Pressure function and limit theorems for almost Anosov flows

Henk Bruin [*] Dalia Terhesiu [†] Mike Todd [‡]

November 23, 2018

Abstract

We obtain limit theorems (Stable Laws and Central Limit Theorems, both standard and non-standard) and thermodynamic properties for a class of non-uniformly hyperbolic flows: almost Anosov flows, constructed here. The proofs of the limit theorems for these flows are applications of corresponding theorems in an abstract functional analytic framework, which may be applicable to other classes of non-uniformly hyperbolic flows.

1 Introduction and summary of the main results

This paper is a contribution to the theory of limit theorems and thermodynamic formalism in the context of non-uniformly hyperbolic flows on manifolds. In particular we introduce a new class of 'almost Anosov flows', a natural analogue of almost Anosov diffeomorphisms introduced in [HY95], and prove limit theorems for natural observables ψ with respect to the SRB measure, also giving the form of the associated pressure function. This example is presented in the context of a general framework, which may be applicable to a range of other non-uniformly hyperbolic flows. We recall that various statistical properties for several classes of Anosov flows are known [D98, D03, L04], but none of these results apply to the class of 'almost Anosov flows' considered here.

Given a flow $(\Phi_t)_t$ with a real-valued potential ϕ, we suppose that there is an equilibrium state μ_ϕ. A standard way of studying the behaviour of averages of a real-valued observable ψ is to consider a Poincaré section Y and study the induced first return map $F : Y \to Y$, the induced potentials $\bar\phi$ and $\bar\psi$, with the induced measure $\mu_{\bar\phi}$. In the discrete time case, in fact in the setting where all the dynamical systems are countable Markov shifts, [S06, Section 2] showed a one to one relation between stable laws for ψ and the asymptotic form of the pressure function $\mathcal{P}(\phi + s\psi)$,

*Faculty of Mathematics, University of Vienna, Oskar Morgensternplatz 1, 1090 Vienna, Austria, *henk.bruin@univie.ac.at*

†Department of Mathematics, University of Exeter, Streatham Campus, North Park Road, Exeter EX4 4QF, UK, *daliaterhesiu@gmail.com*

‡Mathematical Institute, University of St Andrews North Haugh, St Andrews KY16 9SS, Scotland, *m.todd@st-andrews.ac.uk*

as $s \to 0$. This function experiences a phase transition at $s = 0$, and its precise form determines the index of the stable law. Furthermore, under certain conditions on ψ, [S06, Theorem 8] gave an asymptotically linear relation between the induced pressure $\mathcal{P}(\overline{\phi + s\psi})$ and $\mathcal{P}(\phi + s\psi)$. On the limit theorems side, [MTo04, Z07] and [S06, Theorem 7] show how one can go between results on the induced and on the original system: the first of these also applies in the flow setting. In these cases, the tail of the roof function/return time to the Poincaré section (both for the flow and the map) plays a major role in determining the form of the results. Here we will follow this paradigm in the setting of a new class of flows. The proofs of the main theorems are facilitated by corresponding theorems in an abstract functional analytic framework. Applying this to the considered example requires precise estimates on the tails of the roof function, which we prove for our main example.

Our central example is an almost Anosov flow, which is a flow having a continuous flow-invariant splitting of the tangent bundle $T\mathcal{M} = E^{\mathbf{u}} \oplus E^{\mathbf{c}} \oplus E^{\mathbf{s}}$ (where $E^{\mathbf{c}}_q$ is the one-dimension flow direction) such that we have exponential expansion/contraction in the direction of $E^{\mathbf{u}}_q, E^{\mathbf{s}}_q$, except for a finite number of periodic orbits (in our case a single orbit). We can think of these as perturbed Anosov flows, where the perturbation is local around these periodic orbits, making them neutral. A precise description is given in Section 2. Almost Anosov diffeomorphisms have been introduced in [HY95] and sufficiently precise estimates on the tail of the return function to a 'good' set have been obtained in [BT17]. (These estimates are for both finite and infinite measure preserving almost Anosov diffeomorphisms.) We emphasise that the roof function is not constant on the stable direction, which is a main source of difficulty. When considering these examples we choose ϕ so that μ_ϕ is the SRB measure. We build on the construction in [BT17] to obtain the asymptotic of the tail behaviour of the roof functions for almost Anosov flows (when viewed as suspension flows). The main technical results for these systems are Propositions 2.2 and 2.4. These are then used to prove the main results for these systems, Theorems 2.5 and 2.7.

For a major part of the statements of the abstract theorems in this paper (in contexts not restricted to almost Anosov flows) we do not require any Markov structure for our system: it is only when we want to see our results in terms of the pressure function (making the connection between the leading eigenvalue of the twisted transfer operator of the base map and pressure as in [S99, Theorem4]) that this is needed. Our setup requires good functional analytic properties of the Poincaré map in terms of abstract Banach spaces of distributions. Using the rather mild abstract functional assumptions described in Section 4, in Sections 5 we obtain stable laws, standard and non-standard CLT; this is the content of Theorem 5.1. In Section 6 we recall [S06, Theorem 7] and [MTo04, Theorem 1.3] to lift Theorem 5.1 to the flow, which allow us to prove Proposition 6.1.

In Section 7 we do exploit the assumption of the Markov structure to relate the definition of the pressure $\mathcal{P}(\bar{\phi} + s\bar{\psi}), s \geq 0$, with that of the family of eigenvalues of the family of twisted transfer operators of the Poincaré map; the twist is in terms of roof function of the suspension flow and potential ψ. Using this type of identification, in Theorem 7.1 we relate the induced pressure $\mathcal{P}(\bar{\phi} + s\bar{\psi})$ with the original pressure. Using the main result in Section 7, in Section 8 we summarise the results for the abstract framework in the concluding Theorem 8.2, which gives the equivalence between the asymptotic behaviour of the pressure function $P(\phi + s\psi)$ and limit theorems. It is this summarising result that can be viewed as a version of Theorems 2–4 and Theorem 7 of [S06] combined, for flows.

In Appendix C we verify the abstract hypotheses of Theorems 6.1 and 8.2 for the almost Anosov flows introduced in Section 2 which allows us to complete the proofs of Theorems 2.5 and 2.7.

We conclude our introduction by noting some possible extensions: we do not approach the correspondence between the shape of the pressure and decay of correlations, but believe that such a correspondence can be established using the present results and arguments/estimates in [MTe17, BMT18, BMT]. Moreover, for a large class of discrete time, infinite measure preserving systems, a version of [S06, Theorem 8] along with connection with limit theorems has been obtained in [BTT18]: we expect this to carry over to case where the almost Anosov flow has infinite measure.

Notation We use "big O" and $\ll$ notation interchangeably, writing $a_n = O(b_n)$ or $a_n \ll b_n$ if there is a constant $C > 0$ such that $a_n \le Cb_n$ for all $n \ge 1$. We write $a_n \sim b_n$ if $\lim_n a_n/b_n = 1$. Throughout we let $\to^d$ stand for convergence in distribution. By F being piecewise Hölder, etc. we mean that F is Hölder on the elements $a \in \mathcal{A}$, with uniform exponent, but the Hölder norm of $F|_a$ is allowed to depend on a.

Acknowledgements: The authors would like to thank the Erwin Schrödinger Institute where this paper was initiated during a "Research in Teams" project.

2 The setup for almost Anosov flows

2.1 Description via ODEs

An almost Anosov flow is a flow having continuous flow-invariant splitting of the tangent bundle $T\mathcal{M} = E^{\mathbf{u}} \oplus E^{\mathbf{c}} \oplus E^{\mathbf{s}}$ (where $E^{\mathbf{c}}_q$ is the one-dimension flow direction) such that we have exponential expansion/contraction in the direction of $E^{\mathbf{u}}_q, E^{\mathbf{s}}_q$, except for a finite number of periodic orbits (in our case a single orbit Γ). As noted in the introduction, we can think of almost Anosov flows as perturbed Anosov flows, where the perturbation is local around Γ, making Γ neutral.

Let $\Phi_t : \mathcal{M} \times \mathbb{R} \to \mathcal{M}$ be an almost Anosov flow on a 3-dimensional manifold. We assume that the flow in local Cartesian coordinates near the neutral periodic orbit $\Gamma := (0, 0) \times \mathbb{T}$ is determined by a vector field $X : \mathcal{M} \to T\mathcal{M}$ defined as:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = X \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x(a_0x^2 + a_2y^2) \\ -y(b_0x^2 + b_2y^2) \\ 1 + w(x, y) \end{pmatrix} \tag{2.1}$$

where[1] $a_0, a_2, b_0, b_2 \ge 0$ with $\Delta = a_2b_0 - a_0b_2 \ne 0$, and w has a homogeneous function with exponent $\rho \ge 0$ as leading term, cf. Remark 2.8. Let us call the horizontal (i.e., (x, y)-component) of the flow Φ_t^{hor}. This is the vector field of [BT17, Equation (4)], with $\kappa = 2$ and the vertical

[1] The notation and absence of mixed terms a_1xy and b_1xy goes back to [H00], who could not treat mixed terms. In [BT17], the mixed terms are absent too, in order to make the explicit form of the first integrals possible. We believe that it is possible to treat mixed terms to some extent, but the additional required technicalities are beyond the purpose of this paper.

component is added as a skew product. Therefore we can take some crucial estimates from the estimates of Φ_t^{hor} in [BT17, Proposition 2.1].

The flow Φ_t has a periodic orbit $\Gamma = \{p\} \times \mathbb{T}$ of period 1 for $p = (0,0)$ (which is neutral because DX is zero on Γ), and it has local stable/unstable manifolds $W^{\mathbf{s}}_{loc}(\Gamma) = \{0\} \times (-\epsilon, \epsilon) \times \mathbb{T}^1$ and $W^{\mathbf{u}}_{loc}(\Gamma) = (-\epsilon, \epsilon) \times \{0\} \times \mathbb{T}^1$. It is an equilibrium point of neutral saddle type if we only consider Φ_t^{hor}. The time-1 map $\hat{f}$ of Φ_t^{hor} is an almost Anosov map, with Markov partition $\{P_i\}_{i \geq 0}$, where we assume that p is an interior point of $\hat{P}_0$.

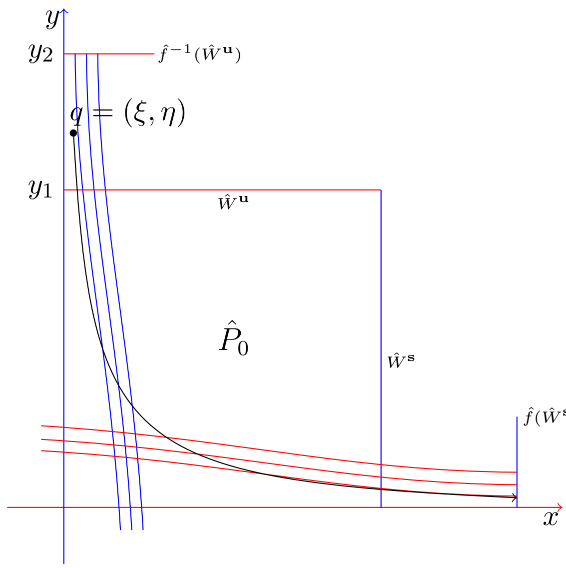

Figure 1: The first quadrant of the rectangle $\hat{P}_0$, with stable and unstable foliations of time-1 map $\hat{f} = \Phi_1^{hor}$ drawn vertically and horizontally, respectively. Also the integral curve of q is drawn.

Given $q \in \hat{f}^{-1}(\hat{P}_0)$, define

$$\hat{\tau}(q) := \min\{t > 0 : \Phi_t^{hor}(q) \in \hat{W}^{\mathbf{s}}\}, \tag{2.2}$$

where $\hat{W}^{\mathbf{s}}$ is the stable boundary leaf of $\hat{P}_0$, see Figure 1. Let $\hat{W}^{\mathbf{u}}(y)$ denote the unstable leaf of $\hat{f}$ intersecting $(0, y) \in \hat{W}^{\mathbf{s}}(p)$. The function $\hat{\tau}$ is strictly monotone on $\hat{W}^{\mathbf{u}}(y)$. For $y > 0$ and $T \geq 1$, let $\xi(y, T)$ denote the distance between $(0, y)$ and the (unique) point $q \in \hat{W}^{\mathbf{u}}(y)$ with $\hat{\tau}(q) = T$. The crucial information from [BT17, Proposition 2.1] is

$$|\xi(y, T) - \xi(y, T+1)| = \beta \xi_0(y)^{-\beta} T^{-(\beta+1)}(1 + o(1)) \quad \text{as } T \to \infty, \tag{2.3}$$

for $\beta = (a_2 + b_2)/(2b_2)$ and specific values $\xi_0(y)$ given in [BT17, Proof of Proposition 2.1]. The parameters $a_2, b_2 > 0$ can be chosen freely, so (2.3) holds for $\beta \in (\frac{1}{2}, \infty)$, but in this paper we choose $a_2 > b_2$, and therefore $\beta \in (1, \infty)$.

Remark 2.1 In fact, for the most general vector fields treated in [BT17, Equation (3)] (i.e., vector fields with higher order terms), we cannot do better than "big tails" estimates:

$$\xi(y,T) = \xi_0(y)T^{-\beta}(1 + O(T^{-\beta_*} + T^{-\frac{1}{2}}\log T)), \tag{2.4}$$

for $\beta_* = \min\{1, a_2/b_2, b_0/a_0\}$, see [BT17, Lemma 2.3]. This is due to the perturbation arguments in [BT17, Section 2.4] that become necessary when higher order terms are present and a precise first integral from [BT17, Lemma 2.2] is not available. Since the horizontal part of vector field of (2.1) here is the 'ideal' vector field from [BT17, Equation (4)], our small tail estimate (2.3) becomes possible.

The next proposition gives an estimate of integrals along such curves. This allows us to estimate the tail of the roof function (when viewing Φ_t as a suspension flow) in Lemma 2.4 and also, the tail of induced potentials (see Remark 2.8). In the first instance we use it to estimate the vertical component of the flow Φ_t (compared to t).

Proposition 2.2 *Let $\theta : \mathbb{R}^2 \to \mathbb{R}$ be a homogeneous function with exponent $\rho \in \mathbb{R}$ such that $\theta(x,0) \not\equiv 0 \not\equiv \theta(0,y)$ for $x \neq 0 \neq y$. Then there is a constant $C_\rho > 0$ (given explicitly in the proof) such that for every q with $\hat{\tau}(q) = T$ as in* (2.2),

$$\Theta(T) := \int_0^T \theta(\Phi_t(q,0))\,dt = \begin{cases} C_\rho T^{1-\frac{\rho}{2}}(1 + o(1)) & \text{if } \rho < 2, \\ C_\rho \log(T)(1 + o(1)) & \text{if } \rho = 2, \\ C_\rho(1 + o(1)) & \text{if } \rho > 2. \end{cases} \tag{2.5}$$

The proof of Proposition 2.2 is deferred to Appendix B.

2.2 Description via suspension flows, tail estimates of the roof function

The 3-dimensional time-1 map Φ_1 preserves no 2-dimensional submanifold of $\mathcal{M}$. Yet in order to model Φ_t as a suspension flow over a 2-dimensional map, we need a genuine Poincaré map. For this we choose a section Σ transversal to Γ and containing a neighbourhood U of p. As an example, Σ could be $\mathbb{T}^2 \times \{0\}$, and the Poincaré map to $\mathbb{T}^2 \times \{0\}$ could be (a local perturbation of) Arnol'd's cat map; in this case (and most cases) $\mathcal{M}$ is not homeomorphic to $\mathbb{T}^3$ because the homology is more complicated, see [BF13, N76].

Let $h : \Sigma \to \mathbb{R}^+$, $h(q) = \min\{t > 0 : \Phi_t(q) \in \Sigma\}$ be the first return time. Assuming that $\sup_\Sigma |w(x,y)| < 1$, the first return time h is bounded and bounded away from zero, say $0 < \inf_\Sigma h < \sup_\Sigma h$.

The Poincaré map $f := \Phi_h : \Sigma \to \Sigma$ has a neutral saddle point p at the origin. Its local stable/unstable manifolds are $W^{\mathbf{s}}_{loc}(p) = \{0\} \times (-\epsilon, \epsilon)$ and $W^{\mathbf{u}}_{loc}(p) = (-\epsilon, \epsilon) \times \{0\}$. Because the flow Φ_t is a perturbation of an Anosov flow, and f is a Poincaré map, it has a finite Markov partition $\{P_i\}_{i \geq 0}$ and we can assume that p is in the interior of P_0. In the sequel, let U be a neighbourhood of p that is small enough that (2.1) is valid on $U \times [0,1]$ but also that $f(U) \supset \hat{P}_0 \cup P_0$.

In order to regain the hyperbolicity lacking in f, let

$$r(q) := \min\{n \geq 1 : f^n(q) \in Y\} \tag{2.6}$$

be the first return time to $Y := \Sigma \setminus P_0$. Then the Poincaré map $F = f^r = \Phi_\tau$ of Φ_t to $Y \times \{0\}$ is hyperbolic, where

$$\tau(q) = \min\{t > 0 : \Phi_t((q,0)) \in Y \times \{0\}\} = \sum_{j=0}^{r-1} h \circ f^j$$

is the corresponding first return time.

Consequently, the flow $\Phi_t : \mathcal{M} \times \mathbb{R} \to \mathcal{M}$ can be modelled as a suspension flow on $Y^\tau = \left(\bigcup_{q \in Y} \{q\} \times [0, \tau(q))\right) / (q, \tau(q)) \sim (F(q), 0)$. Since the flow and section $Y \times \{0\}$ are C^1 smooth, τ is C^1 on each piece $\{r = k\}$.

Lemma 2.3 *With the notation from Proposition 2.2 with $\theta = w$, we have $\tau(q) = \hat{\tau}(q) + O(1)$ and $r = \hat{\tau}(q) + \Theta(\hat{\tau}(q)) + O(1)$.*

Proof By the definition of $\hat{\tau}$ we have $\Phi^{hor}_{\hat{\tau}}(q) \in \hat{W}^{\mathbf{s}}$. Therefore it takes a bounded amount of time (positive or negative) for $\Phi_{\hat{\tau}}(q, 0)$ to hit $Y \times \{0\}$, so $|\tau(q) - \hat{\tau}(q)| = O(1)$.

If in (2.5) we set $\theta = w$, then $\hat{\tau}(q) + \Theta(\hat{\tau}(q))$ indicates the vertical displacement under the flow Φ_t. In particular, it gives the number of times the flow-line intersects Σ, and hence $r = \hat{\tau}(q) + \Theta(\hat{\tau}(q)) + O(1)$. ∎

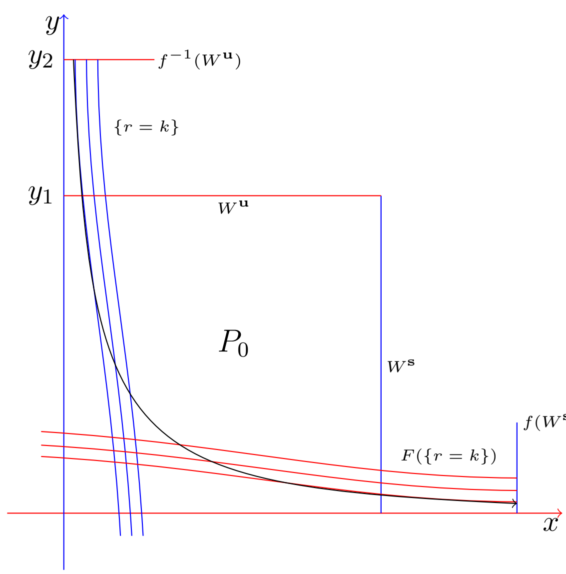

Figure 2: The first quadrant of the rectangle P_0, with stable and unstable foliations of Poincaré map $f = \Phi_h$ drawn vertically and horizontally, respectively. Also one of the integral curves is drawn.

Let $\phi : Y^\tau \to \mathbb{R}$, $\phi(q) = \lim_{t \to 0} \frac{1}{t} \log |D\Phi_t|_{W^{\mathbf{u}}}(q)|$ be the potential for the SRB measure of the flow. Let $\mu_{\bar{\phi}}$ be the F-invariant equilibrium measure of the potential $\bar{\phi} : Y \to \mathbb{R}$, $\bar{\phi}(q) =$

$\int_0^{\tau(q)} \phi \circ \Phi_t(q)\, dt = -\log |DF|_{W^{\mathbf{u}}}(q)|$; so $\bar\phi$ is logarithm of the derivative in the unstable direction of the Poincaré map F. This is at the same time the SRB-measure for F and thus is absolutely continuous conditioned to unstable leaves.

Proposition 2.4 *Recall that $\beta = \frac{a_2+b_2}{2b_2} \in (\frac{1}{2}, \infty)$. There exists $C^* > 0$ such that*

$$\mu_{\bar\phi}(\{\tau > t\}) = C^* t^{-\beta}(1 + o(1)) \tag{2.7}$$

for the F-invariant SRB measure $\mu_{\bar\phi}$.

Proof The function τ is defined on $\Sigma \setminus P_0$ and $\tau \geq h_2 = h + h \circ f$ on $Y_{\{r\geq 2\}} := f^{-1}(P_0) \setminus P_0$. The set $Y_{\{r\geq 2\}}$ is a rectangle with boundaries consisting of two stable and two unstable leaves of the Poincaré map f. Let $W^{\mathbf{u}}(y)$ denote the unstable leaf of f inside $Y_{\{r\geq 2\}}$ with $(0, y)$ as (left) boundary point. Let $y_1 < y_2$ be such that $W^{\mathbf{u}}(y_1)$ and $W^{\mathbf{u}}(y_2)$ are the unstable boundary leaves of $Y_{\{r\geq 2\}}$.

The unstable foliation of $\hat f = \Phi_1^{hor}$ does not entirely coincide with the unstable foliation of f. Let $\hat W^{\mathbf{u}}(y)$ denote the unstable leaf of $\hat f$ with $(0, y)$ as (left) boundary point. Both $\hat W^{\mathbf{u}}(y)$ and $W^{\mathbf{u}}(y)$ are C^1 curves emanating from $(0, y)$; let $\gamma(y)$ denote the angle between them. Then the lengths

$$\begin{aligned} \mathrm{Leb}(W^{\mathbf{u}}(y) \cap \{\tau > t\}) &= |\cos\gamma(y)|\ \mathrm{Leb}(\hat W^{\mathbf{u}}(y) \cap \{\tau > t\})(1 + o(1)) \\ &= |\cos\gamma(y)|\ \xi_0(y)\ t^{-\beta}(1 + o(1)) \end{aligned}$$

as $t \to \infty$, where the last equality and the notation $\xi_0(y)$ and $\beta = (a_2 + b_2)/(2b_2)$ come from (2.3).

We decompose the measure $\mu_{\bar\phi}$ on $Y_{\{r\geq 2\}}$ as

$$\int_{Y_{\{r\geq 2\}}} v\, d\mu_{\bar\phi} = \int_{y_1}^{y_2} \left(\int_{W^{\mathbf{u}}(y)} v\, d\mu^{\mathbf{s}}_{W^{\mathbf{u}}(y)} \right) d\nu^{\mathbf{u}}(y).$$

The conditional measures $\mu_{W^{\mathbf{u}}(y)}$ on $W^{\mathbf{u}}(y)$ are equivalent to Lebesgue $m_{W^{\mathbf{u}}(y)}$ with density $h_0^{\mathbf{u}} = \frac{d\mu_{W^{\mathbf{u}}(y)}}{d\mu^{\mathbf{s}}_{W^{\mathbf{u}}(y)}}$ tending to a constant $h_*(y)$ at the boundary point $(0, y)$. Therefore, as $t \to \infty$,

$$\begin{aligned} \mu_{\bar\phi}(\tau > t) &= \int_{y_1}^{y_2} \mu_{W^{\mathbf{u}}(y)}(W^{\mathbf{u}}(y) \cap \{\tau > t\})\, d\nu^{\mathbf{u}}(y) \\ &= \int_{y_1}^{y_2} h_*(y)\ m_{W^{\mathbf{u}}(y)}(W^{\mathbf{u}}(y) \cap \{\tau > t\})\, d\nu^{\mathbf{u}}(y) \\ &= \int_{y_1}^{y_2} h_*(y) |\cos\gamma(y)|\ m_{\hat W^{\mathbf{u}}(y)}(\hat W^{\mathbf{u}}(y) \cap \{\tau > t\})(1 + o(1))\, d\nu^{\mathbf{u}}(y) \\ &= \int_{y_1}^{y_2} h_*(y) |\cos\gamma(y)|\ \xi_0(y)\ t^{-\beta}(1 + o(1))\, d\nu^{\mathbf{u}}(y) = C^* t^{-\beta}(1 + o(1)), \end{aligned}$$

for $C^* = \int_{y_1}^{y_2} h_*(y) |\cos\gamma(y)|\ \xi_0(y)\, d\nu^{\mathbf{u}}(y)$. This proves the result. ∎

2.3 Main results for the Poincaré map F and flow Φ_t

Throughout this section we assume the setup and notation of Subsection 2.2. We emphasise that we are in the finite measure setting, so

$$\tau^* := \int_Y \tau \, d\mu_{\bar{\phi}} < \infty. \tag{2.8}$$

The first result below gives limit theorems for (F, τ).

Theorem 2.5 *Set $\tau_n = \sum_{j=0}^{n-1} \tau \circ F^j$ and let β and C^* be as in Proposition 2.4.*
The following hold as $n \to \infty$, w.r.t. $\mu_{\bar{\phi}}$.

