ASYMPTOTIC ARC-COMPONENTS OF UNIMODAL INVERSE LIMIT SPACES

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Abstract. We consider the inverse limit space (I, f) of a unimodal bonding map f as fixed bonding map. If f has a periodic turning point, then (I, f) has a finite non-empty set of asymptotic arc-components. We show how asymptotic arc-components can be determined from the kneading sequence of f. This gives an alternative to the substitution tiling space approach taken by Barge & Diamond [4].

1. Introduction

Inverse limit spaces of endomorphisms appear as the global attractors of many dynamical systems [22]. For instance, the relevant example for this paper is the global attractor of Hénon maps, cf. [6]. We will study inverse limit spaces for which a unimodal map of the interval is the (fixed) bonding map. Since two conjugate maps gives rise to homeomorphic inverse limit spaces, it suffices to consider quadratic maps $f(x) = 1 - ax^2$ with $a \in [0, 2]$. Then f has a unique critical point $c = 0$ and maps the interval $I := [1-a, 1]$ onto itself. Write $X := (I, f)$ for the inverse limit space with f as single bonding map. For values of a below the Feigenbaum parameter ($a \leq a_{Feig} \approx 1.40115\dots$), the structure of the inverse limit space X is relatively simple, see [7]. For $a > a_{Feig}$, the known results are largely restricted to parameters for which the critical orbit is finite. In this case, X consists of uncountably many arc-components which (assuming f is non-renormalizable) lie dense in X. If c is periodic of period N, then X contains N *endpoints*, cf. [8], and the arc-component of these endpoints are continuous images of the half-line $[0, \infty)$. All other arc-components are continuous images of $\mathbb{R}$. If c is strictly preperiodic, no endpoints exists [9]; instead there are turnlink points at which X is not homogeneous, [11]. Each point that is neither endpoint nor turnlink point has a neighborhood homeomorphic to the product of a Cantor set and an interval.

There are still more inhomogeneities. In [1], Barge & Diamond point out the existence of asymptotic arc-components. Two arc-components C and

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$\tilde{C}$ are *asymptotic* if there exists parametrizations $\varphi, \tilde{\varphi} : \mathbb{R} \to C, \tilde{C}$ such that $\lim_{t\to\infty} d(\varphi(t), \tilde{\varphi}(t)) = 0$. Here d is a metric on X compatible with the topology. In this paper we use symbolic dynamics to describe the asymptotic arc-components. Depending on the kneading sequence, we can algorithmically determine the pattern of asymptotic arc-components. They appear in *k-fans* (i.e., k arc-components which are all asymptotic in one direction), or *k-cycles* (i.e., k arc-components each of which is asymptotic in either direction to a neighboring arc-component), or more complicated combinations of these two, see Figure 1.

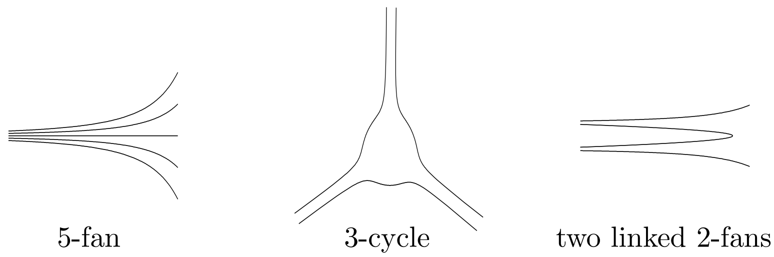

FIGURE 1. Some patterns of asymptotic arc-components. Note that the arc-components accumulate on themselves, which, for simplicity, is not shown in the picture,

The induced homeomorphism permutes the asymptotic arc-components and from the structure of this permutation, the pattern of the asymptotic arc-components can be further analyzed. The pattern gives some visualization of why inverse limit spaces of non-conjugate "periodic" unimodal maps are non-homeomorphic, contributing to the partial results on the classification in [4, 12, 20, 16]. For example, it seems that for any given period $N \geq 6$, the four N-periodic kneading sequences appearing last in the parity-lexicographical order, all lead to non-homeomorphic inverse limit spaces, cf. [20, Theorem 5.1]. However, our results provide no complete classification, and hence cannot replace the claim of [17, 19] that all "periodic" inverse limit spaces are non-homeomorphic.

The paper is organized as follows. The next section gives preliminaries on inverse limit spaces of unimodal maps and how to track their arc-components using kneading theory. In Section 3, we determine when two arc-components are asymptotic. Section 4 presents the results for the unimodal maps with periodic critical point up to period 8, and gives the theorems that led to this classification. The proofs are given in the final section.

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2. Preliminaries

Let $f(x) = 1 - ax^2$ be a quadratic map on the interval $I := [1-a, 1]$. The critical point is $c = 0$, and we write $c_i = f^i(c)$ for the i-th image of c. Therefore the interval $[1-a, 0] = [c_2, c_1]$ is the *core* of the map; throughout this paper, we will always restrict unimodal maps to their cores. The inverse limit space is

$$X := (I, f) = \{(\dots x_{-3}, x_{-2}, x_{-1}); x_i = f(x_{i-1}) \in I \text{ for all } i < 0\}.$$

Endow X with product topology and metric $d(x, y) = \sum_{i<0} 2^i |x_i - y_i|$. Let $\pi_i(x) = x_i$ be the projection on the i-th coordinate for $i < 0$ and $\pi(x) := f(x_{-1})$ for the projection on the 0-th coordinate. The *induced homeomorphism* is

$$\hat{f}((\dots x_{-3}, x_{-2}, x_{-1})) = (\dots x_{-3}, x_{-2}, x_{-1}, f(x_{-1})),$$

with the right-shift as inverse $\hat{f}^{-1}$.

We use the standard symbolic dynamics known as *kneading theory* for f. Given $x \in I$, the *itinerary* of x is the sequence $e(x) = e_0 e_1 e_2 \dots$, with

$$e_i = \begin{cases} 1 & \text{if } f^i(x) > c; \\ \star & \text{if } f^i(x) = c; \\ 0 & \text{if } f^i(x) < c. \end{cases}$$

The itinerary of the critical value c_1 is called the *kneading sequence* and will be denoted as $\nu = \nu_1 \nu_2 \nu_3 \dots$ Let $\vartheta(e_1 \dots e_n)$ denote the number of ones in $e_1 \dots e_n$. We use the shorthand $\vartheta(n) = \vartheta(\nu_1 \dots \nu_n)$. If c is periodic, say of period N, then let by convention

$$\nu_N \in \{0, 1\} \text{ be such that } \vartheta(N) \text{ is even.} \tag{1}$$

