On volume preserving almost Anosov flows

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Abstract

The purpose of this paper is to establish limit laws for volume preserving almost Anosov maps of $\mathbb{T}^2$ flows on 3-three manifolds having a transversally neutral periodic orbit of cubic saddle type. In the process, we derive estimates for the Dulac maps for neutral saddles in planar vector fields.

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1 Introduction

A flow $\phi^t : \mathcal{M} \times \mathbb{R} \to \mathcal{M}$ on a (in our setting 3-dimensional) compact differentiable manifold $\mathcal{M}$ is called *Anosov* if its tangent bundle has a continuous flow-invariant mutually transversal splitting into a neutral flow direction E^c, a hyperbolically stable direction E^s and a hyperbolically unstable direction E^u. The uniform hyperbolicity of such flows enables one to show various ergodic and statistical properties, such as ergodicity (if the flow is topologically mixing) and the Central Limit Theorem (CLT) for Hölder continuous observables.

We obtain an almost Anosov flow (see Definition 1.1 below) by inserting a neutral orbit $\Gamma \simeq \{(0,0)\} \times \mathbb{S}^1$ near which the flow has the following form in local Euclidean coordinates:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = X \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x(a_0x^2 + a_1xy + a_2y^2) \\ -y(b_0x^2 + b_1xy + b_2y^2) \\ 1 + w(x,y) \end{pmatrix} + \mathcal{O}(4) \tag{1}$$

where the parameters satisfy

$$a_1, b_1 \in \mathbb{R},\ a_0, a_2, b_0, b_2 \geq 0 \text{ with } \Delta := a_2b_0 - a_0b_2 \neq 0 \text{ and } a_1^2 < 4a_0a_2,\ b_1^2 < 4b_0b_2. \tag{2}$$

The last two conditions of (2) imply that the first component in (1) is non-negative and the second non-positive for $x, y \geq 0$. The term $\mathcal{O}(4)$ indicates terms of order four and higher, under the condition that they are of the form $x^2\mathcal{O}(2)$ near the yz-plane and of

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the form $y^2\mathcal{O}(2)$ near the xz-plane, because otherwise these are not a small perturbation compared to the leading terms of (1). These leading terms are cubic in the direction transversal to Γ, but this is the only source of non-hyperbolicity. Finally, w is a linear combination of homogeneous functions in x and y, vanishing at $(0,0)$. Thus the period of Γ is its length.

The original motivation to study such systems was to find a class of natural examples of non-uniformly hyperbolic invertible maps where operator renewal theory can be applied to establish precise statistical laws. The map in this context can be the Poincaré map on a section $\Sigma \subset \mathbb{R}^2 \times \{0\}$ or the time-1 map $f_{hor} = \phi^1_{hor}$ for the horizontal flow where only the x- and y-coordinates are taken into account:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = X_{hor} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x(a_0x^2 + a_1xy + a_2y^2) \\ -y(b_0x^2 + b_1xy + b_2y^2) \end{pmatrix} + \mathcal{O}(4), \tag{3}$$

with the restrictions (2).

Initially, in [2] for the parameter range $\beta_2 := \frac{a_2+b_2}{2b_2} \leq 1$ where f_{hor} preserves an infinite Sinaĭ-Bowen-Ruelle (SRB) measure, we gave mixing rates for C^1 observables. Later [3], and more relevant to this paper, in the parameter range $\beta_2 > 1$ where the flow ϕ^t preserves a finite SRB-measure, we established limit laws (Stable Laws and the CLT with standard or non-standard scaling, depending on whether $\beta_2 \in (1,2)$, $\beta = 2$ or $\beta_2 > 2$).

All these results were obtained in the absence of *mixed terms*, i.e., $a_1 = b_1 = 0$ in (1). This is of course not a natural assumption, and to our knowledge there is no change of coordinates that allows one to remove the mixed terms. In fact, if $a_1^2 > 4a_0a_2$ or $b_1^2 > 4b_0b_2$, then the dynamics near the saddle is not locally conjugate to the dynamics under the current condition (2).

The purpose of this paper is to perform the analysis when mixed terms are present, and also the treatment of the perturbation terms (see Section 3) is substantially different and more straightforward than in [2, 3]. The crux of the analysis is the existence of a local first integral (and its explicit form when $\mathcal{O}(4)$-terms are absent in (9)), which allows us to reduce the ODE to dimension one. We will show in Lemma 2.1 that the first integral L can be found if

$$\frac{b_1}{a_1} = \frac{b_2a_0 + a_2 + 2b_0b_2}{b_2a_0 + a_2 + 2a_0a_2}. \tag{4}$$

This is a co-dimension one condition in parameter space. However, if we also stipulate that the flow ϕ^t is volume preserving, we must assume that $\operatorname{div} X = 0$ in (1), which is equivalent to $\operatorname{div} \mathcal{O}(4) = 0$ together with

$$3a_0 = b_0, a_2 = 3b_2, a_1 = b_1. \tag{5}$$

Condition (4) follows automatically from (5), and therefore (1) describes a generic volume preserving almost Anosov flow with a single neutral periodic orbit of cubic saddle type. We present the results on limit laws in the volume preserving setting, see Theorem 1.2.

Central to the proof of the main theorem of this paper (as well as precise mixing rates) is the analysis of the Dulac map near the neutral equilibrium of (3). This means that we take an incoming and an outgoing transversal to the flow, in our case an unstable leaf $W^u(0,\eta)$, $\eta \in [\eta_0, \eta_1]$, and a stable leaf $W^s(\zeta_0, 0)$, see Figure 1. The *Dulac map* $D : W^u(0,\eta) \to W^s(\zeta_0, 0)$ assigns to (ξ,η) the first intersection $\phi^T_{hor}(\xi,\eta)$ of the integral curve through (ξ,η) with the outgoing transversal $W^s(\zeta_0, 0)$. The corresponding flow-time is denoted by T. As main technical result of this paper we obtain precise estimates of the Dulac map when (3) contains mixed terms, but using the assumption (4).

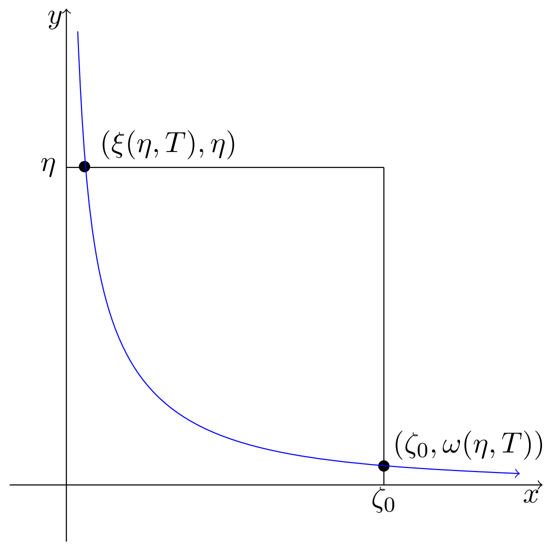

Figure 1: The Dulac map $D : (\xi(\eta_0, T), \eta_0) \mapsto (\zeta_0, \omega(\eta_0, T))$ with Dulac time T.

Dulac [5] introduced his map as an ingredient to prove that polynomial vector fields in the plane have at most finitely many limit cycles, thus making a major contribution to the solution of Hilbert's 16th problem. Il'yashenko [7] corrected some weak parts in Dulac's arguments, see also the summary in Roussarie's book [12, Section 3.3]. Further contributions are due to e.g. Dumortier and more recently Mardešić and collaborators [4, 8, 9, 13, 10]. Specifically, polycycles (i.e., heteroclinic saddle connections) are not accumulated upon by limit cycles. Whereas our estimates only concern a single neutral saddle, it is to our knowledge the first estimation of the Dulac times (and hence the Dulac map, see (6)), at cubic saddle of this generality.

1.1 Main results

The crucial estimates here are of the Dulac times, i.e., the times that orbits take to pass from an "incoming" unstable transversal to an "outgoing" unstable transversal to the flow, see Figure 1.

Theorem 1.1 *Consider a C^3 vector field of local form* (3) *with parameters satisfying* (2) *and* (4)*. Define*

$$\beta_0 := \frac{a_0 + b_0}{2a_0}, \qquad \beta_2 := \frac{a_2 + b_2}{2b_2}$$

Then there constants[1] $\xi_0(\eta), \omega_0(\eta)$ *such that the following asymptotics hold:*

$$\xi(\eta, \tilde{T}) = \xi_0(\eta)\tilde{T}^{-\beta_2}(1 + O(\tilde{T}^{-\frac{1}{2}}))$$

and

$$\omega(\eta, \tilde{T}) = \omega_0(\eta)\tilde{T}^{-\beta_0}(1 + O(\tilde{T}^{-\frac{1}{2}})).$$

as $T \to \infty$.

[1] The precise values of $\xi_0(\eta)$ and $\omega_0(\eta)$ are given in in the proof Proposition 2.1.

In particular, the functions ξ and ω are *regularly varying* of order $-\beta_2$ and $-\beta_0$ in T, respectively. That is: $\lim_{T\to\infty} \frac{\xi(\eta,cT)}{\xi(\eta,T)} = c^{\beta_2}$ for every $c > 0$ and analogous for $\omega(\eta,T)$. Moreover, the Dulac map $D : W^u(0,\eta) \to W^s(\zeta_0,0)$ itself has the form (as $\xi \to 0$)

$$\omega = D(\xi) = \omega_0(\eta)\xi_0(\eta)^{-\frac{\beta_0}{\beta_2}}\, \xi^{\frac{\beta_0}{\beta_2}} \left(1 + \mathcal{O}(\xi^{\frac{1}{2\beta_2}})\right). \tag{6}$$

With assumptions (5) and $c_1^2 < 4c_0c_2$ in place, we can use the change of coordinates $\bar{x} = \sqrt{a_0}x$, $\bar{y} = \sqrt{b_2}y$ and $\bar{\gamma} = a_1/\sqrt{a_0b_2} \in (-4,4)$ to transform (1) into the one-parameter family

$$\begin{pmatrix} \dot{\bar{x}} \\ \dot{\bar{y}} \\ \dot{\bar{z}} \end{pmatrix} = \begin{pmatrix} \bar{x}(\bar{x}^2 + \bar{\gamma}\bar{x}\bar{y} + 3\bar{y}^2) \\ -\bar{y}(3\bar{x}^2 + \bar{\gamma}\bar{x}\bar{y} + \bar{y}^2)) \\ 1 + \bar{w}(\bar{x},\bar{y}) \end{pmatrix} + \mathcal{O}(4), \tag{7}$$

for some transformed function $\bar{w}$.

Because of this genericity and reduced number of technicalities that Lebesgue measure gives as opposed to the SRB-measure, we state our statistical result for volume preserving flows. Theorem 1.1 is used to estimate the measures of the strips $\{\varphi = n\}$, see Figure 2, which in turn, together with the spectral properties of an induced Poincaré map $\hat{f}$, are crucial ingredients for the analysis required to establish the following stochastic limit properties of the flow ϕ^t.