(i) When $\beta < 2$, set b_n such that $b_n \sim (C^*)^{-1/\beta} n^{1/\beta}$. Then $\frac{1}{b_n}(\tau_n - \tau^* \cdot n) \to^d G_\beta$, where G_β is a stable law of index β.

When $\beta = 2$, set b_n such that $b_n \sim (C^*)^{-1/2}\sqrt{n \log n}$. Then $\frac{1}{b_n}(\tau_n - \tau^* \cdot n) \to^d \mathcal{N}(0,1)$.

(ii) If $\beta > 2$ then there exists $\sigma \neq 0$ such that $\frac{1}{\sqrt{n}}(\tau_n - \tau^* \cdot n) \to^d \mathcal{N}(0, \sigma^2)$.

Remark 2.6 The above result holds for all piecewise $C^1(Y)$ observables v in the space $\mathcal{B}_0(Y)$ defined in Appendix C.3. For item (i), we need to assume that v is in the domain of a stable law with index $\beta \in (1, 2]$ with $v \notin L^2(\mu_{\bar{\phi}})$, while for item (ii) we assume $v \in L^2(\mu_{\bar{\phi}})$.

We recall that the natural potential associated to the SRB-measure for F is $\bar{\phi} = -\log |DF|_{W^{\mathbf{u}}}$, which is the induced version of $\phi = \lim_{t \to 0} \frac{-\log |D\Phi_t|_{W^{\mathbf{u}}}}{t}$. The next result can be viewed as a version of the results in [S06] for the flow Φ_t; it gives gives the link between limit theorems for (Φ_t, ψ) for (unbounded from below) potentials ψ on $\mathcal{M}$ and pressure function $\mathcal{P}(\phi + s\psi), s \geq 0$. We let $\overline{\phi + s\psi}, s \geq 0$ be the family of induced potentials and denote the associate pressure function by $\mathcal{P}(\overline{\phi + s\psi})$. Some background on equilibrium states and pressure function (along with their induced versions) is recalled in Section 3.

Theorem 2.7 *Suppose $\bar{\psi}(y) = C' - \psi_0$ where $C' > 0$ and ψ_0 is positive piecewise $C^1(Y)$. Furthermore, suppose that for some $c > 0$ and β be as in Proposition 2.4, $\mu_{\bar{\phi}}(\psi_0 > t) \sim ct^{-\frac{\beta}{\kappa}}$ for $\kappa \in (\frac{1}{\beta}, \beta)$. Set $\psi^* = \int_Y \bar{\psi} \, d\mu_{\bar{\phi}}$ and let $\psi_T = \int_0^T \psi \circ \Phi_t \, dt$. The following hold as $T \to \infty$, w.r.t. $\mu_{\bar{\phi}}$.*

(a) (i) When $\beta < 2$ and $\kappa \in (\beta/2, \beta)$, set $b(T)$ such that $\frac{T}{(cb(T))^{\frac{\beta}{\kappa}}} \to 1$.

Then $\frac{1}{b(T)}(\psi_T - \psi^* \cdot T) \to^d G_{\beta/\kappa}$, where $G_{\beta/\kappa}$ is a stable law of index β/κ and this is, further, equivalent to $\mathcal{P}(\overline{\phi + s\psi}) = \psi^* s + cs^{\beta/\kappa}(1 + o(1))$, as $s \to 0$.

(ii) When $\beta = 2$ but $\kappa = \beta/2$, set $b(T) \sim c^{-1/2}\sqrt{T \log T}$.

Then $\frac{1}{b(T)}(\psi_T - \psi^* \cdot T) \to^d \mathcal{N}(0,1)$ and this is, further, equivalent to $\mathcal{P}(\overline{\phi + s\psi}) = \psi^* s + cs^2 \log(1/s)(1 + o(1))$, as $s \to 0$.

(b) Suppose that $\beta > 2$. Then there exists $\sigma \neq 0$ such that $\frac{1}{\sqrt{T}}(\psi_T - \psi^* \cdot T) \to^d \mathcal{N}(0, \sigma^2)$ and this is, further, equivalent to $\mathcal{P}(\overline{\phi + s\psi}) = \psi^* s + \frac{\sigma^2}{2} s^2(1 + o(1))$ as $s \to 0$.

Moreover, $\mathcal{P}(\phi + s\psi) = \frac{1}{\tau^*}\mathcal{P}(\overline{\phi + s\psi})(1 + o(1))$.

Remark 2.8 If $\psi : \mathcal{M} \to \mathbb{R}$ satisfies the following properties, then the condition on $\bar{\psi}$ in Theorem 2.5 holds. In local tubular coordinates near $\{p\} \times \mathbb{T}$, $\psi = g_0 - g(W(x, y))$ where g_0 is Hölder function vanishing on a neighbourhood of p, and $W(x, y)$ is a linear combination of homogeneous functions depending only on (x, y), such that the term $W_\rho(x, y)$ with lowest exponent ρ satisfies $W_\rho(0, 1) \neq 0 \neq W_\rho(1, 0)$, and $g : \mathbb{R} \to \mathbb{R}$ is analytic with $g(0) = 0$, $g'(0) \neq 0$. Then $\kappa = 1 - \frac{\rho}{2}$ (see Proposition 2.2), so assuming that this lowest exponent $\rho \in (2(1-\beta), 2(\beta-1)/\beta)$, then $\kappa \in (\frac{1}{\beta}, \beta)$.

Proof of Theorems 2.5 and 2.7 These follow directly from applying Theorems 6.1 and 8.2 to (F, τ) and (Φ_t, ϕ). Appendix C verifies all the abstract hypotheses of Theorems 6.1 and 8.2 for the flow Φ_t with base map F and roof function τ introduced in Section 2 and potential ψ defined in the statement of Theorem 2.7. The extra assumptions (including coboundary assumptions) that ensure $\sigma \neq 0$ in Theorems 6.1 (b) and 8.2 (b) are verified in Appendix C.3. ∎

3 Background on equilibrium states for suspension flows

Here we outline the standard general theory of thermodynamic formalism for suspension flows. We start with a discrete time dynamical system $F : Y \to Y$. Defining $\mathcal{M}_F$ to be the set of F-invariant probability measures, for a potential $\psi : Y \to [-\infty, \infty]$ we define the pressure as

$$\mathcal{P}(\psi) := \sup\left\{h(\mu) + \int \psi \, d\mu : \ \mu \in \mathcal{M}_F \text{ and } -\int \psi \, d\mu < \infty\right\}.$$

Given a roof function $\tau : Y \to \mathbb{R}^+$, let $Y^\tau = \{(y, z) \in Y \times \mathbb{R} : 0 \leq z \leq \tau(y)\}/\sim$ where $(y, \tau(y)) \sim (Fy, 0)$. Then the suspension flow $F_t : Y^\tau \to Y^\tau$ is given by $F_t(y, z) = (y, z + t)$ computed modulo identifications. We will suppose throughout that $\inf \tau > 0$, but note that the case $\inf \tau = 0$ can also be handled, see for example [Sav98, IJT15]. Barreira and Iommi [BI06] define pressure as

$$\mathcal{P}(\phi) := \inf\{u \in \mathbb{R} : \mathcal{P}(\bar{\phi} - u\tau) \leq 0\} \tag{3.1}$$

in the case that (Y, F) is a countable Markov shift and the potential $\phi : Y^\tau \to \mathbb{R}$ induces to $\bar{\phi} : Y \to \mathbb{R}$, where

$$\bar{\phi}(x) := \int_0^{\tau(x)} \phi(x, t) \, dt, \tag{3.2}$$

has summable variations. This also makes sense for general suspension flows, so we will take (3.1) as our definition. In [AK42], there is a bijection between F_t-invariant probability measures μ, and the corresponding F-invariant probability measures $\bar{\mu}$ given by the identification

$$\mu = \frac{\bar{\mu} \times m}{(\bar{\mu} \times m)(Y^\tau)},$$

where m is Lebesgue measure. That is, whenever there is such a μ there is such a $\bar{\mu}$, and vice versa. Moreover, Abramov's formula for flows [Ab59b] gives the following characterisation of entropy:

$$h(\mu) = \frac{h(\bar{\mu})}{(\bar{\mu} \times m)(Y^\tau)}, \tag{3.3}$$

and clearly

$$\int \phi \, d\mu = \frac{\int \bar{\phi} \, d\bar{\mu}}{(\bar{\mu} \times m)(Y^\tau)} = \frac{\int \bar{\phi} \, d\bar{\mu}}{\int \tau \, d\bar{\mu}}. \tag{3.4}$$

One consequence of these formulas is that the pressure in (3.1) is independent of the choice of cross section Y which essentially follows from the fact that if we choose a subset of Y and reinduce there, then Abramov's formula gives the same value for pressure (here we can use the discrete version of Abramov's formula [Ab59a]). Note that we say 'Abramov's formula' in both the continuous and discrete time cases as the formulas are analogous [Ab59a, Ab59b], and similarly for integrals, where the formula holds by the ergodic and ratio ergodic theorems.

We say that μ_ϕ is an *equilibrium state* for ϕ if $h(\mu_\phi) + \int \phi \, d\mu_\phi = \mathcal{P}(\phi)$. The same notion extends to the induced system $(Y, F, \bar{\phi})$. In that setting we may also have a *conformal measure* $m_{\bar{\phi}}$ for $\bar{\phi}$. This means that $m_{\bar{\phi}}(F(A)) = \int_A e^{-\bar{\phi}} \, dm_{\bar{\phi}}$ for any measurable set A on which F is injective. Later on, in order to link our limit theorem results with pressure, we will assume that $F : Y \to Y$ is Markov. In that setting our assumptions on ϕ will be equivalent to assuming $\mathcal{P}(\phi) = 0$, in which case we will have a equilibrium states and conformal measures $\mu_{\bar{\phi}}$ and $m_{\bar{\phi}}$.

4 Abstract setup

We start with a flow $f_t : \mathcal{M} \to \mathcal{M}$, where $\mathcal{M}$ is a manifold. Let Y be a co-dimension 1 section of $\mathcal{M}$ and define $\tau : Y \to \mathbb{R}_+$ to be a roof function. We think of τ as a first return of f_t to Y and define $F : Y \to Y$ by $F = f_\tau$. The flow f_t is isomorphic with the suspension flow $F_t : Y^\tau \to Y^\tau$, $Y^\tau = \{(y, z) \in Y \times \mathbb{R} : 0 \le z \le \tau(z)\}/\sim$, as described in Section 3. Throughout, we assume that τ is bounded from below.

Given the potential $\phi : \mathcal{M} \to \mathbb{R}$ and its induced version $\bar{\phi} : Y \to \mathbb{R}$ defined in (3.2), we assume that there is a conformal measure $m_{\bar{\phi}}$ for $(F, \bar{\phi})$. In the rest of this section we recall the abstract framework and hypotheses in [LT16] as relevant for studying limit theorems.

4.1 Banach spaces and equilibrium measures for $(F, \bar{\phi})$ and (f_t, ϕ).

Let $R_0 : L^1(m_{\bar{\phi}}) \to L^1(m_{\bar{\phi}})$ be the transfer operator for the first return map $F : Y \to Y$ w.r.t. conformal measure $m_{\bar{\phi}}$, defined by duality on $L^1(m_{\bar{\phi}})$ via the formula $\int_Y R_0 v \, w \, dm_{\bar{\phi}} = \int_Y v \, w \circ F \, dm_{\bar{\phi}}$ for every bounded measurable w.

We assume that there exist two Banach spaces of distributions $\mathcal{B}$, $\mathcal{B}_w$ supported on Y such that for some $\alpha, \gamma > 0$

(H1) (i) $C^\alpha \subset \mathcal{B} \subset \mathcal{B}_w \subset (C^{\alpha_1})'$, where $(C^{\alpha_1})'$ is the dual of $C^{\alpha_1} = C^{\alpha_1}(Y, \mathbb{C})$.[2]

[2]We will use systematically a "prime" to denote the topological dual.

(ii) There exists $C > 0$ such that for all $h \in \mathcal{B}$, $g \in C^{\alpha}$, we have $hg \in \mathcal{B}$.

(iii) The transfer operator R_0 associated with F maps continuously from C^{α} to $\mathcal{B}$ and R_0 admits a continuous extension to an operator from $\mathcal{B}$ to $\mathcal{B}$, which we still call R_0.

(iv) The operator $R_0 : \mathcal{B} \to \mathcal{B}$ has a simple eigenvalue at 1 and the rest of the spectrum is contained in a disc of radius less than 1.

A few comments on **(H1)** are in order and here we just recall the similar ones in [LT16]. We note that **(H1)**(i) should be understood in terms of the following usual convention (see, for instance, [GL06, DL08]): there exist continuous injective linear maps π_i such that $\pi_1(C^{\alpha}) \subset \mathcal{B}$, $\pi_2(\mathcal{B}) \subset \mathcal{B}_w$ and $\pi_3(\mathcal{B}_w) \subset (C^{\alpha_1})'$. Throughout, we leave such maps implicit, but recall their meaning here. In particular, we assume that $\pi = \pi_3 \circ \pi_2 \circ \pi_1$ is the usual embedding, i.e., for all $h \in C^{\alpha}$ and $g \in C^{\alpha_1}$

$$\langle \pi(h), g \rangle_0 = \int_Y hg \, dm_{\bar{\phi}}.$$

Via the above identification, the conformal measure $m_{\bar{\phi}}$ can be identified with the constant function 1 both in $(C^{\alpha_1})'$ and in $\mathcal{B}$ (i.e., $\pi(1) = m_{\bar{\phi}}$). Also, by **(H1)**(i), $\mathcal{B}' \subset (C^{\alpha})'$, hence the dual space can be naturally viewed as a space of distributions. Next, note that $\mathcal{B}' \supset (C^{\alpha_1})'' \supset C^{\alpha_1} \ni 1$, thus we have $m_{\bar{\phi}} \in \mathcal{B}'$ as well. Moreover, since $m_{\bar{\phi}} \in \mathcal{B}$ and $\langle 1, g \rangle_0 = \langle g, 1 \rangle_0 = \int g \, d\mu^{\mathbf{s}}_{\bar{\phi}}$, $m_{\bar{\phi}}$ can be viewed as the element 1 of both spaces $\mathcal{B}$ and $(C^{\alpha_1})'$.

By **(H1)**, the spectral projection P_0 associated with the eigenvalue 1 is defined by $P_0 = \lim_{n\to\infty} R_0^n$. Note that by **(H1)**(ii), for each $g \in C^{\alpha}$, $P_0 g \in \mathcal{B}$ and

$$\langle P_0 g, 1 \rangle_0 = m_{\bar{\phi}}(P_0 g) = \lim_{n\to\infty} m_{\bar{\phi}}(1 \cdot R_0^n g) = m_{\bar{\phi}}(g) = \langle g, 1 \rangle_0.$$

By **(H1)**(iv), there exists a unique $\mu_{\bar{\phi}} \in \mathcal{B}$ such that $R_0 \mu_{\bar{\phi}} = \mu_{\bar{\phi}}$ and $\langle \mu_{\bar{\phi}}, 1 \rangle = 1$. Thus, $P_0 g = \mu_{\bar{\phi}} \langle g, 1 \rangle_0$. Also $R_0' m_{\bar{\phi}} = m_{\bar{\phi}}$ where R_0' is dual operator acting on $\mathcal{B}'$. Note that for any $g \in C^{\alpha_1}$,

$$|\langle \mu_{\bar{\phi}}, g \rangle_0| = |\langle P_0 1, g \rangle_0| = \left| \lim_{n\to\infty} R_0^n m_{\bar{\phi}}(g) \right| = \lim_{n\to\infty} \left| m_{\bar{\phi}}(g \circ F^n) \right| \leq |g|_{\infty}.$$

Hence $\mu_{\bar{\phi}}$ is a measure. For each $g \geq 0$,

$$\langle P_0 1, g \rangle_0 = \lim_{n\to\infty} \langle R_0^n 1, g \rangle = \lim_{n\to\infty} \langle 1, g \circ F^n \rangle_0 \geq 0.$$

It follows that $\mu_{\bar{\phi}}$ is a probability measure.

Summarising the above, the eigenfunction associated with the eigenvalue $\lambda = 1$ is an invariant probability measure for F and we can write $P_0 1 = \mu_{\bar{\phi}}$ and $P_0 1 = h_0$, for $d\mu_{\bar{\phi}} = h_0 dm_{\bar{\phi}}$. Under **(H1)**, the standard theory summarised in Section 3 applies $\mathcal{P}(\bar{\phi}) = \log \lambda = 0$. So, under **(H1)**, $\mu_{\bar{\phi}}$ is an equilibrium (probability) measure for $(F, \bar{\phi})$. Further, via (3.4),

$$\mu_{\phi} = \frac{\mu_{\bar{\phi}} \times m}{(\mu_{\bar{\phi}} \times m)(Y^{\tau})}$$

gives an equilibrium (probability) measure for (f_t, ϕ).

4.2 Transfer operator R defined w.r.t. the equilibrium measure $\mu_{\bar\phi}$

Given the equilibrium measure $\mu_{\bar\phi} \in \mathcal{B}$, we let $R : L^1(\mu_{\bar\phi}) \to L^1(\mu_{\bar\phi})$ be the transfer operator for the first return map $F : Y \to Y$ w.r.t. the invariant measure $\mu_{\bar\phi}$ given by $\int_Y Rv\, w\, d\mu_{\bar\phi} = \int_Y v\, w\circ F\, d\mu_{\bar\phi}$ for every bounded measurable w.

Under **(H1)**(i)-(iv), we further assume

(H1)(v) The transfer operator R maps continuously from C^α to $\mathcal{B}$ and R admits a continuous extension to an operator from $\mathcal{B}$ to $\mathcal{B}$, which we still call R.

By **(H1)**(iv) and **H1**(v), the operator $R : \mathcal{B} \to \mathcal{B}$ has a simple eigenvalue at 1 and the rest of the spectrum is contained in a disc of radius less than 1. While the spectra of R_0 and R are the same, the spectral projection $P = \lim_{n\to\infty} R^n$ acts differently on $\mathcal{B}, \mathcal{B}_w$. In particular, $P1 = 1$, while $P_0 1 = \mu_{\bar\phi}$, $P_0 1 = h_0$ Throughout the rest of this paper, for any $g \in C^\alpha$, we let

$$\langle g, 1\rangle := \langle g, P_0 1\rangle_0 = \langle P_0 1, g\rangle_0 = \int_Y g\, d\mu_{\bar\phi}$$

and note that $P_0 g = h\langle g, 1\rangle_0$ and $Pg = \langle g, 1\rangle$.

4.3 Further assumptions on the transfer operator R

Given R as defined in Subsection 4.2, for $u \geq 0$ and $\tau : Y \to \mathbb{R}_+$ we define the perturbed transfer operator

$$\hat{R}(u)v = R(e^{-u\tau}v).$$

By [BMT, Proposition 4.1], a general proposition on twisted transfer operators that holds independently of the specific properties of F (see also Section A.1), we can write

$$\hat{R}(u) = g_0(u)\int_0^\infty R(\omega(t-\tau))e^{-ut}\, dt := g_0(u)\int_0^\infty M(t)\, e^{-ut}\, dt \tag{4.1}$$

where $\omega : \mathbb{R} \to [0, 1]$ is an integrable function with $\operatorname{supp}\omega \subset [-1, 1]$ and g_0 is analytic such that $g_0(0) = 1$.

For most results we require that

(H2) There exists a function ω satisfying (4.1) and $C_\omega > 0$ such that

(i) for any $t \in \mathbb{R}_+$ and for all $v \in C^\alpha$, $h \in \mathcal{B}$, we have $\omega(t-\tau)h \in \mathcal{B}_w$ and

$$\langle \omega(t-\tau)vh, 1\rangle \leq C_\omega \|v\|_{C^\alpha}\|h\|_{\mathcal{B}_w}\mu_{\bar\phi}(\omega(t-\tau));$$

(ii) for all $T > 0$,

$$\int_T^\infty \|M(t)\|_{\mathcal{B}\to\mathcal{B}_w}\, dt \leq C_\omega \mu_{\bar\phi}(\tau > T).$$

Further, we assume the usual Doeblin-Fortet inequality:

(H3) There exist $\sigma_0 > 1$ and $C_0, C_1 > 0$ such that for all $h \in \mathcal{B}$, for all $n \in \mathbb{N}$ and for all $u > 0$,

$$\|\hat{R}(u)^n h\|_{\mathcal{B}_w} \leq C_1\|h\|_{\mathcal{B}_w}, \quad \|\hat{R}(u)^n h\|_{\mathcal{B}} \leq C_0\sigma_0^{-n}\|h\|_{\mathcal{B}} + C_1\|h\|_{\mathcal{B}_w}.$$

4.4 Refined assumptions on τ

For the purpose of obtaining limit laws for F (and in the end f_t) we assume that

(H4) One of the following holds as $t \to \infty$,

(i) $\mu_{\bar{\phi}}(\tau > t) = \ell(t)t^{-\beta}$, for some slowly varying function ℓ and $\beta \in (1, 2]$. When $\beta = 2$, we assume $\tau \notin L^2(\mu_{\bar{\phi}})$ and ℓ is such that the function $\tilde{\ell}(t) = \int_1^t \frac{\ell(x)}{x}\, dx$ is unbounded and slowly varying.

(ii) $\tau \in L^2(\mu_{\bar{\phi}})$. We do not assume[3] $\tau \in \mathcal{B}$, but require that for any $h \in \mathcal{B}$, $R(\tau h) \in \mathcal{B}$.

5 Limit laws for (τ, F)

Theorem 5.1 *Assume* **(H1)** *and* **(H2)** *and let* $\tau_n = \sum_{j=0}^{n-1} \tau \circ F^j$. *The following hold as* $n \to \infty$, *w.r.t. any probability measure* ν *such that* $\nu \ll \mu_{\bar{\phi}}$.

(i) Assume **(H4)***(i).*

When $\beta < 2$, *set* b_n *such that* $\frac{n\ell(b_n)}{b_n^\beta} \to 1$. *Then* $\frac{1}{b_n}(\tau_n - \frac{\tau^* \cdot n}{\nu(Y)}) \to^d G_\beta$, *where* G_β *is a stable law of index* β.

When $\beta = 2$, *set* b_n *such that* $\frac{n\tilde{\ell}(b_n)}{b_n^2} \to c > 0$. *Then* $\frac{1}{b_n}(\tau_n - \frac{\tau^* \cdot n}{\nu(Y)}) \to^d \mathcal{N}(0, c)$.

(ii) Assume **(H4)***(ii) and* $\tau - \tau^* \neq h - h \circ F$ *for any* $h \in \mathcal{B}$. *Then there exists* $\sigma \neq 0$ *such that* $\frac{1}{\sqrt{n}}(\tau_n - \frac{\tau^* \cdot n}{\nu(Y)}) \to^d \mathcal{N}(0, \sigma^2)$.

The rest of this section is devoted to the proof of the above result.

5.1 Family of eigenvalues for $\hat{R}(u)$

By (4.1) and **(H2)**(ii), $u \to \hat{R}(u)$ is an analytic family of bounded linear operators on $\mathcal{B}_w$ for $u > 0$ and this family can be continuously extended as $u \to 0$. We note that by (4.1), $\lim_{u\to 0} \frac{d}{du} g_0(u), \frac{d^2}{du^2} g_0(u)$ exist and for simplicity of notation, when used, we assume $\frac{d}{du} g_0(u), \frac{d^2}{du^2} g_0(u) \to 1$ as $u \to 0$.

Lemma 5.2 *Assume* **(H2)**. *Then*

$$\|\hat{R}(u) - \hat{R}(0)\|_{\mathcal{B} \to \mathcal{B}_w} \ll u + \mu_{\bar{\phi}}(\tau > 1/u) := q(u).$$

Proof Using (4.1), we write

$$\begin{aligned} \hat{R}(u) &= (g_0(u) - 1) \int_0^\infty M(t) e^{-ut}\, dt + g_0(0) \int_0^\infty M(t)(e^{-ut} - 1)\, dt + g_0(0) \int_0^\infty M(t)\, dt \\ &= (g_0(u) - 1) \int_0^\infty M(t) e^{-ut}\, dt + \int_0^\infty M(t)(e^{-ut} - 1)\, dt + \hat{R}(0). \end{aligned}$$

[3] In our example, τ is only piecewise $C^1(Y)$.