We need the parity-lexicographical ordering $\preceq$: if $e = e_1 e_2 e_3 \dots$ and $\tilde{e} = \tilde{e}_1 \tilde{e}_2 \tilde{e}_3 \dots$ are finite or infinite words of the symbols $0, \star, 1$ and their first difference is at entry k, then

$$e \prec \tilde{e} \text{ if } \begin{cases} e_k < \tilde{e}_k \text{ and } \vartheta(e_1 \dots e_{k-1}) \text{ is even, or} \\ e_k > \tilde{e}_k \text{ and } \vartheta(e_1 \dots e_{k-1}) \text{ is odd,} \end{cases}$$

where $0 < \star < 1$. Assuming that $e_i = \star$ implies that $e_{i+1} e_{i+2} \dots = \nu_1 \nu_2 \dots$, this is a total ordering on the set of $0 \star 1$-words. A kneading sequence $\nu \in \{0, 1\}^{\mathbb{N}}$ with $\nu_1 = 1$ is called *admissible* if there exists a unimodal map with ν as kneading sequence. Admissible sequences can be characterized by the fact that they are shift-maximal with respect to the left-shift σ and the parity-lexicographical ordering:

$$\sigma\nu \preceq \sigma^n \nu \preceq \nu \text{ for all } n \geq 0.$$

Equivalent admissibility conditions for kneading sequences are given in Section 5. Given a kneading sequence ν, we say that a $0\star1$-word A is *admissible* (cf. [13, Chapter 1.18]), if

$$\sigma\nu \preceq B\nu \preceq \nu \text{ for all subwords } B \text{ of } A.$$

Note that a finite subword B corresponds to an interval in I whose itinerary starts with B. We extend the symbolic dynamics to the inverse limit space, by giving $x \in X$ the *backward itinerary* $e(x) = \dots e_{-3}e_{-2}e_{-1}$, where

$$e_i = \begin{cases} 1 & \text{if } x_i > c; \\ \star & \text{if } x_i = c; \\ 0 & \text{if } x_i < c. \end{cases}$$

Lemma 1. *(cf.* [13, Chapter 1.18]*) The sequence e is the backward itinerary of some point $x \in X$ if and only if*

(1) *if $e_{-k} = \star$, then $e_{-k+1} \dots e_{-1} = \nu_1 \dots \nu_{k-1}$;*
(2) *$\sigma\nu \preceq B\nu \preceq \nu$ for all subwords B of $\dots e_{-3}e_{-2}e_{-1}$.*

Lemma 2. *Let f have a periodic critical point. Let $p, \tilde{p} \in X$ have backward itineraries $e, \tilde{e}$. Then p and $\tilde{p}$ belong to the same arc-component if and only if e and $\tilde{e}$ have the same tail, i.e. $e_{-i} = \tilde{e}_{-i}$ for all i sufficiently large.*

Proof. This is contained [9]. In fact, assuming that f is long-branched (or equivalently, its kneading map if bounded, see Lemma 4 is sufficient for this result. In general, the "if" direction is true: if backward itineraries e and $\tilde{e}$ agree from entry $-N$ onwards, then the arc-component can be parametrized by the $-N$-th coordinate. The "only if" direction fails, however. There can be arcs whose endpoint(s) have backward itineraries with different tails than the backward itineraries of the rest of the arc, cf. the endpoint characterizations in [11]. □

To describe the folding pattern of an arc-component, we define

$$\tau_R(e) = \sup\{n \geq 1; e_{-(n-1)} \cdots e_{-1} = \nu_1 \cdots \nu_{n-1},\ \vartheta(n-1) \text{ is even}\}$$

and

$$\tau_L(e) = \sup\{n \geq 1; e_{-(n-1)} \cdots e_{-1} = \nu_1 \cdots \nu_{n-1},\ \vartheta(n-1) \text{ is odd}\}.$$

It was shown in [11] (cf. [9]), that the set of points $x \in X$ with given sequence e as itinerary is an arc A which projects to

$$\pi(\overline{A}) = [c_{\tau_L(e)}, c_{\tau_R(e)}]. \tag{2}$$

(Here we assume that τ_L and τ_R both are finite; for examples otherwise, see [11].) Moreover, if A and $\tilde{A}$ are two such adjacent arcs in the same arc-component of X, then for the corresponding backward itineraries e and $\tilde{e}$, $e_i = \tilde{e}_i$, except for a single $i < 0$ which is either $i = \tau_L(e) = \tau_L(\tilde{e})$ or $i = \tau_R(e) = \tau_R(\tilde{e})$. This gives an algorithm to compute the folding pattern of an arc-component.

$\dots 1111111$	$[c_2, c_1]$
$\dots 1110111$	$[c_2, c_1]$
$\dots 1101011$	$[c_5, c_1]$
$\dots 1111011$	$[c_5, c_1]$
$\dots 1110101$	$[c_2, c_1]$
$\dots 1111101$	$[c_2, c_4]$
$\dots 1111010$	$[c_3, c_1]$
$\dots 1111110$	$[c_3, c_1]$

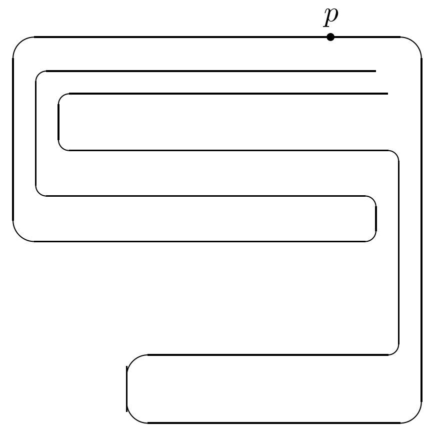

FIGURE 2. Embedding in the plane of the arc-component through the fixed point p for $\nu = \overline{101}$. The horizontal lines denote adjacent arcs A with constant backward itinerary. To the left, at the same horizontal line, these backward itineraries are given, and the projection $\pi(\overline{A})$ according to equation (2).

Define $\{\alpha_n\}_{n\in\mathbb{Z}\setminus\{0\}}$, $\alpha_n = \alpha_n(e) \in \mathbb{Z}$ as follows.

$$\alpha_1(e) = \tau_R(e) \text{ and } \alpha_{-1}(e) = \tau_L(e).$$

The coordinate α_0 remains undefined. Next define Re by

$$(Re)_i = \begin{cases} e_i & \text{if } i \neq \tau_R(e), \text{ i.e. } i \neq \alpha_1(e), \\ 1 - e_i & \text{if } i = \tau_R(e), \text{ i.e. } i = \alpha_1(e). \end{cases}$$

Similarly, $R^{-1}e$ is defined as

$$(R^{-1}e)_i = \begin{cases} e_i & \text{if } i \neq \tau_L(e), \text{ i.e. } i \neq \alpha_{-1}(e), \\ 1 - e_i & \text{if } i = \tau_L(e), \text{ i.e. } i = \alpha_{-1}(e). \end{cases}$$

We continue to define $\alpha_2(e) = \tau_L(Re)$, $\alpha_3(e) = \tau_R(R^2e)$, etc., and for negative subscripts $\alpha_{-2}(e) = \tau_R(R^{-1}e)$, $\alpha_{-3}(e) = \tau_L(R^{-2}e)$, etc.[1] In other words, the numbers $\alpha_1, \alpha_2, \dots$ record at which entries the backward itineraries change as we follow the arc-component through x to the right, while $\alpha_{-1}, \alpha_{-2}, \dots$ record changes as we follow the arc-component through x to the left. Figure 2 gives a suggestive embedding into the plane of the arc-component through the fixed point p (with backward itinerary $\dots 111$) of the induced homeomorphism for $\nu = \overline{101}$.