Theorem 1.2 *Consider a volume preserving almost Anosov flow* (7) *on* $\mathcal{M}$ *with* $\bar{\gamma} \in (-4,4)$ *and an observables* $v : \mathcal{M} \to \mathbb{R}$ *that is* C^1 *on* $\mathcal{M} \setminus \Gamma$ *and has the form* $v = v_0 + o(\rho)$ *where* $\int_0^\tau v_0 \circ \phi^t\, dt$ *is homogeneous of order* $\rho > -2$ *in local coordinates* (x,y) *near* p *and* $o(\rho)$ *stands for terms of order* $> \rho$.

1. *If* $\rho \in (0,\infty)$, *then* v *satisfies the Central Limit Theorem, i.e.,*

$$\frac{\int_0^t v \circ \phi^s\, ds - t \int v\ dVol}{\sigma\sqrt{t}} \Rightarrow_{dist} \mathcal{N}(0,1) \qquad \text{as } t \to \infty,$$

where the variance $\sigma^2 > 0$ *unless* $\int_0^\tau v \circ \phi^t\, dt$ *is a coboundary.*

2. *If* $\rho = 0$, *then* v *satisfies the Central Limit Theorem with non-standard scaling* $\sqrt{t \log t}$, *i.e.,*

$$\frac{\int_0^t v \circ \phi^s\, ds - t \int v\ dVol}{\sigma\sqrt{t \log t}} \Rightarrow_{dist} \mathcal{N}(0,1) \qquad \text{as } t \to \infty,$$

and the variance $\sigma^2 > 0$ *unless* $\int_0^\tau v \circ \phi^t\, dt$ *is a coboundary.*

3. *If* $\rho \in (-2,0)$ *then* v *satisfies a Stable Law of order* $\frac{4}{2-\rho} \in (1,2)$.

Theorem 1.1 also allows us to derive other limit theorems such as in the infinite measure setting of [2], but with mixed terms. Since we restrict to flows preserving Lebesgue measure (not just an SRB-measure), we don't give any further details here.

1.2 Set-up

The set-up here is largely taken over from [3]. Our phase space will be the 3-dimensional compact manifold $\mathcal{M}$.

Definition 1.1 *[6, Definition 1] A diffeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ is called* almost Anosov *if there exists two continuous families of non-trivial cones $x \to \mathcal{C}^u_x, \mathcal{C}^s_x$ such that except for a finite set S,*

i) $Df_x \mathcal{C}^u_x \subseteq \mathcal{C}^u_{f(x)}$ and $Df_x \mathcal{C}^s_x \supseteq \mathcal{C}^s_{f(x)}$;

ii) $|Df_x v| > |v|$ for any $0 \neq v \in \mathcal{C}^u_x$ and $|Df_x v| < |v|$ for any $0 \neq v \in \mathcal{C}^s_x$.

For $x \in S$, Df_x is the identity.

A flow f^t on 3*-torus $\mathbb{T}^3$ is called* almost Anosov flow *if it has a finite set S of neutral periodic orbits, but everywhere else observes the condition of an Anosov flow in that there is a continuous splitting of the tangent bundle into a stable, an unstable and a neutral (flow) direction. For $x \in S$, the derivative at the return time τ is Df^τ_x is the identity.*

The time-1 map f of the flow ϕ^t of (1) has the form of a skew-product

$$f \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x(1 + a_0x^2 + a_1xy + a_2y^2) \\ y(1 - b_0x^2 - b_1xy - b_2y^2) \\ z + O(|w(x,y)|) \end{pmatrix} + \mathcal{O}(4), \tag{8}$$

see [2, Section 2.1]. Restricted to the (x, y)-coordinates, this restriction f_{hor} of f to the first two coordinates is a smooth almost Anosov map with a single neutral fixed point $p = (0, 0)$. Let $\{P_i\}_{i=0}^k$ be the Markov partition for f_{hor} (which we can assume to exist since f_{hor} is a local perturbation of a Anosov diffeomorphism on $\mathbb{T}^2$). We assume that p belongs to the interior of P_0. Clearly, the x- and y-axis are the unstable and stable manifolds of p respectively. We assume that the Markov partition element $P_0 \subset U$ is a small rectangle such that $f^{-1}_{hor}(P_0) \cup P_0 \cup f_{hor}(P_0) \subset U$. Due to the symmetries $(x, y) \mapsto (\pm x, \pm y)$, it suffices to do the analysis only in the first quadrant $Q = [0, \zeta_0] \times [0, \eta_0]$ of P_0, see Figure 2. Without loss of generality (see [2, Lemma 2.1]) we can think of $[0, \zeta_0] \times \{\eta_0\}$ as a local unstable leaf and $\{\zeta_0\} \times [0, \eta_0]$ as a local stable leaf of the global diffeomorphism.

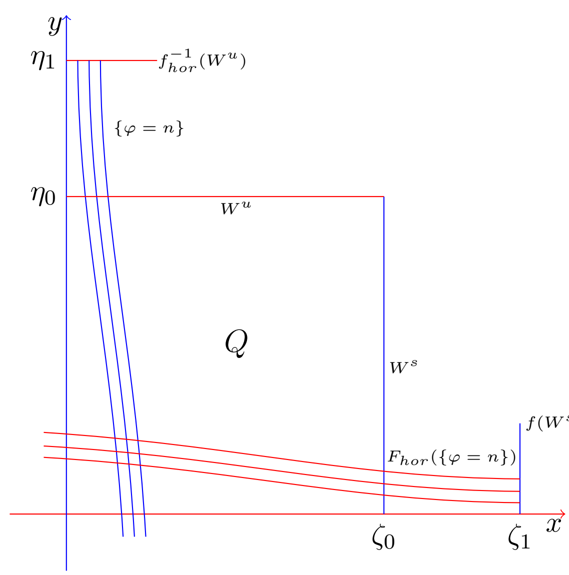

Figure 2: The first quadrant Q of the rectangle P_0, with stable and unstable foliations drawn vertically and horizontally, respectively.

We consider an induced map $F_{hor} = f_{hor}^{\varphi} : Y \to Y$ for $Y := \mathbb{T}^2 \setminus P_0$, where

$$\varphi(z) = \min\{n \geq 1 : f_{hor}^n(z) \notin P_0\}$$

is the first return time to Y. Note that F_{hor} is invertible because f_{hor} is. In the first quadrant of $U \setminus P_0$, the sets $\{\varphi = n\} := \{z \in f^{-1}(Q) \setminus Q : \varphi(z) = n\}$, $n \geq 2$, are vertical strips (see Figure 2) adjacent to the local unstable leaf $[0, \zeta_0] \times \{\eta_0\}$, and converging to $\{0\} \times [\eta_0, \eta_1]$ as $n \to \infty$. The images $F_{hor}(\{\varphi = n\})$ are horizontal strips, adjacent to the local stable leaf $\{\zeta_0\} \times [0, \eta_0]$, and converging to $[\zeta_0, \zeta_1] \times \{0\}$ as $n \to \infty$.

In contrast to f_{hor}, the induced map F_{hor} is uniformly hyperbolic, but only piecewise continuous. Indeed, continuity fails at the boundaries of the strips $\{\varphi = n\}$, $n \geq 2$ (and F is undefined on $W^s(p)$), but these boundaries are local stable and unstable leaves, and it is possible to create a countable Markov partition refining $\{P_i\}_{i=1}^k$ of Y for F, in which all the strips $\{\varphi = n\}$ are partition elements.

2 Regular variation of $\mu(\varphi > n)$ with mixed terms

In this section, we allow quadratic mixed terms in (3), but for the moment leave out the $\mathcal{O}(4)$-terms. That is, we consider

$$\begin{cases} \dot{x} = x(a_0x^2 + a_1xy + a_2y^2), \\ \dot{y} = -y(b_0x^2 + b_1xy + b_2y^2), \end{cases} \tag{9}$$

with the restrictions (2) and (4). The condition $c_1^2 < 4c_0c_2$ avoids the formation of invariant lines $y = px$, but in the below proofs it is used to guarantee that expressions as $c_0 + c_1M + c_2M^2$ for $M = y/x$ are positive. Our exposition closely follows [2], but since the mixed terms require adjustments throughout the proof, we will give it in full.

Set $c_i := a_i + b_i,\ i = 0, 1, 2$. The conditions $a_1^2 < 4a_0a_2, b_1^2 < 4b_0b_2$ imply that $a_1^2b_1^2 - 16a_0a_2b_0b_2 \leq 0 \leq 4(a_0b_0 - a_2b_2)^2$, and thus $a_1^2b_1^2 \leq 4(a_0b_2 + a_2b_0)^2$. Since $a_0, b_0, a_2, b_2 \geq 0$, we also have $a_1b_1 \leq 2(a_0b_2 + a_2b_0)$, which implies that $c_1^2 \leq 4c_0c_2$. Let $u, v \in \mathbb{R}$ be the solutions of the linear equations

$$\begin{cases} (u+2)a_0 = vb_0 \\ (v+2)b_2 = ua_2 \end{cases} \qquad \text{that is:} \qquad \begin{cases} u = \frac{2b_2c_0}{\Delta}, \\ v = \frac{2a_0c_2}{\Delta}. \end{cases} \tag{10}$$

Note that u, v and Δ (recall $\Delta \neq 0$) all have the same sign and (4) implies that $\frac{b_1}{a_1} = \frac{u+1}{v+1}$. Compute that

$$\beta_0 := \frac{a_0 + b_0}{2a_0} = \frac{u+v+2}{2v}, \quad \beta_2 := \frac{a_2 + b_2}{2b_2} = \frac{u+v+2}{2u}, \quad \frac{\beta_0}{\beta_2} = \frac{u}{v} = \frac{b_2c_0}{a_0c_2}, \tag{11}$$

and note that $\beta_0, \beta_2 > \frac{1}{2}$ (or $= \frac{1}{2}$ if we allow $b_0 = 0$ or $a_2 = 0$ respectively). Under the extra assumption (5) we obtain $\beta_0 = \beta_2 = 2$ and $u = v = 1$.

The first estimates is about the Dulac map of (3).

Proposition 2.1 *Consider a vector field on the 2-torus with local form (3) for $a_0, a_2, b_0, b_2 \geq 0$ and $\Delta \neq 0$. There are functions $\xi_0(\eta), \omega_0(\eta), \xi_1(\eta), \omega_1(\eta) > 0$ independent of T (with exact expressions given in the proof) such that*

$$\xi(\eta, T) = \xi_0(\eta)T^{-\beta_2}\left(1 - \xi_1(\eta)T^{-1} + O(T^{-2}, T^{-2\beta_2})\right)$$

and

$$\omega(\eta, T) = \omega_0(\eta)T^{-\beta_0}\left(1 - \omega_1(\eta)T^{-1} + O(T^{-2}, T^{-2\beta_0})\right).$$

Lemma 2.1 *The function*

$$L(x,y) = \begin{cases} x^u y^v (\frac{a_0}{v}\ x^2 + \frac{a_1}{v+1} xy + \frac{b_2}{u}\ y^2) & \text{if } \Delta > 0, \\ x^{-u} y^{-v} (\frac{a_0}{v}\ x^2 + \frac{a_1}{v+1} xy \frac{b_2}{u}\ y^2)^{-1} & \text{if } \Delta < 0, \end{cases} \tag{12}$$

is a first integral of (9).