By the properties of g_0 and **(H2)**(ii), we have that as $u \to 0$,

$$\begin{aligned} \|\hat{R}(u) - \hat{R}(0)\|_{\mathcal{B}\to\mathcal{B}_w} &\ll u \int_0^\infty \|M(t)\|_{\mathcal{B}\to\mathcal{B}_w}\, dt + \int_0^{1/u} (e^{-ut}-1)\|M(t)\|_{\mathcal{B}\to\mathcal{B}_w}\, dt \\ &+ \int_{1/u}^\infty \|M(t)\|_{\mathcal{B}\to\mathcal{B}_w}\, dt \ll u + \mu_{\bar\phi}(\tau > 1/u) + u \int_0^{1/u} t\|M(t)\|_{\mathcal{B}\to\mathcal{B}_w}\, dt. \end{aligned}$$

Let $S(t) = \int_t^\infty \|M(x)\|_{\mathcal{B}\to\mathcal{B}_w}\, dx$ and note that

$$\|M(t)\|_{\mathcal{B}\to\mathcal{B}_w} \ll \int_t^{t+1} \|M(x)\|_{\mathcal{B}\to\mathcal{B}_w}\, dx \ll S(t+1) - S(t). \tag{5.1}$$

Rearranging,

$$\begin{aligned} u \int_0^{1/u} t\|M(t)\|_{\mathcal{B}\to\mathcal{B}_w}\, dt &\ll u \int_0^{1/u} t(S(t+1) - S(t))\, dt \\ &= u\Big(\int_1^{1/u+1} (t-1)S(t))\, dt - \int_0^{1/u} tS(t)\, dt \Big) \\ &= u \int_0^1 tS(t)\, dt - u \int_1^{1/u+1} S(t)\, dt - u \int_{1/u}^{1/u+1} tS(t)\, dt \ll u + S(1/u). \end{aligned}$$

By **(H2)**(ii), $S(1/u) \ll \mu_{\bar\phi}(\tau > 1/u)$ and the conclusion follows. ∎

By **(H1)**(iv), 1 is an eigenvalue for $\hat{R}(0)$. By **(H3)**, there exists a family of eigenvalues $\lambda(u)$ well-defined for $u \in [0, \delta_0)$, for some $\delta_0 > 0$. Also, for $u \in [0, \delta_0)$, $\hat{R}(u)$ has a spectral decomposition

$$\hat{R}(u) = \lambda(u)P(u) + Q(u), \quad \|Q(u)^n\|_{\mathcal{B}} \le C\sigma_0^n, \tag{5.2}$$

for all $n \in \mathbb{N}$, some fixed $C > 0$ and some $\sigma_0 < 1$. Here, $P(u)$ is a family of rank one projectors and we write $P(u) = \mu_{\bar\phi}(u) \otimes v(u)$, normalising such that $\mu_{\bar\phi}(u)(v(u)) = 1$, where $\mu_{\bar\phi}(0)$ is the invariant measure and $v(0) = 1 \in \mathcal{B}$. Equivalently, we normalise such that $\langle v(u), 1\rangle = 1$ and write

$$\lambda(u) = \langle \hat{R}(u)v(u), 1\rangle. \tag{5.3}$$

The result below gives the continuity of the families $P(u)$ and $v(u)$ (in $\mathcal{B}_w$).

Lemma 5.3 *Assume* **(H1)**, **(H2)** *and* **(H3)**. *Let $q(u)$ be as defined in Lemma 5.2. Then there exist $0 < \delta_1 \le \delta_0$ and $C > 0$ such that*

$$\|P(u) - P(0)\|_{\mathcal{B}\to\mathcal{B}_w} \le Cq(u)|\log(q(u))|$$

for all $0 \le u < \delta_1$. The same continuity estimate holds for $v(u)$ (viewed as an element of $\mathcal{B}_w$).

Proof Given the continuity estimate in Lemma 5.2, we can apply [KL99, Remark 5] (an improved version of [KL99, Corollary 1] under less general assumptions). Assumption **(H3)** here corresponds to [KL99, Hypotheses 2, 3] and the estimate in Lemma 5.2 corresponds to [KL99, Hypothesis 5]. The extra assumptions of [KL99, Remark 5] are satisfied in the present setup, since by **(H3)**, $\hat{R}(u)$ is a family of $\|\cdot\|_{\mathcal{B}}$ contractions. Going from $P(u)$ to $v(u)$ is a standard step (see, for instance, [LT16, Proof of Lemma 3.5]). ∎

Since $\lambda(0) = 1$ is a simple eigenvalue (by **(H1)**), Lemma 5.3 ensures that $\lambda(u)$ is a family of simple eigenvalues (see [KL99, Remark 4]). The next result gives precise information on the asymptotic and continuity properties of $\lambda(u)$, $u \to 0$ (a version of the results in [AD01a]) in the present abstract framework.

Lemma 5.4 *Assume* **(H1)** *and* **(H2)**. *Let $q(u)$ be as defined in Lemma 5.2. Set $\Pi(u) = \mu_{\bar{\phi}}(1 - e^{-u\tau})$. Then as $u \to 0$,*

$$1 - \lambda(u) = \Pi(u)\Big(1 + O\big(q(u)|\log(q(u))|\big)\Big).$$

Proof By (5.3), $\lambda(u)v(u) = \hat{R}(u)v(u)$, $u \in [0, \delta_0)$ with $\lambda(0) = 1$ and $v(0) = \mu_{\bar{\phi}}$. Then

$$\lambda(u) = \langle \hat{R}(u)v(u), 1\rangle = \mu_{\bar{\phi}}(e^{-u\tau}) + \langle (\hat{R}(u) - \hat{R}(0))(v(u) - v(0)), 1\rangle.$$

because $\langle v(u), 1\rangle = 1$. Since **(H2)** holds (so, (4.1) holds),

$$\begin{aligned}\lambda(u) = \mu_{\bar{\phi}}(e^{-u\tau}) + V(u) := \mu_{\bar{\phi}}(e^{-u\tau}) &+ \int_0^\infty (e^{-ut} - 1)\langle \omega(t-\tau)[v(u) - v(0)], 1\rangle\, dt \\ &+ (g_0(u) - 1)\int_0^\infty e^{-ut}\langle \omega(t-\tau)[v(u) - v(0)], 1\rangle\, dt.\end{aligned}$$

Thus, $1 - \lambda(u) = \Pi(u) - V(u)$. By **(H2)**(i),

$$\langle \omega(t-\tau)[v(u) - v(0)], 1\rangle \le C\|v(u) - v(0)\|_{\mathcal{B}_w}\mu_{\bar{\phi}}(\omega(t-\tau)).$$

Recalling that $\operatorname{supp}\omega \subset [-1, 1]$, for all $u \in [0, \delta_0)$ and some $C_1 > 0$,

$$\begin{aligned}\left|\int_0^\infty e^{-ut}\langle \omega(t-\tau)[v(u) - v(0)], 1\rangle\, dt\right| &\le C\|v(u) - v(0)\|_{\mathcal{B}_w}\left|\int_0^\infty e^{-ut}\int_Y \omega(t-\tau)\, d\mu_{\bar{\phi}}\, dt\right| \\ &\le C\|v(u) - v(0)\|_{\mathcal{B}_w}\left|\int_Y \int_{\tau-1}^{\tau+1} e^{-ut}\omega(t-\tau)\, dt\, d\mu_{\bar{\phi}}\right| \\ &\le C\|v(u) - v(0)\|_{\mathcal{B}_w}\int_Y e^{-u\tau}\left|\int_{-1}^1 \omega(t)e^{ut}\, dt\right|\, d\mu_{\bar{\phi}} \\ &\le C_1\|v(u) - v(0)\|_{\mathcal{B}_w}\int_Y e^{-u\tau}\, d\mu_{\bar{\phi}} = C_1\|v(u) - v(0)\|_{\mathcal{B}_w}(\Pi(u) + 1).\end{aligned}$$

In the previous to last inequality, we have used that $\left|\int_{-1}^{1}\omega(t)e^{ut}\,dt\right| \ll 1$ for $u \in [0,\delta_0)$. By a similar argument,

$$\Big|\int_0^\infty (e^{-ut}-1)\langle\omega(t-\tau)[v(u)-v(0)],1\rangle\,dt\Big| \le C_1\|v(u)-v(0)\|_{\mathcal{B}_w}\Pi(u).$$

The previous two displayed equations imply that for $u \in [0,\delta_0)$,

$$\begin{aligned}|V(u)| &\le C_1\|v(u)-v(0)\|_{\mathcal{B}_w}(g_0(u)-1)(\Pi(u)+1) + C_1\|v(u)-v(0)\|_{\mathcal{B}_w}\Pi(u)\\ &= C_1 g_0(u)\|v(u)-v(0)\|_{\mathcal{B}_w}\Pi(u).\end{aligned}$$

By Lemma 5.3, $\|v(u)-v(0)\|_{\mathcal{B}_w} = O(q(u)|\log(q(u))|)$. Hence $V(u) = O(\Pi(u)q(u)|\log(q(u)|)$. ∎

A first consequence of the above result is

Corollary 5.5 *Assume* **(H1)** *and* **(H2)***. The following hold as* $u \to 0$.

(i) If **(H4)***(i) holds with* $\beta < 2$*, then* $1-\lambda(u) = \tau^* u + u^\beta \ell(1/u)(1+o(1))$.

(ii) If **(H4)***(i) holds with* $\beta = 2$*, then* $1-\lambda(u) = \tau^* u + u^2 L(1/u)(1+o(1))$*, where* $L(t) = \frac{1}{2}\tilde{\ell}(t)$.

Proof Under **(H4)**(i) with $\beta < 2$, $\Pi(u) = \tau^* u + u^\beta\ell(1/u)(1+o(1))$; we refer to, for instance, [F66, Ch. XIII] and [AD01b]. This together with Lemma 5.4 (with $q(u) \sim u$) ensures that $1-\lambda(u) = \tau^* u + u^\beta \ell(1/u) + O(u^2\log(1/u)) + O(u^{\beta+1}\ell(1/u))$. Item (i) follows.

Under **(H4)**(i) with $\beta = 2$, it follows from [AD01b] that $\Pi(u) = \tau^* u + u^2 L(1/u)(1+o(1))$, where $L(t) \sim \ell(t)\log(t)$ with $\tilde{\ell}(t) = \int_1^t \ell(x)/x\,dx$ as in **(H4)**(i) for $\beta = 2$. The estimate for $\Pi(u)$ together with Lemma 5.4 (with $q(u) \sim u$) implies that $1-\lambda(u) = \tau^* u + u^2 L(1/u) + O(u\log(1/u)\Pi(u))$.

In the case of **(H4)**(i) with $\beta = 2$ and $\ell(1/u) \to \infty$ as $u \to 0$, we have $\log(1/u) = o(L(1/u))$. As a consequence, $u\log(1/u)\Pi(u) = O(u^2\log(1/u)) = o(u^2 L(1/u))$ and the conclusion follows. In the general case (which allows ℓ to be asymptotically constant), for fixed small $\delta > 0$ and $\epsilon > 0$, we write

$$\begin{aligned}u\log(1/u)\Pi(u) &= u\log(1/u)\Big(\Pi(u) - \int_{\tau\le\delta/u}(e^{-u^{2+\epsilon}\tau^{2+\epsilon}}-1)\,d\mu_{\bar\phi}\Big)\\ &\quad + u\log(1/u)\int_{\tau\le\delta/u}(e^{-u^{2+\epsilon}\tau^{2+\epsilon}}-1)\,d\mu_{\bar\phi} = u\log(1/u)(I_1(u)+I_2(u)).\end{aligned}$$

First,

$$\begin{aligned}u\log(1/u)|I_2(u)| &\ll u\log(1/u)u^{2+\epsilon}\int_{\tau\le\delta/u}\tau^{2+\epsilon}\,d\mu_{\bar\phi}\\ &\le \log(1/u)u^{3+\epsilon}u^{-2\epsilon}\delta^{2\epsilon}\int_Y \tau^{2-\epsilon}\,d\mu_{\bar\phi} = o(u^2 L(1/u)).\end{aligned}$$

Next, compute that

$$\begin{aligned} I_1(u) &= \int_{\tau\le\delta/u}(e^{-u\tau}-e^{-u^{2+\epsilon}\tau^{2+\epsilon}})\,d\mu_{\bar\phi} + \int_{\tau>\delta/u}(e^{-u\tau}-1)\,d\mu_{\bar\phi} \\ &= \int_{\tau\le\delta/u}(e^{-u\tau}-e^{-u^{2+\epsilon}\tau^{2+\epsilon}})\,d\mu_{\bar\phi} + O(\mu_{\bar\phi}(\tau>\delta/u)) \\ &= \int_{\tau\le\delta/u} e^{-u\tau}(1-e^{-u^{1+\epsilon}\tau^{1+\epsilon}})\,d\mu_{\bar\phi} + O(u^2\ell(\delta/u)). \end{aligned}$$

Let $G(x)=\mu_{\bar\phi}(\tau<x)$ and compute that

$$\begin{aligned} &\int_{\tau\le\delta/u} e^{-u\tau}(1-e^{-u^{1+\epsilon}\tau^{1+\epsilon}})\,d\mu_{\bar\phi} \ll u^{1+\epsilon}\int_{\tau\le\delta/u}\tau^{1+\epsilon}e^{-u\tau}\,d\mu_{\bar\phi} \\ &\quad \ll u^{1+\epsilon}\int_0^\infty x^{1+\epsilon}e^{-ux}\,dG(x) = -u^{1+\epsilon}\int_0^\infty x^{1+\epsilon}e^{-ux}\,d(1-G(x)) \\ &\quad = u^{2+\epsilon}\int_0^\infty x^{1-\epsilon}(1-G(x))\,e^{-ux}\,dx + u^{1+\epsilon}\int_0^\infty x^{-\epsilon}(1-G(x))\,e^{-ux}\,dx \ll u^{1+\epsilon}. \end{aligned}$$

Altogether, $I_1(u)\ll u^{1+\epsilon}$. Hence, $u\log(1/u)|I_1(u)| = o(u^2L(1/u))$. This together with the estimate for $u\log(1/u)|I_2(u)$ implies that $u\log(1/u)\Pi(u) = o(u^2L(1/u))$ and item (ii) follows. ∎

5.2 Proof of item (i) of Theorem 5.1: Stable law and non standard Gaussian

Let τ_n and b_n be as in Theorem 5.1. Using (5.2) and Corollary 5.5 (based on [AD01b]) we obtain that the Laplace transform $\mathbb{E}_{\mu_{\bar\phi}}(e^{-ub_n^{-1}\tau_n})$ converges (as $n\to\infty$) to the Laplace transform of either the stable law or of the $\mathcal{N}(0,1)$ law. The conclusion w.r.t. $\mu_{\bar\phi}$ follows from the theory of Laplace transforms (as in [F66, Ch. XIII]). The conclusion w.r.t. ν follows from [E04, Theorem 4].

5.3 Estimates required for the CLT under (H4)(ii)

For the CLT case we need the following

Proposition 5.6 *Assume* **(H1)***,* **(H2)** *and* **(H4)***(ii). Suppose that $\tau\ne h\circ F-h$, for any $h\in\mathcal{B}$. Then there exists $\sigma\ne 0$ such that $1-\lambda(u)=\tau^*u+\frac{\sigma^2}{2}u^2(1+o(1))$.*

We need to ensure that under the assumptions of Proposition 5.6, which do not require that $\tau\in\mathcal{B}$, there exists the required $\sigma\ne 0$. The argument goes by and large as [G04b, Proof of Theorem 3.7] (which works with a different Banach space) with the exception of estimating the second derivative of the eigenvalue $\tilde\lambda$ defined below. The argument in [G04b, Proof of Theorem 3.7] for estimating such a derivative does not directly apply to our setup due to: i) the two Banach spaces $\mathcal{B},\mathcal{B}_w$ at our disposal are not embedded in L^p, $p>1$; ii) the presence of $\log q(u)$ in Lemma 5.3. Our estimates below rely heavily on **(H2)** and **(H4)**(ii).

As usual, we can can reduce the proof to the case of mean zero observables. Let $\tilde\tau = \tau - \tau^*$. We recall that under **(H4)**(ii), $R(\tau h) \in \mathcal{B}$, for every in $h \in \mathcal{B}$ and the same holds for $\tilde\tau$. By **H1**(v), $(I - R)$ is invertible on the space of functions inside $\mathcal{B}$ of zero integral. Thus, as in [G04b, Proof of Theorem 3.7], we can set

$$\chi := (I - R)^{-1} R\tilde\tau \in \mathcal{B}. \tag{5.4}$$

Define $\tilde R(u) = R(e^{-u\tilde\tau})$. Clearly, $\tilde R(u)$ has the same continuity properties as $\hat R(u)$. Let $\tilde\lambda(u)$ be the associated family of eigenvalues. Recall that $\lambda(u)$ is the family of eigenvalues associated with $\hat R(u)$. Hence, $\lambda(u) = e^{u\tau^*}\tilde\lambda(u)$. As in the previous sections, let $v(u)$ be the family of associated eigenvectors normalised such that $\langle v(u), 1\rangle = 1$. The next two results are technical tools required in the proof of Proposition 5.6; the longer proof is postponed to Subsection 5.3.1.

Lemma 5.7 *Suppose that* **(H1)**–**(H3)** *and* **(H4)***(ii) hold, and recall* $\tilde\tau = \tau - \tau^*$ *Then, as* $u \to 0$,

$$\frac{d^2}{du^2}\tilde\lambda(u) + \int_Y \tilde\tau^2\, d\mu_{\bar\phi} + \left\langle \frac{d}{du}\tilde R(u)\frac{d}{du}v(u), 1\right\rangle \to 0.$$

Proof Set $\tilde\Pi(u) = \mu_{\bar\phi}(1 - e^{-u\tilde\tau})$. By the calculation used in the proof of Lemma 5.4, $1 - \tilde\lambda(u) = \tilde\Pi(u) - \langle(\tilde R(u) - \tilde R(0))(\tilde v(u) - v(0)), 1\rangle$. Differentiating twice,

$$\frac{d^2}{du^2}\tilde\lambda(u) - \frac{d^2}{du^2}\tilde\Pi(u) = \left\langle \frac{d^2}{du^2}\tilde R(u)(v(u) - v(0), 1\right\rangle + 2\left\langle \frac{d}{du}\tilde R(u)\frac{d}{du}v(u), 1\right\rangle$$
$$+ \left\langle (\tilde R(u) - \tilde R(0))\frac{d^2}{du^2}v(u), 1\right\rangle.$$

Under **(H2)**(i), arguments similar to the ones used in Lemma 5.4 together with Lemma A.1 (ii) imply that as $u \to 0$,

$$\left|\left\langle (\tilde R(u) - \tilde R(0))\frac{d^2}{du^2}v(u), 1\right\rangle\right| \ll \left\|\frac{d^2}{du^2}v(u)\|_{\mathcal{B}_w}\int_0^\infty (e^{-ut} - 1)\mu_{\bar\phi}(\omega(t - \tilde\tau))\, dt\right.$$
$$+(g_0(u) - 1)\|\frac{d^2}{du^2}v(u)\bigg\|_{\mathcal{B}_w}\int_0^\infty e^{-ut}\mu_{\bar\phi}(\omega(t - \tilde\tau))\, dt \ll u\log(1/u) = o(1).$$

Lemmas A.1 (ii) and 5.3 together with $\tilde\tau^2 \in L^1(\mu_{\bar\phi})$ imply that $|\langle\frac{d^2}{du^2}\tilde R(u)(v(u) - v(0), 1\rangle| \ll u\log(1/u) = o(1)$. Finally, $\frac{d^2}{du^2}\tilde\Pi(u) = -\int_Y \tilde\tau^2\, d\mu_{\bar\phi}(1 + o(1))$. The conclusion follows by putting the above estimates together. ∎

Lemma 5.8 *Assume the setup of of Proposition 5.6. Then*

$$\left\langle \frac{d}{du}\tilde R(u)\frac{d}{du}v(u), 1\right\rangle = -\int \tilde\tau\chi\, d\mu_{\bar\phi} + D(u),$$

where $D(u) = o(1)$ *as* $u \to 0$, *and* χ *is defined as in* (5.4).

Proof of Proposition 5.6 By Lemmas 5.8 and A.1

$$\frac{d^2}{du^2}\tilde{\lambda}(u) = -\int_Y \tilde{\tau}^2\, d\mu_{\bar{\phi}} - 2\int \tilde{\tau}\chi\, d\mu_{\bar{\phi}} + T(u), \tag{5.5}$$

where $T(u) \to 0$, as $u \to 0$. Hence, $-\frac{d^2}{du^2}\tilde{\lambda}(u)|_{u=0} = \int_Y \tilde{\tau}^2\, d\mu_{\bar{\phi}} + 2\int \tilde{\tau}\chi\, d\mu_{\bar{\phi}}$ and we can set $\sigma^2 = \int_Y \tilde{\tau}^2\, d\mu_{\bar{\phi}} + 2\int \tilde{\tau}\chi\, d\mu_{\bar{\phi}}$. From here on the proof goes word for word as [G04b, Proof of Theorem 3.7], which shows that given the previous formula for σ, the only possibility for $\sigma = 0$ is when $\tilde{\tau} = h - h \circ F$, for some density $h \in \mathcal{B}$. This is ruled out by assumption.

To conclude recall that $\lambda(u) = e^{u\tau^*}\tilde{\lambda}(u)$. By (5.5), $1 - \tilde{\lambda}(u) = \frac{\sigma^2}{2}u^2(1 + o(1))$, as required. ∎

5.3.1 Proof of Lemma 5.8

Proof of Lemma 5.8 Although $\frac{d}{du}\tilde{R}(u)$ is bounded in $\mathcal{B}_w$, a priori $\frac{d}{du}v(u)$ is not; this a consequence of Lemma A.1 (i). However, we argue that due to **(H4)**(ii), $\frac{d}{du}v(u)$ is well-defined at 0 and as such, we get the claimed formula for $\langle \frac{d}{du}\tilde{R}(u)\frac{d}{du}v(u), 1\rangle$. Let $W(u) := \frac{d}{du}\tilde{R}(u)$. Since $\tilde{R}(u) = R(e^{-u\tilde{\tau}})$, we have that for any $h \in \mathcal{B}$,

$$W(0)h := \left.\frac{d}{du}\tilde{R}(u)\right|_{u=0} h = -R(\tilde{\tau}h). \tag{5.6}$$

By **(H4)**(ii), $W(0)h \in \mathcal{B}$. Recall that $P(u)$ is the eigenprojection for $\hat{R}(u)$, so for $\tilde{R}(u)$ as well. For δ small enough (independent of u), we can write

$$P(u) = \int_{|\xi-1|=\delta} (\xi I - \tilde{R}(u))^{-1}\, d\xi.$$

Recall that $v(u) = \frac{P(u)1}{\langle P(u)1, 1\rangle} = \frac{P(u)1}{m_{\bar{\phi}}(u)1}$; in particular, $\mu_{\bar{\phi}}(u)1 = \langle P(u)1, 1\rangle$. Compute

$$\frac{d}{du}v(u) = \frac{\frac{d}{du}P(u)1}{\mu_{\bar{\phi}}(u)1} - \frac{P(u)1}{(\mu_{\bar{\phi}}(u)1)^2}\frac{d}{du}\mu_{\bar{\phi}}(u)1. \tag{5.7}$$

Also, compute that by (5.6)

$$\begin{aligned}\frac{d}{du}P(u) &= \int_{|\xi-1|=\delta} (\xi I - \tilde{R}(u))^{-1} W(u) (\xi I - \tilde{R}(u))^{-1}\, d\xi \\ &= \int_{|\xi-1|=\delta} (\xi I - R)^{-1} W(0) (\xi I - R)^{-1}\, d\xi + Z_0(u) \\ &= -\int_{|\xi-1|=\delta} (\xi I - R)^{-1} R\tilde{\tau} (\xi I - R)^{-1}\, d\xi + Z_0(u),\end{aligned} \tag{5.8}$$

where the first term is well defined in $\mathcal{B}$ by **(H4)**(ii) and

$$\begin{aligned}\|Z_0(u)\|_{\mathcal{B}_w} \ll &\left\| \int_{|\xi-1|=\delta} (\xi I - R)^{-1}\Big(W(u) - W(0)\Big)(\xi I - R)^{-1}\, d\xi \right\|_{\mathcal{B}_w} \\ &+ \left\| \int_{|\xi-1|=\delta} \Big((\xi I - R(u))^{-1} - (\xi I - R)^{-1}\Big) W(u)(\xi I - R)^{-1}\, d\xi \right\|_{\mathcal{B}_w} \\ &+ \left\| \int_{|\xi-1|=\delta} (\xi I - R)^{-1} W(u)\Big((\xi I - R(u))^{-1} - (\xi I - R)^{-1}\Big)\, d\xi \right\|_{\mathcal{B}_w}.\end{aligned}$$

By Lemma A.1 (ii), $\frac{d^2}{du^2}\tilde{R}(u)$ is bounded in $\|\cdot\|_{\mathcal{B}_w}$; so, $\|W(u) - W(0)\|_{\mathcal{B}_w} \ll u$. Hence, each one of three terms of the previous displayed equation contains an $O(u)$ factor. By the argument recalled in the proof of Lemma 5.3, for any ξ such that $|\xi - 1| = \delta$, the integrand of each one of these three terms is $o(1)$. Altogether, $\|Z_0(u)\|_{\mathcal{B}_w} = o(1)$. Also, by (5.8) and Lemma 5.9 below,

$$\frac{d}{du} P(u)1 = -\chi - \xi_0 \int_Y \chi\, d\mu_{\bar{\phi}} + Z_0(u), \tag{5.9}$$

for some $\xi_0 \in \mathbb{C}$ and $\chi = (I - R)^{-1} R\tilde{\tau}$ as in (5.4). Plugging (5.9) into (5.7),

$$\begin{aligned}\frac{d}{du} v(u) &= \frac{1}{\mu_{\bar{\phi}}(0)1}\left(-\chi - \xi_0 \int_Y \chi\, d\mu_{\bar{\phi}}\right) - \frac{P(0)1}{(\mu_{\bar{\phi}}(0)1)^2}\left\langle -\chi - \xi_0 \int_Y \chi\, d\mu_{\bar{\phi}}, 1\right\rangle + Z_1(u) + Z_0(u) \\ &= -\chi + \tilde{\xi} \int_Y \chi\, d\mu_{\bar{\phi}} + Z_1(u) + Z_0(u),\end{aligned}$$

where $\tilde{\xi}$ is linear combination of ξ_0 and

$$\begin{aligned}\|Z_1(u)\|_{\mathcal{B}_w} \ll &\frac{|\mu_{\bar{\phi}}(u)1 - \mu_{\bar{\phi}}(0)1|}{m_{\bar{\phi}}(u)1} + \frac{\|P(u)1 - P(0)1\|_{\mathcal{B}_w}}{(m_{\bar{\phi}}(u)1)^2} + \left|\frac{1}{\mu_{\bar{\phi}}(u)1} - \frac{1}{\mu_{\bar{\phi}}(0)1}\right| \\ &+ \left|\frac{1}{(\mu_{\bar{\phi}}(u)1)^2} - \frac{1}{(\mu_{\bar{\phi}}(0)1)^2}\right|.\end{aligned}$$