[1]Note that $(R^n)_{n\in\mathbb{Z}}$ fails the usual group property of dynamical systems, since $R^{n+m}(e) \neq R^n(R^m(e))$ if m is odd, see (3).

If $p \neq \tilde{p} \in X$ lie on the same arc-component C (and hence their backward itineraries e and $\tilde{e}$ have the same tail), then we say that $p \lhd \tilde{p}$ if $\tilde{p}$ is reached from p by following C to the right. This means that

$$\begin{cases} \pi(p) < \pi(\tilde{p}) & \text{if } e = \tilde{e}, \\ \tilde{e} = R^k e \text{ for some } k > 0 & \text{if } e \neq \tilde{e}. \end{cases}$$

By abuse of notation, we say that $e \lhd \tilde{e}$ in this case. Note that $\lhd$ is not a transitive order relation: Since e and Re meet at their 'right' endpoints (and hence both $e \lhd Re$ and $Re \lhd e$ are true), what is to the right of Re turns out to be to the left of e. We use this repeatedly in the following form:

$$Re \lhd R^m e \text{ implies } e \ntriangleleft R^m e \text{ and hence } m < 0. \tag{3}$$

Define the *last discrepancy* of e and $\tilde{e}$ as

$$\overline{d} = \overline{d}(e, \tilde{e}) = \sup\{i; e_{-i} \neq \tilde{e}_{-i}\}.$$

Obviously, $\overline{d} < \infty$ if e and $\tilde{e}$ have different tails.

Lemma 3. *Let $e, \tilde{e}$ be two backward itineraries with the same tail. Then $e \lhd \tilde{e}$ if and only if $\sum_{i=1}^{\overline{d}-1} e_{-i}$ is even.*

Proof. Assume that $p, \tilde{p} \in X$ have backward itineraries e and $\tilde{e}$. Without loss of generality, $\pi_i(p), \pi_i(\tilde{p}) \neq c$ for all i. Let B be the arc connecting p and $\tilde{p}$. It is easy to see that $\pi_{\overline{d}} : B \to I$ is a homeomorphism, and that the critical point lies between $p_{-\overline{d}} = \pi_{-\overline{d}}(p)$ and $\tilde{p}_{-\overline{d}} = \pi_{-\overline{d}}(\tilde{p})$.

Assume $p_{-\overline{d}} < c < \tilde{p}_{-\overline{d}}$. Take q in the interior of B such that q and p have the same itinerary. Then $\pi(p) < \pi(q)$ if and only if $f^{\overline{d}}$ preserves orientation on $(p_{-\overline{d}}, q_{-\overline{d}})$ if and only if $\sum_{i=1}^{\overline{d}-1} e_{-i}$ is even. In this case we have indeed $\tilde{p} > p$.

The other case $\tilde{p}_{-\overline{d}} < c < p_{-\overline{d}}$ is treated in the same way. $\square$

Corollary 1. *Suppose two backward itineraries e and $R^n e$ coincide on the rightmost k entries: $e_{-k} \dots e_{-1} = (R^n e)_{-k} \dots (R^n e)_{-1}$. Then there exists $m < n$ such that $\alpha_m(e) > k$.*

3. Asymptotic arc-components

Definition 1. *Two arc-components C and $\tilde{C}$ of X are* asymptotic *if there exists parametrizations $\varphi, \tilde{\varphi}$ of $C, \tilde{C}$ such that* $\lim_{t\to\infty} d(\varphi(t), \tilde{\varphi}(t)) = 0$.

Given two itineraries e and $\tilde{e}$, let

$$\underline{d}_0 := \underline{d}(e, \tilde{e}) := \min\{n; e_{-n} \neq \tilde{e}_{-n}\}$$

be the *first discrepancy*. Similarly, $\underline{d}_n = \underline{d}(R^n(e), R^n(\tilde{e}))$.

Proposition 1. *Let C and $\tilde{C}$ be two arc-components, containing points with backward itineraries e and $\tilde{e}$. Then C and $\tilde{C}$ are asymptotic if and only if the following holds:*

1. *There exists $k, \tilde{k} \in \mathbb{Z}$ such that $\underline{d}_n(R^k(e), R^{\tilde{k}}(\tilde{e})) \to \infty$ as $n \to \infty$.*
2. *Let α and $\tilde{\alpha}$ be the folding patterns of $R^k(e)$ and $R^{\tilde{k}}(\tilde{e})$ respectively. Then $|c_{\alpha_n} - c_{\tilde{\alpha}_n}| \to 0$ as $n \to \infty$.*

Proof. We carry out the proof for the case that c is periodic of period N. This means that Condition (2) is equivalent to $c_{\alpha_n} = c_{\tilde{\alpha}_n}$ for all n sufficiently large.

Let $\varepsilon = \frac{1}{8}\{|c_i - c_j|; 0 < i < j \leq N\}$. If C and $\tilde{C}$ are indeed asymptotic, then they have parametrizations φ and $\tilde{\varphi}$ such that $d(\varphi(t), \tilde{\varphi}(t)) \to 0$ as $t \to \infty$. Take t_0 so large that $d(\varphi(t), \tilde{\varphi}(t)) \leq 4|\pi \circ \varphi(t) - \pi \circ \tilde{\varphi}(t)| < \varepsilon$ for all $t \geq t_0$. Then the right folding patterns of $\varphi(t_0)$ and $\tilde{\varphi}(t_0)$ must be the same modulo N. For any backward itinerary e one can find $k \in \mathbb{Z}$ such that the backward itinerary of $\varphi(t_0)$ coincides with $R^k(e)$. The analogous statement holds for $\tilde{e}$. Condition (2) follow immediately. Suppose by contradiction that $\underline{d}_n(R^k(e), R^{\tilde{k}}(\tilde{e})) = K < \infty$ infinitely often. For each such n, the middle points on the arcs corresponding to $R^{k+n}(e)$ and $R^{\tilde{k}+n}(\tilde{e})$ are some definite distance away. This contradicts that $d(\varphi(t), \tilde{\varphi}(t)) \to 0$.

Conversely, if the folding patterns of $R^k(e)$ and $R^{\tilde{k}}(\tilde{e})$ are the same modulo N, then there are parametrizations φ of C and $\tilde{\varphi}$ of $\tilde{C}$ such that $\varphi(0)$ has backward itinerary e, $\tilde{\varphi}(0)$ has backward itinerary $\tilde{e}$, and $\pi \circ \varphi(t) = \pi \circ \tilde{\varphi}(t)$ for all $t \geq 0$. Condition (1) implies that also $d(\varphi(t), \tilde{\varphi}(t)) \to 0$.