Proof of Lemma 2.1. First assume $\Delta > 0$, so $u, v > 0$ as well. By (10), we can write $L(x,y)$ as

$$L(x,y) = x^u y^v (\frac{b_0}{u+2}\ x^2 + \frac{a_1}{v+1} xy + \frac{b_2}{u}\ y^2) = x^u y^v (\frac{a_0}{v}\ x^2 + \frac{b_1}{u+1} xy + \frac{a_2}{v+2}\ y^2).$$

Using these two equivalent expressions we compute the Lie derivative directly

$$\begin{aligned} \dot{L} &= \langle \nabla L, X \rangle \\ &= x^{u-1} y^v \left(\frac{b_0}{u+2}(u+2)x^2 + \frac{a_1}{v+1} xy + \frac{b_2}{u} u x^{u-1} y^2 \right) x(a_0x^2 + a_1xy + a_2y^2) \\ &\quad - x^u y^{v-1} \left(\frac{a_0}{v} x^2 v + \frac{a_1}{v+1} xy + \frac{a_2}{v+2}(v+2)y^2 \right) y(b_0x^2 + b_1xy + b_2y^2) \\ &= 0. \end{aligned}$$

Any function of a first integral is a first integral; in particular this holds for $1/L$. Therefore the conclusion is immediate for $\Delta < 0$ too. $\square$

Corollary 2.1 *Every $(x(y), y)$ on the integral curve from (ξ, η) to (δ, δ) satisfies*

$$x(y) = U(y)\delta^{1+\frac{a_2}{b_2}} \eta^{-\frac{a_2}{b_2}},$$

for

$$U(y) := (c_2 + \frac{a_1 u c_2}{b_2(v+1)} + c_0)^{\frac{1}{u}} (c_2 + \frac{a_1 u c_2}{b_2(v+1)} \frac{x}{y} + c_0 \frac{x^2}{y^2})^{-\frac{1}{u}}. \tag{13}$$

In terms $M = y/x$, we have

$$\begin{aligned} x(M)^2 &= \xi^{\frac{2u}{u+v+2}} \eta^{\frac{2v}{u+v+2}} M^{-\frac{2v}{u+v+2}} \left(\frac{c_0\xi^2 + c_1\xi\eta + c_2\eta^2}{c_0 + c_1 M + c_2 M^2} \right)^{\frac{2}{u+v+2}} \\ &= \xi^{\frac{1}{\beta_2}} \eta^{\frac{1}{\beta_0}} M^{-\frac{1}{\beta_0}} \left(\frac{c_0\xi^2 + c_1\xi\eta + c_2\eta^2}{c_0 + c_1 M + c_2 M^2} \right)^{1-\frac{1}{2\beta_0}-\frac{1}{2\beta_2}}. \end{aligned} \tag{14}$$

Proof. We carry out the proof for $\Delta > 0$, so $L(x,y) = x^u y^v (\frac{a_0}{v}\ x^2 + \frac{a_1}{v+1} xy + \frac{b_2}{u}\ y^2)$ as in Lemma 2.1. The case $\Delta < 0$ goes likewise.

We solve for x from $L(x,y) = L(\delta,\delta) = \delta^{u+v+2}(\frac{a_0}{v} + \frac{a_1}{v+1} + \frac{b_2}{u})$:

$$\begin{aligned} x = x(y) &= \delta^{\frac{u+v+2}{u}} y^{-\frac{v+2}{u}} (1 + \frac{a_1 u}{b_1(v+1)} + \frac{u a_0}{v b_2})^{\frac{1}{u}} (1 + \frac{a_1 u}{b_1(v+1)} \frac{x}{y} + \frac{c_0}{c_2} \frac{x^2}{y^2})^{-\frac{1}{u}} \\ &= U(y)\delta^{1+\frac{a_2}{b_2}} y^{-\frac{a_2}{b_2}}, \end{aligned}$$

giving the required expression with $U(y)$ as in (13). Note that $\lim_{y\to\delta} U(y) = 1$ (i.e., the limit as (x,y) approaches the diagonal), and $U(y)$ is differentiable.

For the other expression, we use that $L(x,y) = L(\xi,\eta) = \xi^u\eta^v(\frac{a_0}{v}\xi^2 + \frac{a_1}{v+1}\xi\eta + \frac{b_2}{u}\eta^2)$, and solve for x^2:

$$\begin{aligned}
\xi^u\eta^v(\frac{a_0}{v}\xi^2 + \frac{a_1}{v+1}\xi\eta + \frac{b_2}{u}\eta^2) &= x^u y^v(\frac{a_0}{v}x^2 + \frac{a_1}{v+1}xy + \frac{b_2}{u}y^2) \\
&= x^{u+v+2}M^v(\frac{a_0}{v} + \frac{a_1}{v+1}M + \frac{b_2}{u}M^2).
\end{aligned}$$

Here we used

$$\begin{aligned}
\xi^u\eta^v(\frac{a_0}{v}\xi^2 + \frac{a_1}{v+1}\xi\eta + \frac{b_2}{u}\eta^2) &= \xi^u\eta^v(\frac{a_0\Delta}{2a_0c_2}\xi^2 + \frac{a_1\Delta}{2a_0c_2+\Delta}\xi\eta + \frac{b_2\Delta}{2b_2c_0}\eta^2) \\
&= \frac{\Delta}{2c_0c_2}\left(c_0\xi^2 + \frac{2a_1c_0c_2}{2a_0c_2+\Delta}\xi\eta + c_2\eta^2\right) \\
&= \frac{\Delta}{2c_0c_2}\left(c_0\xi^2 + c_1\xi\eta + c_2\eta^2\right), \qquad (15)
\end{aligned}$$

(where the last step follows from (4)) and a similar computation for the term with x, y. Use (10) and (11) to obtain

$$\begin{cases}
\frac{a_0}{v} + \frac{a_1}{v+1}M + \frac{b_2}{u}M^2 = \frac{\Delta}{2c_0c_2}(c_0 + c_1M + c_2M^2), \\
\frac{a_0\xi^2}{v} + \frac{a_1}{v+1}\xi\eta + \frac{b_2\eta^2}{u} = \frac{\Delta}{2c_0c_2}(c_0\xi^2 + c_1\xi\eta + c_2\eta^2).
\end{cases}$$

This gives

$$\begin{aligned}
x^2 &= \xi^{\frac{2u}{u+v+2}}\eta^{\frac{2v}{u+v+2}}M^{-\frac{2v}{u+v+2}}\left(\frac{c_0\xi^2 + c_1\xi\eta + c_2\eta^2}{c_0 + c_1M + c_2M^2}\right)^{\frac{2}{u+v+2}} \\
&= \xi^{\frac{1}{\beta_2}}\eta^{\frac{1}{\beta_0}}M^{-\frac{1}{\beta_0}}\left(\frac{c_0\xi^2 + c_1\xi\eta + c_2\eta^2}{c_0 + c_1M + c_2M^2}\right)^{1-\frac{1}{2\beta_0}-\frac{1}{2\beta_2}},
\end{aligned}$$

where we recall that $\beta_0 = \frac{u+v+2}{2v}$ and $\beta_2 = \frac{u+v+2}{2u}$ from (11), which also gives $\frac{2}{u+v+2} = 1 - \frac{1}{2\beta_0} - \frac{1}{2\beta_2}$. □

Proof of Proposition 2.1. We carry out the proof for $\Delta > 0$, so $L(x,y) = x^u y^v(\frac{a_0}{v}\ x^2 + \frac{a_1}{v+1}xy + \frac{b_2}{u}\ y^2)$ as in Lemma 2.1. The case $\Delta < 0$ goes likewise. Fix η such that $(\xi(\eta,T),\eta) \in \overline{\phi^{-1}(Q) \setminus Q}$. We use the variable $M = y/x$, so $y = Mx$ and differentiating gives $\dot{y} = \dot{M}x + M\dot{x}$. Recalling that $c_i = a_i + b_i$ and inserting the values for $\dot{x}$ and $\dot{y}$ from (9), we get

$$\dot{M} = -M(c_0 + c_1M + c_2M^2)x^2. \qquad (16)$$

Combined with (14), this gives

$$\dot{M} = -G(\xi,\eta)M^{1-\frac{1}{\beta_0}}\left(c_0 + c_1M + c_2M^2\right)^{\frac{1}{2\beta_0}+\frac{1}{2\beta_2}} \qquad (17)$$

with

$$G(\xi,\eta) := \xi^{\frac{1}{\beta_2}}\eta^{\frac{1}{\beta_0}}\left(c_0\xi^2 + c_1\xi\eta + c_2\eta^2\right)^{1-\frac{1}{2\beta_0}-\frac{1}{2\beta_2}}. \qquad (18)$$

For the exit time $T \geq 0$, recall that $\xi(\eta,T)$ and $\omega(\eta,T)$ are such that the solution of (3) satisfies $(x(0),y(0)) = (\xi(\eta,T),\eta)$ and $(x(T),y(T)) = (\zeta_0,\omega(\eta,T))$. This implies $M(0) = \eta/\xi(\eta,T)$ and $M(T) = \omega(\eta,T)/\zeta_0$. Inserting this in (17), separating variables, and integrating we get

$$\int_{\omega(\eta,T)/\zeta_0}^{\eta/\xi(\eta,T)} \frac{M^{\frac{1}{\beta_0}-1}\,dM}{(c_0 + c_1M + c_2M^2)^{\frac{1}{2\beta_0}+\frac{1}{2\beta_2}}} = G(\xi(\eta,T),\eta)T. \qquad (19)$$

In the rest of the proof, we will frequently suppress the dependence on η and T in $\xi(\eta, T)$ and $\omega(\eta, T)$. We know that $L(\xi(\eta, T), \eta) = \xi^u \eta^v (\frac{a_0}{v}\xi^2 + \frac{b_2}{u}\eta^2) = \zeta_0^u \omega^v (\frac{a_0}{v}\zeta_0^2 + \frac{b_2}{u}\omega^2) = L(\eta, \omega(\eta, T))$, which gives

$$\xi^u \eta^v (c_0\xi^2 + c_1\xi\eta + c_2\eta^2) = \zeta_0^u \omega^v (c_0\zeta_0^2 + c_1\zeta_0\omega + c_2\omega^2). \tag{20}$$