By Lemma 5.3, $\|P(u) - P(0)\|_{\mathcal{B}_w} \ll u\log(1/u)$ and as a consequence, $|\mu_{\bar{\phi}}(u)1 - \mu_{\bar{\phi}}(0)1| \ll u\log(1/u)$. Thus, $\|Z_1(u)\|_{\mathcal{B}_w} = o(1)$. Recalling that $\|Z_0(u)\|_{\mathcal{B}_w} = o(1)$, we have

$$\frac{d}{du} v(u) = -\chi + \tilde{\xi} \int_Y \chi\, d\mu_{\bar{\phi}} + Z(u),$$

where $\|Z(u)\|_{\mathcal{B}_w} = o(1)$. Using this expression for $\frac{d}{du}v(u)$, we obtain

$$\begin{aligned}\left\langle \frac{d}{du}\tilde{R}(u)\frac{d}{du}v(u), 1\right\rangle &= -\left\langle R\tilde{\tau}\frac{d}{du}v(u), 1\right\rangle + \left\langle (W(u) - W(0))\,\frac{d}{du}v(u), 1\right\rangle \\ &= -\int_Y R\tilde{\tau}\chi\, d\mu_{\bar{\phi}} + \tilde{\xi}\int \tilde{\tau}\, d\mu_{\bar{\phi}} \int_Y \chi\, d\mu_{\bar{\phi}} + \left\langle RZ(u), 1\rangle + \langle (W(u) - W(0))\,\frac{d}{du}v(u), 1\right\rangle \\ &= -\int_Y \tilde{\tau}\chi\, d\mu_{\bar{\phi}} + \langle RZ(u), 1\rangle + \left\langle (W(u) - W(0))\,\frac{d}{du}v(u), 1\right\rangle,\end{aligned}$$

where we have used $\int \tilde{\tau}\, d\mu_{\bar{\phi}} = 0$. Finally, using **(H2)**, we compute that

$$|\langle RZ(u), 1\rangle| \ll \|Z(u)\|_{\mathcal{B}_w} \int_0^\infty \int_Y \omega(t-\tilde{\tau})\, d\mu_{\bar{\phi}}\, dt \ll \|Z(u)\|_{\mathcal{B}_w} = o(1).$$

Similarly, $|\langle (W(u) - W(0)\, w(u), 1\rangle| = o(1)$, since we already know that $\|W(u) - W(0)\|_{\mathcal{B}_w} = o(1)$. Thus,

$$\left\langle \frac{d}{du}\tilde{R}(u)\frac{d}{du}v(u), 1\right\rangle = -\int_Y \tilde{\tau}\chi\, d\mu_{\bar{\phi}} + o(1),$$

ending the proof. ∎

Lemma 5.9 *Assume the setup and notation of Lemma 5.8. Then there exists $\xi_0 \in \mathbb{C}$ with such that*

$$\int_{|\xi-1|=\delta} (\xi I - R)^{-1} R\tilde{\tau}\, (\xi I - R)^{-1} 1\, d\xi = \chi + \xi_0 \int_Y \chi\, d\mu_{\bar{\phi}}.$$

Proof We note that for each ξ such that $\xi - 1| = \delta$, $R\tilde{\tau}\,(\xi I - R)^{-1}1$ is well defined in $\mathcal{B}$ by **(H4)**(ii) and that $\int_{|\xi-1|=\delta} R\tilde{\tau}\,(\xi I - R)^{-1} 1\, d\xi = 0$. Hence, $\int_{|\xi-1|=\delta}(I-R)^{-1}R\tilde{\tau}\,(\xi I - R)^{-1}1\, d\xi \in \mathcal{B}$.

By the first resolvent identity and the Mean Value Theorem,

$$\begin{aligned}
&\int_{|\xi-1|=\delta} (\xi I - R)^{-1} R\tilde{\tau}\, (\xi I - R)^{-1} 1\, d\xi \\
&= \int_{|\xi-1|=\delta} (I - R)^{-1} R\tilde{\tau}\, (\xi I - R)^{-1} 1\, d\xi + \int_{|\xi-1|=\delta} \Big((\xi I - R)^{-1} - (I-R)^{-1}\Big) R\tilde{\tau}\, (\xi I - R)^{-1} 1\, d\xi \\
&= \chi P(0) 1 + \int_{|\xi-1|=\delta} (1-\xi)(\xi I - R)^{-1} \chi\, (\xi I - R)^{-1} 1\, d\xi \\
&= \chi + \xi_1 \int_{|\xi-1|=\delta} (\xi I - R)^{-1} \chi\, (\xi I - R)^{-1} 1\, d\xi,
\end{aligned}$$

where ξ_1 is a complex constant with $0 < |\xi_1| \le \delta$.

Write $(\xi I - R)^{-1} - I = -(\xi I - R)^{-1}((\xi - 1)I - R)$. The Mean Value Theorem gives

$$\begin{aligned}
\int_{|\xi-1|=\delta} (\xi I - R)^{-1} \chi\, (\xi I - R)^{-1} 1\, d\xi &= \int_{|\xi-1|=\delta} (\xi I - R)^{-1} a \Big(I + (\xi I - R)^{-1} - I\Big) 1\, d\xi \\
&= \int_{|\xi-1|=\delta} (\xi I - R)^{-1} \chi\, d\xi - \int_{|\xi-1|=\delta} (\xi I - R)^{-1} \chi\, (\xi I - R)^{-1}((\xi - 1)I - R) 1\, d\xi \\
&= P(0)\chi - \int_{|\xi-1|=\delta} (\xi I - R)^{-1} \chi\, (\xi I - R)^{-1} (\xi - 2)\, d\xi \\
&= \int_Y \chi\, d\mu_{\bar{\phi}} - (\xi_2 - 2) \int_{|\xi-1|=\delta} (\xi I - R)^{-1} \chi\, (\xi I - R)^{-1} 1\, d\xi,
\end{aligned}$$

for some $\xi_2 \in \mathbb{C}$ with $|\xi_2 - 1| \le \delta$. Thus,

$$(\xi_2 - 1)\int_{|\xi-1|=\delta} (\xi I - R)^{-1}\chi\,(\xi I - R)^{-1}1\,d\xi = \int_Y \chi\,d\mu_{\bar\phi}.$$

Since $\int_Y \chi\,d\mu_{\bar\phi} > 0$, we have $\xi_2 \ne 1$. The conclusion follows with $\xi_0 = \xi_1(\xi_2 - 1)^{-1}$. ∎

5.4 Proof of item (ii) of Theorem 5.1: CLT

Let τ_n and b_n be as in Theorem 5.1 under **(H4)**(ii). Proposition 5.6 shows that the Laplace transform $\mathbb{E}_{\mu_{\bar\phi}}(e^{-ub_n^{-1}\tau_n})$ converges (as $n \to \infty$) to the Laplace transform of $\mathcal{N}(0,\sigma^2)$. The conclusion w.r.t. $\mu_{\bar\phi}$ follows from the theory of Laplace transforms (as in [F66, Ch. XIII]). The conclusion w.r.t. ν follows from [E04, Theorem 4].

6 Limit laws for the flow f_t

The result below generalises Theorem 5.1 to the flow f_t. Given an observable $g : \mathcal{M} \to \mathbb{R}$, define $\bar g : Y \to \mathbb{R}$ as in (3.2) and let $g_T = \int_0^T g \circ f_t\,dt$.

Proposition 6.1 *Assume* **(H1)***. Let $g : \mathcal{M} \to \mathbb{R}$ and let $g^* = \int_Y \bar w\,d\mu_{\bar\phi}$. Suppose that the twisted operator $\hat R_g(u)v = R(e^{-ug}v)$ satisfies* **(H2)** *(with τ replaced by g). Then the following hold as $T \to \infty$, w.r.t. $\mu_{\bar\phi}$ (or any probability measure $\nu \ll \mu_{\bar\phi}$).*

(i) Assume **(H4)***(i) and $\mu_{\bar\phi}(\bar g > T) \sim \mu_{\bar\phi}(\tau > T)$.*

When $\beta < 2$, set $b(T)$ such that $\frac{T\ell(b(T))}{b(T)^\beta} \to 1$. Then $\frac{1}{b(T)}(g_T - g^*\cdot T) \to^d G_\beta$, where G_β is a stable law of index β.

When $\beta = 2$, set $b(T)$ such that $\frac{T\tilde\ell(b(T))}{b(T)^2} \to c > 0$. then $\frac{1}{b(T)}(g_T - g^*\cdot T) \to^d \mathcal{N}(0,c)$.

(ii) Suppose that **(H4)***(ii) holds. If $\bar g \notin B$, we assume that $R\bar w \in \mathcal{B}$. We further assume that $\bar g \ne h - h\circ F$ with $h \in \mathcal{B}$. Then there exists $\sigma \ne 0$ such that $\frac{1}{\sqrt{T}}(g_T - g^*\cdot T) \to^d \mathcal{N}(0,\sigma^2)$.*

Remark 6.2 We note that the assumption $\mu_{\bar\phi}(\bar g > T) \sim \mu_{\bar\phi}(\tau > T)$ is satisfied for all g bounded above and uniformly bounded away from 0 and such that $\bar g$ is C^α on the elements of the (Markov) partition $\mathcal{A}$.

In Theorem 8.2 below, we consider potentials of the form $\bar g = C' - C\tau^\kappa(1 + o(1))$, $\kappa \in (0,\beta)$ and obtain specific form of (i) and/or (ii) for different values of κ.

Proof We use that $f_t : \mathcal{M} \to \mathcal{M}$ can be represented as a suspension flow $F_t : Y^\tau \to Y^\tau$. Under the present assumptions, $\bar g$ satisfies all the assumptions of Theorem 5.1 (with τ replaced by $\bar g$).

Let $\bar g_n = \sum_{j=0}^{n-1} \bar g \circ F^j$ and recall $\tau_n = \sum_{j=0}^{n-1} \tau \circ F^j$. Theorem 5.1 (i) applies to $\bar g_n$; the argument goes word for word as in the proof of Theorem 5.1 (i) for τ_n. Under the present assumptions on

$\bar{g}$, item (ii) of Theorem 5.1 applies to $\bar{g}_n$ with the argument used in the proof of Proposition 5.6 applying word for word with $\bar{g}$ instead of τ.

Item (i) follows from this together with Lemma 6.3 below (correspondence between stable laws/non standard Gaussian for the base map F and suspension flow. Item (ii) follows in the same way using [MTo04, Theorem 1.3] instead of Lemma 6.3. ∎

The next result is a version of [S06, Theorem 7] (generalising [MTo04, Theorem 1.3]) for suspension flows which holds in a very general setup; in particular, it is totally independent of method used to prove limit theorems for the base map.

Lemma 6.3 *Assume $\int_Y \tau d\mu_{\bar{\phi}} < \infty$ and let $g \in L^q(\mu_\phi)$, for $q > 1$. Suppose that there exists a sequence $b_n = n^{-\rho}\ell(n)$ for $\rho \in (1, 2]$ and ℓ a slowly varying function such that $b_n^{-1}(\tau_n - n\mu_\phi(Y))$ is tight on (Y, μ_ϕ). Then the following are equivalent:*

(a) $b_n^{-1}\bar{g}_n$ converges in distribution on $(Y, \mu_{\bar{\phi}})$.

(b) $b(T)^{-1}g_T$ converges in distribution on (Y^τ, μ_ϕ), where $b(T) = T^{-\rho}\ell(T)$.

Proof The fact that (b) implies (a) is obvious. The implication from (a) to (b) is contained in the proofs of [MTo04, Theorem 1.3] for the standard CLT case and in [S06, Theorem 7] for the stable and nonstandard CLT. ∎

7 Asymptotics of $\mathcal{P}(\phi + s\psi)$ for the flow f_t

In this section and the next we shall assume that $F : Y \to Y$ is Markov, which allows us to express our results in terms of pressure. [S06, Theorem 8] gives a link between the shape of the pressure of a given discrete time finite measure dynamical system and an induced version (this was extended in [BTT18] to some infinite measure settings). Here we give a version of this result in the abstract setup of Section 4 along with suitable assumptions on a second potential ψ. Note that our assumptions are not directly comparable with those in [S06, Theorem 8].

Throughout this section, we let $\phi : \mathcal{M} \to \mathbb{R}$ and μ_ϕ be as Section 4. In particular, we recall that $\tau^* = \int_Y \tau \, d\mu_{\bar{\phi}} < \infty$ and assume that the conformal measure $m_{\bar{\phi}}$ satisfies **(H1)**. This ensures that $\mu_{\bar{\phi}}$ is an equilibrium measure for $(F, \bar{\phi})$ and μ_ϕ is an equilibrium measure for (f_t, ϕ). We consider the potential $\psi : \mathcal{M} \to \mathbb{R}$ and its induced version $\bar{\psi} : Y \to \mathbb{R}$ defined in (3.2) and require:

(H5) There exists $C, C' > 0$ such that for $y \in Y$, $\bar{\psi}(y) = C' - \psi_0(y)$, where $C^{-1}\tau^\kappa(y) \le \psi_0(y) \le C\tau^\kappa(y)$, for some $\kappa > 0$. For $\kappa > 1$ we further require $\int_Y \tau^\kappa \, d\mu_{\bar{\phi}} < \infty$.

Given $\psi : \mathcal{M} \to \mathbb{R}$ and its induced version $\bar{\psi}$ on Y, we define the 'doubly perturbed' operator

$$\hat{R}(u, s)v = R(e^{-u\tau}e^{s\bar{\psi}}v).$$

Under **(H5)**, we require the following extended version of **(H3)**.

(H6) There exist a range of κ, $\sigma_1 > 1$, constants $C_0, C_1, \delta > 0$ such that for all $h \in \mathcal{B}$, for all $n \in \mathbb{N}$ and $u \geq 0$, $s \in [0, \delta)$,

$$\|\hat{R}(u,s)^n h\|_{\mathcal{B}_w} \leq C_1 \|h\|_{\mathcal{B}_w}, \quad \|\hat{R}(u,s)^n h\|_{\mathcal{B}} \leq C_0 \sigma_1^{-n} \|h\|_{\mathcal{B}} + C_1 \|h\|_{\mathcal{B}_w}.$$

To ensure that we can understand the pressure $\mathcal{P}(\overline{\phi + s\psi})$ in terms of the eigenvalues of $\hat{R}(u,s)$, under **(H1)** (in particular, for α as in **(H1)**(i)) and **(H5)** we require that

(H7) There exist $\gamma \in (0,1]$ with $\gamma < 1$ for $\kappa > 1$ and $C_2, C_3, \delta > 0$ such that for all $u \geq 0$, $s \in (0, \delta)$, $h \in \mathcal{B}$ and every element a of the (Markov) partition $\mathcal{A}$ we have:

$$\|(e^{-u\tau + s\bar{\psi}} - 1)1_a h\|_{\mathcal{B}_w} \leq \left(C_2 s^\gamma \sup_a \psi_0^\gamma + C_3 u^\gamma \sup_a \tau^\gamma \right) \|h\|_{\mathcal{B}_w}. \tag{7.1}$$

In (7.1), multiplication by 1_a means that we restrict $\tau, \bar{\psi}$ to a. The main result of this section reads as follows.

Theorem 7.1 *Assume* **(H1)***(ii) (for $m_{\bar{\phi}}$),* **(H1)***(iv) (for $\mu_{\bar{\phi}}$) and suppose that* **(H2)** *holds. Suppose that $\psi : \mathcal{M} \to \mathbb{R}$ satisfies* **(H5)** *with $C' > C \int \tau^\kappa \, d\mu_{\bar{\phi}}$. Assume* **(H6)** *and* **(H7)**. *Then*

$$\mathcal{P}(\phi + s\psi) = \frac{1}{\tau^*} \mathcal{P}(\overline{\phi + s\psi})(1 + o(1)) \text{ as } s \to 0^+.$$

7.1 Technical tools, family of eigenvalues of $\hat{R}(u,s)$

Note that $\hat{R}(u,0)v = \hat{R}(u)v$ and recall from Subsection 5.1 that under **(H1)**–**(H3)**, the family of eigenvalues $\lambda(u)$ is well-defined on $[0, \delta_0)$ with $\lambda(0) = 1$. Recall that **(H6)** and **(H7)** hold for some $\delta > 0$. Throughout the rest of this work we let $\delta_1 = \min\{\delta, \delta_0\}$.

Lemma 7.2 *Assume* **(H1)**, **(H2)**, **(H5)** *and* **(H7)**. *Then for all $s \in (0, \delta_1)$, there exists $c > 0$ such that*

$$\|\hat{R}(u,s) - \hat{R}(u,0)\|_{\mathcal{B} \to \mathcal{B}_w} \leq c s^\gamma.$$

Proof Using (4.1), write

$$\hat{R}(u,s) - \hat{R}(u,0) = g_0(u) \int_0^\infty R(\omega(t-\tau)(e^{s\bar{\psi}} - 1)) e^{-ut} \, dt. \tag{7.2}$$

By **(H7)**, $(e^{s\bar{\psi}} - 1)1_a \in \mathcal{B}$ for any element a in the (Markov) partition $\mathcal{A}$. Without loss of generality we assume that $\operatorname{supp} \omega(t-\tau)$ is a subset of a finite union $\cup_{a \in A_t} a$ of elements in $\mathcal{A}$. Thus, $\|(e^{s\bar{\psi}} - 1)1_{\operatorname{supp}\, \omega(t-\tau)}\|_{\mathcal{B}_w} \ll \max_{a \in A_t} \|(e^{s\bar{\psi}} - 1)1_a\|_{\mathcal{B}_w}$.

Note that

$$\|R(\omega(t-\tau)(e^{s\bar{\psi}} - 1))v\|_{\mathcal{B}_w} \leq C \|R(\omega(t-\tau))\|_{\mathcal{B} \to \mathcal{B}_w} \max_{a \in A_t} \|(e^{s\bar{\psi}} - 1)1_a\|_{\mathcal{B}_w}.$$

By **(H5)** and **(H7)** (with $u = 0$),

$$\|(e^{s\bar{\psi}} - 1)1_{\text{supp }\omega(t-\tau)}\|_{\mathcal{B}_w} \ll \|(e^{s\bar{\psi}} - 1)1_a\|_{\mathcal{B}_w} \ll s^\gamma \sup_a \psi_0^\gamma \ll s^\gamma t^{\kappa\gamma}.$$

Recall that $\gamma < 1$ if $\kappa > 1$. Putting the above together and recalling $R(\omega(t-\tau)) = M(t)$, we obtain that for any $\epsilon \in (0, 1-\gamma)$,

$$\|\hat{R}(u,s) - \hat{R}(u,0)\|_{\mathcal{B}\to\mathcal{B}_w} \ll s^\gamma \int_0^\infty t^{\kappa\gamma}\|M(t)\|_{\mathcal{B}\to\mathcal{B}_w}\, dt \ll s^\gamma \int_0^\infty t^{\kappa-\epsilon}\|M(t)\|_{\mathcal{B}\to\mathcal{B}_w}\, dt.$$

Proceeding as in the proof of Lemma 5.2, in particular using (5.1), we have $\int_0^\infty t^{\kappa-\epsilon}\|M(t)\|_{\mathcal{B}\to\mathcal{B}_w}\, dt \ll \int_0^\infty t^{\kappa-\epsilon}(S(t+1) - S(t))\, dt$, where $S(t) = \int_t^\infty \|M(x)\|_{\mathcal{B}\to\mathcal{B}_w}\, dx$. Thus,

$$\int_0^\infty t^{\kappa-\epsilon}\|M(t)\|_{\mathcal{B}\to\mathcal{B}_w}\, dt \ll \int_0^\infty t^{\kappa-\epsilon} S(t+1)\, dt - \int_0^\infty t^{\kappa-\epsilon} S(t)\, dt \ll \int_1^\infty t^{\kappa-1-\epsilon} S(t)\, dt$$

By assumption, $\tau \in L^\kappa(\mu_{\bar{\phi}})$, so $t^\kappa \mu_{\bar{\phi}}(\tau > t) \le \int_{\tau\ge t} \tau^\kappa\, d\mu_{\bar{\phi}} \le \int_Y \tau^\kappa\, d\mu_{\bar{\phi}} < \infty$. By **(H2)**(ii), $S(t) \ll \mu_{\bar{\phi}}(\tau > t)$. Thus,

$$\int_0^\infty t^{\kappa-\epsilon}\|M(t)\|_{\mathcal{B}\to\mathcal{B}_w}\, dt \ll \int_1^\infty t^{\kappa-1-\epsilon}\mu_{\bar{\phi}}(\tau > t)\, dt \ll \int_1^\infty t^{-(1+\epsilon)}\, dt < \infty, \tag{7.3}$$

which ends the proof. ∎

Lemmas 5.2 and 7.2 ensure that $(u,s) \to \hat{R}(u,s)$ is analytic in u and continuous in s, for all $s \in [0, \delta_0)$, when viewed as an operator from $\mathcal{B}$ to $\mathcal{B}_w$ with

$$\|\hat{R}(u,s) - \hat{R}(0,0)\|_{\mathcal{B}\to\mathcal{B}_w} \ll s^\gamma + q(u),$$

where $q(u)$ is as defined in Lemma 5.2.

Recall that $\lambda(u)$ is well-defined for all $u \in [0, \delta_1)$ with $\delta_1 = \min\{\delta, \delta_0\}$. By **(H6)**, there exists a family of eigenvalues $\lambda(u,s)$ well-defined for $u, s \in [0, \delta_1)$ with $\lambda(0,0) = 1$. Throughout, let $v(u,s)$ be a family of eigenfunctions associated with $\lambda(u,s)$. The next result gives the continuity properties of $v(u,s)$.

Lemma 7.3 *Assume* **(H1)**, **(H2)**, **(H5)** *and* **(H6)**. *Then there exists $\delta_2 \le \delta_1$ such that for all $u, s \in [0, \delta_2)$, there exists $c > 0$ such that*

$$\|v(u,s) - v(u,0)\|_{\mathcal{B}_w} \le cs^\gamma \log(1/s).$$

Proof This follows from Lemma 7.2 and **(H6)** (which holds for $u \ge 0$ and $s \in [0, \delta_1)$) together with the argument recalled in the proof of Lemma 5.3. ∎

We recall $\Pi(u) = \mu_{\bar{\phi}}(1 - e^{-u\tau})$and set $\Pi_0(s) = \mu_{\bar{\phi}}(1 - e^{s\bar{\psi}})$. The result below gives the asymptotic behaviour and continuity properties of $\lambda(u,s)$.

Lemma 7.4 *Assume* **(H1)**, **(H2)**, **(H5)**, **(H6)** *(for some range of* κ*) and* **(H7)**. *Then as* $u, s \to 0$,

$$1 - \lambda(u,s) = \Pi(u) + \Pi_0(s) + D(u,s),$$

where $|\frac{d}{du}D(u,s)| \ll (\log(1/u))^{\kappa\gamma+2}(s^\gamma + q(u)) + \mu_{\bar\phi}(\tau > \log(1/u))$ *with* $q(u)$ *as defined in Lemma 5.2.*

Proof Proceeding as in the proof of Lemma 5.4, write $\lambda(u,s)v(u,s) = \hat R(u,s)v(u,s)$ for all $u, s \in [0, \delta_1)$ with $\lambda(0,0) = 1$ and $v(0,0) = \mu_{\bar\phi}$. Normalising such that $\langle v(u,s), 1\rangle = 1$,

$$1 - \lambda(u,s) = \Pi(u) + \Pi_0(s) - W(u,s) - V(u,s) \quad \text{for}$$
$$W = \int_Y (e^{-u\tau} - 1)(e^{s\bar\psi} - 1)\, d\mu_{\bar\phi} \quad \text{and} \quad V(u,s) = \langle (\hat R(u,s) - \hat R(0,0))(v(u,s) - v(0,0)), 1\rangle.$$

We first deal with $V(u,s)$. Let $Q(u,s) := \langle (\hat R(u,0) - \hat R(0,0))(v(u,s) - v(0,0)), 1\rangle$. Using (7.2),

$$V(u,s) = g_0(u) \int_0^\infty e^{-ut}\langle (e^{s\bar\psi} - 1)\omega(t-\tau)[v(u) - v(0)], 1\rangle\, dt + Q(u,s).$$

By the argument used in the proof of Lemma 5.4,

$$|Q(u,s)| \ll \|v(u,s) - v(0,0)\|_{\mathcal{B}_w} \Pi(u).$$

By **(H2)**(i),

$$\langle (e^{s\bar\psi} - 1)\omega(t-\tau)[v(u) - v(0)], 1\rangle \ll \|(e^{s\bar\psi} - 1)(v(u,s) - v(0,0)\|_{\mathcal{B}_w} \mu_{\bar\phi}(\omega(t-\tau)),$$

which together with **(H7)** (with $u = 0$) gives

$$\langle (e^{s\bar\psi} - 1)\omega(t-\tau)[v(u) - v(0)], 1\rangle \ll s^\gamma t^{\kappa\gamma} \|v(u,s) - v(0,0)\|_{\mathcal{B}_w} \mu_{\bar\phi}(\omega(t-\tau)).$$

Proceeding as in the proof of Lemma 5.4 and using that $\int_Y \tau^\kappa\, d\mu_{\bar\phi} < \infty$,

$$\Big| \int_0^\infty e^{-ut}\langle (e^{s\bar\psi} - 1)\omega(t-\tau)[v(u,s) - v(0,0)], 1\rangle\, dt \Big|$$
$$\ll s^\gamma \|v(u,s) - v(0,0)\|_{\mathcal{B}_w} \int_Y \tau^{\kappa\gamma} e^{-u\tau}\, d\mu_{\bar\phi} \ll s^\gamma \|v(u,s) - v(0,0)\|_{\mathcal{B}_w}.$$

By a similar argument,

$$\Big| \int_0^\infty (e^{-ut} - 1)\langle (e^{s\bar\psi} - 1)\omega(t-\tau)[v(u,s) - v(0,0)], 1\rangle\, dt \Big| \ll s^\gamma \|v(u,s) - v(0,0)\|_{\mathcal{B}_w}.$$

By Lemmas 5.3 and 7.3, $\|v(u,s) - v(0,0)\|_{\mathcal{B}_w} \ll s^\gamma \log(1/s) + q(u)|\log q(u)|$. This together with the previous two displayed equations gives $|V(u,s)| \ll s^\gamma(s^\gamma \log(1/s) + q(u)|\log q(u)|)$.

