THE WORK OF Y. SINAĬ; ON THE OCCASION OF HIM RECEIVING THE ABEL PRIZE 2014

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ABSTRACT. On May 19 2014, Professor Yakov Sinaĭ (Moscow State University) was awarded the Abel Prize for his contributions to mathematical physics. We sketch some of his main achievements.

Receiving the Abel prize entailed for Professor Yakov Sinaĭ and his wife Elena Vul (also a mathematician) in a varied program of journeys (Oslo, Stavanger and Stockholm), lectures, receptions and interviews (with the Press and Martin Raussen & Christian Skau: the latter traditionally appears in the Notices of the AMS). The highlight was of course the award ceremony itself, at the hands of Crown Prince Haakon of Norway, on May 20 2014.

Yakov Grigorevich Sinaĭ was born in 1935 in a family of scientists. His parents were both microbiologists in Moscow and his grandfather a prominent mathematician, head of department of the differential geometry at Moscow State University. Sinaĭ obtained his first degree in 1957, which was also the year of his first publication. His master (equivalent to PhD) degree followed in 1960. The academic landscape in Moscow, within the rapidly developing fields of Ergodic Theory and Statistical Mechanics was truly remarkable: Chataev, Dynkyn and Kolmogorov were his advisers, and the faculty included Anosov, Krylov, Dobrushin, Gel'fand and others. The fact that he came from Jewish family, however, restricted his possibilities in the Russian system, and he was unable to get a full position at the Mathematics Department. Instead, he accepted a position at the Landau Institute of Theoretical Physics of the USSR Academy of Sciences. This enabled him collaborate with physicists as well as mathematicians, and to bridge the two disciplines, as he would continue to do in an unparalleled way. He introduced fundamental concepts of statistical physics into mathematics (Kolmogorov-Sinaĭ entropy, thermodynamic formalism, renormalization groups) giving them a rigorous basis.

Sinaĭ was a crucial figure in spread of ergodic theory. At the time, it was common for talented mathematicians in Eastern Europe to study in Moscow, and this is how Fritz, Krámli and Szász from Budapest and Krzyzewski and Szlenk from Warsaw first came in contact with the emerging field, and in due time founded schools in their home countries.

Sinaĭ's work was well-known in the West early on, and in 1962 he was invited to give plenary lecture at the ICM in Stockholm. Russian mathematicians being allowed to travel to the West were more the exception than the rule. For example, the support of the dissident poet-mathematician Esenin, resulted in Sinaĭ being barred from giving an address at the ICM in Nice in 1970. He was not alone in this; for example, Novikov was unable to come to Nice and accept the Fields' medal in person.

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In 1992, Sinaĭ joined the faculty of Princeton University, dividing his time between Moscow and Princeton from that time onwards, in addition to several guest professorships (such as CalTech in 2005). The Abel Prize is currently the last of a long list of awards: the Boltzmann Gold Medal (1986); the Heineman Prize (1989); the Markov Prize (1990); the Dirac Medal (1992); the Wolf Prize in Mathematics (1997); the Brazilian Award of Merits in Sciences (2000); the Moser Prize (2001); the Nemmers Prize in Mathematics (2002); the Henri Poincaré Prize (2009); the Dobrushin International Prize (2009).

Sinaĭ major work lies in Statistical Mechanics. The aim of this area is to derive the statistical "macroscopic" behaviour of material (which can be gases, liquids, but also fixed atoms or moving electrons in a grid) from the behaviour of the individual particles it consists of. This goes back to James Clerk Maxwell (1831 - 1879) and Ludwig Boltzmann (1844 - 1906), who applied the notions of *ergodicity* and *entropy*, although the modern form of these notions (partly due to Sinaĭ) is quite different from Boltzmann's original approach. The Laws of Thermodynamics predict that a system of particles strives to minimal energy and maximal entropy (= disorder) and the work of Josiah Willard Gibbs (1839 - 1903) united this by the introduction of equilibrium states. These are measures assigning probabilities to configurations of the system, where the weights are inverse proportional to the exponential of the potential energy of the configuration. The fact that particular energy levels are achieved by vastly more configurations than other energy levels creates an equilibrium between energy and entropy. In principle, the system can move away from the equilibrium, but if the number of particles is large[1] this becomes astronomically unlikely.

In his derivation of the H-Theorem (now called second law of thermodynamics) Boltzmann needed the so-called *Ergodic Hypothesis*:

> The trajectory of the point representing the state of the system in phase space passes through every point on the constant-energy hypersurface of the phase space.