From their definition, $\xi(\eta, T)$ and $\omega(\eta, T)$ are clearly decreasing in T, so their T-derivatives $\xi'(\eta, T), \omega'(\eta, T) \leq 0$. Since $c_0, c_2 > 0$ (otherwise $\Delta = 0$), the integrand of (19) is $O(M^{\frac{1}{\beta_0}-1})$ as $M \to 0$ and $O(M^{-\frac{1}{\beta_2}-1})$ as $M \to \infty$. Hence the integral is increasing and bounded in T. But this means that $G(\xi(\eta, T), \eta)T$ is increasing in T and bounded as well. Let $g(\eta, T) = \xi(\eta, T)T^{\beta_2}$. Since

$$G(\xi(\eta, T), \eta)T = g(\eta, T)^{\frac{1}{\beta_2}} \eta^{\frac{1}{\beta_0}} (c_0 g(\eta, T)^2 T^{-2\beta_2} + c_1 g(\eta, T)T^{-\beta_2} + c_2\eta^2)^{1-\frac{1}{2\beta_0}-\frac{1}{2\beta_2}},$$

and $1 - \frac{1}{2\beta_0} - \frac{1}{2\beta_2} > 0$, we find that $g(\eta, T)$ converges[2]:

$$\xi_0(\eta) := \lim_{T\to\infty} g(\eta, T) = c_2^{-\frac{1}{u}} \eta^{-\frac{a_2}{b_2}} \left(\int_0^\infty \frac{M^{\frac{1}{\beta_0}-1}\, dM}{(c_0 + c_1 M + c_2 M^2)^{\frac{1}{2\beta_0}+\frac{1}{2\beta_2}}} \right)^{\beta_2}, \tag{21}$$

where we have used $-\beta_2(1 - \frac{1}{2\beta_0} - \frac{1}{2\beta_2}) = -\frac{2\beta_2}{u+v+2} = -\frac{1}{u}$ for the exponent of c_2, and $\frac{2}{u} + \frac{\beta_2}{\beta_0} = \frac{v+2}{u} = \frac{a_2}{b_2}$ for the exponent of η.

We continue the proof to get higher asymptotics. Differentiating (19) w.r.t. T gives

$$-\frac{\eta^{\frac{1}{\beta_0}} \xi^{\frac{1}{\beta_2}-1} \xi'}{(c_0\xi^2 + c_1\xi\eta + c_2\eta^2)^{\frac{1}{2\beta_0}+\frac{1}{2\beta_2}}} - \frac{\zeta_0^{\frac{1}{\beta_2}} \omega^{\frac{1}{\beta_0}-1} \omega'}{(c_0\zeta_0^2 + c_1\zeta_0\omega + c_2\omega^2)^{\frac{1}{2\beta_0}+\frac{1}{2\beta_2}}} = \frac{\partial G(\xi, \eta)}{\partial \xi} T\xi' + G(\xi, \eta), \tag{22}$$

where (by differentiating (18))

$$\frac{\partial G(\xi, \eta)}{\partial \xi} = (2b_0\xi^2 + c_1(\frac{b_0}{c_0} + \frac{a_2}{c_2})\xi\eta + b_2\eta^2)\xi^{\frac{1}{\beta_2}-1} \eta^{\frac{1}{\beta_0}} (c_0\xi^2 + c_1\xi\eta + c_2\eta^2)^{-\frac{1}{2\beta_0}-\frac{1}{2\beta_2}}.$$

Combined with (18), (20) and (22), this gives

$$\begin{aligned} -\eta^{\frac{1}{\beta_0}} \xi' - \zeta_0^{\frac{1}{\beta_2}} \left(\frac{\zeta_0^u \omega^v}{\xi^u \eta^v} \right)^{\frac{1}{2\beta_0}+\frac{1}{2\beta_2}} \frac{\omega^{\frac{1}{\beta_0}-1}}{\xi^{\frac{1}{\beta_2}-1}} \omega' \\ = (2b_0\xi^2 + c_1(\frac{b_0}{c_0} + \frac{a_2}{c_2})\xi\eta + b_2\eta^2)T\eta^{\frac{1}{\beta_0}} \xi' + \eta^{\frac{1}{\beta_0}} \xi (c_0\xi^2 + c_1\xi\eta + c_2\eta^2). \end{aligned} \tag{23}$$

Because $\frac{1}{2\beta_0} + \frac{1}{2\beta_2} - 1 = -\frac{2}{u+v+2}$, using (11) and dividing by $\eta^{\frac{1}{\beta_0}}$, we can simplify (23) to

$$-\xi' - \frac{\zeta_0^u}{\eta^v} \frac{\omega^{v-1}}{\xi^{u-1}} \omega' = (2b_0\xi^2 + c_1(\frac{b_0}{c_0} + \frac{a_2}{c_2})\xi\eta + b_2\eta^2)T\xi' + \xi(c_0\xi^2 + c_1\xi\eta + c_2\eta^2). \tag{24}$$

Taking the derivative of (20) w.r.t. T and multiplying with $\Delta/(c_0c_2)$ gives

$$(2b_0\xi^2 + c_1(\frac{b_0}{c_0} + \frac{a_2}{c_2})\xi\eta + b_2\eta^2)\eta^v \xi^{u-1} \xi' = (2a_0\zeta_0^2 + +c_1(\frac{b_0}{c_0} + \frac{a_2}{c_2})\zeta_0\omega + 2a_2\omega^2)\zeta_0^u \omega^{v-1} \omega'.$$

[2] For the symmetric statement on $\omega(\eta, T)$, define $\hat{g}(\eta, T) = \omega(\eta, T)T^{\beta_0}$. Then $\lim_{T\to\infty} \hat{g}(\eta, T) = \lim_{T\to\infty} g(\eta, T)^{\beta_0/\beta_2} \eta^{1+2/v} \zeta_0^{-b_0/a_0} (\frac{c_2}{c_0})^{1/v}$.

Hence, we can rewrite (24) as

$$
\begin{aligned}
-\Big(1+&\frac{2b_0\xi^2+c_1(\frac{b_0}{c_0}+\frac{a_2}{c_2})\xi\eta+2b_2\eta^2}{2a_0\zeta_0^2+c_1(\frac{b_0}{c_0}+\frac{a_2}{c_2})\zeta_0\omega+2a_2\omega^2}\Big)\xi' \\
&= \ (2b_0\xi^2+c_1(\frac{b_0}{c_0}+\frac{a_2}{c_2})\xi\eta+b_2\eta^2)T\xi'+\xi(c_0\xi^2+c_1\xi\eta+c_2\eta^2).
\end{aligned}
$$

We insert $\xi'=g'(T)T^{-\beta_2}-\beta_2 g(T)T^{-(1+\beta_2)}$ and multiply with T^{β_2}, which leads to

$$
\begin{aligned}
-\Big(1+&\frac{2b_0\xi^2+c_1(\frac{b_0}{c_0}+\frac{a_2}{c_2})\xi\eta+2b_2\eta^2}{2a_0\zeta_0^2+c_1(\frac{b_0}{c_0}+\frac{a_2}{c_2})\zeta_0\omega+2a_2\omega^2}\Big)(g'(T)-\beta_2 g(T)T^{-1}) \\
&= \ (2b_0\xi^2+c_1(\frac{b_0}{c_0}+\frac{a_2}{c_2})\xi\eta+2b_2\eta^2)g'(T)\ T-\frac{\Delta}{b_2}g(T)^3T^{-2\beta_2}.
\end{aligned}
$$

Since $\xi=O(T^{-\beta_2})$ and $\omega=O(T^{-\beta_0})$, we can write this differential equation as

$$
\frac{g'}{g}=\frac{1}{T^2}\frac{\beta_2}{2}\frac{\frac{a_0\zeta_0^2+b_2\eta^2+O(T^{-2\beta_2})}{a_0\zeta_0^2+O(T^{-2\beta_0})}-\frac{\Delta}{b_2}g(T)^2T^{-\frac{a_2}{b_2}}}{b_2\eta^2+O(T^{-2\beta_2})+O(T^{-1})}.
$$

Keeping the leading terms only (where we use that $2\beta_2, 2\beta_0>1$), we get the differential equation

$$
\frac{g'}{g}=(\xi_1(\eta)+O(\max\{T^{-1},T^{-\frac{a_2}{b_2}}\}))\frac{1}{T^2}\qquad \text{for } \xi_1(\eta):=\frac{\beta_2}{2}\left(\frac{1}{a_0\zeta_0^2}+\frac{1}{b_2\eta^2}\right).
$$

Using the limit boundary value $\xi_0=\xi_0(\eta)=\lim_{T\to\infty}g(\eta,T)$, we find the solution

$$
g(\eta,T)=\xi_0 e^{-(\xi_1+O(\max\{T^{-1},T^{-\frac{a_2}{b_2}}\}))T^{-1}}=\xi_0(1-\xi_1T^{-1}+O(\max\{T^{-2},T^{-2\beta_2}\}))
$$

as required. The analogous asymptotics for ω and the constants ω_0 and ω_1 can be derived by changing the time direction and the roles $(a_0,a_2)\leftrightarrow(b_2,b_0)$, and also by the relation $\xi^u\eta^{v+2}c_2\sim\zeta_0^{u+2}\omega^v c_0$ from (20):

$$
\omega_0(\eta)=c_0^{-\frac{1}{v}}\zeta_0^{-\frac{b_0}{a_0}}\left(\int_0^\infty\frac{M^{\frac{1}{\beta_2}-1}\,dM}{(c_0M^2+c_1M+c_2)^{\frac{1}{2\beta_0}+\frac{1}{2\beta_2}}}\right)^{\beta_0},\quad \omega_1(\eta)=\frac{\beta_0}{2}\left(\frac{1}{b_2\zeta_0^2}+\frac{1}{a_0\eta^2}\right).
$$

This concludes the proof. □

3 Proof of Theorem 1.1

To prove that the regular variation established in Proposition 2.1 is robust under perturbations of the vector field, we put the $\mathcal{O}(4)$ terms back into (3), but since we consider it as a perturbation of (9), we write $\tilde X$ instead:

$$
\tilde X=\begin{pmatrix}\tilde X_1\\ \tilde X_2\end{pmatrix}=\begin{pmatrix}x(a_0x^2+a_1xy+a_2y^2)\\ -y(b_0x^2+b_1xy+b_2y^2)\end{pmatrix}+\mathcal{O}(4), \tag{25}
$$

so that $|\tilde X-X|=\mathcal{O}(4)$. The quantities $\xi(\eta,T),\omega(\eta,T)$ will be written as $\tilde\xi(\eta,T),\tilde\omega(\eta,T)$ etc., and the goal is to show that $\tilde\xi(\eta,T)$ is still regularly varying.