We continue with $|\frac{d}{du}V(u,s)|$. Compute that

$$\frac{d}{du}V(u,s) = \left\langle \frac{d}{du}\hat{R}(u,s)(v(u,s)-v(0,0)), 1 \right\rangle + \left\langle \left(\hat{R}(u,s)-\hat{R}(0,0)\right)\frac{d}{du}v(u,s), 1 \right\rangle.$$

By **(H5)**, **(H7)** and the argument used above in estimating $V(u,s)$, we get

$$\begin{aligned}\left|\left\langle \frac{d}{du}\hat{R}(u,s)(v(u,s)-v(0,0)), 1 \right\rangle\right| &\ll \left|\left\langle \frac{d}{du}\hat{R}(u,0)(v(u,s)-v(0,0)), 1 \right\rangle\right| \\ &\ll \|v(u,s)-v(0,0)\|_{\mathcal{B}_w}\left|\int_0^\infty t\omega(t-\tau)e^{-ut}\,dt\right| \\ &\ll \|v(u,s)-v(0,0)\|_{\mathcal{B}_w} \ll \log(1/u)(s+q(u)).\end{aligned}$$

By **(H5)**, **(H7)** and Lemma A.1 (i), $\|\frac{d}{du}v(u,s)\|_{\mathcal{B}_w} \ll \|\frac{d}{du}v(u,0)\|_{\mathcal{B}_w} \ll \log(1/u)$. Thus,

$$\left|\left\langle (\hat{R}(u,s)-\hat{R}(0,0))\frac{d}{du}v(u,s), 1 \right\rangle\right| \ll (s^\gamma + q(u))\log(1/u)$$

and $|\frac{d}{du}V(u,s)| \ll (s^\gamma + q(u))\log(1/u)$. We briefly estimate $W(u,s)$. Compute that

$$\begin{aligned}\left|\frac{d}{du}W(u,s)\right| &\ll \int_{\{\tau<\log(1/u)\}} \tau e^{-\tau}|1-e^{s\bar{\psi}}|\,d\mu_{\bar{\phi}} + \int_{\{\tau\ge\log(1/u)\}} \tau\,d\mu_{\bar{\phi}} \\ &\ll s^\gamma \int_{\{\tau<\log(1/u)\}} \tau^{\kappa\gamma+1}\,d\mu_{\bar{\phi}} + \mu_{\bar{\phi}}(\{\tau \ge \log(1/u)\}) \\ &\ll s^\gamma(\log(1/u))^{\kappa\gamma+2} + \mu_{\bar{\phi}}(\tau \ge \log(1/u)).\end{aligned}$$

The conclusion follows with $D(u,s) = -(W(u,s)+V(u,s))$. ∎

For a further technical result exploiting **(H7)** we introduce the following notation. For $u,s\ge 0$, set $g = s\bar{\psi} - u\tau$, so $\hat{R}(u,s) = R(e^g)$. For $N>1$ and $x\in[a]$ define the 'lower' and 'upper' flattened versions of g as

$$g_N^-(x) = \begin{cases} g(x) & \text{if } \tau(y)\le N \text{ for all } y\in a, \\ \inf_{y\in[a]} g(y) & \text{otherwise;}\end{cases}$$

and

$$g_N^+(x) = \begin{cases} g(x) & \text{if } \tau(y)\le N \text{ for all } y\in a, \\ \sup_{y\in[a]} g(y) & \text{otherwise.}\end{cases}$$

Since ψ is bounded above by **(H5)**, monotonicity of the pressure function gives

$$\mathcal{P}(\bar{\phi}+g_N^-) \le \mathcal{P}(\bar{\phi}+g) \le \mathcal{P}(\bar{\phi}+g_N^+) \le \mathcal{P}(\bar{\phi}) + s\,\sup\bar{\psi} < \infty \tag{7.4}$$

for all $u,s\ge 0$. For fixed $u,s\in[0,\delta_0)$, set $\hat{R}_N^\pm(u,s) = R(e^{g_N^\pm})$. By **(H6)**, the associated eigenvalues $\lambda_N^\pm(u,s)$ exist for all N and $u,s\in[0,\delta_0)$.

Lemma 7.5 *Assume* **(H2)**, **(H5)**, **(H6)** *and* **(H7)**. *Then for fixed* $u, s \in [0, \delta_0)$,

$$\lim_{N\to\infty} |\lambda(u,s) - \lambda_N^{\pm}(u,s)| \to 0.$$

The reason to introduce $g_N^{\pm}$ is that, unlike g, the potentials $g_N^{\pm}$ have summable variation, and therefore, $P(\bar{\phi} + g_N^{\pm}) = \log \lambda_N^{\pm}(u,s)$ by [S99, Lemma 6] (where [S99, Theorem 3] is used to equate Gurevich pressure in [S99, Lemma 6] and variational pressure in this paper, and then [S99, Theorem 4] together with the existence of the eigenfunctions $v_N^{\pm}(u,s)$ and eigenmeasures $m_N^{\pm}(u,s)$ of $R_N^{\pm}(u,s)$ and its dual, to conclude that $\phi + g_N^{\pm}$ are positively recurrent). From this, together with (7.4) and Lemma 7.5, we conclude

$$\mathcal{P}(\overline{\phi + s\psi - u}) = \log \lambda(u,s). \tag{7.5}$$

Proof of Lemma 7.5 Assume $u \neq 0$ and/or $s \neq 0$. Proceeding similarly to the argument of Lemma 7.2, we write

$$\hat{R}(u,s) - \hat{R}_N^{\pm}(u,s) = \int_0^\infty R(\omega(t-\tau)(e^g - e^{\tilde{g}_N}))\,dt. \tag{7.6}$$

Without loss of generality we assume that $\operatorname{supp}\omega(t-\tau)$ is a subset of a finite union $\cup_{b\in B_t} b$ of elements $b \in \mathcal{A}$. Note that for any $v \in C^\alpha(Y)$,

$$\|R(\omega(t-\tau)(e^g - e^{\pm g_N}))v\|_{\mathcal{B}\to\mathcal{B}_w} \le C\|R(\omega(t-\tau))\|_{\mathcal{B}\to\mathcal{B}_w} \max_{b\in B_t} \|(e^g - e^{\tilde{g}_N})1_b v\|_{\mathcal{B}_w}.$$

We only have to consider those $b \in B_t$ with $b \cap \{\tau > N\} \neq \emptyset$, otherwise $(e^g - e^{g_N^{\pm}})1_b = 0$. This together with **(H7)** gives

$$\max_{b\in B_t} \|(e^g - e^{\pm g_N})1_b v\|_{\mathcal{B}_w} \ll \max_{b\in B_t} \sup_b |g - \pm g_N|^\gamma \ll t^{\gamma\kappa} 1_{\{t>N\}}.$$

Recall $\gamma < 1$ for $\kappa > 1$. By **(H2)** and the previous displayed equation, for any $\epsilon \in (0, 1-\gamma)$,

$$\begin{aligned}\|\hat{R}(u,s) - \hat{R}_N^{\pm}(u,s)\|_{\mathcal{B}\to\mathcal{B}_w} &\ll \int_N^\infty t^{\gamma\kappa}\|M(t)\|_{\mathcal{B}\to\mathcal{B}_w}\,dt = \int_N^\infty t^{-(1-\gamma-\epsilon)\kappa} t^{\kappa-\epsilon}\|M(t)\|_{\mathcal{B}\to\mathcal{B}_w}\,dt \\ &\le N^{-(1-\gamma-\epsilon)\kappa} \int_N^\infty t^{\kappa-\epsilon}\|M(t)\|_{\mathcal{B}\to\mathcal{B}_w}\,dt. \end{aligned} \tag{7.7}$$

By equation (7.3), $\int_0^\infty t^{\kappa-\epsilon}\|M(t)\|_{\mathcal{B}\to\mathcal{B}_w}\,dt < \infty$. Using (7.7) and (7.3),

$$\|\hat{R}(u,s) - \hat{R}_N^{\pm}(u,s)\|_{\mathcal{B}\to\mathcal{B}_w} \ll N^{-(1-\gamma-\epsilon)\kappa}.$$

This together with the argument recalled in the proof of Lemma 5.3 gives

$$\|v(u,s) - v_N^{\pm}(u,s)\|_{\mathcal{B}\to\mathcal{B}_w} \ll N^{-(1-\gamma-\epsilon)\kappa}\log N.$$

where for fixed $u, s \in [0, \delta_0)$, $v(u,s)$ and $v_N^{\pm}(u,s)$ are the eigenfunctions associated with $\lambda(u,s)$ and $\lambda_N^{\pm}(u,s)$, respectively. Thus,

$$\|\hat{R}(u,s) - \hat{R}_N^{\pm}(u,s)\|_{\mathcal{B}\to\mathcal{B}_w} \to 0, \quad \|v(u,s) - v_N^{\pm}(u,s)\|_{\mathcal{B}_w} \to 0 \text{ as } N \to \infty.$$

The conclusion follows from these estimates together with the argument used in the first part of the proof of Lemma 7.4. ∎

7.2 Proof of Theorem 7.1

Lemma 7.6 *Assume* **(H5)** *and that* $\mathcal{P}(\phi+s\psi) > 0$ *for* $s > 0$. *Then* $\mathcal{P}(\overline{\phi + s\psi - \mathcal{P}(\phi + s\psi)}) = 0$.

Proof Set $u_0 = P(\phi + s\psi)$. To show that $\mathcal{P}(\overline{\phi + s\psi - u_0}) \leq 0$, suppose by contradiction that $\mathcal{P}(\overline{\phi + s\psi - u_0}) > 0$. We use the ideas of [S99, e.g. Theorem 2] to truncate the whole system, i.e., restrict to the system to the first n elements of the partition $\mathcal{A}$ and the corresponding flow. We write the pressure of this system as $P_n(\cdot)$, so [S99, Theorem 2] says that $P_n(\overline{\phi + s\psi - u_0}) > 0$ for all large n. By the definition of pressure, there exists a measure $\bar{\mu}$ so that

$$h(\bar{\mu}) + \int \overline{\phi + s\psi - u_0}\, d\bar{\mu} > 0.$$

Note that $\int \tau \, d\bar{\mu} < \infty$ since τ is bounded on this subsystem. Abramov's formula (3.3) implies that the projected measure μ (which relates to $\bar{\mu}$ via (3.4)) has

$$h(\mu) + \int \phi + s\psi - u_0 \; d\mu > 0,$$

contradicting the choice $u_0 = P(\phi + s\psi)$.

To show that $\mathcal{P}(\overline{\phi + s\psi - u_0}) \geq 0$, we will use the continuity of the pressure function, where finite. By the definition of pressure, for any $\epsilon \in (0, u_0)$, there exists μ' with

$$h(\mu') + \int \phi + s\psi - (u_0 - \epsilon) \; d\mu' > 0.$$

By (3.4), any invariant probability measure for the flow corresponds to an invariant probability measure $\bar{\mu}'$ for the map F. By Abramov's formula,

$$h(\bar{\mu}') + \int \overline{\phi + s\psi - (u_0 - \epsilon)} \; d\bar{\mu}' = \left(h(\mu') + \int \phi + s\psi - (u_0 - \epsilon) \; d\mu' \right) \int \tau \; d\bar{\mu}' > 0$$

regardless of whether $\int \tau \; d\bar{\mu}'$ is finite or not. By continuity in u, $\mathcal{P}(\overline{\phi + s\psi - u_0}) \geq 0$ as required. ∎

The following result is a consequence of (7.5) and of Lemma 7.4.

Lemma 7.7 *Assume the setting of Lemma 7.4. Suppose that there exists* $a > 0$ *such that* $s = O(u^a)$, *as* $u \to 0$. *Then as* $u \to 0^+$,

$$\frac{d}{du}\mathcal{P}(\overline{\phi + s\psi - u}) = -\tau^*(1 + o(1)).$$

Proof By (7.5) and Lemma 7.4,

$$\frac{d}{du}\mathcal{P}(\overline{\phi + s\psi - u}) = \frac{d}{du} \log \lambda(u, s) = -\frac{d}{du}\Pi(u) + D(u, s),$$

where $D(u,s) = O\Big((\log(1/u))^{\kappa\gamma+2}(s^\gamma + q(u)) + \mu_{\bar\phi}(\tau > \log(1/u))\Big)$. Also, compute that

$$\frac{d}{du}\Pi(u) = \int_Y \tau e^{-u\tau}\, d\mu_{\bar\phi} = \int_Y \tau\, d\mu_{\bar\phi} + \int_Y \tau(e^{-u\tau} - 1)\, d\mu_{\bar\phi}.$$

Since $\tau|e^{-u\tau} - 1|$ is bounded by the integrable function τ and it converges pointwise to 0, it follows from the Dominated Convergence Theorem that $\int_Y \tau(e^{-u\tau} - 1)\, d\mu_{\bar\phi} \to 0$, as $u \to 0^+$. Hence, $\frac{d}{du}\Pi(u) = \tau^*(1 + o(1))$.

Since $\tau \in L^1(\mu_{\bar\phi})$ by assumption, $\mu_{\bar\phi}(\tau > \log(1/u)) \to 0$, as $u \to 0$. To conclude, note that since for some $a > 0$, $s = O(u^a)$, as $u \to 0$, we have $D(u,s) = o(1)$. ■

We can now complete

Proof of Theorem 7.1 Set $r(u,s) = \frac{d}{du}\mathcal{P}(\overline{\phi + s\psi - u})$. By Lemma 7.7, as $u \to 0$ and $s = O(u^a)$ for some $a > 0$,

$$r(u,s) = -\tau^*(1 + o(1)).$$

For any small $u_0 > 0$, integration gives

$$\mathcal{P}(\overline{\phi + s\psi - u_0}) - \mathcal{P}(\overline{\phi + s\psi}) = \int_0^{u_0} r(u,s)\, du = -\tau^* u_0(1 + o(1)). \tag{7.8}$$

The convexity of $s \mapsto \mathcal{P}(\phi + s\psi)$ implies that $\frac{d}{ds}\mathcal{P}(\phi + s\psi) = \int \psi\, d\mu_\phi = \frac{1}{c}\int \bar\psi\, d\mu_{\bar\phi}$. Thus $\mathcal{P}(\phi + s\psi) \geq \frac{s}{\tau^*}\int \bar\psi\, d\mu_{\bar\phi}$ since $\int \bar\psi\, d\mu_{\bar\phi} > 0$; this is guaranteed by our assumption that $C' > C\int \tau^\kappa\, d\mu_{\bar\phi}$. Hence, $u_0 = u_0(s) = \mathcal{P}(\phi + s\psi) \gg s$ and such u_0 satisfies the assumptions of Lemma 7.7 with $a = 1$. Thus, (7.8) holds for this u_0.

By Lemma 7.6 applied to $u_0 = u_0(s) = \mathcal{P}(\phi + s\psi)$, we obtain $\mathcal{P}(\overline{\phi + s\psi - u_0}) = 0$. Thus, the left hand side of (7.8) is $-\mathcal{P}(\overline{\phi + s\psi})$. By assumption, $u_0(s) > 0$, for $s > 0$. The continuity property of the pressure function gives $u_0(s) \to 0$ as $s \to 0$. Hence, (7.8) applies to $u_0(s) = \mathcal{P}(\phi + s\psi)$. Thus, $\mathcal{P}(\phi + s\psi) = \frac{1}{\tau^*}\mathcal{P}(\overline{\phi + s\psi})(1 + o(1))$, which ends the proof. ■

8 Pressure function and limit theorems

For $\psi : \mathcal{M} \to \mathbb{R}$, define $\bar\psi : Y \to \mathbb{R}$ as in (3.2) and let $\psi_T = \int_0^T \psi \circ f_t\, dt$. Throughout this section we assume that $\bar\psi$ satisfies **(H5)**; in particular, we require that $\bar\psi = C' - \psi_0$, with $C^{-1}\tau^\kappa \leq \psi_0 \leq C\tau^\kappa$, $\int \tau^\kappa\, d\mu_{\bar\phi} < \infty$ for some $C, C' > 0$. For s real, $s \geq 0$, define

$$\hat R(s)v = R(e^{s\bar\psi}v) = e^{sC'}R(e^{-s\psi_0}v) := e^{sC'}\hat R_1(s).$$

As in Subsection 4.3 (when making assumptions on $\hat R(u)$), we write

$$\hat R_1(s) = g_0(s)\int_0^\infty R(\omega(t - \psi_0))e^{-st}\, dt := g_0(s)\int_0^\infty M_0(t)\, e^{-st}\, dt, \tag{8.1}$$

where $\omega : \mathbb{R} \to [0,1]$ is an integrable function with $\operatorname{supp}\omega \subset [-1,1]$ and $g_0(0) = 1$. Similarly to **(H2)**, we require that

(H8) There exists a function ω satisfying (8.1) and $C_\omega > 0$ such that

(i) for any $t \in \mathbb{R}_+$ and for all $h \in \mathcal{B}$, we have $\omega(t-\psi_0)h \in \mathcal{B}_w$ and for all $v \in C^\alpha(Y)$,

$$\langle \omega(t-\psi_0)vh, 1\rangle \le C_\omega \|v\|_{C^\alpha(Y)} \|h\|_{\mathcal{B}_w} \mu_{\bar\phi}(\omega(t-\psi_0)).$$

(ii) there exists $C > 0$ such that for all $T > 0$,

$$\int_T^\infty \|M_0(t)\|_{\mathcal{B}\to\mathcal{B}_w}\, dt \le C\mu_{\bar\phi}(\psi_0 > T).$$

Remark 8.1 In practice, checking **(H8)** requires no extra difficulty compared to checking **(H2)**. But in the generality of the present abstract framework, there is no obvious way of deriving **(H8)** from **(H2)**.

Summarising the arguments used in the proof of Theorems 5.1 and 7.1 together with Proposition 6.1, in this section we obtain

Theorem 8.2 *Assume* **(H1)** *and* **(H8)**. *Let $\psi : \mathcal{M} \to \mathbb{R}$ and assume that $\bar\psi$ satisfies* **(H5)**, **(H6)** *and* **(H7)** *(with $u = 0$). Set $\psi^* = \int_Y \bar\psi\, d\mu_{\bar\phi}$. The following hold as $T \to \infty$.*

(a) Suppose that **(H4)***(i) holds and that $\mu(\psi_0 > t) = \mu(\tau^\kappa > t)$ with κ in (some subset of) $(0, \beta)$.*

(i) When $\beta < 2$ and $\kappa \in (\beta/2, \beta)$, set $b(T)$ such that $T\ell(b(T))/b(T)^{\frac{\beta}{\kappa}} \to 1$. Then $\frac{1}{b(T)}(\psi_T - \psi^* \cdot T) \to^d G_{\beta/\kappa}$, where $G_{\beta/\kappa}$ is a stable law of index β/κ. This is further equivalent to $\mathcal{P}(\overline{\phi + s\psi}) = \psi^* s + s^{\beta/\kappa}\ell(1/s)(1 + o(1))$, as $s \to 0$.

(ii) If $\beta \le 2$ but $\kappa = \beta/2$, set $b(T)$ such that $T\tilde\ell(b(T))/b(T)^{\frac{2}{\kappa}} \to c > 0$. Then $\frac{1}{b(T)}(\psi_T - \psi^* \cdot T) \to^d \mathcal{N}(0, c)$ and this is further equivalent to $\mathcal{P}(\overline{\phi + s\psi}) = \psi^* s + s^2 L(1/s)(1 + o(1))$, as $s \to 0$ with $L(x) = \frac{1}{2}\tilde\ell(x)$.

(b) Suppose that **(H4)***(ii) holds and further assume that $\mu(\psi_0 > t) \ll \mu(\tau^\kappa > t)$ with $\kappa \in (0, \beta/2)$, and $R\bar\psi \in \mathcal{B}$. Then there exists $\sigma \ne 0$ such that $\frac{1}{\sqrt{T}}(\psi_T - \psi^* \cdot T) \to^d \mathcal{N}(0, \sigma^2)$ and this is further equivalent to $\mathcal{P}(\overline{\phi + s\psi}) = \psi^* s + \frac{\sigma^2}{2}s^2(1 + o(1))$ as $s \to 0$.*

Moreover, if **(H6)** *and* **(H7)** *hold for all $u, s \in [0, \delta)$, then in both cases (a) and (b) we have $\mathcal{P}(\phi + s\psi) = \frac{1}{\tau^*}\mathcal{P}(\overline{\phi + s\psi})(1 + o(1))$, for $\tau^* = \int_Y \tau\, d\mu_{\bar\phi}$.*

Proof For $s > 0$, let $\lambda(s)$, $\lambda_1(s)$ be the families of eigenvalues associated with $\hat R(s)$ and $\hat R_1(s)$, respectively. Note that $\lambda(s) = e^{sC'}\lambda_1(s)$. Further, by (7.5), $\mathcal{P}(\overline{\phi + s\psi}) = \log\lambda(s)$. Similar to the proofs of Theorem 5.1 and Proposition 6.1, to conclude that (a) and/or (b) hold, we just need to estimate $1 - \lambda(s)$.

For the proof of (a) and (b), we note that under **(H5)** –**(H8)** (where **(H8)** and **(H6)** with $u = 0$ are the analogues of **(H2)** and **(H3)** with τ there replaced by ψ_0), the argument used in the proof of Lemma 5.4 gives

$$1 - \lambda_1(s) = \Pi_1(s)(1 + O(q_1(s)|\log(q_1(s))|)),$$

where $q_1(s) = s + \mu(\psi_0 > 1/s) = s + s^{\beta/\kappa} \ll s^{\beta/\kappa}$. Hence,

$$1 - \lambda_1(s) = \Pi_1(s)(1 + O(s^{\beta/\kappa} \log(1/s))).$$

In case (a) (i), the argument recalled in the proof of Corollary 5.5 (i) gives $\Pi_1(s) = \psi^* s + s^{\beta/\kappa}\ell(1/s)(1+o(1))$ and thus, $1-\lambda_1(s) = \psi^* s + s^{\beta/\kappa}\ell(1/s)(1+o(1))$. As a consequence, $\mathcal{P}(\overline{\phi + s\psi}) = \psi^* s + s^{\beta/\kappa}\ell(1/s)(1+o(1))$. Similarly, by the argument used in the proof of Theorem 5.1 (i), this expansion of the eigenvalue/pressure is equivalent to $\frac{1}{b(n)}(\bar{\psi}_n - \psi^* \cdot n) \to^d G_{\beta/\kappa}$, where $\bar{\psi}_n = \sum_{j=0}^{n-1} \bar{\psi} \circ F^j$. Further, by the argument used in Proposition 6.1 (i) with $\beta < 2$ this is further equivalent to $\frac{1}{b(T)}(\psi_T - \psi^* \cdot T) \to^d G_{\beta/\kappa}$, which completes the proof of (a) (i).

The proof of (a) (ii) goes similarly with the arguments used in the proofs of Theorem 5.1 (i), Proposition 6.1 (i) with $\beta < 2$ replaced by the argument used in the proofs of Theorem 5.1 (i), Proposition 6.1 (i) with $\beta = 2$.

The proof of (b) goes again similarly with the arguments used in the proofs of Theorem 5.1 (i), Proposition 6.1 (i) replaced by the argument used in the proofs of Theorem 5.1 (ii), Proposition 6.1 (ii) .

