y,u) := \varphi_0(y) + \begin{cases} u, & u \in [0,\frac{1}{2}); \\ u-1, & u \in [\frac{1}{2},1). \end{cases} \tag{9.8}$$

The first inequality of (9.6) is part (b) of Lemma 4.1. in [21], where we used (9.2) and (9.3). The only difference is that in our case $\varphi(y,u) = \varphi_0(y) + \psi(y,u)$, where $\varphi_0(y)$ plays the role of the roof function in [21, Lemma 4.1.]. Since ψ is linear and continuous on partition elements of $\tilde{\mathcal{P}}_n$, there are no additional difficulties here.

Showing that $\|\hat{R}^n(s)v\|_\infty \le \|v\|_\infty$ is standard. □

Remark 9.3. *Redefining the strong seminorm as* $\| \ \|_{\mathcal{B}'} = \alpha|\ |_\theta + \beta\| \ \|_\infty$ *for coefficients* $\alpha = 1/(4K_2(1+|s|^\epsilon))$ *and* $\beta = 1/(4(K_3 - K_1\theta^n))$ *transforms the Lasota-Yorke inequality into* $\|\hat{R}^n(s)v\|_{\mathcal{B}'} \le K_1\theta^n|v|_\theta^* + \frac{1}{2}\|v\|_\infty$*, so condition (H5) i) indeed follows.*

Remark 9.4. *Hölderness implies equicontinuity. Hence the Arzela Ascoli Theorem guarantees that every infinite subset the unit ball in* $(\mathcal{B}, \| \ \|_{\mathcal{B}})$ *has a subsequence* $(v_n)_{n\ge 1}$ *that converges in* $L^\infty(\mu_\Phi)$ *to some limit* v*. Therefore* $v \in L^\infty(\mu_\Phi)$*, and by [1, Chapter IV],* $\mathcal{B}$ *is indeed compactly embedded in* $L^\infty(\mu_\Phi)$*.*

Now we turn to the tail estimates of $R_{t,a}$.

Proposition 9.5. *Assume tail condition (4) and* $\tilde{h} \equiv 1$*. Fix* $\theta \in (0,1)$ *and let* $\mathcal{B}$ *be the Banach space as above. Then for* $\tau < \beta$*,*

$$\int_0^\infty \sigma^\tau \|R_{\sigma,a}\|_{\mathcal{B}\to L^\infty(\mu_\Phi)}\, d\sigma < \infty.$$

Proof. Assumption (9.4) implies that $|t - h_n| \le |\varphi - \varphi_0| + a + |\varphi_0 - h_n| \le C_{\varphi_0} + 1 + a$, so there is $C' = C'(a)$ such that

$$\lfloor \eta(t) - C' \rfloor \le n < \lfloor \eta(t) + C' \rfloor, \tag{9.9}$$

where we recall from property (3) that η is the asymptotic inverse of h_n. Using (9.7), we have

$$R_{t,a}v = \sum_{W\in\tilde{\mathcal{P}}} \frac{1}{2} e^{p(y'_W)}\, 1_{S_{t,a}}(y'_W, u'_W)\, v(y'_W, u'_W),$$

where $(y'_W, u'_W) = \Phi^{-1}(y,u) \cap W$. Each $W \in \tilde{\mathcal{P}}$ has the form $Z \times [0,\frac{1}{2})$ or $Z \times [\frac{1}{2},1)$ for $Z \in \mathcal{P}$. This gives by (9.2)

$$\begin{aligned} \|R_{t,a}v\|_\infty &\le \sum_{W\in\tilde{\mathcal{P}}, W\cap S_{t,a}\ne\emptyset} \frac{1}{2} e^{p(y'_W)}\, \|v\|_\infty \le \sum_{|h_{\tau_F(Z)} - t| \le C_{\varphi_0}+1+a} C_p \mu(Z)\, \|v\|_\infty \\ &= C_p\|v\|_\infty \mu(y \in Y : |h_{\tau_F(y)} - t| \le C_{\varphi_0} + 1 + a) \\ &\le C_p\|v\|_\infty \mu(y \in Y : |\tau_F(y) - \eta(t)| \le C'). \end{aligned} \tag{9.10}$$

Since $\eta(t) = t/h(0) + o(t)$, we obtain

$$\begin{aligned}\int_0^\infty \sigma^\tau \|R_{\sigma,a}\|_\infty \, d\sigma &\ll 2C'C_p h(0)^{\tau+1} \sum_n n^\tau \mu(y \in Y : \tau_F(y) = n) \\ &\ll 2C'C_p h(0)^{\tau+1} \sum_n n^{\tau-1} \mu(y \in Y : \tau_F(y) \geq n) < \infty,\end{aligned}$$

by assumption (9.5) and since $\tau < \beta$. □

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