As before, let $\xi = \xi(\eta, T)$ be such that for the unperturbed flow, $\phi^T(\xi, \eta) = (\zeta_0, \omega(\eta, T))$. Proposition 2.1 gives the asymptotics of $\xi(\eta, T)$ as $T \to \infty$. At the same time, under the perturbed flow associated to (25), $\phi^{\tilde{T}}(\xi, \eta) = (\zeta_0, \tilde{\omega}(\eta, \tilde{T}))$ for some $\tilde{T}$. Therefore we can write $\xi(\eta, T) = \tilde{\xi}(\eta, \tilde{T})$, and once we estimated $\tilde{T}$ as function of T, we can express $\tilde{\xi}(\eta, \tilde{T})$ explicitly as function of $\tilde{T}$. We follow the argument of the proof of Proposition 2.1, keeping track of the effect of the higher order terms.

The perturbed first integral: To start, we construct a first integral $\tilde{L}$ on $Q = [0, \zeta_0] \times [0, \eta_0]$ by defining

$$\tilde{L}(\tilde{\phi}^t(\delta, \delta)) = L(\delta, \delta) = \begin{cases} \delta^{u+v+2} \left(\frac{a_0}{v} + \frac{a_1}{v+1} + \frac{b_2}{u}\right) & \text{if } \Delta > 0, \\ \delta^{-(u+v+2)} \left(\frac{a_0}{v} + \frac{a_1}{v+1} + \frac{b_2}{u}\right)^{-1} & \text{if } \Delta < 0, \end{cases}$$

for $0 < \delta \leq \min\{\zeta_0, \eta_0\}$ and $t \in \mathbb{R}$. (We continue the argument for the case $\Delta > 0$; the other case goes analogously.)

By construction, $\tilde{L}$ is constant on integral curves of $\dot{z} = \tilde{X}(z)$. Because $\tilde{X}$ is C^3, the integral curves are C^3 curves, and form a C^3 foliation of P_0, see e.g. [14, Theorem 2.10]. Note that the coordinate axes consist of the stationary point $(0, 0)$ and its stable and unstable manifold; we put $\tilde{L}(x, 0) = \tilde{L}(0, y) = 0$. Then $\tilde{L}$ is continuous on Q and C^3 on the interior of Q.

Now we compare $\tilde{L}$ with L on a small neighbourhood U of $\phi_{hor}^{-1}(Q) \cup Q \cup \phi_{hor}^{1}(Q)$. Take $y_0 = \eta_0$ and $x_0 = x_0(\delta)$ such that the integral curve of $\dot{z} = X(z)$ through $z_0 := (x_0, y_0)$ intersects the diagonal at (δ, δ). Then the integral curve of $\dot{z} = \tilde{X}(z)$ through z_0 intersects the diagonal at $(\tilde{\delta}, \tilde{\delta})$ for some $\tilde{\delta} = \tilde{\delta}(\delta)$, see Figure 3.

For some z_1 on this integral curve between z_0 and $(\tilde{\delta}, \tilde{\delta})$, the integral curve through z_1 of the unperturbed system intersects the diagonal in some point (δ', δ') between (δ, δ) and $(\tilde{\delta}, \tilde{\delta})$. Therefore

$$\tilde{L}(z_1) = \tilde{L}(\tilde{\delta}, \tilde{\delta}) = L(\tilde{\delta}, \tilde{\delta}) = L(\delta', \delta') \left(\frac{\tilde{\delta}}{\delta'}\right)^{u+v+2} = L(z_1) \left(\frac{\tilde{\delta}}{\delta'}\right)^{u+v+2}. \tag{26}$$

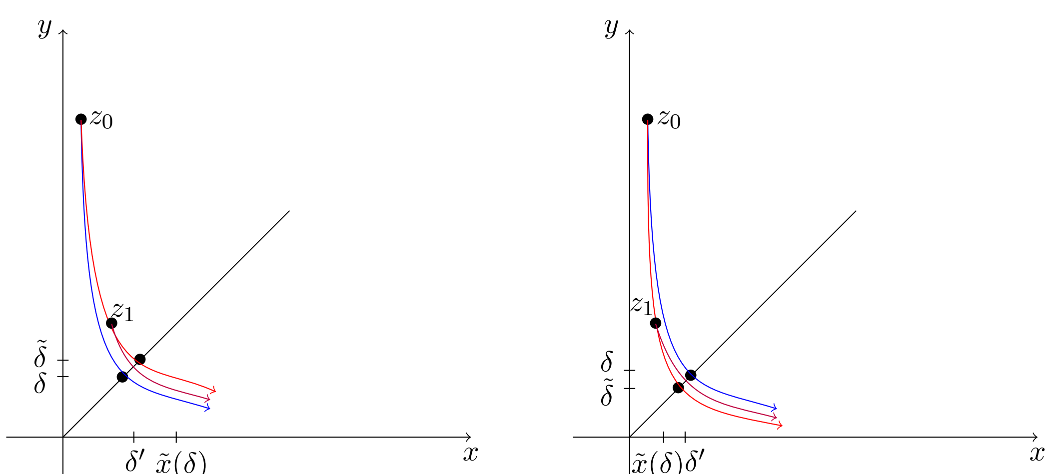

Figure 3: Solutions of (27) and (28), starting from the same point $z_0 = (x_0, y_0)$.

Estimating $|\tilde{\delta} - \delta|$**:** Parametrise the integral curve of X through z_0 as $(x(y), y)$ for $\min\{\delta, \tilde{\delta}\} \le y \le y_0$. (So $x \le y$; the case $y \le x$ can be dealt with by switching the roles of x and y.) Then by (9):

$$x'(y) = F(x, y) := -\frac{x(a_0x^2 + a_1xy + a_2y^2)}{y(b_0x^2 + b_1xy + b_2y^2)}. \tag{27}$$

For the perturbed vector field (25) we parametrise the integral curve through z_0 as $(\tilde{x}(y), y)$ and we have the analogue of (27):

$$\tilde{x}'(y) = \tilde{F}(\tilde{x}, y) := -\frac{\tilde{x}(a_0\tilde{x}^2 + a_1\tilde{x}y + a_2y^2 + \sum_{j=1}^{3} \hat{a}_j\tilde{x}^j y^{3-j})}{y(b_0\tilde{x}^2 + b_1\tilde{x}y + b_2y^2 + \sum_{j=1}^{3} \hat{b}_j\tilde{x}^j y^{3-j})}, \tag{28}$$

where $\hat{a}_j, \hat{b}_j$ are bounded functions in $\tilde{x}$ and y. The absence of the terms $\hat{a}_0y^3$ and $\hat{b}_0y^3$ is because of the assumption that $\mathcal{O}(4) = x^2\mathcal{O}(2)$ near the yz-plane. For the region $\tilde{x} > y$ we need to leave out the terms $\hat{a}_3\tilde{x}^3$ and $\hat{b}_3\tilde{x}^3$ instead.

Combining (27) and (28) we have

$$\tilde{x}'(y) = -\frac{\tilde{x}(a_0\tilde{x}^2 + a_1\tilde{x}y + a_2y^2)}{y(b_0\tilde{x}^2 + b_1\tilde{x}y + b_2y^2)}(1 + \tilde{x}\, q_0(\tilde{x}, y)) \tag{29}$$

for the bounded function $q_0 : [0, \max\{\delta, \tilde{\delta}\}] \times [\delta, y_0] \to \mathbb{R}$, given by

$$q_0(\tilde{x}, y) = \frac{1}{\tilde{x}}\left(\left(1 + \frac{\tilde{x}\sum_{j=1}^{3} \hat{a}_j\tilde{x}^{j-1}y^{3-j}}{a_0\tilde{x}^2 + a_1\tilde{x}y + a_2y^2}\right)\left(1 + \frac{\tilde{x}\sum_{j=1}^{3} \hat{b}_j\tilde{x}^{j-1}y^{3-j}}{b_0\tilde{x}^2 + b_1\tilde{x}y + b_2y^2}\right)^{-1} - 1\right).$$

From Grönwall's Lemma, we can bound the speed at which $\tilde{x}(y)$ and $x(y)$ diverge from each other, see [14, Theorem 2.8]:

$$u(y) := |\tilde{x}(y) - x(y)| \le |\tilde{x}(y_0) - x(y_0)|e^{(y_0-y)\,\mathrm{Lip}} + \frac{V}{\mathrm{Lip}}\left(e^{(y_0-y)\,\mathrm{Lip}} - 1\right), \tag{30}$$

for $y \in [\delta, y_0]$, where Lip is the Lipschitz constant of $F(x, y)$ in x maximized over $y \in [\delta, y_0]$ and

$$V = \sup_{y\in[\delta,y_0]} \sup_{x\in[0,\max\{\delta,\tilde{\delta}\}]} |F(x, y)|q_0(x, y)x.$$

The Lipschitz constant of F is $\sup_{x,y} \frac{\partial}{\partial x}F(x, y)$, which is of order $1/\delta$. Therefore (30) gives a very poor estimate if we apply it at once to the whole interval $[\delta, y_0]$, but we can improve it by partitioning $[\delta, y_0]$ into smaller intervals.

Proposition 3.1 *Assume the equation $\tilde{x}'(y) = F(\tilde{x}, y)(1 + q_0\tilde{x})$ for some bounded function $q_0 = q_0(\tilde{x}, y)$ and initial value $\tilde{x}(y_0) = x(y_0) = x_0$. Then there is a constant K for all $y \in [\delta, y_0]$ we have $|\tilde{x}(y) - x(y)| \le K\delta^2$.*

Proof. We divide the interval $[\delta, y_0]$ into N interval $[y_{j+1} - y_j]$ of the same length, so $y_j = \delta + \frac{y_0-\delta}{N}(N - j)$ and set $x_j = x(y_j)$. The Lipschitz constant $[y_{j+1}, y_j]$ is $\sup_{y\in[y_{j+1},y_j]} \sup_{x\in[0,\delta^2]} \frac{\partial}{\partial x}F(x, y) \le \frac{1}{y}(\gamma + \varepsilon)$ for $\gamma = \frac{a_2}{b_2}$. Therefore

$$\mathrm{Lip}(y_j)(y_j - y_{j+1}) \le \frac{y_0 - \delta}{N}\frac{\gamma + \varepsilon}{\delta + \frac{y_0-\delta}{N}(N - j - 1)} = \frac{\gamma + \varepsilon}{(N - (j + 1)) + \delta N/(y_0 - \delta)}$$

Recall from (13) that $x(y) = U(y)\delta^{1+\frac{a_2}{b_2}} y^{-\frac{a_2}{b_2}}$. Let

$$C = 2 \sup_{y\in[y_{j+1},y_j]} \sup_{x\in[0,2x(y)]} q_0(x,y)\, \frac{a_0x^2 + a_1xy + a_2y^2}{b_0x^2 + b_1xy + b_2y^2}\, U(y)^2.$$