Finally, if **(H6)** for all $u, s \in [0, \delta_0)$ and **(H7)** also hold, then Theorem 7.1 applies and ensures that $\mathcal{P}(\phi + s\psi) = \frac{1}{\tau^*}\mathcal{P}(\overline{\phi + s\psi})(1 + o(1))$, ending the proof. ∎

A Technical results used in proofs of the abstract results

A.1 Proof of Equation (4.1)

Since it is short, for the reader's convenience we include the proof of (4.1) (as in the proof of [BMT, Proposition 2.1])

Let ω be an integrable function supported on $[-1, 1]$ such that $\int_{-1}^1 \omega(t)\,dt = 1$, and set $\hat{\omega}(u) = \int_{-1}^1 e^{-ut}\omega(t)\,dt$. Note that $\hat{\omega}(u)$ is analytic and $\hat{\omega}(0) = 1$. Since $\tau \geq 1$ and $\operatorname{supp}\omega \subset [-1, 1]$,

$$\int_0^\infty \omega(t - \tau)e^{-ut}\,dt = e^{-u\tau}\int_{-\tau}^\infty \omega(t)\,e^{-ut}\,dt = e^{-u\tau}\hat{\omega}(u).$$

Hence,

$$\int_0^\infty R(\omega(t-\tau)v)e^{-ut}\,dt = R(\int_0^\infty \omega(t-\tau)ve^{-ut}\,dt) = \hat{\omega}(u)\hat{R}(u)v.$$

Formula (4.1) follows with $g_0(u) = 1/\hat{\omega}(u)$, so $g_0(0) = 1$ and $\frac{d}{du}g_0(u)$ and $\frac{d}{du}g_0(u)$ are continuous functions, and by easy additional properties on ω, we get $\frac{d}{du}g_0(u), \frac{d^2}{du^2}g_0(u) \to 1$ as $u \to 1$.

A.2 Derivatives of $\hat{R}(u)$ and $v(u)$

Lemma A.1 *Let $\kappa \geq 1$ and assume that $\tau^\kappa \in L^1(\mu_{\bar{\phi}})$. Suppose that* **(H1)**–**(H3)** *hold. The following hold as $u \to 0$.*

(i) $\|\frac{d}{du}\hat{R}(u)\|_{\mathcal{B}_w} \ll 1$ *and* $\|\frac{d}{du}v(u)\|_{\mathcal{B}_w} \ll \log(\log(1/u))$.

(ii) If $\kappa \geq 2$ *then* $\|\frac{d^2}{du^2}\hat{R}(u)\|_{\mathcal{B}_w} \ll 1$ *and* $\|\frac{d^2}{du^2}v(u)\|_{\mathcal{B}_w} \ll \log(1/u)$.

Proof Under **(H2)**,

$$\frac{d}{du}\hat{R}(u) = \frac{d}{du}g_0(u)\int_0^\infty M(t)e^{-ut}\,dt + g_0(u)\int_0^\infty tM(t)e^{-ut}\,dt.$$

Recall that given(3.2), we take $\frac{d}{du}g_0(u) \to 1$. So, the first term is bounded in $\|\cdot\|_{\mathcal{B}_w}$. Also, by (7.3) with $\kappa = 1$, $\int_0^\infty t\|M(t)\|_{\mathcal{B}\to\mathcal{B}_w}\,dt < \infty$. Item (i) follows.

For the estimate on $\frac{d}{du}v(u)$, we recall $P(u) = \mu_{\bar{\phi}}(u) \otimes v(u)$, with $\mu_{\bar{\phi}}(u)(v(u)) = 1$. Hence, we can write $v(u) = \frac{P(u)1}{\mu_{\bar{\phi}}(u)(1)}$. Since $\|\frac{d}{du}\hat{R}(u)\|_{\mathcal{B}_w} \ll 1$, using **(H1)**, **(H3)** and arguments similar to [KL99] (see also [LT16, Proof of Corollary 3.11]) we obtain

$$\|\frac{d}{du}P(u)\|_{\mathcal{B}_w} \ll \log(\log(1/u)).$$

The bound $\log(\log(1/u))$ is far from optimal (given that $\|\frac{d}{du}\hat{R}(u)\|_{\mathcal{B}_w} \ll 1$), but this suffices for the present purpose. The same estimate holds for $|\frac{d}{du}\mu_{\bar{\phi}}(u)(1)|$. These together with the formula $v(u) = \frac{P(u)1}{\mu_{\bar{\phi}}(u)(1)}$ give the second part of item (i).

For the estimates on the second derivatives of $\hat{R}(u)$, we just need to differentiate once more, recall that $\frac{d^2}{du^2}g_0(u) \to 1$ and repeat the argument above with $\kappa = 2$. The estimate $\|\frac{d^2}{du^2}v(u)\|_{\mathcal{B}_w} \ll \log(1/u)$ follows from item (i) together with **(H1)**, **(H3)** and, again, arguments similar to [KL99] and [LT16, Proof of Corollary 3.11]. ∎

B Proof of Proposition 2.2

We recall some notation and the form of the first integral $L(x,y)$ from [BT17, Lemma 2.2], replacing κ from that paper by 2.

Assume that $\Delta := a_2b_0 - a_0b_2 \neq 0$. Let $u, v \in \mathbb{R}$ be the solutions of the linear equation

$$\begin{cases}(u+2)a_0 = vb_0\\ (v+2)b_2 = ua_2\end{cases} \qquad \text{that is:} \qquad \begin{cases}u = \frac{2b_2}{\Delta}(a_0+b_0),\\ v = \frac{2a_0}{\Delta}(a_2+b_2).\end{cases} \tag{B.1}$$

Note that u, v and Δ all have the same sign. Define:

$$\beta_0 := \frac{a_0+b_0}{2a_0} = \frac{u+v+2}{2v}, \quad \beta := \beta_2 := \frac{a_2+b_2}{2b_2} = \frac{u+v+2}{2u}, \qquad \begin{matrix}c_0 = a_0+b_0\\ c_2 = a_2+b_2\end{matrix}\ . \tag{B.2}$$

Note that $\frac{\beta_0}{\beta_2} = \frac{u}{v}$, $\frac{a_0u}{b_2v} = \frac{c_0}{c_2}$ and $\beta_0, \beta_2 > \frac{1}{2}$ (or $= \frac{1}{2}$ if we allow $b_0 = 0$ or $a_2 = 0$ respectively). The content of [BT17, Lemma 2.2] is that

$$L(x,y) = \begin{cases}x^uy^v(\frac{a_0}{v}\ x^2 + \frac{b_2}{u}\ y^2) & \text{if } \Delta > 0;\\ x^{-u}y^{-v}(\frac{a_0}{v}\ x^2 + \frac{b_2}{u}\ y^2)^{-1} & \text{if } \Delta < 0.\end{cases} \tag{B.3}$$

is a first integral of Φ_t^{hor} (and therefore of Φ_t).

For the proof of Proposition 2.2, we follow the proof of [BT17, Proposition 2.1]. In comparison, we have κ from [BT17, Proposition 2.1] equal to 2, our current integrand is more complicated, but we only need first order error terms.

Proof of Proposition 2.2 Fix η such that the local unstable leaf $\hat{W}^{\mathbf{u}}_{loc}(0,\eta)$ intersects $\overline{\hat{f}^{-1}(\hat{P}_0) \setminus \hat{P}_0}$. Recall from the text below (2.2) that, given T large enough, there is a unique point $\xi(\eta,T),\eta') \in \hat{W}^{\mathbf{u}}_{loc}(0,\eta)$ such that

$$(\zeta_0, \omega(\eta,T)) := \Phi_T^{hor}((\xi(\eta,T),\eta')) \in \hat{W}^s_{loc}(\zeta_0,0),$$

where $\hat{W}^s_{loc}(\zeta_0,0)$ is the local stable leave forming the right boundary of P_0, see Figure 3. Assuming $\hat{W}^{\mathbf{u}}_{loc}(0,\eta)$ and $\hat{W}^s_{loc}(\zeta_0,0)$ are straight horizontal and vertical lines respectively, induces a negligible error in these vertical coordinates, so in the rest of the proof, we write $\eta' = \eta$ and $\zeta_0' = \zeta_0$.

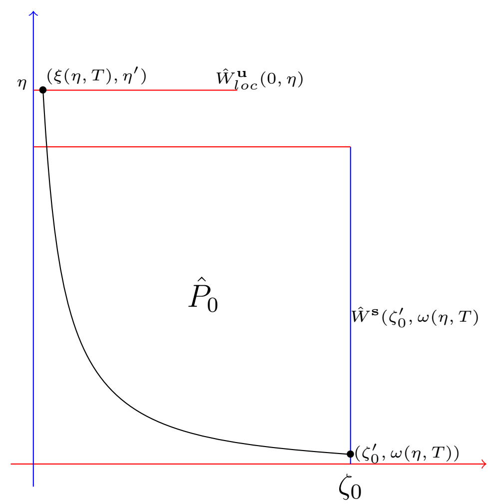

Figure 3: The first quadrant of the rectangle $\hat{P}_0$, with the integral curve $I(T)$ connecting $(\xi(\eta,T),\eta) \in \hat{W}^{\mathbf{u}}_{loc}(0,\eta))$ and $(\zeta_0',\omega(\eta,T)) \in \hat{W}^{\mathbf{s}}(\zeta_0',\omega(\eta,T))$.

For simplicity of notation, we will suppress the η and T in $\xi(\eta,T)$ and $\omega(\eta,T)$. For $0 \le t \le T$, let $\Phi_t^{hor}(\xi,\eta) = (x(t),y(t))$, so $(x(0)),y(0)) = (\xi,\eta)$ and $(x(T),y(T)) = (\zeta_0,\omega)$. We need to compute

$$\Theta(T) = \int_0^T \theta(x(t),y(t))\,dt,$$

where $\theta : \mathbb{R}^2 \to \mathbb{R}$ is homogeneous of exponent ρ.

New coordinates: We compute $\Theta(T)$ by introducing new coordinates (L,M) where L is the first integral from (B.3) (and hence independent of t), and $M(t) = y(t)/x(t)$, so $y(t) = M(t)x(t)$. For simplicity of notation, we suppress t in x,y and M. Differentiating to $\dot{y} = \dot{M}x + M\dot{x}(t)$, and inserting the values for $\dot{x}$ and $\dot{y}$ from (2.1), we get

$$\dot{M} = -M(c_0 + c_2M^2)x^2. \tag{B.4}$$

We carry out the proof for $\Delta > 0$ (the case $\Delta < 0$ goes likewise), so the first integral is $L(x, y) = x^u y^v(\frac{a_0}{v}\, x^2 + \frac{b_2}{u}\, y^2)$ in (B.3). Since (x, y) is in the level set $L(x, y) = L(\xi, \eta) = \xi^u\eta^v(\frac{a_0}{v}\xi^2 + \frac{b_2}{u}\eta^2)$, we can solve for x^2 in the expression

$$\xi^u\eta^v\left(\frac{a_0}{v}\xi^2 + \frac{b_2}{u}\eta^2\right) = x^u y^v\left(\frac{a_0}{v}x^2 + \frac{b_2}{u}y^2\right) = x^{u+v+2}M^v\left(\frac{a_0}{v} + \frac{b_2}{u}M^2\right).$$

Use (B.1) and (B.2) to obtain

$$\frac{a_0}{v} + \frac{b_2}{u}M^2 = \frac{\Delta}{2c_0c_2}(c_0 + c_2M^2) \quad \text{and} \quad \frac{a_0\xi^2}{v} + \frac{b_2\eta^2}{u} = \frac{\Delta}{2c_0c_2}(c_0\xi^2 + c_2\eta^2).$$

Note also from (B.2) that $\frac{2v}{u+v+2} = \frac{1}{\beta_0}$, $\frac{2u}{u+v+2} = \frac{1}{\beta_2}$ and $\frac{2}{u+v+2} = 1 - \frac{1}{2\beta_0} - \frac{1}{2\beta_2}$. This plus the previous two equations together gives

$$x^2 = G(T)M^{-\frac{1}{\beta_0}}(c_0 + c_2M^2)^{\frac{1}{2\beta_0}+\frac{1}{2\beta_2}-1}, \tag{B.5}$$

with

$$G(T) := G(\xi(\eta, T), \eta) = \xi(\eta, T)^{\frac{1}{\beta_2}}\eta^{\frac{1}{\beta_0}}(c_0\xi(\eta, T)^2 + c_2\eta^2)^{1-\frac{1}{2\beta_0}-\frac{1}{2\beta_2}}. \tag{B.6}$$

We claim that $G(T) \sim G_0T^{-1}$ as $T \to \infty$ for some $G_0 > 0$.

Proving the claim: First note that

$$T = \int_0^T dt = \int_{M(0)}^{M(T)} dM = -\int_{M(T)}^{M(0)} dM. \tag{B.7}$$

Recall that $M(0) = \eta/\xi$ and $M(T) = \omega/\zeta_0$. Inserting this and (B.4) and (B.5) into (B.7) gives

$$\int_{\omega/\zeta_0}^{\eta/\xi} \frac{dM}{M^{1-\frac{1}{\beta_0}}(c_0 + c_2M^2)^{\frac{1}{2\beta_0}+\frac{1}{2\beta_2}}} = G(T)T. \tag{B.8}$$

As T increases, the integral curve connecting $(\xi(\eta, T), \eta)$ to $(\zeta_0, \omega(\eta, T)$ tends to the union of the local stable and unstable manifolds of $(0, 0)$, whilst $M(T) = \omega(\eta, T)/\zeta_0 \to 0$ and $M(0) = \eta/\xi(\eta, T) \to \infty$. From their definition, $\xi(\eta, T)$ and $\omega(\eta, T)$ are decreasing in T, so their T-derivatives $\xi'(\eta, T), \omega'(\eta, T) \le 0$.

Since $c_0, c_2 > 0$ (otherwise $\Delta = 0$), the integrand of (B.8) is $O(M^{\frac{1}{\beta_0}-1})$ as $M \to 0$ and $O(M^{-\frac{1}{\beta_2}-1})$ as $M \to \infty$. Hence the integral is increasing and bounded in T. But this means that $G(T)T$ is increasing in T and bounded as well. Since by (B.6)

$$G(T)T = \left(\xi(\eta, T)T^{\beta_2}\right)^{\frac{1}{\beta_2}}\eta^{\frac{1}{\beta_0}}(c_0\xi(\eta, T)^2 + c_2\eta^2)^{1-\frac{1}{2\beta_0}-\frac{1}{2\beta_2}},$$

we find by combining with (B.8) that

$$\begin{aligned}
\xi_0(\eta) := \lim_{T\to\infty}\xi(\eta, T)T^{\beta_2} &= \lim_{T\to\infty}(G(T)T)^{\beta_2}\eta^{-\frac{\beta_2}{\beta_0}}(c_0\xi(\eta, T)^2 + c_2\eta^2)^{-\beta_2(1-\frac{1}{2\beta_0}-\frac{1}{2\beta_2})} \\
&= c_2^{-\frac{1}{u}}\eta^{-\frac{a_2}{b_2}}\left(\int_0^\infty \frac{dM}{M^{1-\frac{1}{\beta_0}}(c_0 + c_2M^2)^{\frac{1}{2\beta_0}+\frac{1}{2\beta_2}}}\right)^{\beta_2},
\end{aligned}$$

where we have used $-\beta_2(1-\frac{1}{2\beta_0}-\frac{1}{2\beta_2}) = -\frac{2\beta_2}{u+v+2} = -\frac{1}{u}$ for the exponent of c_2, and $-\frac{\beta_2}{\beta_0} - 2\beta_2(1-\frac{1}{2\beta_0}-\frac{1}{2\beta_2}) = 1-2\beta_2 = -\frac{a_2}{b_2}$ for the exponent of η. This shows that

$$G(T) \sim c_2^{1-\frac{1}{2\beta_0}-\frac{1}{2\beta_2}} \xi_0(\eta)^{\frac{1}{\beta_2}} \eta^{2-\frac{1}{\beta_2}} T^{-1} =: G_0 T^{-1} \quad \text{as } T \to \infty. \tag{B.9}$$

Estimating the integral $\Theta(T)$: Recalling $M = y/x$ and using homogeneity of θ, we have $\theta(x,y) = x^\rho \theta(1,M)$, and $\theta_0 := \theta(1,0)$ and $\theta_\infty := \theta(0,1) = \lim_{M\to\infty} M^{-\rho}\theta(1,M)$ are non-zero by assumption. Inserting the above into the integral of (B.8), and using (B.5) to rewrite x, we obtain

$$\Theta(T) = G(T)^{\frac{\rho}{2}-1} \int_{\omega/\zeta_0}^{\eta/\xi} \frac{M^{-\frac{\rho}{2\beta_0}} (c_0 + c_2 M^2)^{-\frac{\rho}{2}(1-\frac{1}{2\beta_0}-\frac{1}{2\beta_0})} \theta(1,M)}{M^{1-\frac{1}{\beta_0}} (c_0 + c_2 M^2)^{\frac{1}{2\beta_0}+\frac{1}{2\beta_2}}} \, dM. \tag{B.10}$$

Set $\rho_0 := \frac{\rho}{2} - (1+\frac{\rho}{2})(\frac{1}{2\beta_0}+\frac{1}{2\beta_2})$. The leading terms in the integrand of (B.10) are

$$\begin{cases} \theta_0 c_0^{\rho_0} M^{-1+(1-\frac{\rho}{2})\frac{1}{\beta_0}} & \text{as } M \to 0; \\ \theta_\infty c_2^{\rho_0} M^{-1-(1-\frac{\rho}{2})\frac{1}{\beta_2}} & \text{as } M \to \infty. \end{cases} \tag{B.11}$$

The case $\rho < 2$: By (B.11), we have for $\rho < 2$ that the exponent of M is > -1 as $M \to 0$ and < -1 as $M \to \infty$. This means that the integral in (B.10) converges to some constant

$$C^* = \int_0^\infty \frac{M^{-\frac{\rho}{2\beta_0}} (c_0 + c_2 M^2)^{-\frac{\rho}{2}(1-\frac{1}{2\beta_0}-\frac{1}{2\beta_0})} \theta(1,M)}{M^{1-\frac{1}{\beta_0}} (c_0 + c_2 M^2)^{\frac{1}{2\beta_0}+\frac{1}{2\beta_2}}} \, dM$$

as $T \to \infty$, and $\Theta \sim C^* G(T)^{\frac{\rho}{2}-1} \sim C_\rho T^{1-\frac{\rho}{2}}$ for $C_\rho = \left(c_2^{1-\frac{1}{2\beta_0}-\frac{1}{2\beta_2}} \xi_0(\eta)^{\frac{1}{\beta_0}} \eta^{2-\frac{1}{\beta_2}} \right)^{\frac{\rho}{2}-1} C^*$ by (B.6). This finishes the proof for $\rho < 2$.

The case $\rho > 2$: The value of $\Theta(T)$ based on the leading terms (B.11) of the integrand only is

$$\Theta(T) \sim \frac{G(T)^{\frac{\rho}{2}-1}}{\frac{\rho}{2}-1} \left(\beta_0 \theta_0 c_0^{\rho_0} \left(\frac{\omega}{\zeta_0}\right)^{(1-\frac{\rho}{2})\frac{1}{\beta_0}} + \beta_2 \theta_\infty c_2^{\rho_0} \left(\frac{\eta}{\xi}\right)^{-(1-\frac{\rho}{2})\frac{1}{\beta_2}} \right).$$

Insert $\xi = \xi_0(\eta) T^{-\beta_2}(1+o(1))$ and $\omega = \omega_0(\eta) T^{-\beta_0}(1+o(1))$ from [BT17, Proposition 2.1]:

$$\Theta(T) \sim \frac{G(T)^{\frac{\rho}{2}-1}}{\frac{\rho}{2}-1} \left(\beta_0 \theta_0 c_0^{\rho_0} \left(\frac{\zeta_0}{\omega_0(\eta)}\right)^{\frac{\rho/2-1}{\beta_0}} + \beta_2 \theta_\infty c_2^{\rho_0} \left(\frac{\eta}{\xi_0(\eta)}\right)^{\frac{\rho/2-1}{\beta_2}} \right) T^{\frac{\rho}{2}-1}.$$

Next, by inserting the asymptotics of $G(T)$ from (B.9), the factor $T^{\frac{\rho}{2}-1}$ cancels:

$$\Theta(T) \sim C_\rho := \frac{G_0^{\frac{\rho}{2}-1}}{\frac{\rho}{2}-1} \left(\beta_0 \theta_0 c_0^{\rho_0} \left(\frac{\zeta_0}{\omega_0(\eta)}\right)^{\frac{\rho/2-1}{\beta_0}} + \beta_2 \theta_\infty c_2^{\rho_0} \left(\frac{\eta}{\xi_0(\eta)}\right)^{\frac{\rho/2-1}{\beta_2}} \right).$$

The case $\rho = 2$**:** The factor $G(T)^{\frac{\rho}{2}-1}$ in (B.10) now disappears and the leading terms (B.11) are $\theta_0 c_0^{\rho_0} M^{-1}$ and $\theta_\infty c_2^{\rho_0} M^{-1}$ respectively. This gives

$$\Theta(T) \sim \int_{\omega/\zeta_0}^{\eta/\xi} \frac{\theta_0 c_0^{\rho_0} + \theta_\infty c_2^{\rho_0}}{M} \, dM = (\theta_0 c_0^{\rho_0} + \theta_\infty c_2^{\rho_0}) \left(\log \frac{\eta}{\xi} - \log \frac{\omega}{\zeta_0} \right).$$

Inserting again the values of ξ and ω from [BT17, Proposition 2.1] gives

$$\Theta(T) \sim (\theta_0 c_0^{\rho_0} + \theta_\infty c_2^{\rho_0}) (\beta_0 + \beta_2) \log T \qquad \text{as } T \to \infty.$$

This completes the proof. ∎

C Checking (H1)–(H8) for the almost Anosov flow

We start with a technical result that will be essential in verifying **(H2)** and **(H6)**.

Lemma C.1 *Assume that* $w(x,y)$ *is a homogeneous function of degree* ρ *as in Proposition 2.2. Then*

$$\sup_k \left\{ \sup_{\{r=k\}} \tau - \inf_{\{r=k\}} \tau \right\} < \infty.$$

Proof Recall that $\hat{\tau}(x,y) = \min\{t > 0 : \Phi_t^{hor}(x,y) \in \hat{W}^{\mathbf{s}}\}$ and that $|\tau - \hat{\tau}| = O(1)$. From Lemma 2.3 we know that $r = \hat{\tau} + \Theta(\hat{\tau}) + O(1)$. We will first show that if $\hat{\tau}$ varies between, say, n and $n+1$, then the corresponding r varies by an amount of $O(1)$ as well.

Indeed, take (x,y) and (x',y') arbitrary such that $n \leq \hat{\tau}(x,y), \hat{\tau}(x',y') \leq n+1$. Define $T = \hat{\tau}(x,y)$ and abbreviate $q(t) = (x(t),y(t)) = \Phi_t^{hor}(x,y)$ for $0 \leq t \leq T$. There is $t_0 \in \mathbb{R}$ with $|t_0| \leq 1$ such that the y-component $\Phi_{-t_0}^{hor}(\hat{x},\hat{y})$ is y. Also write $q'(t) = (x'(t),y'(t)) = \Phi_{t-t_0}^{hor}(x',y')$, and the Euclidean distance $\epsilon(t) = |q'(t) - q(t)|$.

Clearly $|w(q'(t)) - w(q(t))| \ll |\nabla w(q(t))| \epsilon(t)$, so $\epsilon := \epsilon(0) \ll T^{-(1+\beta)}$ by (2.3). However, since $\hat{q}(0) - q(0)$ is roughly in the unstable direction, $\epsilon(t)$ will increase to size $O(1)$ as t increases to $\hat{\tau}(q')$. This is too large, but we can set $T_1 := \min\{t > 0 : x(t) = y(t)\}$, and then estimate $\epsilon(T_1)$ as follows. Take $T_1' = \min\{t > 0 : x'(t) = y'(t)\}$, then $|T_1' - T_1| = O(1)$ and we can write $q(T_1) = (\delta,\delta)$, $q'(T_1') = (\delta',\delta')$ with $\epsilon' := \delta' - \delta$. Furthermore, since L from (B.3) is constant on integral curves, we have

$$L(x,y) = x^u y^v \left(\frac{a_0}{v} x^2 + \frac{b_2}{u} y^2 \right) = \delta^{u+v+2} \left(\frac{a_0}{v} + \frac{b_2}{u} \right), \tag{C.1}$$

and

$$L(q'(0)) = L(x+\epsilon, y) = (x+\epsilon)^u y^v \left(\frac{a_0}{v} (x+\epsilon)^2 + \frac{b_2}{u} y^2 \right) = (\delta + \epsilon')^{u+v+2} \left(\frac{a_0}{v} + \frac{b_2}{u} \right).$$

Taking the difference of both expressions, we obtain

$$\frac{\epsilon}{x}\, u\, L(x,y) \left(1 + \frac{2a_0}{ua_0x^2 + vb_2y^2} + O\left(\frac{\epsilon}{x}\right)\right) = \frac{\epsilon'}{\delta}\, (u+v+2)\, L(x,y) \left(1 + O\left(\frac{\epsilon'}{\delta}\right)\right).$$

Divide by $(u+v+2)\, L(x,y)/\delta$ and ignore the quadratic error terms. Then we have

$$\epsilon' \sim \frac{u}{u+v+2} \left(1 + \frac{2a_0}{vb_2y^2}\right) \frac{\delta\ \epsilon}{x}.$$

But δ can be estimated in terms of x using (C.1), and since $\frac{u}{u+v+2} = \frac{1}{2\beta}$ and $\frac{a_0u}{b_2v} = \frac{c_0}{c_2}$, we obtain

$$\epsilon' \sim \frac{\epsilon}{2\beta} x^{\frac{1}{2\beta}-1} y^{1-\frac{1}{2\beta}} \left(\left(1 + \frac{c_0}{c_2}\right)\left(1 + \frac{c_0}{c_2y^2}\right)\right)^{-\frac{1}{u+v+2}} \qquad \text{as } n \to \infty.$$

Rewriting x and ϵ in terms of T using (2.3), we obtain $\epsilon' \ll T^{-\frac{3}{2}}$. Next, because the vector-valued function ∇w is homogeneous with exponent $\rho - 1$, Proposition 2.2 gives for $\rho < 2$:

$$\begin{aligned} \int_0^{T_1} |w(q'(t)) - w(q(t))|\, dt &\ll \sup_{t\in[0,T_1]} \epsilon(t) \int_0^{T_1} |\nabla w(q(t))|\, dt + O(1) \\ &\ll \epsilon'\ T^{1-\frac{\rho-1}{2}} + O(1) = T^{-\frac{\rho}{2}} + O(1). \end{aligned}$$

The integral $\int_{T_1}^T \|w(q'(t)) - w(q(t))\|\, dt$ is dealt with in the same way, but now integrating backwards from $\Phi^{hor}_{\hat\tau(x,y)}(x,y)$ and $\Phi^{hor}_{\hat\tau(x',y')}(x',y')$. Therefore $\int_0^T w(q'(t)) - w(q(t))\, dt$ varies at most $O(1)$ on the region $\{(x,y) : n \le \hat\tau(x,y) \le n+1\}$.