The proof for when c is not periodic is similar. □

Example 1. *Let $\nu = \overline{101}$. The arc-component C through the fixed point p of $\hat{f}$ is shown Figure 2, whereas Figure 3 shows its folding pattern; more precisely, points x with backward itinerary $e = \dots 11111101$ and $\tilde{x}$ with backward itinerary $\tilde{e} = \dots 11111010$ are compared. (If $\pi(x) < c$, then we could even take $\tilde{x} = \hat{f}(x)$, which lies in the same arc-component because $\hat{f}(C) = C$.) The table shows that the two components of $C \setminus \{p\}$ are asymptotic to each other. For this reason, we want to call the arc-component C* self-asymptotic*. (This is stronger than C and C being asymptotic. In the sense of Definition 1, every arc-component is trivially asyptotic to itself.) To prove that C is self-asymptotic, observe that $\vartheta(\alpha_n) - \vartheta(\tilde{\alpha}_n)$ is always a multiple of 3 and $\underline{d}_n(e, \tilde{e}) \to \infty$. Note that $R^2 e = e11$ and $R^2 \tilde{e} = \tilde{e}11$ (and $R^9 e = e1111$, $R^9 \tilde{e} = \tilde{e}1111$, and also $Re = \tilde{e}1$, $R\tilde{e} = e1$ and $R^6 e = \tilde{e}111$, $R^6 \tilde{e} = e111$). This similarity is used to prove the above statements, see Case I in Section 5.*

Corollary 2. *If f has a strictly preperiodic critical point, then X has no asymptotic pair of arc-components.*

n	α_n	$\vartheta(\alpha_n)$	$R^{n-1}e$	$\tilde\alpha_n$	$\vartheta(\tilde\alpha_n)$	$R^{n-1}\tilde e$
1	4	2	$\dots 11111101$	1	0	$\dots 11111010$
2	2	1	$\dots 11110101$	5	3	$\dots 11111011$
3	1	0	$\dots 11110111$	1	0	$\dots 11101011$
4	6	3	$\dots 11110110$	3	1	$\dots 11101010$
5	1	0	$\dots 11010110$	1	0	$\dots 11101110$
6	2	1	$\dots 11010111$	2	1	$\dots 11101111$
7	4	2	$\dots 11010101$	7	4	$\dots 11101101$
8	2	1	$\dots 11011101$	2	1	$\dots 10101101$
9	1	0	$\dots 11011111$	1	0	$\dots 10101111$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

FIGURE 3. Folding pattern of the asymptotic arc-components for $\nu = \overline{101}$. Note that 3 divides $\alpha_n - \tilde\alpha_n$ for all $n \geq 0$.

Proof. Let f have period N and preperiod M. Suppose that $C \neq \tilde C$ are asymptotic. Choose corresponding itineraries e and $\tilde e$ such that for the folding patterns $|c_{\alpha_n} - c_{\tilde\alpha_n}| \to 0$ as $n \to \infty$. Hence for some $n_0 > 0$, N divides $\alpha_n - \tilde\alpha_n$ for all $n \geq n_0$. At the same time, the discrepancies $\underline{d}_n \to \infty$. Fix some $n_1 \geq n_0$ such that $\underline{d}_{n_1} \gg N + M$. Let $n_2 \geq n_1$ be the smallest integer such that $\alpha_{n_2} = \underline{d}_{n_1} < \tilde\alpha_{n_2}$. (The case $\tilde\alpha_{n_2} = \underline{d}_{n_1} < \alpha_{n_2}$ can be treated by the same argument.) Note that such n_2 must exists, because otherwise the discrepancy $\underline{d}_{n_1}$ is never resolved, contradicting that $\underline{d}_n \to \infty$.

Now we have $\tilde\alpha_{n_2} - \alpha_{n_1} = iN$ for some $i \geq 1$. Let $\eta = R^{n_2}(e)$ and $\tilde\eta = R^{n_2}(\tilde e)$. Then

$$\eta_{-\alpha_{n_2}} \cdots \eta_{-1} = \tilde\eta_{-\alpha_{n_2}} \cdots \tilde\eta_{-1} = \nu_1 \cdots \nu_{\alpha_{n_2}-1}.$$

On the other hand,

$$\tilde\eta_{-\tilde\alpha_{n_2}} \cdots \tilde\eta_{-1} = \nu_1 \cdots \nu_{\tilde\alpha_{n_2}-1} = \nu_1 \dots \nu_{iN} \nu_1 \cdots \nu_{\alpha_{n_2}-1}.$$

This contradicts that ν is preperiodic. □

A similar proof shows that X has no asymptotic arc-components when f has a non-recurrent critical point.

4. Periodic kneading sequences

Let from now on c be periodic with period N. Recall that by convention we write $\nu = \overline{\nu_1 \dots \nu_N}$, where $\nu_N \in \{0, 1\}$ is such that $\vartheta(N)$ is even. In Figure 4, we list the admissible periodic kneading sequences up to period 8, the tails of their asymptotic arc-components and the pattern these arc-components make. The kneading sequences for $a \leq a_{Feig}$ are left out, because their inverse limit spaces possess no asymptotic arc-components, see [7].

	ν	type	periodic tail(s)	k	$Case$
1	101	1-cycle	1	2	I
2	1001	3-fan	101	3	I
3	10001	4-fan	1001	4	I
4	10010	3-cycle	101	3	II
5	10111	three 2-fans	101110	3	III
6	100001	5-fan	10001	5	I
7	100010	4-cycle	1001	4	II
8	100111	four 2-fans	10010011	4	III
9	101110	two linked 3-fans[2]	10, 1	4	II, IV
10	1000001	6-fan	100001	6	I
11	1000010	5-cycle	10001	5	II
12	1000111	five 2-fans	1000100011	5	III
13	1000100	four 2-fans (l.i.p.)	10, 1001	4	II
14	1001101	four 2-fans	10011010	4	III
15	1001110	five 2-fans	10010, 10111	5	II
16	1001011	five 2-fans	1001011011	5	III
17	1011010	5-cycle	10111	5	II
18	1011111	five 2-fans	1011111110	5	III
19	10000001	7-fan	1000001	7	I
20	10000010	6-cycle	100001	6	II
21	10000111	six 2-fans	100001110000	6	III
22	10000100	five 2-fans	10001, 10010	5	II
23	10001101	five 2-fans	1000110100	5	III
24	10001110	six 2-fans	100010, 100111	6	II
25	10001011	six 2-fans	100010110011	6	III
26	10011010	six 2-fans (l.i.p.)	101, 100111	6	II
27	10011111	six 2-fans	100111110110	6	III
28	10011100	five 2-fans	10010, 10111	5	II
29	10010101	five 2-fans	1001010111	5	III
30	10010110	six 2-fans (l.i.p.)	100, 101110	6	II
31	10110111	three 3-cycles	101101110	3	III
32	10111110	two linked 4-fans	$101110, 1$	5	II, IV

FIGURE 4. Periodic kneading sequences and corresponding asymptotic arc-components. (l.i.p. = linked in pairs.)