This was criticised, not just as mathematically unfeasible, but also it was unknown if any system satisfied this hypothesis, even in a weakened form proposed in the influential survey paper by Tatiana and Paul Ehrenfest [7]. A rigorous proof for even the simplest systems of a few colliding particles (mathematical billiards) was beyond the state of mathematics at the time, and rigorous definitions of ergodicity (and entropy) had yet to be formulated.

In the 1930s, progress was made in the study of geodesic flow on surfaces of negative curvature. These are non-Euclidean "hyperbolic" surfaces on which initially close trajectories diverge at an exponential rate. This can be considered as a continuous version of dispersion as opposed to the dispersion at discrete time collisions of the particles in the billiard system. The first proofs of ergodicity (Hadamard [8], Artin [1]) for certain hyperbolic geodesic flows relied on number theoretic properties (continued fractions, the Gauß map), but in 1939, Eberhard Hopf (1902 - 1983) designed a general method of proving ergodicity for hyperbolic flows Φ_t. This became known as *Hopf Chains*, and relies on the fact that ergodic averages $\lim_{T\to\infty} \frac{1}{T}\int_0^T \psi \circ \Phi_t dt$ are constant on stable and unstable sets of points in configuration space. However, this method required smoothness, with stable and unstable sets stretching sufficiently far so as to create net spanning the entire configuration space. This condition is fulfilled for many geodesic flows, but not for systems of colliding particles.

[1]which is of course the case in practical situations; in fact the sheer number of particles makes the system completely intractable by deterministic methods

After about 20 years of no progress, the Russian school started to get involved. Using methods from measure theory, Andrej Kolmogorov (1903 - 1987) and Sinaĭ were able to formalize entropy in a effective way [12, 18]. A measure μ on the a (configuration) space $(X, \mathcal{B})$ is called flow-invariant if $\mu(A) = \mu(\Phi_t A)$ for all sets $A \subset \mathcal{B}$ and time $t \in \mathbb{R}$. Energy preserving (Hamiltonian) flows have a natural invariant measure, called Liouville measure, but there are many others.

Given a measure μ and a partition $\mathcal{Q}$ of the configuration space, the *entropy* of this partition is given by the sum

$$H(\mathcal{Q}, \mu) = -\sum_{Q \in \mathcal{Q}} \mu(Q) \log \mu(Q),$$

and it takes its maximal value $\log \#\mathcal{Q}$ when μ distributes the mass evenly over all partition elements $Q \in \mathcal{Q}$. This reflects that the entropy becomes largest when the probability of finding yourself in a particular state is spread the most. Assuming a discrete-time flow Φ_n for simplicity, let $\mathcal{Q}_n = \vee_{k=0}^{n-1} \Phi_{-k} \mathcal{Q}$ be the n-th joint of the partition; elements in $\mathcal{Q}_n$ are those sets of x that visit the same elements of $\mathcal{Q}$ in time steps $k = 0, \dots, n-1$. The Kolmogorov-Sinaĭ-entropy is now computed as the growth rate of $H(Q_n, \mu)$, and then maximized over all finite partitions $\mathcal{Q}$. That is:

$$h_\mu(\Phi_n) = \sup_{\mathcal{Q}} \lim_{n \to \infty} \frac{1}{n} H(\mathcal{Q}_n, \mu).$$

Kolmogorov-Sinaĭ-entropy became a wide-spread tool, also beyond statistical mechanics. There are parallels to Information Theory developed by Shannon [16] in the 1940s; I would also like to mention Ornstein's remarkable theorem [13] that in the context of two-sided Bernoulli shifts, entropy is a complete isomorphism invariant: two Bernoulli shifts are isomorphic if and only if they have the same entropy.

In the early 1970s, Sinaĭ, combined entropy with the potential energy U of Gibbs' approach to what is known as thermodynamic pressure[2]:

$$P(\beta U) = \sup\{h_\mu(\Phi_t) - \beta \int U d\mu\}.$$

Here the supremum is taken over all flow-invariant measures μ and the parameter $\beta = 1/kT$ for Boltzmann's constant k and absolute temperature T. Those flow-invariant measures μ that achieve this supremum play the role of equilibrium states in Gibbs' approach, in the sense that they realize equilibrium between the (maximal) entropy and (minimal) potential energy. Under some regularity conditions, equilibrium states satisfy the *Gibbs property* which means that the mass of sets $Q \in \mathcal{Q}_n$ scales as the exponential of the ergodic sum of $U - P$, i.e., $\mu(Q) \sim \exp(\sum_{k=0}^{n-1} U \circ \Phi_k - P(\beta U))$. As function of the inverse temperature parameter β, equilibrium states can vary continuously, or abruptly. The latter case is referred to as *phase transition*, in analogy between abrupt chance of equilibrium describing e.g. water in liquid versus frozen form, or a piece of iron in magnetised versus demagnetised form (cf. the Ising model). With this, Sinaĭ [21] laid the foundation for thermodynamic formalism in dynamics. Further contributions come from Rufus Bowen [2] and David Ruelle[3] [14]. A modern text book in ergodic theory with emphasis on this material was written by G. Keller [11].