Since also $\int_0^T w(q(t))\, dt = o(T)$, we can find $N \in \mathbb{N}$ independent of n such that $\tau = \int_0^T 1 + w(q(t))\, dt + O(1)$ varies by at least 1 (but at most by $O(N)$) over $\{n \le \hat\tau \le n+N\}$, and therefore r varies by at least 1 on this region. It follows that for each k, $\{r = k\} \subset \{n \le \hat\tau \le n+N\}$ for some $n = n(k)$ and $\sup_{\{r=k\}} \tau - \inf_{\{r=k\}} \tau$ is bounded, uniformly in k.

The proof for $\rho \ge 2$ goes likewise. ∎

C.1 Verifying (H1): recalling previously used Banach spaces

Charts and distances: Throughout, $\mathcal{W}^s$ denotes the set of *admissible leaves*, which consists of maximal stable leaves in elements of the partition $\mathcal{Y} = \{Y_j\}_j = \{P_i\}_i \vee \{\{r = k\}\}_{k\ge 2}$. It is convenient to arrange the enumeration of $\mathcal{Y}$ such that $Y_j = \{r = j\}$ for $j \ge 2$, and use indices $j \le 1$ for the remaining (parts of) P_i. To define distances between leaves, as in [BT17, Section 3.1], we use charts $\chi_j : [0, L_{\mathbf{u}}(Y_j)] \times [0,1] \to Y_j$, where $L_{\mathbf{u}}(Y_j)$ is the length of the (largest) unstable leaf in Y_j. For any leaf $W \subset Y_j$, we have the parametrisation

$$\chi_j^{-1}(W_{Y_j}) = \{(g_{Y_j,W}(\rho)), \rho \in [0,1]\}. \tag{C.2}$$

Next we define the distance between stable leaves $W \subset Y_k$ and $\tilde{W} \in \mathcal{W}^{\mathbf{s}} \in Y_\ell$ by

$$d(W, \tilde{W}) = \begin{cases} \sup_{x\in[0,1]} |g(x) - \tilde{g}(x)| & \text{if } k = \ell; \\ \sup_{x\in[0,1]} |g(x) - L_{\mathbf{u}}(Y_k) - \tilde{g}(x)| + \sum_{j=k+1}^{l-1} L_{\mathbf{u}}(Y_j) & \text{if } 2 \le k < \ell; \\ \infty & \text{otherwise.} \end{cases}$$

Here an empty sum $\sum_{j=k+1}^{k}$ is 0 by convention.

From here on, in the notation $\int_W \cdot \, d\mu^{\mathbf{s}}$, the measure $\mu^{\mathbf{s}}$ refers to the SRB measure $\mu_{\bar{\phi}}$ conditioned on the stable leaf W: similarly for $m^{\mathbf{s}}$ and Lebesgue measure. Hence the norms defined below are w.r.t. the invariant measure, not the conformal measure as in [BT17]. It is possible to do this due to Lemma C.2, specifically (C.7).

Definition of the norms: Given $h \in C^1(Y, \mathbb{C})$, define the *weak norm* by

$$\|h\|_{\mathcal{B}_w} := \sup_{W\in\mathcal{W}^{\mathbf{s}}} \sup_{|\varphi|_{C^1(W)}\le 1} \int_W h\varphi \, d\mu^{\mathbf{s}}. \tag{C.3}$$

Given $q \in [0, 1)$ we define the *strong stable norm* by

$$\|h\|_{\mathbf{s}} := \sup_{W\in\mathcal{W}^{\mathbf{s}}} \sup_{|\varphi|_{C^q(W)}\le 1} \int_W h\varphi \, d\mu^{\mathbf{s}}. \tag{C.4}$$

Finally, we define the *strong unstable norm* by

$$\|h\|_{\mathbf{u}} := \sup_{\ell} \sup_{W,\tilde{W}\in\mathcal{W}^{\mathbf{s}}\cap Y_\ell} \sup_{\substack{|\varphi|_{C^1(W)}, |\tilde{\varphi}|_{C^1(\tilde{W})}\le 1 \\ d(\varphi,\tilde{\varphi})\le d(W,\tilde{W})}} \frac{1}{d(W,\tilde{W})} \left| \int_W h\varphi \, d\mu^{\mathbf{s}} - \int_{\tilde{W}} h\tilde{\varphi} \, d\mu^{\mathbf{s}} \right|, \tag{C.5}$$

where $d(\varphi, \tilde{\varphi}) = |\varphi \circ \chi_\ell(g(\eta), \eta) - \tilde{\varphi} \circ \chi_\ell(\tilde{g}(\eta), \eta)|_{C^1([0,1])}$.

The *strong norm* is defined by $\|h\|_{\mathcal{B}} = \|h\|_{\mathbf{s}} + \|h\|_{\mathbf{u}}$.

Definition of the Banach spaces: We define $\mathcal{B}$ to be the completion of C^1 in the strong norm and $\mathcal{B}_w$ to be the completion in the weak norm. As clarified in [BT17, Lemma 3.2, Lemma 3.3, Proposition 3.2], **(H1)** holds with $\alpha = \alpha_1 = 1$.

The spaces $\mathcal{B}$ and $\mathcal{B}_w$ defined above are simplified versions of functional spaces defined in [DL08], adapted to the setting of (F, Y). The main difference in the present setting is the simpler definition of admissible leaves and the absence of a control on short leaves. This is possible due to the Markov structure of the diffeomorphism.

As this is the same Banach space as in [BT17], most parts of **(H1)** can be taken from there. For **H1**(v), which is concerned with the transfer operator w.r.t. $\mu_{\bar{\phi}}$, we first need a lemma.

Lemma C.2 *Let $\gamma = \gamma(q)$ be the angle between the stable and unstable leaf at q and $h_0^{\mathbf{u}} = \frac{d\mu^{\mathbf{s}}}{dm^{\mathbf{u}}}$ be the density of $\mu_{\bar{\phi}}$ conditioned on unstable leaves. Then*

$$\int_{W^{\mathbf{s}}} Rv \cdot \varphi \, d\mu^{\mathbf{s}} = \sum_j \int_{W_j} v \cdot \varphi \circ F \cdot K_{W_j}^{\mathbf{u}} \, d\mu^{\mathbf{s}}, \tag{C.6}$$

where the sum is over all preimage leaves $W_j = F^{-1}(W^{\mathbf{s}}) \cap \{r = j\}$ *and*

$$K^{\mathbf{u}}_{W_j} = \frac{J^{\mathbf{s}}_{\mu_{\bar\phi}} F}{J_{\mu_{\bar\phi}} F} = \frac{1}{J^{\mathbf{u}}_{m_{\bar\phi}} F} \frac{(h^{\mathbf{u}}_0 \sin\gamma) \circ F}{h^{\mathbf{u}}_0 \sin\gamma} \tag{C.7}$$

is piecewise C^1 *on unstable leaves.*

Proof We have the pointwise formulas for $R_0 : L^1(m_{\bar\phi}) \to L^1(m_{\bar\phi})$ and $R : L^1(\mu_{\bar\phi}) \to L^1(\mu_{\bar\phi})$:

$$R_0 v = \mathbf{1}_Y \frac{v}{|DF|} \circ F^{-1}, \qquad Rv = \mathbf{1}_Y \frac{h_0 \circ F^{-1}}{h_0} \frac{v}{|DF|} \circ F^{-1},$$

where $|DF| = J_{m_{\bar\phi}} = |\det(DF)|$. This also shows that $J_{\mu_{\bar\phi}} F = J_{m_{\bar\phi}} \frac{h_0}{h_0 \circ F^{-1}}$ and analogous formulas hold for $J_{\mu^{\mathbf{s}}} F$ and $J_{\mu^{\mathbf{u}}} F$. Integration over a stable leaf $W \in \mathcal{W}^{\mathbf{s}}$ with preimage leaves W_j gives

$$\begin{aligned}
\int_W R_{\mu_{\bar\phi}} v \,\varphi \, d\mu^{\mathbf{s}} &= \int_W \frac{h_0 \circ F^{-1}}{h_0} \frac{v}{|DF|} \circ F^{-1} \,\varphi \, h^{\mathbf{s}}_0 \, dm^{\mathbf{s}} \\
&= \sum_j \int_{W_j} \frac{J_{m^{\mathbf{s}}} F}{J_{m_{\bar\phi}} F} \frac{h_0}{h_0 \circ F} \, v \,\varphi \circ F \, \frac{h^{\mathbf{s}}_0 \circ F}{h^{\mathbf{s}}_0} \, h^{\mathbf{s}}_0 dm^{\mathbf{s}} \\
&= \sum_j \int_{W_j} \frac{J_{\mu^{\mathbf{s}}} F}{J_{\mu_{\bar\phi}} F} \, v \,\varphi \circ F \, d\mu^{\mathbf{s}} = \sum_j \int_{W_j} v \,\varphi \circ F \, K^{\mathbf{u}}_{W_j} \, d\mu^{\mathbf{s}}.
\end{aligned}$$

Since $d\mu^{\mathbf{u}} = h^{\mathbf{u}}_0 \, dm^{\mathbf{u}}$ for a C^1 density $h^{\mathbf{u}}_0$, and $dm_{\bar\phi} = dm^{\mathbf{s}} dm^{\mathbf{u}} \sin\gamma$, whence $J_{m_{\bar\phi}} F = J_{m^{\mathbf{s}}} F \cdot J_{m^{\mathbf{u}}} F \cdot \frac{\sin\gamma \circ F}{\sin\gamma}$, the other formula $K^{\mathbf{u}}_W = \frac{1}{J^{\mathbf{u}}_{m_{\bar\phi}} F} \frac{(h^{\mathbf{u}}_0 \sin\gamma) \circ F}{h^{\mathbf{u}}_0 \sin\gamma}$ follows as well. Then [BT17, Equation (18)], leading to the parametrisation of the unstable foliation of the induced map over $\hat f = \Phi^{hor}_1$, shows that this foliation is C^1. The analogous statement holds for the stable foliation. To obtain the unstable/stable foliations of F, we need to flow a bounded and piecewise C^1 amount of time (see Lemma C.1). Therefore $q \mapsto \gamma(q)$ is piecewise C^1. Hence the latter expression of $K^{\mathbf{u}}_W$ shows that it is piecewise C^1 on unstable leaves. ∎

Now **H1**(v) follows as in [BT17, Lemma 3.3], with $J_{\mu_{\bar\phi}}$ and $K^{\mathbf{u}}_{W_j}$ instead of $|DF|^{-1}$ and $|DF|^{-1} J_{W_j} F$; this can be done because $K^{\mathbf{u}}_{W_j}$ is piecewise C^1 on unstable leaves by Lemma C.2.

C.2 Verifying (H2) and (H8)

Let $\omega : \mathbb{R} \to [0,1]$ be a function with $\operatorname{supp} \omega \subset [-1,1]$ and $\int \omega(x)\, dx = 1$. To fix a choice that suffices for the present purpose we take $\omega = 1_{[0,1]}$; in particular, $\omega(t-\tau) = 1_{\{t \le \tau \le t+1\}}$. The first result below verifies assumption **(H2)**(i).

Proposition C.3 *There is* C_ω *such that* $\int_Y \omega(t-\tau) v \, d\mu_{\bar\phi} \le C_\omega \|v\|_{\mathcal{B}_w} \int_Y \omega(t-\tau) d\mu_{\bar\phi}$ *for all* $t \ge 0$ *and* $v \in \mathcal{B}_w$.

Proof By Lemma C.1, there is N (independent of t) and $k(t)$ such that

$$E := \{q \in f^{-1}(P_0) \setminus P_0 : k(t) \leq r(q) \leq k(t) + N\}$$

contains $\mathrm{supp}(\omega(t-\tau))$. We split $v = v^+ - v^-$ into its positive and negative parts and treat them separately. Also we assume without loss of generality that $0 \leq \omega \leq 1$ (otherwise we split ω in a positive and negative part as well). Then $\int \omega(t-\tau)\, v^+ \, d\mu_{\bar\phi} \leq \int_E v^+ \, d\mu_{\bar\phi}$.

Let $\mathcal{W}^{\mathbf{s}}$ denote the stable foliation of F and decompose the measure $\mu_{\bar\phi}$ as $\int v^+ \, d\mu_{\bar\phi} = \int_{\mathcal{W}^{\mathbf{s}}} \int_{W^{\mathbf{s}}} v^+ d\mu_{W^{\mathbf{s}}} \, d\nu^{\mathbf{u}}$. Note that E is the union of leaves in $\tilde{\mathcal{W}}^{\mathbf{s}}$, so $\mathbf{1}_E$ is constant on each $W^{\mathbf{s}} \in \mathcal{W}^{\mathbf{s}}$. Take $\varphi : E \to \mathbb{R}^+$ arbitrary such that $\varphi|_{W^{\mathbf{s}}} \in C^1(W^{\mathbf{s}})$ with $\|\varphi\|_{C^1(W^{\mathbf{s}})} \leq 1$ for each $W^{\mathbf{s}} \in \mathcal{W}^{\mathbf{s}} \cap E$. Similar to [BT17, Proposition 3.1], we get

$$\begin{aligned}
\int_E v^+ \varphi \, d\mu_{\bar\phi} &= \int_{\mathcal{W}^{\mathbf{s}} \cap E} \int_{W^{\mathbf{s}}} v^+ \varphi \, d\mu_{W^{\mathbf{s}}} \, d\nu^{\mathbf{u}} \leq \int_{\mathcal{W}^{\mathbf{s}} \cap E} \|v^+\|_{\mathcal{B}_w} \|\varphi\|_{C^1(W^{\mathbf{u}})} \, d\nu^{\mathbf{u}} \\
&\leq \|v^+\|_{\mathcal{B}_w} \mu_{\bar\phi}(E) \ll \|v^+\|_{\mathcal{B}_w} \int_{k(t)}^{k(t)+N} \int_Y \omega(s-\tau) \, d\mu_{\bar\phi} \, ds \\
&\ll \|v^+\|_{\mathcal{B}_w} N \int_Y \omega(t-\tau) \, d\mu_{\bar\phi}.
\end{aligned}$$

The same holds for v^-, and this ends the proof. ∎

The next result verifies assumption **(H2)**(ii).

Proposition C.4 *Let $M(t) = R(\omega(t-\tau))$. Then $\int_T^\infty \|M(t)\|_{\mathcal{B} \to \mathcal{B}_w} \, dt \ll T^{-\beta}$.*

Proof We need to estimate the $\mathcal{B}_w$-norm of $M(t)v = R(\omega(t-\tau)v)$. This means that we need to take some leaf $W^{\mathbf{s}} \in \mathcal{W}^{\mathbf{s}}$, the collection of stable leaves in Y stretching across an element of the Markov partition of F, and compute (as in Lemma C.2) that

$$\int_{W^{\mathbf{s}}} M(t)v \cdot \varphi \, d\mu^{\mathbf{s}} = \sum_j \int_{W_j^{\mathbf{s}}} \omega(t-\tau) v \cdot \varphi \circ F \cdot K^{\mathbf{u}}_{W_j} \, d\mu^{\mathbf{s}}.$$

As in the proof of Proposition C.3, we can split $v = v^+ - v^-$ and use that $\{t-1 \leq \tau \leq t+1\} \subset E := \{k(t) \leq r \leq k(t) + N\}$. Thus, following [BT17, Proposition 3.2 (*weak norm*)] and [BT17, Equation (42)], the integral for v^+ is bounded by

$$\begin{aligned}
\sum_{j=k(t)}^{k(t)+N} \int_{W_j} v^+ J_{W^j} F |DF|^{-1} \cdot \varphi \circ F \, d\mu^{\mathbf{s}} &\leq \|v^+\|_{\mathcal{B}_w} |\varphi|_{C^1(W^{\mathbf{s}})} \sum_{j=k(t)}^{k(t)+N} \int_{W_j} K^{\mathbf{u}}_{W_j} \, d\mu^{\mathbf{s}} \\
&\ll \|v^+\|_{\mathcal{B}} \sum_{j=k(t)}^{k(t)+N} \mu_{\bar\phi}(\{r = j\}) \\
&\ll \|v^+\|_{\mathcal{B}} \, N \, \mu_{\bar\phi}(\{r = k(t)\}).
\end{aligned}$$

For v^+ and v^- together, this gives $\int_T^\infty \|M(t)\|_{\mathcal{B} \to \mathcal{B}_w} dt \ll \int_T^\infty \mu_{\bar\phi}(\{r = k(t)\}) \, dt \ll \mu_{\bar\phi}(\{\tau \geq T\})$ and the proposition follows. ∎

Assumption **(H8)** is the same as **(H2)**, with τ replaced by ψ_0 and $e^{-u\tau}$ by e^{-st}. Its verification is entirely analogous to the above.

C.3 Verifying that $R\zeta \in \mathcal{B}$ for a large class of ζ (including $\bar{\psi}$ in Theorem 2.7 (b)) and completing the verification of (H4)

Assumption **(H4)**(i) and the tail estimate part of **(H4)**(ii) is verified in Proposition 2.4. To check the remaining part of **(H4)**(ii) and assumption on $\bar{\psi}$ in Theorem 2.7 (b) we give a more general result in Lemma C.5 below. This result ensures that given $\bar{\psi}$ as in Theorem 2.7 (b) we have $R\bar{\psi} \in \mathcal{B}$. This is needed to verify the abstract assumptions of Theorem 8.2(b) for $\bar{\psi}$ as in Theorem 2.7 (b).

Lemma C.5 *For $0 < \kappa < \beta$ and $\zeta : Y \to \mathbb{R}$ piecewise C^1, define*

$$\|\zeta\|_0 := \sup_{j \geq 1} \frac{1}{j^\kappa} \left(\|\zeta\|_{\mathcal{B}_w} + \frac{1}{j^{1+\beta}} \|\zeta\|_{\mathbf{u}} \right)$$

and the Banach space $\mathcal{B}_0 = \{\zeta : Y \to \mathbb{R} : \zeta$ is piecewise C^1 and $\|\zeta\|_0 < \infty\}$. Then $R(\mathcal{B}_0) \subset \mathcal{B}$.

Proof Recall that, by Proposition 2.2 and Lemma 2.3, $r = O(\tau)$ and vice versa. Abbreviate $Y_j = \{r = j\}$; for large j these are strips close to the stable manifold $W_p^{\mathbf{s}}$ of the neutral fixed point, and bounded by stable and unstable curves and both F^{-1} and $|DF|^{-1}$ are C^1 on each $\overline{F(Y_j)}$.

For the stable norm $\| \, \|_{\mathbf{s}}$, choose an arbitrary stable leaf W and q-Hölder function $\varphi \in C^q(W)$ with $|\varphi|_{C^q(W)} \leq 1$. Let $W_j = F^{-1}(W) \cap Y_j$. By Lemma C.2, and the fact that $K^{\mathbf{u}} \ll j^{-(1+\beta)}$ and $|\varphi \circ F|_{C^1} \ll |\varphi|_{C^1}$ on each W_j, we have

$$\begin{aligned} \int_W R\zeta \, \varphi \, d\mu^{\mathbf{s}} &= \sum_j \int_{W_j} \zeta \, \varphi \circ F \, K^{\mathbf{u}} \, d\mu^{\mathbf{s}} \ll \sum_j j^{-(1+\beta)} \int_{W_j} \zeta \, \varphi \circ F \, d\mu^{\mathbf{s}} \\ &\ll \sum_j j^{-(1+\beta)} \|\zeta\|_{\mathcal{B}_w} \ll \sum_j j^{-1+\kappa-\beta} < \infty. \end{aligned} \tag{C.8}$$

The stable part $\| \, \|_{\mathbf{s}}$ of $\| \, \|_{\mathcal{B}}$ is treated in the same way.

Now for the unstable part $\| \, \|_{\mathbf{u}}$, let φ such that $|\varphi|_{C^1(W)} \leq 1$. Using [BT17, Equation (43)], for any nearby leaves $W, \tilde{W} \in Y$, we define a diffeomorphism $v : \tilde{W} \to W$ as in [BT17, Proof of Proposition 3.2 (*unstable norm part*)]. Let $v_j = F^{-1} \circ v \circ F : \tilde{W}_j \to W_j$ be the corresponding bijection between the preimage leaves $W_j, \tilde{W}_j \subset Y_j$. Then we compute,

$$\begin{aligned} \left| \int_{\tilde{W}} R\zeta\varphi \, d\mu^{\mathbf{s}} - \int_W R\zeta\varphi \, d\mu^{\mathbf{s}} \right| &\leq \sum_j \left| \int_{\tilde{W}_j} \zeta\varphi \circ F \, K^{\mathbf{u}}_{\tilde{W}_j} \, d\mu^{\mathbf{s}} - \int_{W_j} \zeta\varphi \circ F \, K^{\mathbf{u}}_{W_j} \, d\mu^{\mathbf{s}} \right| \\ &\leq \sum_j \int_{\tilde{W}_j} \zeta |\varphi \circ v \circ F - \varphi \circ F| \, K^{\mathbf{u}}_{\tilde{W}_j} \, d\mu^{\mathbf{s}} \\ &\quad + \sum_j \int_{\tilde{W}_j} \zeta \, |\varphi \circ F| \left| K^{\mathbf{u}}_{W_j} \circ v_j - K^{\mathbf{u}}_{\tilde{W}_j} \right| d\mu^{\mathbf{s}} \\ &\quad + \sum_j \int_{W_j} |\zeta \circ v_j - \zeta| |\varphi \circ F| \, K^{\mathbf{u}}_{W_j} \, d\mu^{\mathbf{s}} \\ &= S_1 + S_2 + S_3. \end{aligned}$$

Next

$$S_1 \le |\varphi|_{C^1}\, d(W,\tilde W) \sum_j |\zeta|_{\tilde W_j}|_\infty\, |K^{\mathbf{u}}_{\tilde W_j}|_\infty \ll \|\varphi\|_{C^1}\, d(W,\tilde W),$$

because as in the first part of this proof, the sum in the above expression is bounded.

For the sum S_2, using (C.7) we split

$$\begin{aligned}\left|K^{\mathbf{u}}_{\tilde W_j} - K^{\mathbf{u}}_{W_j}\circ v_j\right| = {} & \frac{1}{J^{\mathbf{u}}_{m_{\bar\phi}}} \frac{|(h^{\mathbf{u}}\sin\gamma)\circ F\circ v_j - (h_0^{\mathbf{u}}\sin\gamma)\circ F|}{h_0^{\mathbf{u}}\sin\gamma} \\ & + \frac{1}{J^{\mathbf{u}}_{m_{\bar\phi}}}\left|\frac{J^{\mathbf{u}}_{m_{\bar\phi}}}{J^{\mathbf{u}}_{m_{\bar\phi}}\circ v_j} - 1\right| \frac{(h_0^{\mathbf{u}}\sin\gamma)\circ F\circ v_j}{h_0^{\mathbf{u}}\sin\gamma} \\ & + \frac{1}{J^{\mathbf{u}}_{m_{\bar\phi}}} \frac{(h^{\mathbf{u}}\sin\gamma)\circ F\circ v_j}{(h_0^{\mathbf{u}}\sin\gamma)\circ v_j}\left|\frac{(h^{\mathbf{u}}\sin\gamma)\circ v_j}{h_0^{\mathbf{u}}\sin\gamma} - 1\right|.\end{aligned}$$

By distortion estimate [BT17, Equation (40)], this is bounded by $Cd(W,\tilde W)\frac{1}{J^{\mathbf{u}}_{m_{\bar\phi}}}$ for some uniform distortion constant $C>0$. Therefore

$$S_2 \le C\, d(W,\tilde W)\sum_j \|\zeta\|_{\mathcal{B}_w}\left|\frac{1}{J^{\mathbf{u}}_{m_{\bar\phi}}}\right|_\infty \ll d(W,\tilde W)\sum_j j^\kappa j^{-(1+\beta)} < \infty$$

as before.