Suspensions of substitution shifts (also called *substitution tilings spaces*) have frequently been studied in the 20th century, see e.g. monographs of Hedlund & Gottschalk [14], and Queffelec [18]. It is in this context that asymptotic arc-components (their existence and finiteness) were noticed first,

[2]The strands of each fan that is not asymptotic to the other fan belong to one arc-component, and hence are self-asymptotic.

cf. [18, Theorem V.21]. Since such spaces appear as orientable 2-to-1 coverings of unimodal inverse limit spaces, one can conclude that the collection $\mathcal{A}$ of asymptotic arc-components is non-empty and finite. In [5, Proposition 4], it is shown (after subtracting the N arc-components with endpoints), that the cardinality $\#\mathcal{A} \leq 2(N-2)$. As $\hat{f}$ permutes the asymptotic arc-components, they can be viewed as the unstable manifolds of periodic points of $\hat{f}$, and their backward itineraries have periodic tails. The maximal (in parity-lexicographical order) shifts of these tails are shown in the fourth column. Let us write $s \sim t$ if the tails s and t are tails of asymptotic arc-components. If only one tail t is given, it means that $t \sim \sigma^n t$ for some $n \geq 1$: the tail is asymptotic to a shift of itself. For instance, in line 2, all three shifts of $t = \overline{101}$ are simultaneously asymptotic, see Case I of Theorem 1. This leads to a 3-fan of asymptotic arc-components (cf. Figure 1).

In line 4, the shift of the same tail $t = \overline{101}$ are only pairwise asymptotic: $t \sim \sigma t$, $\sigma t \sim \sigma^2 t$ and $\sigma^2 t \sim t$. The resulting arrangement of asymptotic arc-components is called a 3-cycle, see Figure 1).

In line 12, the tail $t = \overline{1000100011}$ is asymptotic to its fifth shift: $t \sim \sigma^5 t$, and similarly $\sigma t \sim \sigma^6 t$, $\sigma^2 t \sim \sigma^7 t$, $\sigma^3 t \sim \sigma^8 t$ and $\sigma^4 t \sim \sigma^9 t$. Hence we see five 2-fans, and $\hat{f}$ permutes all of the 10 strands in one cycle. In line 15, we see two tails $t = \overline{10010}$ and $s = \overline{10111}$. Here $t \sim s$, $\sigma t \sim \sigma s$, etc., so again we see five 2-fans, but this time $\hat{f}$ permutes them in two separate cycles. We do not know if all homeomorphisms of X into itself are homotopic to iterates of $\hat{f}$, but if this is true, it would follow that the inverse limit spaces of line 12 and line 15 are non-homeomorphic[3]. In any case, a homeomorphism between these spaces cannot commute with the induced homeomorphisms.

In line 13, we find two different tails $t = \overline{10}$ and $s = \overline{1001}$. Since $t \sim s$, $\sigma t \sim \sigma s$ and also $t = \sigma^2 t \sim \sigma^2 s$, $\sigma t = \sigma^3 t \sim \sigma^3 s$, we find that the four resulting 2-fans must actually be linked in pairs. The pattern consists of two copies of a linked pair, see Figure 1).

In line 31, we have $t = \overline{101101110}$ which is asymptotic to $\sigma^3 t$. Moreover, it turns out that $\sigma^3 \sim \sigma^6$ and $\sigma^6 t \sim t$. The same happens for $\sigma t \sim \sigma^4 t$, $\sigma^4 t \sim \sigma^7 t$, $\sigma^7 t \sim \sigma t$ and $\sigma^2 t \sim \sigma^5 t$, $\sigma^5 t \sim \sigma^8 t$, $\sigma^8 t \sim \sigma^2 t$. This gives three copies of 3-cycles.

The following theorem predicts the asymptotic arc-components and their tails. We write $\rho : \mathbb{N} \to \mathbb{N}$,

$$\rho(n) = \min\{k > n; \nu_k \neq \nu_{k-n}\}.$$

and we use the notation $a' := 1 - a$ for $a \in \{0, 1\}$.

[3]Swanson & Volkmer [20] showed that these inverse limit spaces are indeed non-homeomorphic, but for a different reason.

Theorem 1. *Given a periodic kneading sequence $\nu = \overline{\nu_1 \dots \nu_N}$ which is not of $*$-product structure (i.e., f is not renormalizable, see below),*

- **Case I:** $\nu = \overline{10^{N-2}1}$. *Then*
$$\begin{cases} e &= \overline{10^{N-3}1}10^{N-2}1 \\ \tilde{e} &= \overline{10^{N-3}1}10^{N-2}10^l \quad \textit{for} \quad l = 1, \dots, N-2 \end{cases}$$
are asymptotic to the right.

Assume that $k < N$ is such that $\rho(k) \geq N$.

- **Case II:** $\vartheta(k)$ *is odd. Then*
$$\begin{cases} e &= \overline{\nu_1 \dots \nu'_k}\nu_1 \dots \nu_N \\ \tilde{e} &= \overline{\nu_1 \dots \nu'_{N-k}\nu_{N-k+1} \dots \nu_k}\nu_1 \dots \nu'_{N-k}\nu_{N-k+1} \dots \nu'_N \end{cases}$$
are asymptotic to the right (if admissible).

- **Case III:** $\vartheta(k)$ *is even and $N = ak + r$ for $0 \leq r < N$. Then*
$$\begin{cases} e &= \overline{\nu_{r+1} \dots \nu'_k \nu_1 \dots \nu_N} \\ \tilde{e} &= \overline{\nu_{r+1} \dots \nu'_k \nu_1 \dots \nu_N}\nu_{N-k+1} \dots \nu'_N \end{cases}$$
are asymptotic to the right (if admissible).

The resulting basic pattern of asymptotic arc-components is a $k-1$-fan in Case I, k 2-fans in Cases II and III, provided $a = 1$. However, if the backward itineraries e and $\tilde{e}$ have some additional symmetry, a more complicated pattern may arise.

Theorem 1 does not predict, for Cases II and III, whether the backward itineraries e and $\tilde{e}$ are admissible. Corollary 3 shows that in each case, only one value of k is possible. For most lines in Figure 4, the choice $k = \min\{i < N; \rho(i) \geq N\}$ works, and other choices of k give non-admissible backward itineraries. An exception is line 17. Here $k = 5$ is taken, whereas the minimal value $k = 3$ leads to a non-admissible backward itinerary.

Cases I-III are mutually exclusive, see Proposition 2 in Section 5. Note that, if an N-periodic kneading sequence ν satisfies Case I, then it also satisfies Case II, which leads to one more asymptotic arc-component with tail $\dots 0000$. However, this tail is only admissible if we enlarge the interval I on which f is defined to $[q, 1]$, where $q = \frac{1}{2a}(-1 - \sqrt{4a+1})$ is the orientation preserving fixed point of f. The arc-component with tail $\dots 000$ is the arc-component of $(\dots, q, q, q)$, which coils into (I, f) in such a way that it is asymptotic to the $N-1$-fan.

A unimodal map is *renormalizable* if there exists a closed neighborhood $J \ni c$ and an integer $m > 1$ such that $J, f(J), \dots f^{m-1}(J)$ have pairwise disjoint interiors, while $f^m(J) \subset J$. Let $\mu = \overline{\mu_1 \dots \mu_m}$ be the itinerary of

$f(J)$, where $\mu_m \in \{0,1\}$, using the same convention as (1). Also let $\tilde{\mu}$ be the kneading sequence of $f^m|J$. The kneading sequence of ν can be written as $*$-product $\nu = \mu * \tilde{\mu}$, defined as

$$\nu_i = \begin{cases} \mu_{i \bmod m} & \text{if } i \bmod m \neq 0, \\ \mu_m & \text{if } i \bmod m = 0, \text{ and } \vartheta(\mu_1 \dots \mu_{m-1}) \text{ is even,} \\ \mu'_m & \text{if } i \bmod m = 0, \text{ and } \vartheta(\mu_1 \dots \mu_{m-1}) \text{ is odd.} \end{cases}$$

The following theorem takes care of these renormalizable examples.