Take $K = \frac{2C}{\gamma} y_0^{-2\gamma}$ and assume by induction that $u(y_j) \le K\delta^2 \le 1$ (so we assume that δ is sufficiently small). Clearly this induction hypothesis holds for $j = 0$ because $u(y_0) = 0$. Then

$$\begin{aligned} V(y_j) &:= \sup_{y\in[y_{j+1},y_j]} \sup_{x\in[0,x(y)(1+K\delta^2)]} q_0(x,y)\, x\, F(x,y) \\ &\le \sup_{x,y} \frac{a_0x^2 + a_1xy + a_2y^2}{b_0x^2 + b_1xy + b_2y^2}\, q_0(x,y) \sup_{y\in[y_{j+1},y_j]} \frac{x(y)^2(1+K\delta^2)^2}{y} \\ &\le \sup_{x,y} \frac{a_0x^2 + a_1xy + a_2y^2}{b_0x^2 + b_1xy + b_2y^2}\, q_0(x,y)\; 2U(y)^2 \sup_{y\in[y_{j+1},y_j]} \delta^{2(1+\gamma)} y^{-(1+2\gamma)} \\ &= C\delta^{2(1+\gamma)} y_{j+1}^{-(1+2\gamma)}. \end{aligned}$$

The estimate (30) applied to $[y_{j+1}, y_j]$ gives the recursive formula

$$u(y_{j+1}) \le u(y_j) e^{(\gamma+\varepsilon)\frac{y_j - y_{j+1}}{y_{j+1}}} + C\delta^{2(1+\gamma)} y_{j+1}^{-(1+2\gamma)} (y_j - y_{j+1}). \tag{31}$$

Therefore, for $1 \le M < N$,

$$u(y_M) \le C\delta^{2(1+\gamma)} \sum_{j=1}^{M} y_j^{-(1+2\gamma)} (y_{j-1} - y_j) e^{(\gamma+\varepsilon)\sum_{k=j}^{M} \frac{y_{k-1}-y_k}{y_k}}. \tag{32}$$

Then (32) gives for $N = \lceil \delta^{-1} \rceil$ and $M < N$,

$$\begin{aligned} u(y_M) &\le C\delta^{2(1+\gamma)} y_0^{-2\gamma} N^{2\gamma} \sum_{j=1}^{M} \frac{e^{(\gamma+\varepsilon)\sum_{k=j}^{M} 1/(N-k)}}{(N-j)^{1+2\gamma}} \\ &\le C\delta^2 y_0^{-2\gamma} \frac{1}{(N-M)^{\gamma+\varepsilon}} \sum_{j=1}^{M} \frac{1}{(N-j)^{\gamma-\varepsilon+1}} \le \frac{C}{\gamma - \varepsilon} \delta^2 y_0^{-2\gamma}. \end{aligned}$$

Therefore $|\tilde{x}(y_M) - x(y_M)| = u(y_M) \le K\delta^2$ as claimed. $\square$

By (29) for $y = \delta = \tilde{x}(\delta) + O(\delta^2)$, the derivative $\tilde{x}'(\delta) = \frac{a_0+a_1+a_2}{b_0+b_1+b_2} + O(\delta)$. Since $\tilde{\delta}$ lies between δ and $\tilde{x}(\delta)$ (see Figure 3), we have by a Taylor approximation

$$\begin{aligned} |\tilde{x}(\delta) - \delta| &= |\tilde{x}(\delta) - \tilde{\delta}| + |\tilde{\delta} - \delta| = \left(1 + \frac{a_0 + a_1 + a_2}{b_0 + b_1 + b_2} + O(\delta)\right) |\tilde{\delta} - \delta| \\ &= \frac{c_0 + c_1 + c_2 + O(\delta)}{b_0 + b_1 + b_2} |\tilde{\delta} - \delta|. \end{aligned} \tag{33}$$

and therefore $|\tilde{\delta} - \delta| = O(|\tilde{x}(\delta) - \delta|) = O(\delta^2)$. Now that we have this relation between $\tilde{\delta}$ and δ, we can estimate $\tilde{T}$ in terms of T, and effectively finish of Theorem 1.1.

Proof of Theorem 1.1. Let $z_0 = (x_0, y_0) = (\xi(\eta, T), \eta) = (\tilde{\xi}(\eta, \tilde{T}), \eta)$ as in Figure 3 be the point such that $\phi_{hor}^{T}(z_0) = (\zeta_0, \omega(\eta, T))$ under the **unperturbed** flow and $\tilde{\phi}_{hor}^{\tilde{T}}(z_0) = (\zeta_0, \tilde{\omega}(\eta, \tilde{T}))$ under the **perturbed** flow. We estimate $\tilde{T}$ in terms of T.

Combining the estimate for $\xi(\eta, T)$ from Proposition 2.1 with $L(\delta,\delta) = L(\xi(\eta,T),\eta)$, we can find the relation between δ and T:

$$\delta = \delta_0 T^{-\frac{1}{2}}(1 - \frac{\xi_1}{2\beta_2}T^{-1} + O(T^{-2}, T^{-2\beta_2})), \tag{34}$$

for

$$\delta_0 = \xi_0^{\frac{1}{2\beta_2}} \eta^{1-\frac{1}{2\beta_2}} \left(\frac{c_0 \frac{\xi^2}{\eta^2} + c_1 \frac{\xi}{\eta} + c_2}{c_0 + c_1 + c_2} \right)^{\frac{1}{u+v+2}}.$$

To estimate $\tilde{T}$, we divide the trajectory $\tilde{\phi}^t(z_0) = (\tilde{x}(t), \tilde{y}(t))$ of z_0 for the region $\tilde{x} \le \tilde{y}$ into two parts delimited by points in time:

$$\tilde{T}_1 = \min\{t > 0 : 2\tilde{x}(t) = \tilde{y}(t)\}, \qquad \tilde{T}_2 = \min\{t > 0 : \tilde{x}(\tilde{T}_1 + t) = \tilde{y}(\tilde{T}_1 + t)\}, \tag{35}$$

and symmetric quantities for the region $\tilde{y} \le \tilde{x}$. Let T_1, T_2 be the analogous quantities for the unperturbed trajectory. For $y_1 := y(T_1)$ and $\tilde{y}_1 := y(\tilde{T}_1)$, we have $|\tilde{y}_1 - y_1| = O(\delta^2)$. Assume that $\tilde{y}_1 \ge y_1$ (the case $\tilde{y}_1 < y_1$ goes likewise). We compute

$$\begin{aligned} T_1 &= \int_{y_1}^{y_0} \frac{dy}{\dot{y}} = \int_{\tilde{y}_1}^{y_0} \frac{dy}{\dot{y}} + \int_{y_1}^{\tilde{y}_1} \frac{dy}{\dot{y}} \\ &= \int_{\tilde{y}_1}^{y_0} \frac{dy}{-y(b_0 x(y)^2 + b_1 x(y) y + b_2 y^2)} + \int_{y_1}^{\tilde{y}_1} \frac{dy}{\dot{y}}. \end{aligned}$$

The first term is $O(\delta^{-2})$ as $\delta \to 0$ and the second is $O(1)$ (and therefore small compared to the first term) because $|\tilde{y}_1 - y_1| = O(\delta^2)$. Hence $T_1 = O(\delta^{-2})$. Similarly,

$$\tilde{T}_1 - T_1 = \int_{\tilde{y}_1}^{y_0} \frac{dy}{\dot{\tilde{y}}} - \int_{\tilde{y}_1}^{y_0} \frac{dy}{\dot{y}} - \int_{y_1}^{\tilde{y}_1} \frac{dy}{\dot{y}} = \int_{\tilde{y}_1}^{y_0} \frac{1}{\dot{\tilde{y}}} - \frac{1}{\dot{y}}\, dy + O(1). \tag{36}$$

Using that $\tilde{x}(y) - x(y) \le \delta^2$ and abbreviating $\Sigma = \sum_{j=1}^{3} \hat{b}_j \tilde{x}^j y^{3-j} = O(y^3)$ for $\tilde{x} \le \tilde{y}$, we compute the last integral of (36) as

$$\begin{aligned} &\int_{\tilde{y}_1}^{y_0} \frac{1}{-y(b_0\tilde{x}(y)^2 + b_1\tilde{x}(y)y + b_2y^2 + \Sigma)} - \frac{1}{-y(b_0x(y)^2 + b_1x(y)y + b_2y^2)}\, dy \\ &= \int_{\tilde{y}_1}^{y_0} \frac{(\tilde{x} - x)(b_0(\tilde{x} + x) + b_1 y) + \Sigma}{y(b_0\tilde{x}(y)^2 + b_1\tilde{x}(y)y + b_2y^2 + \Sigma)(b_0x(y)^2 + b_1x(y)y + b_2y^2)}\, dy \\ &\le \delta^2 \int_{\tilde{y}_1}^{y_0} \frac{3b_0 + 2b_1}{\max_i b_i\, y^2}\, dy + \int_{\tilde{y}_1}^{y_0} O(y^{-1})\, dy = O(\delta^{-1}). \end{aligned}$$

Together with (34) and the fact that $\int_{y_1}^{\tilde{y}_1} \frac{dy}{\dot{y}} = O(1)$, this gives

$$\tilde{T}_1 = T_1(1 + O(\delta)) = T_1(1 + O(T^{-\frac{1}{2}})). \tag{37}$$

For $\tilde{M} = y/\tilde{x}$, computations analogous to (16) show that $\dot{\tilde{M}} = -\tilde{M}(c_0 + c_1\tilde{M} + c_2\tilde{M}^2 + \tilde{x}\Psi)\tilde{x}^2$ for $\hat{c}_j = \hat{a}_j + \hat{b}_j$ and $\Psi := \sum_{j=0}^{2} \hat{c}_{3-j}\tilde{M}^j$. For every $(\tilde{x}, y) = (\tilde{x}, \tilde{x}\tilde{M})$ on the $\tilde{\phi}$-trajectory of z_0 (which is a level set of $\tilde{L}$), we have

$$\xi^u \eta^v (\frac{a_0}{v}\xi^2 + \frac{a_1}{v+1}\xi\eta + \frac{b_2}{u}\eta^2)\frac{\tilde{\delta}}{\delta'} = \tilde{x}^{u+v+2}\tilde{M}^v(\frac{a_0}{v} + \frac{a_1}{v+1}\tilde{M} + \frac{b_2}{u}\tilde{M}^2),$$

with δ' as in (26). This gives the analogue of [2, formula (32)]:

$$\dot{\tilde{M}} = -G(\xi,\eta)\tilde{M}^{1-\frac{1}{\beta_0}}\left(c_0 + c_1\tilde{M} + c_2\tilde{M}^2 + \tilde{x}\Psi(\tilde{x},\tilde{M})\right)^{\frac{1}{2\beta_0}+\frac{1}{2\beta_2}}\left(\frac{\tilde{\delta}}{\delta'}\right)^{1-\frac{1}{2\beta_0}-\frac{1}{2\beta_2}}, \qquad (38)$$

where $G(\xi,\eta)$ is as in (18). Because δ' lies between δ and $\tilde{\delta}$, the conclusion $|\tilde{\delta}-\delta| = O(\delta^2)$ of (33) gives

$$\frac{\tilde{\delta}}{\delta'} \leq 1 + \frac{|\tilde{\delta}-\delta'|}{\delta'} \leq 1 + \frac{|\tilde{\delta}-\delta|}{\delta} = 1 + O(\delta).$$