Now for S_3, the weighted $\|\ \|_{\mathbf{u}}$ part of the norm $\|\ \|_0$ gives $|\zeta\circ v_j - \zeta| \ll j^{\kappa+1+\beta} d(W_j,\tilde W_j) \ll j^{\kappa+1+\beta} d(W,\tilde W) L_{\mathbf{u}}(Y_j)$. As in the first half of the proof, $|K^{\mathbf{u}}_{W_j}|_\infty \ll j^{-(1+\beta)}$. We have $L_{\mathbf{u}}(Y_j) \ll \tau^{1+\beta}$ due to the small tail estimates (2.3), so

$$S_3 \ll |\varphi|_\infty d(W,\tilde W)\sum_j j^\kappa L_{\mathbf{u}}(Y_j) \ll |\varphi|_\infty d(W,\tilde W).$$

Therefore $\|R\zeta\|_{\mathcal{B}} < \infty$ and the proof is complete. ∎

Corollary C.6 *$R(\tau^\kappa h)\in\mathcal{B}$ for each $0<\kappa<\beta$ and $h\in\mathcal{B}$ (in particular $R(\tau h)\in\mathcal{B}$ as required for* **(H4)***(ii)).*

Proof Since τ is the return time of a C^1 flow to a C^1 Poincaré section, it is piecewise C^1, and we know $\tau|_{Y_j} \ll j$. Taking $h\in\mathcal{B}$ and $\varphi\in C^1(W)$ for an arbitrary stable leaf $W\subset Y_j$, we have $\int_W R(\tau^h)\,\varphi\, d\mu^{\mathbf{s}} \le j^\kappa \int_W h\varphi\, d\mu^{\mathbf{s}} = j^\kappa |h\|_{\mathcal{B}_w}$, so $\frac{1}{j^\kappa}\|\tau^\kappa h\|_{\mathcal{B}_w} \ll \|h\|_{\mathcal{B}_w} < \infty$.

Using (2.3), we have $\epsilon \le \xi(y,T+1) - \xi(y,T) \ll T^{-(1+\beta)}$ for $T=\tau(x,y)+O(1)$ and $\xi(y,T)$ plays the role of x. By (2.4), $x=\xi_0(y)\tau(x,y)^{-\beta}(1+o(1))$ as $x\to 0$, so $\epsilon \ll x^{\frac{1+\beta}{\beta}}$. Using (2.4) again, we can estimate that as $\tau=\tau(x)\to\infty$ (so as $x\to 0$ and $\epsilon=o(x)$),

$$\begin{aligned}|\tau^\kappa(x+\epsilon,y) - \tau^\kappa(x,y)| &= \xi_0(y)^\kappa |(x+\epsilon)^{-\frac{\kappa}{\beta}} - x^{-\frac{\kappa}{\beta}}|(1+o(1)) \\ &= \xi_0(y)^\kappa x^{-\frac{\kappa}{\beta}}\frac{\kappa}{\beta}\frac{\epsilon}{x}\left(1+o(1)+O\left(\frac{\epsilon}{x}\right)\right) \\ &= \frac{\kappa}{\beta}\xi_0(y)^{-\frac{\kappa}{\beta}}\tau(x,y)^{\kappa+\beta}\epsilon\left(1+o(1)+O\left(\frac{\epsilon}{x}\right)\right).\end{aligned} \tag{C.9}$$

Let $W, \tilde{W}$ nearby stable leaves in Y_j with $\epsilon = d(W, \tilde{W})$ and $v : \tilde{W} \to W$ a diffeomorphism. Then, for $\varphi \in C^1(W)$, $\tilde{\varphi} \in C^1(\tilde{W})$ with $d(\varphi, \tilde{\varphi}) \le d(W, \tilde{W})$,

$$\begin{aligned}\int_{\tilde{W}} \tau^k\, h\, \tilde{\varphi}\, d\mu^{\mathbf{s}} - \int_W \tau^k\, h\, \varphi\, d\mu^{\mathbf{s}} &\ll \int_{\tilde{W}} |\tau^\kappa - \tau^\kappa \circ v_j|\, h\, \varphi\, d\mu^{\mathbf{s}} + j^\kappa \left(\int_{\tilde{W}} h\, \tilde{\varphi}\, d\mu^{\mathbf{s}} - \int_W h\, \varphi\, d\mu^{\mathbf{s}} \right) \\ &\ll j^{\kappa+\beta} \|h\|_{\mathcal{B}_w} + j^\kappa \|h\|_{\mathbf{u}}.\end{aligned}$$

Therefore $\frac{1}{j^{\kappa+\beta}} \|\tau^\kappa h\|_{\mathbf{u}} \ll \|h\|_{\mathcal{B}_w} < \infty$, showing that $\tau^\kappa h \in \mathcal{B}_0$. Combined with Lemma C.5, the result follows. ∎

As in the statement of Theorem 5.1 (ii) and Proposition 5.6 we need

Corollary C.7 *$\tilde{\tau} := \tau - \tau^*$ cannot be written as $h \circ F - h$ for any $h \in \mathcal{B}$.*

Proof By Corollary C.6, $R\tilde{\tau} \in \mathcal{B}$. Because $\sup_{W \in \mathcal{W}^{\mathbf{s}}} \int_W \tilde{\tau}\, d\mu^{\mathbf{s}} = \infty$, $\tilde{\tau} \notin \mathcal{B}_w$ and hence not a coboundary. ∎

C.4 Verifying (H7)

Extending the inequality $|1 - e^{-x}| \le x^\gamma$ for all $\gamma \in (0, 1]$ and $x \ge 0$, we can find C_γ depending only on $s \sup_a \bar{\psi}$ such that

$$|e^{-u\tau + s\bar{\psi}} - 1| 1_a \le C_\gamma (u\tau + s\psi_0)^\gamma \le C_\gamma \left(u^\gamma \sup_a \tau^\gamma + s^\gamma \sup_a \psi_0^\gamma \right).$$

Therefore, for each $h \in \mathcal{B}$, and $\varphi \in C^1(W)$, $W \in \mathcal{W}^{\mathbf{s}}$,

$$\int_W \left| (e^{-u\tau + s\bar{\psi}} - 1) 1_a h\, \varphi \right| d\mu^{\mathbf{s}} \le C_\gamma \left(u^\gamma \sup_a \tau^\gamma + s^\gamma \sup_a \psi_0^\gamma \right) \int_W h\, \varphi\, d\mu^{\mathbf{s}},$$

so **(H7)** follows for $C_2 = C_3 = C_\gamma$.

C.5 Verifying (H6) (and thus, (H3))

In this section we verify **(H6)** for $\kappa > 1/\beta$.

Proposition C.8 *Assume that $\bar{\psi}$ satisfies* **(H5)** *and let $\hat{R}(u, s)v = R(e^{-u\tau} e^{s\bar{\psi}} v)$. Then there exists $\sigma_1 \in (0, 1)$ and $\delta, C_0, C_1 > 0$ such that for all $h \in \mathcal{B}$, $n \in \mathbb{N}$, $0 \le s < \delta$, and $u \ge 0$,*

$$\|\hat{R}(u, s)^n h\|_{\mathcal{B}_w} \le C_1 e^{-un} \|h\|_{\mathcal{B}_w}, \qquad \|\hat{R}(u, s)^n h\|_{\mathcal{B}} \le e^{-un} (C_0 \sigma_1^{-n} \|h\|_{\mathcal{B}} + C_1 \|h\|_{\mathcal{B}_w}).$$

Before turning to the proof, we need another lemma (which we will apply with $g \equiv 1$, but the general g is needed for the induction in the proof).

Lemma C.9 *Assume that $\bar\psi$ satisfies* **(H5)** *(in particular, $\bar\psi < C' - C^{-1}\tau^\kappa < \infty$ for some $C', C > 0$ and $\kappa > 1/\beta$). There exists $N \in \mathbb{N}$ depending only of $F : Y \to Y$ such that for all positive integrable functions g that are bounded and bounded away from zero, the following holds. There exist $\delta > 0$ such that for all $s \in [0, \delta]$, all admissible stable leaves $W \subset Y$ and all $n \in N$,*

$$\sum_{\substack{W' \text{ component} \\ \text{of } F^{-n}(W)}} \int_{W'} e^{s\bar\psi_n}\, g \circ F^n\, K^{\mathbf{u}}_{W'} F^n\, d\mu^{\mathbf{s}} \le e^{sN \sup \bar\psi} \int_W g\, d\mu^{\mathbf{s}},$$

where $K^{\mathbf{u}}_{W'}F^n$ equals the analogue of $K^{\mathbf{u}}_{W'}$ from (C.7) *for the iterate F^n on the preimage leaf W'.*

Proof Let P_1 be the partition element containing the images $F(\{r = k\})$ of the strips $\{r = k\}_{k \ge 2}$, see Figure 2. Let $N \in \mathbb{N}$ be such that $F^{N-1}(P_1)$ intersects each P_i, $i \ge 1$. Let W be an arbitrary stable leaf in P_i for some $i \ge 1$, and let $\mathcal{W}$ be the collection of preimage leaves under F^{-N}. By a change of coordinates,

$$\sum_{W' \in \mathcal{W}} \int_{W'} g \circ F^N\, K^{\mathbf{u}}_{W'} F^n\, d\mu^{\mathbf{s}} = \int_W g\, d\mu^{\mathbf{s}}$$

for any integrable function g.

Given $a \ge 1$ to be chosen later, set $\mathcal{W}^+ = \{W' \in \mathcal{W} : \inf_{W'} \tau^\kappa \ge 2NaC \sup \bar\psi\}$ and $\mathcal{W}^- = \mathcal{W} \setminus \mathcal{W}^+$. Then there is $\epsilon > 0$ (depending only on the geometry of the Markov map F and $\frac{\sup g}{\inf g}$) such that $\sum_{W' \in \mathcal{W}^+} \int_{W'} g \circ F^N\, K^{\mathbf{u}}_{W'} F^N\, d\mu^{\mathbf{s}} \ge \epsilon \int_W g\, d\mu^{\mathbf{s}}$. Since $\int_{W'} d\mu^{\mathbf{s}}$ is proportional to the measure of the element in $\bigvee_{i=0}^{N-1} F^{-i}(\mathcal{P})$ that W' belongs to, Proposition 2.4 gives $\epsilon \gg \mu_{\bar\phi}(\{\tau > (2NaC \sup \bar\psi)^{\frac{1}{\kappa}}\}) \gg Ba^{-\frac{1}{\beta'}}$ for some $B > 0$ and $\beta' := \kappa\beta > 1$. We have $\bar\psi_N \le (N-1) \sup \bar\psi - C^{-1}\tau^\kappa \le (N - 1 - 2Na) \sup \bar\psi \le -Na \sup \bar\psi$ on $\mathcal{W}^+$, and $\bar\psi_N \le N \sup \bar\psi$ on $\mathcal{W}^-$. Therefore

$$\begin{aligned}
&\sum_{W' \in \mathcal{W}} \int_{W'} e^{s\bar\psi_N}\, g \circ F^N\, K^{\mathbf{u}}_{W'} F^N\, d\mu^{\mathbf{s}} \\
&\quad \le \sum_{W' \in \mathcal{W}^+} \int_{W'} e^{-sNa \sup \bar\psi}\, g \circ F^N\, K^{\mathbf{u}}_{W'} F^N\, d\mu^{\mathbf{s}} + \sum_{W' \in \mathcal{W}^-} \int_{W'} e^{sN \sup \bar\psi}\, g \circ F^N\, K^{\mathbf{u}}_{W'} F^N\, d\mu^{\mathbf{s}} \\
&\quad \le \epsilon e^{-sNa \sup \bar\psi} \int_W g\, d\mu^{\mathbf{s}} + (1-\epsilon) e^{sN \sup \bar\psi} \int_W g\, d\mu^{\mathbf{s}} \\
&\quad = \big(\epsilon x^{-a} + (1-\epsilon)x\big) \int_W g\, d\mu^{\mathbf{s}} =: b(x) \int_W g\, d\mu^{\mathbf{s}}
\end{aligned}$$

for $x = e^{sN \sup \bar\psi}$. Clearly $b(1) = 1$ and $b'(x) = -a\epsilon x^{-(a+1)} + (1-\epsilon) < 0$ whenever $0 < x \le (a\epsilon)^{\frac{1}{1+a}}$. Since $\epsilon \ge Ba^{-\frac{1}{\beta'}}$, we find $b'(x) < 0$ for all $x \le a_0 := B^{-\frac{1}{1+a}} a^{\frac{\beta'-1}{\beta'(1+a)}}$. Choose $a > \max\{1, B^{\frac{\beta'}{\beta'-1}}\}$, so that $a_0 > 1$. This means that $b(x) \le 1$ for all $1 \le x \le a_0$, i.e., for all $0 \le s \le \frac{\log a_0}{N \sup \bar\psi}$.

Now for general $n = pN + q$ with $0 \le q < N$, we use induction on p. Let $\mathcal{W}_p(W)$ be the collection of preimages leaves of W under F^{-pN}. Assume by induction that

$$\sum_{W' \in \mathcal{W}_{p-1}(W_1)} \int_{W'} e^{s\bar\psi_{(p-1)N}}\, g \circ F^{(p-1)N}\, K^{\mathbf{u}}_{W'} F^{(p-1)N}\, d\mu^{\mathbf{s}} \le \int_W g\, d\mu^{\mathbf{s}}$$

for all stable leaves W_1 and g as above. Then

$$\sum_{W'\in\mathcal{W}_p(W)} \int_{W'} e^{s\bar{\psi}_{pN}}\, g\circ F^{pN}\, K^{\mathbf{u}}_{W'} F^{pN}\, d\mu^{\mathbf{s}}$$

$$\leq \sum_{W_1\in\mathcal{W}_1(W)} \sum_{W'\in\mathcal{W}_{p-1}(W_1)} \int_{W'} e^{s\bar{\psi}_{(p-1)N}} K^{\mathbf{u}}_{W'} F^{(p-1)N} \left(e^{s\bar{\psi}_N}\, g\circ F^N K^{\mathbf{u}}_{W'} F^N\right)\circ F^{(p-1)N}\, d\mu^{\mathbf{s}}$$

$$\leq \sum_{W_1\in\mathcal{W}_1(W)} \int_{W_1} e^{s\bar{\psi}_N}\, g\circ F^N\, K^{\mathbf{u}}_{W_1} F^N\, d\mu^{\mathbf{s}} \leq \int_W g\, d\mu^{\mathbf{s}}.$$

This way we proved the lemma for all multiples of N. For $0 < q < N$, we get an extra factor $e^{sq\sup\bar{\psi}}$. This proves the lemma. ∎

Proof of Proposition C.8 This proof goes as in [BT17, Proposition 3.2], but with some changes. The factor z^n is to be replaced with $e^{-u\tau_n}$ where $\tau_n = \sum_{i=0}^{n-1}\tau\circ F^i$ and the factor $e^{s\bar{\psi}_n}$ for $\psi_n = \sum_{i=0}^{n-1}\psi\circ F^i$ is dealt with using Lemma C.9. Let W and $\tilde{W}$ be two stable leaves in the same partition element of $\{P_i\}_{i\geq 1}$. Since τ and ψ are not constant on partition elements, we get a third and fourth term in the *strong unstable norm* part of [BT17, Proposition 3.2] expressing the difference of τ_n on nearby preimage leaves $\tilde{W}_j$ and $W_j = v_j(\tilde{W}_j)$ of W and $\tilde{W}$:

$$S_3 = \sum_j \left| \int_{\tilde{W}_j} h\tilde{\varphi}\circ F^n e^{s\bar{\psi}_n} \left(e^{-u\tau_n} - e^{-u(\tau_n\circ v_j)}\right) K^{\mathbf{u}}_{W_j} F^n\, d\mu^{\mathbf{s}} \right| \tag{C.10}$$

and

$$S_4 = \sum_j \left| \int_{\tilde{W}_j} h\tilde{\varphi}\circ F^n e^{-u\tau_n\circ v_j} \left(e^{s\bar{\psi}_n} - e^{s\bar{\psi}_n\circ v_j}\right) K^{\mathbf{u}}_{W_j} F^n\, d\mu^{\mathbf{s}} \right|, \tag{C.11}$$

for some $\tilde{\varphi}$ with $|\tilde{\varphi}|_{C^1(\tilde{W})} \leq 1$.

For S_3, without loss of generality, we can assume that $\tau_n\circ v_j \geq \tau_n$, so $|e^{-u\tau_n} - e^{-u(\tau_n\circ v_j)}| \leq ue^{-u\tau_n}(\tau_n\circ v_j - \tau_n)$. By (C.9), we have $|\tau(x+\epsilon,y)-\tau(x,y)| = \beta^{-1}\xi_0(y)^{-\frac{1}{\beta}}\tau(x,y)^{1+\beta}\epsilon(1+o(1)+O(\frac{\epsilon}{x}))$. Applied to $\epsilon_i := d(F^i(W_j),F^i(\tilde{W}_j)) \ll \tau^{-(1+\beta)}\circ F^i\epsilon_{i+1}$ this gives

$$\begin{aligned}\sum_{i=0}^{n-1} |\tau\circ F^i\circ v_j - \tau\circ F^i| &\ll \sum_{i=0}^{n-1} \tau^{1+\beta}\circ F^i\ d(F^i(W_j),F^i(\tilde{W}_j)) \\ &\ll \sum_{i=0}^{n-1} \epsilon_{i+1} \ll \epsilon_n = d(W,\tilde{W}),\end{aligned} \tag{C.12}$$

and therefore $|e^{-u\tau_n} - e^{-u\tau_n \circ v_j}| \ll u e^{-u\tau_n} d(W, \tilde{W})$. Combining the above with (C.10), we obtain

$$
\begin{aligned}
S_3 &\le u\ d(W,\tilde{W}) \sum_j \left| \int_{\tilde{W}_j} h\tilde{\varphi} \circ F^n e^{-u\tau_n + s\sup\bar{\psi}_n}\, K^{\mathbf{u}}_{W_j} F^n \, d\mu^{\mathbf{s}} \right| \\
&\ll \|h\|_{\mathcal{B}_w}\, u e^{-un}\, d(W,\tilde{W}) \sum_j \left| \int_{\tilde{W}_j} e^{s\bar{\psi}_n}\, K^{\mathbf{u}}_{W_j} F^n \, d\mu^{\mathbf{s}} \right| \\
&\le \|h\|_{\mathcal{B}_w}\, u e^{-un}\, d(W,\tilde{W}) e^{sN\sup\bar{\psi}},
\end{aligned}
$$

where we used that $\tau_n \ge n$, and the last step follows from Lemma C.9 with the N (independent of n, u, s) taken from that lemma too.

For S_4, using **(H5)** and (C.9), we obtain:

$$
\begin{aligned}
|e^{s\bar{\psi}(x+\epsilon)} - e^{s\bar{\psi}(x))}| &\ll e^{s\sup\bar{\psi}} e^{-s\tau^\kappa/C} s|\tau^\kappa(x+\epsilon,y) - \tau^\kappa(x,y)| \\
&= e^{s\sup\bar{\psi}} s e^{-s\tau^\kappa/C}\, \frac{\kappa}{\beta}\, \xi_0(y)\tau^{\kappa+\beta}\epsilon(1+o(1)+O(\epsilon)) \\
&\ll e^{s\sup\bar{\psi}} \underbrace{s e^{-s\tau^\kappa/C}\tau^{\kappa-1}}_{K}\, \tau^{1+\beta}\epsilon,
\end{aligned}
$$

where we compute the supremum over τ to conclude that $K \le e^{-\kappa}(C\kappa)^{\frac{\kappa-1}{\kappa}} s^{\frac{1}{\kappa}}$. Apply the above with $\epsilon = \epsilon_i := d(F^i(W_j), F^i(\tilde{W}_j))$ as in (C.12). Then we can estimate S_4 in the same way as S_3:

$$
\begin{aligned}
S_4 &\le s^{\frac{1}{\kappa}}\ d(W,\tilde{W}) \sum_j \left| \int_{\tilde{W}_j} h\tilde{\varphi} \circ F^n e^{-u\tau_n + s\bar{\psi}_n}\, K^{\mathbf{u}}_{W_j} F^n \, d\mu^{\mathbf{s}} \right| \\
&\ll \|h\|_{\mathcal{B}_w} e^{-un} e^{sN\sup\bar{\psi}}\, s^{\frac{1}{\kappa}}\, d(W,\tilde{W}).
\end{aligned}
$$

∎

References

[AD01a] J. Aaronson, M. Denker. Local limit theorems for partial sums of stationary sequences generated by Gibbs-Markov maps. *Stoch. Dyn.* **1** (2001) 193–237.

[AD01b] J. Aaronson, M. Denker. *A local limit theorem for stationary processes in the domain of attraction of a normal distribution.* In N. Balakrishnan, I.A. Ibragimov, V.B. Nevzorov, eds., Asymptotic methods in probability and statistics with applications. International conference, St. Petersburg, Russia, 1998, Basel: Birkhäuser, (2001) 215–224.

[Ab59a] L.M. Abramov. The entropy of a derived automorphism. *Dokl. Akad. Nauk SSSR* **128** (1959) 647–650.

[Ab59b] L.M. Abramov. On the entropy of a flow. *Dokl. Akad. Nauk SSSR* **128** (1959) 873–875.

[AK42] W. Ambrose, S. Kakutani. Structure and continuity of measurable flows. *Duke Math. J.* **9** (1942) 25–42.

[BF13] T. Barbot, S. Fenley, Pseudo-Anosov flows in toroidal manifolds *Geom. Topol.* **17** (2013), 1877–1954.

[BI06] L. Barreira, G. Iommi, Suspension flows over countable Markov shifts. *J. Stat. Phys.* **124** (2006), 207–230.

[BMT18] H. Bruin, I. Melbourne, D. Terhesiu. Rates of mixing for nonMarkov infinite measure semiflows. Preprint arXiv:1607.08711, To appear in *Trans. Amer. Math. Soc.*

[BMT] H. Bruin, I. Melbourne, D. Terhesiu. Lower bounds for non-Markovian finite measure semiflows and flows. In progress.

[BT17] H. Bruin, D. Terhesiu, Regular variation and rates of mixing for infinite measure preserving almost Anosov diffeomorphisms. *Ergodic Theory Dynam. Systems* (appeared online on August 10 2018, doi.org/10.1017/etds.2018.58).

[BTT18] H. Bruin, D. Terhesiu, M. Todd. The pressure function for infinite equilibrium measures. Preprint arXiv:1711.05069, To appear in Israel J. Math.

[DL08] M. Demers, C. Liverani. Stability of statistical properties in two dimensional piecewise hyperbolic maps. *Trans. Amer. Math. Soc.* **360** (2008) 4777–4814.

[D98] D. Dolgopyat. On the decay of correlations in Anosov flows. *Ann. of Math.* **147** (1998) 357–390.

[D03] D. Dolgopyat. Limit theorems for partially hyperbolic systems. *Trans. Amer. Math. Soc.* **356** (2003) 1637–1689.

[E04] G. K. Eagleson. Some simple conditions for limit theorems to be mixing. *Teor. Verojatnost.* **356** (2004) 1637–1689.

[F66] W. Feller. *An Introduction to Probability Theory and its Applications, II*, Wiley, New York, 1966.

[G04a] S. Gouëzel, Sharp polynomial estimates for the decay of correlations. *Israel J. Math.* **139** (2004) 29–65.

[G04b] S. Gouëzel. Central limit theorems and stable laws for intermittent maps. *Prob. Th. and Rel. Fields* **1** (2004) 82–122.

[GL06] S. Gouëzel, C. Liverani. Banach spaces adapted to Anosov systems. *Ergodic Theory Dynam. Systems* **26** (2006) 189–217.

[H00] H. Hu. Conditions for the existence of SBR measures of "almost Anosov" diffeomorphisms. *Trans. Amer. Math. Soc.* **352** (2000) 2331–2367.

[HY95] H. Hu, L-S. Young, Nonexistence of SBR measures for some diffeomorphisms that are "almost Anosov", *Ergodic Theory Dynam. Systems* **15** (1995) 67–76.

[IJT15] G. Iommi, T. Jordan, M. Todd. Recurrence and transience for suspension flows. *Israel J. Math.* **209** (2015) 547–592.

[KL99] G. Keller, C. Liverani. Stability of the spectrum for transfer operators. *Annali Della Scuola Normale Superiore di Pisa, Classe di Scienze* **19** (1999) 141–152.

[L04] C. Liverani. On contact Anosov flows. *Ann. of Math.* **159** (2004) 1275–1312.

[LT16] C. Liverani, D. Terhesiu. Mixing for some non-uniformly hyperbolic systems. *Annales Henri Poincaré* **17** no. 1 (2016) 179-226.

[MTe17] I. Melbourne, D. Terhesiu. Operator renewal theory for continuous time dynamical systems with finite and infinite measure. *Monatsh. Math.* **182** (2017) 377–431.

[MTo04] I. Melbourne, A. Török. Statistical limit theorems for suspension flows. *Israel J. Math.* **144** (2004) 191–209.

[N76] D. Naugler. Equivalence of suspensions and manifolds with cross section. Dynamical systems. *(*Proc. Internat. Sympos., Brown Univ., Providence, R.I., 1974) **Vol. II,** pp. 2931. Academic Press, New York, 1976.

[S99] O. Sarig. Thermodynamic formalism for countable Markov shifts. *Ergodic Theory Dynam. Systems* **19** (1999), 1565–1593.

[S06] O. Sarig. Continuous phase transitions for dynamical systems. *Commun. Math. Phys.* **267** (2006) 631–667.

[Sav98] S. Savchenko. Special flows constructed from countable topological Markov chains. *Funct. Anal. Appl.* **32** (1998) 32–41.

[Z07] R. Zweimüller. Mixing limit theorems for ergodic transformations, *J. Theoret. Probab.* **20** (2007) 1059–1071.