Theorem 2. *Let ν be a periodic kneading sequence.*

- **Case IV:** *$\nu = \mu * \tilde{\mu}$ has $*$-product structure, where μ has period m. Then X has m subcontinua, each of them homeomorphic to the inverse limit space of the unimodal map with kneading sequence $\tilde{\mu}$, having corresponding asymptotic arc-components.*

Proof. This is standard. Since f is renormalizable, the set

$$\{x; x_{-im+j} \in f^j(J) \text{ for all } i \geq 1 \text{ and } j = 0, 1, \dots, m-1\}$$

forms a subcontinuum H homeomorphic to $(J, f^m|J)$. The subcontinua $H, \hat{f}(H), \dots \hat{f}^{m-1}(H)$ are pairwise disjoint, and connected to each other by additional arc-components, see [2]. Each H has the same asymptotic arc-components as $(J, f^m|J)$. □

As an example, in line 32 from Figure 4, $\nu = 11 * 1001$. The corresponding inverse limit space has two copies of the inverse limit space of line 2 as subcontinua, and hence two 3-fans. In addition, there is the arc-component with tail $\dots 1111$, which turns out to entwine in the these subcontinua asymptotically to the two 3-fans, rendering them two linked 4-fans. The same mechanism in line 9 gives two linked 3-fans where the non-linking strands of each 3-fan are in fact asymptotic to themselves.

We think that Theorems 1 and 2 give the complete picture of asymptotic arc-components. They lead to at most $2(N-2)$ asymptotic arc-components (obtained in Cases II and III, when $k = N-2$ and no additional symmetries in the asymptotic backward itineraries are present), which confirms the bound from [5].

Conjecture 1. *Given a periodic kneading sequence, let $\mathcal{A}$ be the collection of asymptotic arc-components of the corresponding inverse limit space. If $C, \tilde{C} \in \mathcal{A}$, then C is asymptotic to, or coincides with, $\hat{f}^n \tilde{C}$ for some $n \in \mathbb{Z}$.*

5. Proofs

In this section we give the proof of Theorem 1. Case I expresses the scheme of the proof in its simplest form. For the other cases, several more technical arguments are necessary to verify the basic steps. This involves a discussion of admissible words for a given kneading sequence, and of the structure of cutting and co-cutting times.

Proof. **Case I:** We have

$$\begin{cases} e &= \overline{10^{N-3}1}10^{N-2}1, \\ \tilde{e} &= \overline{10^{N-3}1}10^{N-2}10^l, \end{cases}$$

so $\alpha_1(e) = N+1$ and $\alpha_1(\tilde{e}) = 1$. Consequently,

$$\begin{cases} Re &= \overline{10^{N-3}1}10^{N-2}10^{N-2}1, \\ R\tilde{e} &= \overline{10^{N-3}1}10^{N-2}10^{l-1}1, \end{cases}$$

and $\underline{d}(Re, R\tilde{e}) = l+1$. Let $n > 2$ be minimal such that $R^n e$ ends with 10^{l-1}. We claim that

$$\alpha_m(e), \alpha_m(\tilde{e}) \le l+1 \text{ for } m < n.$$

Indeed, if $\alpha_m(e) > l+1$ for the first time, then $\alpha_m = N+1$ and $R^m e = \dots 10^{N-2}1$ ends in the same way as Re. By Corollary 1 this cannot happen. If, on the other hand, $\alpha_m(\tilde{e}) > l+1$, then $\alpha_m(\tilde{e}) = N+l+1$ and $R^m\tilde{e} = 10^{l-1}$. But then also $R^m e = \dots 10^{l-1}$ and $\alpha_m(e) = l+1$. This proves the claim, and we find $\alpha_n(e) = l+1$ and $\alpha_n(\tilde{e}) = N+l+1$. Therefore

$$\begin{cases} R^{n+1}e &= \overline{10^{N-3}1}10^{N-2}110^{N-l-2}10^{l-1} &= e0^{N-l-2}110^{l-1}, \\ R^{n+1}\tilde{e} &= \overline{10^{N-3}1}10^{N-2}10^{N-2}110^{l-1} &= \tilde{e}0^{N-l-2}110^{l-1}, \end{cases}$$

which are left-shifted copies of the original backward itineraries. Since e and $R^{n+1}e$ differ by an even number of entries, $n+1$ is even. Therefore, we can repeat the argument and find that $\underline{d}_n(e, \tilde{e}) \to \infty$ while $\alpha_n(e) - \alpha_n(\tilde{e})$ is always a multiple of N. This proves Case I. $\square$

Define the *cutting times* $(S_k)_{k\ge 0}$ and *co-cutting times* $(T_l)_{l \ge 0}$ of a kneading sequence ν as:

$$S_0 = 1, \quad S_k = \rho(S_{k-1})$$

and

$$T_0 = \min\{i > 1; \nu_i = 1\}, \quad T_l = \rho(T_{l-1}).$$

We list some facts of (co-)cutting times and admissibility from [10, 15, 21].

Lemma 4. *The following statements are equivalent:*

- *A kneading sequence ν is admissible.*
- $\sigma\nu \preceq \sigma^n\nu \preceq \nu$ *for all* $n \ge 0$.
- $\rho(n) - n$ *is a cutting time for each* $n \ge 1$.
- $\rho(n) - n$ *is a cutting time for each cutting or co-cutting time* n*, and* $(S_k)_{k\ge 0}$ *and* $(T_l)_{l\ge 0}$ *are disjoint sequences.*

- $S_k - S_{k-1} = S_{Q(k)}$ *for some* $Q : \mathbb{N} \to \mathbb{N} \cup \{0\}$ *(this map* Q *is called the* kneading map *of* ν*) and*

$$\{Q(k+j)\}_{j \geq 1} \geq_{lex} Q(Q^2(k)+j\}_{j\geq 1} \tag{4}$$

for all k *(*$\geq_{lex}$ *is lexicographical order).*

The (co-)cutting times serve as natural dividers of the kneading sequence, and they determine which subwords of ν are admissible. We list some of these properties more precisely, writing $\bar{S}(n) = \min\{S_k; S_k > n\}$ and $\bar{T}(n) = \min\{T_l; T_l > n\}$.