Note also on the time interval $[\tilde{T}_1, \tilde{T}_1 + \tilde{T}_2]$, we have $\tilde{x}(y) \leq y \leq 2\tilde{x}(y) \leq 2\max\{\delta,\tilde{\delta}\}$ and therefore

$$\begin{aligned} c_0 + c_1\tilde{M} + c_2\tilde{M}^2 + \tilde{x}\Psi(\tilde{x},\tilde{M}) &= c_0(1+O(\tilde{x})) + c_1\tilde{M}(1+O(y)) + c_2\tilde{M}^2 \\ &= (c_0 + c_1\tilde{M} + c_2\tilde{M}^2)(1+O(\delta)). \end{aligned}$$

This combined with (38) we find

$$\dot{\tilde{M}} = -G(\xi,\eta)\tilde{M}^{1-\frac{1}{\beta_0}}\left(c_0 + c_1\tilde{M} + c_2\tilde{M}^2\right)^{\frac{1}{2\beta_0}+\frac{1}{2\beta_2}}(1+O(\delta)).$$

The choice of $\tilde{T}_1$ and T_1 ensures that $\tilde{M}(\tilde{T}_1) = M(T_1) = 2$ and similarly $\tilde{M}(\tilde{T}_1 + \tilde{T}_2) = M(T_1 + T_2) = 1$. Therefore

$$\begin{aligned} \tilde{T}_2 &= \int_{\tilde{T}_1}^{\tilde{T}_1+\tilde{T}_2} dt = \int_{\tilde{M}(\tilde{T}_1+\tilde{T}_2)}^{\tilde{M}(\tilde{T}_1)} \dot{\tilde{M}}^{-1}\, d\tilde{M} \\ &= \int_1^2 \frac{\tilde{M}^{\frac{1}{\beta_0}-1}(1+O(\delta))^{-1}}{G(\xi,\eta)(c_0 + c_1\tilde{M} + c_2\tilde{M}^2)^{\frac{1}{2\beta_0}+\frac{1}{2\beta_2}}}\, d\tilde{M} \\ &= \int_{M(T_1+T_2)}^{M(T_1)} \frac{M^{\frac{1}{\beta_0}-1}(1+O(\delta))^{-1}}{G(\xi,\eta)(c_0 + c_1 M + c_2 M^2)^{\frac{1}{2\beta_0}+\frac{1}{2\beta_2}}}\, dM \\ &= T_2(1+O(\delta))^{-1} = T_2(1+O(T^{-\frac{1}{2}})). \end{aligned}$$

Combining this with (37) gives $\tilde{T} = T(1+O(T^{-\frac{1}{2}}))$. The estimate of Proposition 2.1 now gives $\tilde{\xi}(\eta,\tilde{T}) = \xi_0(\eta)\tilde{T}^{-\beta_2}(1+O(\tilde{T}^{-\frac{1}{2}}))$ as claimed.

Reversing the roles $(a_0,a_2) \leftrightarrow (b_2,b_0)$ as in the end of the proof of Proposition 2.1 gives $\tilde{\omega}(\eta,\tilde{T}) = \omega_0(\eta)\tilde{T}^{-\beta_0}(1+O(\tilde{T}^{-\frac{1}{2}}))$. □

The formula (6) for the Dulac maps follows directly from Theorem 1.1 by inverting $T \mapsto \xi(\eta,T)$ and inserting this in the formula for $\omega(\eta,T)$. In the special case that $\beta_0 = \beta_2$, formula (6) reduces to

$$\omega = D(\xi) = \left(\frac{b_2}{a_0}\right)^{\frac{1}{\beta_2-1}}\left(\frac{\eta}{\zeta_0}\right)^{2\beta_2-1}\xi\left(1+O(\xi^{\frac{1}{2\beta_2}})\right).$$

Reducing further by assuming (5) (i.e., in the volume preserving setting), we get

$$\omega = D(\xi) = \frac{b_2}{a_0}\left(\frac{\eta}{\zeta_0}\right)^3\xi\left(1+O(\xi^{\frac{1}{4}})\right).$$

This coefficient $\frac{b_2}{a_0}\left(\frac{\eta}{\zeta_0}\right)^3 = \frac{\|X_{hor}(0,\eta)\|}{\|X_{hor}(\zeta_0,0)\|}$ agrees with the fact that for $\omega = D(\xi)$, the flow-boxes $\cup_{t\in[0,\varepsilon]}\phi^t_{hor}([0,\xi]\times\{\eta\})$ and $\cup_{t\in[0,\varepsilon]}\phi^t_{hor}(\{\zeta_0\}\times[0,\omega])$ must have the same volume. If the neutral saddle p is part of a heteroclinic cycle, then it is accumulated by periodic solutions, but these are not limit cycles of course.

4 Time-1 map versus Poincaré map

First we give an estimate of observables integrated over the flow-lines of X_{hor} of (9).

Proposition 4.1 *Let* $r = \sqrt{x^2+y^2}$, $\rho > 0$ *and* $W(T)$ *be the integral curve for* (9) *connecting* $(\xi(\eta_0,T),\eta_0))$ *to* $(\zeta_0,\omega(\eta_0,T))$, *see Figure 1. Then there is a constant* $C = C(\rho) > 0$ *such that*

$$\Theta = \Theta(T) := \int_{W(T)} r(t)^\rho \, dt = \begin{cases} CT^{1-\frac{\rho}{2}}(1+o(1)) & \text{if } \rho < 2, \\ C\log(T)(1+o(1)) & \text{if } \rho = 2, \\ C(1+o(1)) & \text{if } \rho > 2. \end{cases} \tag{39}$$

Proof. We build on the proof of Proposition 2.1 (or in fact Theorem 1.1), and in the integral Θ we change coordinates $M = y/x$. That is, $r^\rho = (x^2+y^2)^{\rho/2} = x^\rho(1+M^2)^{\rho/2}$. Use (14) to get

$$x = G(T)^{\frac{1}{2}} M^{-\frac{v}{u+v+2}} (c_0 + c_1 M + c_2 M^2)^{\frac{-1}{u+v+2}},$$

with $G(T) := G(\xi(\eta_0,T)),\eta_0)$ as in (18). Abbreviate $\xi(\eta_0,T) = \xi(T)$ and $\omega(\eta_0,T) = \omega(T)$. Inserting the above in the integral of (19), we obtain

$$\Theta = G(T)^{\frac{\rho}{2}-1} \int_{\omega(T)/\zeta_0}^{\eta/\xi(T)} \frac{M^{-\frac{\rho v}{u+v+2}} (c_0 + c_1 M + c_2 M^2)^{\frac{-\rho}{u+v+2}} (1+M^2)^{\frac{\rho}{2}}}{M^{1-\frac{1}{\beta_0}} (c_0 + c_1 M + c_2 M^2)^{\frac{1}{2\beta_0}+\frac{1}{2\beta_2}}} \, dM. \tag{40}$$

For $M \to 0$, the leading term in the integrand is

$$c_0^{-1+(1-\frac{\rho}{2})\frac{1}{\beta_0}} M^{\frac{1}{\beta_0}-1-\rho\frac{v}{u+v+2}} = c_0^{-1+(1-\frac{\rho}{2})\frac{1}{\beta_0}} M^{(1-\frac{\rho}{2})\frac{1}{\beta_0}-1},$$

i.e., the exponent is > -1 for $\rho < 2$. For $M \to \infty$, the leading term in the integrand is

$$c_2^{-1+(1-\frac{\rho}{2})\frac{1}{\beta_0}} M^{-\frac{1}{\beta_2}-1-\rho(\frac{v}{u+v+2}+\frac{2}{u+v+2}-1)} = c_0^{-1+(1-\frac{\rho}{2})\frac{1}{\beta_0}} M^{-(1-\frac{\rho}{2})\frac{1}{\beta_2}-1},$$

i.e., the exponent is < -1 for $\rho < 2$. This means that the integral in (40) converges to some constant $C_0 = C_0(\rho)$ as $T \to \infty$, and $\Theta \sim C_0 G(T)^{\frac{\rho}{2}-1} \sim CT^{1-\frac{\rho}{2}}$ for $C = C_0 \left(c_2^{1-\frac{1}{2\beta_0}-\frac{1}{2\beta_2}} \xi_0^{\frac{1}{\beta_0}} \eta_0^{1-\frac{1}{\beta_2}} \right)^{1-\frac{\rho}{2}}$. This finishes the proof for $\rho < 2$.

If $\rho > 2$, then the value of Θ based on the leading terms of the integrand only, is

$$\Theta = G(T)^{\frac{\rho}{2}-1} \frac{c_0^{-1+(1-\frac{\rho}{2})\frac{1}{\beta_0}}}{\frac{\rho}{2}-1} \left(\beta_2 \left(\frac{\eta_0}{\xi(T)}\right)^{-(1-\frac{\rho}{2})\frac{1}{\beta_2}} - \beta_0 \left(\frac{\omega(T)}{\zeta_0}\right)^{(1-\frac{\rho}{2})\frac{1}{\beta_0}} \right).$$

Insert the values of $\xi(T)$ and $\omega(T)$ from Proposition 2.1 as well as the leading term of $G(T)$:

$$\Theta = \left(c_2^{1-\frac{1}{2\beta_0}-\frac{1}{2\beta_2}} \xi_0^{\frac{1}{\beta_0}} \eta_0^{1-\frac{1}{\beta_2}} \right)^{1-\frac{\rho}{2}} T^{1-\frac{\rho}{2}} \frac{c_0^{-1+(1-\frac{\rho}{2})\frac{1}{\beta_0}}}{\frac{\rho}{2}-1} \left(\beta_2 \eta_0^{(\frac{\rho}{2}-1)\frac{1}{\beta_2}} - \beta_0 \zeta_0^{(\frac{\rho}{2}-1)\frac{1}{\beta_0}} \right) T^{\frac{\rho}{2}-1}.$$

The powers of T cancel in this expression, proving the case $\rho > 2$. Finally, if $\rho = 2$, then the factor $G(T)^{\frac{\rho}{2}-1}$ in (40) disappears and the leading terms in the integrand (both as $M \to 0$ and $M \to \infty$), are $c_0^{-1} M^{-1}$. This gives, due to Proposition 2.1,

$$\Theta \sim \frac{1}{c_0} \left(\log \frac{\eta_0}{\xi(T)} - \log \frac{\omega(T)}{\zeta_0} \right) \sim \frac{\beta_2 + \beta_0}{c_0} \log T.$$

□

The 3-dimensional time-1 map ϕ^1 preserves no 2-dimensional submanifold of $\mathcal{M}$. Yet in order to model ϕ^t as a suspension flow over a 2-dimensional map, we need a genuine Poincaré map. For this we choose a section Σ transversal to Γ and containing a neighbourhood U of p. As an example, Σ could be $\mathbb{T}^2 \times \{0\}$, and the Poincaré map to $\mathbb{T}^2 \times \{0\}$ could be (a local perturbation of) Arnol'd's cat map; in this case (and most cases) $\mathcal{M}$ is not homeomorphic to $\mathbb{T}^3$ because the homology is more complicated, see [1, 11].