[2]The connections with pressure from Newtonian physics is all but lost here.

[3]See [15] and [24] for some mutual 65th birthday wishes between the two.

FIGURE 1. Collision rule (left) and the billiard flow with one spherical scatterer on the two-torus (right).

Coming back to the billiard systems and the Hopf argument, which, as we mentioned breaks down for billiard systems. Colliding participles can create *singularities* in the flow when they collide tangentially (grazing collisions) or in three or more at exactly the same time. Such singularities create discontinuities in configuration space and prevent the proper construction of stable and unstable sets, making it impossible to carry out Hopf's argument. It was Sinaĭ who forced the breakthrough by showing that in sense of Liouville measure, stable and unstable sets can be defined and are sufficiently long at "most" points of configuration space.

The basic setup of a billiard flow is a particle (or several particles) moving with constant speed in some region Q (the billiard table) and reflecting elastically against the boundary ∂Q so that no kinetic energy is lost in collisions and the angle of incidence is the angle of reflection, see Figure 1. In formula, the velocity v' after collision is identified with the velocity before collision via

$$v' = v - 2\langle v, n(q)\rangle n(q), \qquad q \in \partial Q, \tag{0.1}$$

where $n(q)$ is the inward pointing normal vector at the collision point q of the boundary of the table, see [25]. The phase space M is the unit tangent bundle of Q, with identifications at the boundary according to (0.1). The billiard flow Φ_t preserves Liouville measure $dqd\omega_q$, where ω_q is uniform measure on the sphere of unit tangent vectors at $q \in Q$. Rather than the flow Φ_t, we can look at the collision map $F : \partial Q \times S \to \partial Q \times S$, where S is the "half-sphere" obtained from the unit sphere by the identification (0.1). The map F preserves a measure $\sin\theta d\theta dr$, for $r \in \partial Q$ and angle $\theta \in S$ with the normal vector $n(q)$. For this, ∂Q has to be piecewise smooth; corners of the billiard table, but also grazing collisions with the boundary, give rise to singularities. Usually, the particles are treated as hard spherical objects: the collision of two particles becomes just part of the regular boundary of the billiard table, but simultaneous collisions of three or more particles become "corners" of the billiard.

The first model that Sinaĭ managed to solve this way is a two-particle system on a two-dimensional torus, or equivalently a single particle colliding with a spherical scatterer in the two-dimensional torus, see Figure 1. The general version of this result is known as the Boltzmann-Sinaĭ postulate [19]:

> The system of N spherical particles with elastic collisions on the d-dimensional torus is ergodic.

In trying to extend this result to more particles and higher dimensions, addition technicalities come into play: finite versus infinite horizons, semi-dispersing versus fully dispersing billiard, cusps and other intricacies of the geometry. Together with Leonid Bunimovich [4] and later Nikolai Chernov, new techniques were introduced, [5, 4]. Gradually the Hungarian school (Szász, Krámli, Simányi, Bálint, ...) got more involved, also with the help of another Sinaĭ student Dolgopyat. This finally led to the full proof of the Sinaĭ-Boltzmann postulate by Nándor Simányi in 2013 [17]. Current directions in this field try to address the question of (rates of) mixing and further statistical properties of billiard flows. Without doubt, Sinaĭ's work, insights and encouragement over the span of sixty years have carved the landscape of billiard flows like nothing else.

In have restricted my discussion to areas that I am familiar with, leaving out Sinaĭ's further work on renormalization groups [23], Schrödinger operators [6], fluid mechanics, Navier-Stokes equations (with K. Khanin, J. Mattingly and D. Li), in fact countless topics in mathematical physics, but also number theory and stochastics (e.g. random walks in random environment [22]. His further contributions to dynamical systems include work on Markov partitions for hyperbolic systems (billiards [5, 3]), SRB-measures for (non-uniformly hyperbolic) systems, and their is his expository work of numerous text books, survey articles, and lecture series. Let me finally mention his prominent role in the mathematics community as a whole, and the impressive list of students that he supervised over the years. These include the already mentioned Bunimovich, Chernov, Mattingly and Dolgopyat, but also Bufetov, Gurevich, Jitomirskaya, Katok, Kornfeld, Margulis, Ratner and Ulcigrai.

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