Lemma 5. **a:** *$\vartheta(S_k)$ is odd for all k and $\vartheta(T_l)$ is even for all l.*

b: *If $\rho(n) > \bar{S}(n)$, then $\vartheta(\nu_{n+1} \dots \nu_{\bar{S}(n)})$ is odd. If $\rho(n) > \bar{T}(n)$, then $\vartheta(\nu_{n+1} \dots \nu_{\bar{T}(n)})$ is odd.*

c: *The word $\nu_1 \dots \nu_{n-1}\nu'_n$ is admissible if and only if n is a cutting time.*

d: *If ν is N-periodic (with convention (1)), then N is a co-cutting time and no cutting time is a multiple of N.*

Proof. **a:** This follows by induction and the fact that $\rho(n) - n$ is a cutting time.

b: We prove the following statement by induction

$$\begin{array}{c} \left.\begin{array}{l} \rho(k) > \bar{S}(k) \\ \rho(k) = \bar{S}(k) \end{array}\right\} \quad \text{implies} \quad \left\{\begin{array}{l} \vartheta(k) \text{ is even.} \\ \vartheta(k) \text{ is odd.} \end{array}\right. \\ \\ \left.\begin{array}{l} \rho(k) > \bar{T}(k) \\ \rho(k) = \bar{T}(k) \end{array}\right\} \quad \text{implies} \quad \left\{\begin{array}{l} \vartheta(k) \text{ is odd.} \\ \vartheta(k) \text{ is even.} \end{array}\right. \end{array} \tag{5}$$

This is obviously true for $k \leq \min\{i > 1; \nu_i = 1\}$. Suppose (5) is true for all integers less than k. Assume $\rho(k) > \bar{S}(k)$ and $\bar{S}(k) = S_i$, so $S_{i-1} \leq k < \bar{S}(k) < \rho(k)$. Then $k > S_{i-1}$ because otherwise $\rho(k) = S_i$. It follows that $\rho(k - S_{i-1}) = S_i - S_{i-1}$ is a cutting time, and by induction, $\vartheta(k - S_{i-1})$ is odd. Therefore $\vartheta(k) = \vartheta(S_{i-1}) + \vartheta(k - S_{i-1})$ is even, as asserted. The remaining three statements of (5) are proven in the same way. Now statement b follows immediately from statement a.

c: The points in I with itinerary starting with $\nu_1 \dots \nu_n$ or $\nu_1 \dots \nu'_n$ form a one-sided neighborhood U_n of c_1; this is the largest neighborhood in I on which f^{n-1} is monotone. Both $\nu_1 \dots \nu_n$ and $\nu_1 \dots \nu'_n$ are admissible if and only if $f^{n-1}(U_n) \ni c$ if and only if n is a cutting time.

d: If f is a quadratic map with an N-periodic critical point, then varying the parameter a shows that both $\nu_1 \dots \nu_{N-1}0$ and $\nu_1 \dots \nu_{N-1}1$ can be realized as kneading sequences. The sequences of cutting and co-cutting times are disjoint (see Lemma 4); therefore using convention (1), we conclude that N is a co-cutting time. Since $\vartheta(\nu_1 \dots \nu_{iN})$ is even for all $i \geq 1$, iN is never a cutting time. $\square$

Corollary 3. *The only possible admissible choices of k are in Case II: $k = \min\{i < N; \rho(i) > N\}$, and in Case III: $k = \min\{i < N; \rho(i) = N\}$.*

Proof. In Case II, the minimal k such that $\rho(k) > N$ is the last cutting time before N. If $k' > k$ is taken such that $\rho(k') > N$, then $\vartheta(k')$ is odd, so k' is neither cutting nor co-cutting time. Then, for the backward itinerary e based on k', the subword $\nu_1 \dots \nu'_{k'}$ is not admissible.

In Case III, if $k' > k$ is such that $\rho(k') = N$ and hence $\vartheta(k')$ is even, then also $\vartheta(\nu_{k+1} \dots \nu_{k'})$ is even. If the backward itinerary e were based on k', then it contains $\nu_{k+1} \dots \nu'_{k'}$ as a subword, but $\nu_{k+1} \dots \nu'_{k'} = \nu_1 \dots \nu'_{k'-k} \succeq \nu_1 \dots \nu_{k'-k}$, so it is not admissible with respect to ν. □

Now we continue the proof of Theorem 1.

Proof. **Case II:** Since $\vartheta(k)$ is odd, Lemma 5b implies that $k \geq S_i$, the last cutting before N. If $k > S_i$, then by Lemma 5c, the word $\nu_1 \dots \nu'_k$ is not admissible, so $k = S_i$. It follows that

$$\nu_1 \dots \nu_k \nu_1 \dots \nu_{N-k} = \nu_1 \dots \nu_N.$$

and $N - k$ is a cutting time too. In addition, $k > N/2$ because otherwise either $S_{i+1} - S_i$ is not a cutting time, or ν is k-periodic, contradicting Lemma 5d. We have the backward itineraries

$$\begin{cases} e &= \overline{\nu_1 \dots \nu'_k} \nu_1 \dots \nu_N, \\ \tilde{e} &= \overline{\nu_1 \dots \nu'_{N-k} \nu_{N-k+1} \dots \nu_k} \nu_1 \dots \nu'_{N-k} \nu_{N-k+1} \dots \nu'_N. \end{cases}$$

Since N is the minimal period of ν, we find $\alpha_1(e) = N + 1$. Now for $\tilde{e}$, let $\alpha := \alpha_1(\tilde{e}) = \tau_R(\tilde{e})$, so $\vartheta(\alpha)$ is even. If $1 < \alpha \leq k$, then

$$\nu_1 \dots \nu_{N-\alpha} \nu_{N-\alpha+1} \dots \nu'_N = \nu_1 \dots \nu_{N-\alpha} \nu_1 \dots \nu_\alpha,$$

and by Lemma 5b, $\vartheta(N-\alpha)$ is even. Therefore $\vartheta(\nu_1 \dots \nu'_N) = \vartheta(\alpha) + \vartheta(N-\alpha)$ is both even and odd, a contradiction. If $k < \alpha \leq N$, then

$$\nu_1 \dots \nu_{N-\alpha} \nu_{N-\alpha+1} \dots \nu'_{N-k} \nu_{N-k+1} \dots \nu'_N = \nu_1 \dots \nu_{N-\alpha} \nu_1 \dots \nu_\alpha.$$

Since $N-k$ is a cutting time, and $\rho(N-\alpha) = N-k$, we have by Lemma 5b that $\vartheta(N-\alpha)$ is odd. Therefore $\vartheta(\nu_1 \dots \nu_{N-\alpha} \nu_1 \dots \nu_\alpha)$ is odd. But at the same time $\vartheta(\nu_1 \dots \nu'_{N-k} \nu_{N-k+1} \dots \nu'_N) = \vartheta(\nu_1 \dots \nu_N)$ is even, again a contradiction. Obviously $\alpha \neq N + 1$, and if $\alpha = iN + \beta$ for some $i \geq 0$, $1 < \beta \leq N + 1$, then $\tilde{e}$ ends with $\nu_1 \dots \nu_\beta$ and $\vartheta(\beta)$ is even, which was excluded by the above arguments.