Let $h : \Sigma \to \mathbb{R}^+$, $h(q) = \min\{t > 0 : \phi^t(q) \in \Sigma\}$ be the first return time. Assuming that $\sup_\Sigma |w(x,y)| < 1$, the first return time h is bounded and bounded away from zero, say $0 < \inf_\Sigma h < \sup_\Sigma h$.

The Poincaré map $f := \phi^h : \Sigma \to \Sigma$ has a neutral saddle point p at the origin. Its local stable/unstable manifolds are $W^s_{loc}(p) = \{0\} \times (-\varepsilon, \varepsilon)$ and $W^u_{loc}(p) = (-\varepsilon, \varepsilon) \times \{0\}$. Because the flow ϕ^t is a perturbation of an Anosov flow, and f is a Poincaré map, it has a finite Markov partition $\{P_i\}_{i \geq 0}$ and we can assume that p is in the interior of P_0. In the sequel, let U be a neighbourhood of p that is small enough that (1) is valid on $U \times [0,1]$ but also that $f(U) \supset \hat{P}_0 \cup P_0$.

In order to regain the hyperbolicity lacking in f, let

$$r(q) := \min\{n \geq 1 : f^n(q) \in Y\} \tag{41}$$

be the first return time to $Y := \Sigma \setminus P_0$. Then the Poincaré map $F = f^r = \phi^\tau$ of ϕ^t to $Y \times \{0\}$ is hyperbolic, where

$$\tau(q) = \min\{t > 0 : \phi^t((q,0)) \in Y \times \{0\}\} = \sum_{j=0}^{r-1} h \circ f^j$$

is the corresponding first return time.

Consequently, the flow $\phi^t : \mathcal{M} \times \mathbb{R} \to \mathcal{M}$ can be modeled as a suspension flow on $Y^\tau = \left(\bigcup_{q \in Y} \{q\} \times [0, \tau(q)) \right) / (q, \tau(q)) \sim (F(q), 0)$. Since the flow and section $Y \times \{0\}$ are C^1 smooth, τ is C^1 on each piece $\{r = k\}$.

Lemma 4.1 *Let $\hat{\tau}(q) = \min\{t > 0 : \phi^t_{hor}((q,\eta)) \in W^s(\zeta_0, 0)\}$ be the time the horizontal flow needs to reach ∂Q. Then $\tau(q) = \hat{\tau}(q) + O(1)$ and $r = \hat{\tau}(q) + \Theta(\hat{\tau}(q)) + O(1)$, where Θ is as in Proposition 4.1*

Proof. By the definition of $\hat{\tau}$ we have $\phi^{\hat{\tau}}_{hor}(q) \in \hat{W}^s$. Therefore it takes a bounded amount of time (positive or negative) for $\phi^{\hat{\tau}}(q, 0)$ to hit $Y \times \{0\}$, so $|\tau(q) - \hat{\tau}(q)| = O(1)$.

The function w in the vertical component of the local vector field X of (1) is a linear combination of homogeneous functions $C_i \cdot (x^2 + y^2)^{\rho_i/2}$. It suffices to take the leading term, i.e., the one with the smallest $\rho_i =: \rho$. Using this term in (39), we obtain that $\hat{\tau}(q) + \Theta(\hat{\tau}(q))$ indicates the vertical displacement by the flow ϕ^t. In particular, it gives the number of times the flow-line intersects Σ, and hence $r = \hat{\tau}(q) + \Theta(\hat{\tau}(q)) + O(1)$. □

Assume that ϕ^t and $\hat{f}$ preserve Lebesgue measure.

Proposition 4.2 *Recall that $\beta_2 = \frac{a_2 + b_2}{2b_2} \in (\frac{1}{2}, \infty)$. There exists $C^* > 0$ such that*

$$Leb(\{\tau > t\}) = C^* t^{-\beta_2}(1 + o(1)) \tag{42}$$

for the F-invariant SRB-measure $\mu_{\bar{\phi}}$.

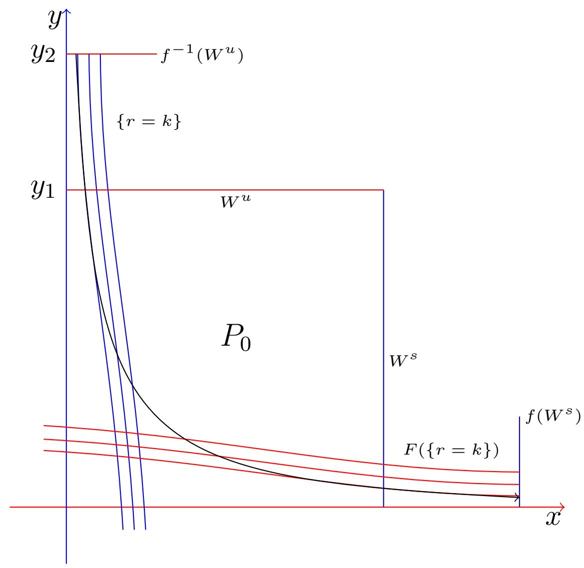

Figure 4: The first quadrant of the rectangle P_0, with stable and unstable foliations of Poincaré map $f = \phi^h$ drawn vertically and horizontally, respectively. Also one of the integral curves is drawn.

Proof. The function τ is defined on $\Sigma \setminus P_0$ and $\tau \geq h_2 = h + h \circ f$ on $Y_{\{r \geq 2\}} := f^{-1}(P_0) \setminus P_0$. The set $Y_{\{r\geq 2\}}$ is a rectangle with boundaries consisting of two stable and two unstable leaves of the Poincaré map f. Let $W^u(y)$ denote the unstable leaf of f inside $Y_{\{r\geq 2\}}$ with $(0, y)$ as (left) boundary point. Let $y_1 < y_2$ be such that $W^u(y_1)$ and $W^u(y_2)$ are the unstable boundary leaves of $Y_{\{r\geq 2\}}$.

The unstable foliation of $\hat{f} = \phi^1_{hor}$ does not entirely coincide with the unstable foliation of f. Let $\hat{W}^u(y)$ denote the unstable leaf of $\hat{f}$ with $(0, y)$ as (left) boundary point. Both $\hat{W}^u(y)$ and $W^u(y)$ are C^1 curves emanating from $(0, y)$; let $\gamma(y)$ denote the angle between them. Then the lengths

$$\begin{aligned} \mathrm{Leb}(W^u(y) \cap \{\tau > t\}) &= |\cos\gamma(y)| \; \mathrm{Leb}(\hat{W}^u(y) \cap \{\tau > t\})(1 + o(1)) \\ &= |\cos\gamma(y)| \; \xi_0(y) \; t^{-\beta_2}(1 + o(1)) \end{aligned}$$

as $t \to \infty$, where the last equality and the notation $\xi_0(y)$ and $\beta_2 = (a_2 + b_2)/(2b_2)$ come from Theorem 1.1

We decompose Lebesgue on $Y_{\{r\geq 2\}}$ as

$$\int_{Y_{\{r\geq 2\}}} v \, d\mu_{\bar{\phi}} = \int_{y_1}^{y_2} \left(\int_{W^u(y)} v \, d\mu^s_{W^u(y)} \right) d\nu^u(y).$$

The conditional measures $\mu_{W^u(y)}$ on $W^u(y)$ equals 1-dimensional Lebesgue $m_{W^u(y)}$ on

$W^u(y)$ Therefore, as $t \to \infty$,

$$\begin{aligned}
\mu_{\bar{\phi}}(\tau > t) &= \int_{y_1}^{y_2} \mu_{W^u(y)}(W^u(y) \cap \{\tau > t\})\, d\nu^u(y) \\
&= \int_{y_1}^{y_2} m_{W^u(y)}(W^u(y) \cap \{\tau > t\})\, d\nu^u(y) \\
&= \int_{y_1}^{y_2} |\cos\gamma(y)|\; m_{\hat{W}^u(y)}(\hat{W}^u(y) \cap \{\tau > t\})(1 + o(1))\, d\nu^u(y) \\
&= \int_{y_1}^{y_2} |\cos\gamma(y)|\; \xi_0(y)\; t^{-\beta_2}(1 + o(1))\, d\nu^u(y) = C^* t^{-\beta_2}(1 + o(1)),
\end{aligned}$$

for $C^* = \int_{y_1}^{y_2} |\cos\gamma(y)|\; \xi_0(y)\, d\nu^u(y)$. This proves the result. $\square$

5 The proof of Theorem 1.2

Proof. The proof of Theorem 1.2 is a direct application of Theorem 2.7 in [3], where $\bar{v} = \int_0^\tau v \circ \phi^t\, dt$ takes the role of $\bar{\psi}$ in [3, Theorem 2.7], but the condition that $\bar{\psi} = C - \psi_0$ for some positive ψ_0 is only important for the results on the shape of the pressure function in [3]. For us, only the tail of $\bar{v}$ matters and since v is C^1 on $\mathcal{M} \setminus \Gamma$, $\bar{v}$ is C^1 on each partition element $\{\phi = n\}$ of the Markov map F. Since Proposition 4.1 applies to v we get $\bar{v}(x, y) \sim C_p T^{1-\frac{\rho}{2}}$ if the Dulac time of (x, y) is T. Since our invariant measure is Lebesgue, and $\beta_2 = 2$, Theorem 1.1 can be immediately used to estimate

$$\mathrm{Leb}(\bar{v} > t) \sim \frac{\int_{\eta_0}^{\eta_1} \xi_0(\eta)\, d\eta}{\mathrm{Vol}\ (\Sigma \setminus P_0)} \left(\frac{t}{C_p}\right)^{\frac{-4}{2-\rho}}, \tag{43}$$

where Σ is the Poincaré section and Vol $(\Sigma \setminus P_0)$ is the normalizing constant for Lebesgue restricted to the domain $\Sigma \setminus P_0$ of F. If $\rho \geq 2$, this asymptotic formula should be interpreted as $\mathrm{Leb}(\bar{v} > t) = 0$ for t large, that is: $\bar{v}$ is bounded.

The exponent of the tail (43) is -2 if and only if $\rho = 0$, and in this case [3, Theorem 2.7(a)(ii)] gives the non-Gaussian CLT.

If $-2 < \rho < 0$, [3, Theorem 2.7(a)(i)] gives a Stable Law of order $4/(2 - \rho) \in (1, 2)$.

Finally, if $0 < \rho < 2$ (or $\rho \geq 2$ when $\bar{v}$ is bounded), then we obtain the CLT provided the variance $\sigma^2 > 0$, and this follows from $\bar{v}$ not being a coboundary. In other words, $\bar{v} \neq h - h \circ F$ for any $h \in \mathcal{B}$, the Banach space used in the proofs of [3], and this we assumed explicitly. $\square$

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