This shows that $\alpha_1(\tilde{e}) = 1$, and hence

$$\begin{cases} Re &= \overline{\nu_1 \dots \nu'_k}\nu_1 \dots \nu_k \nu_1 \dots \nu_N \\ &= \overline{\nu_1 \dots \nu'_k}\nu_1 \dots \nu_N \nu_{N-k+1} \dots \nu_N, \\ R\tilde{e} &= \overline{\nu_1 \dots \nu'_{N-k}\nu_{N-k+1} \dots \nu_k}\nu_1 \dots \nu'_{N-k}\nu_{N-k+1} \dots \nu_N \\ &= \overline{\nu_1 \dots \nu'_{N-k}\nu_{N-k+1} \dots \nu_k} \\ & \quad \nu_1 \dots \nu'_{N-k}\nu_{N-k+1} \dots \nu'_N \nu_{N-k+1} \dots \nu_N. \end{cases}$$

As a result, $\underline{d}_1(e, \tilde{e}) = k+1$. Let $n \geq 1$ be minimal such that $R^n\tilde{e}$ ends with $\nu_1 \dots \nu_k$. We claim:

There is no $1 < m < n$ such that $\alpha_m(e) \geq k+1$ or $\alpha_m(\tilde{e}) \geq k+1$.

Let us start with $\tilde{e}$. Take $1 \leq m \leq n$ minimal such that $\alpha := \alpha_m(\tilde{e}) \geq k+1$. If $k < \alpha \leq 2k$, then by Lemma 5b applied to co-cutting time N, $\vartheta(\nu_{N+k-\alpha+1} \dots \nu'_N)$ is odd. Because $N-k$ is a cutting time, $\vartheta(\nu_{N-k+1} \dots \nu_N)$ is odd. Hence $\vartheta(\nu_{N+k-\alpha+1} \dots \nu'_N \nu_{N-k+1} \dots \nu_N)$ is even. Comparing $R\tilde{e}$ and $\tilde{R}^{m+1}\tilde{e}$ using Lemma 3 gives that $R\tilde{e} \lhd R^{m+1}\tilde{e}$, but by (3), we find that $\tilde{e} \not\lhd R^{m+1}\tilde{e}$, a contradiction. If $2k < \alpha \leq N+k$, then Lemma 5b applied to cutting time $N-k$ shows that $\vartheta(\nu_{N+k-\alpha+1} \dots \nu'_{N-k})$ is odd. Therefore $\vartheta(\nu_{N+k-\alpha+1} \dots \nu'_{N-k}\nu_{N-k+1} \dots \nu'_N \nu_{N-k+1} \dots \nu_N)$ is even. The above argument shows that $\tilde{e} \not\lhd R^{m+1}\tilde{e}$, so again we get a contradiction.

Since k is the last cutting time before N, $\rho(k) > N$, and $\nu_{k+1} \dots \nu_N = \nu_1 \dots \nu_{N-k}$. Hence $\nu_{N-k} = \nu_N$ and $\vartheta(\nu_1 \dots \nu'_{N-k}\nu_{N-k+1} \dots \nu'_N) = \vartheta(N) \pm 2$. Since $\vartheta(\nu_{i+1} \dots \nu_{N+i}) = \vartheta(N)$ for all $i \geq 0$, also $\alpha > N+k$ is not possible. This proves the first half of the claim.

We use similar arguments for e: Suppose that $m > 0$ is minimal such that $\alpha := \alpha_m(e) \geq k+1$. If $k < \alpha \leq N+k$, then by Lemma 5b, $\vartheta(\nu_{N+k-\alpha+1} \dots \nu_N)$ is odd. We already know that $\vartheta(\nu_{N-k+1} \dots \nu_N)$ is odd, so $\vartheta((\nu_{N+k-\alpha+1} \dots \nu_N \nu_{N-k+1} \dots \nu_N)$ is even. By Lemma 3, we find $Re \lhd R^{m+1}e$, so again by (3), $e \not\lhd R^{m+1}e$. The case $\alpha = N+k+1$ is possible, but then $R^m e$ ends with $\nu_1 \dots \nu_k$, so $m = n$. Finally, $\alpha > N+k+1$ is impossible, since N is the smallest period of ν and $\nu_1 \dots \nu'_k \neq \nu_{N-k+1} \dots \nu_N$ (they have different parity). This proves the claim, and hence $\alpha_m(e) = \alpha_m(\tilde{e})$ for all $1 < m < n$. We find

$$\begin{cases} R^n e &= \overline{\nu_1 \dots \nu'_k}\nu_1 \dots \nu_k \nu_1 \dots \nu_{N-k}\nu_1 \dots \nu_k \\ &= \overline{\nu_1 \dots \nu'_k}\nu_1 \dots \nu_N \nu_1 \dots \nu_k, \\ R^n\tilde{e} &= \overline{\nu_1 \dots \nu'_{N-k}\nu_{N-k+1} \dots \nu_k}\nu_1 \dots \nu'_{N-k}\nu_1 \dots \nu_k. \end{cases}$$

By assumption, $\vartheta(k)$ is odd, while $\vartheta(\nu_{N-k+1} \dots \nu'_N)$ is even as we saw before. Therefore $\tilde{e}$ and $R^n\tilde{e}$ differ at an odd number of entries, and hence n is odd. Using the same arguments as for α_1 above, we find $\alpha_n(e) = \tau_L(R^n e) = N+k+1$ and $\alpha_n(\tilde{e}) = \tau_L(R^n\tilde{e}) = k+1$, giving a difference of N. Taking one more iterate, we obtain

$$\begin{aligned} R^{n+1}e &= \overline{\nu_1 \dots \nu_k'}\nu_1 \dots \nu_k \nu_1 \dots \nu_N \nu_1 \dots \nu_k \\ &= \overline{\nu_1 \dots \nu_k'}\nu_1 \dots \nu_N \nu_{N-k+1} \dots \nu_N \nu_1 \dots \nu_k \\ &= e\ \nu_{N-k+1} \dots \nu_N \nu_1 \dots \nu_k. \end{aligned}$$

and

$$\begin{aligned} R^{n+1}\tilde{e} &= \overline{\nu_1 \dots \nu_{N-k}' \nu_{N-k+1} \dots \nu_k}\nu_1 \dots \nu_{N-k}\nu_1 \dots \nu_k \\ &= \overline{\nu_1 \dots \nu_{N-k}' \nu_{N-k+1} \dots \nu_k}\nu_1 \dots \nu_{N-k}'\nu_{N-k+1} \dots \nu_N \nu_1 \dots \nu_k \\ &= \overline{\nu_1 \dots \nu_{N-k}' \nu_{N-k+1} \dots \nu_k} \\ &\qquad \nu_1 \dots \nu_{N-k}' \nu_{N-k+1} \dots \nu_k \nu_1 \dots \nu_{N-k}' \nu_{N-k+1} \dots \nu_N \nu_1 \dots \nu_k \\ &= \overline{\nu_1 \dots \nu_{N-k}' \nu_{N-k+1} \dots \nu_k} \\ &\qquad \nu_1 \dots \nu_{N-k}' \nu_{N-k+1} \dots \nu_N' \nu_{N-k+1} \dots \nu_N \nu_1 \dots \nu_k \\ &= \tilde{e}\ \nu_{N-k+1} \dots \nu_N \nu_1 \dots \nu_k. \end{aligned}$$

Hence $R^{n+1}e$ and $R^{n+1}\tilde{e}$ are left-shifted copies of e and $\tilde{e}$. Since $n+1$ is even, we can repeat the argument and find that $\underline{d}_n(e,\tilde{e}) \to \infty$ while $\alpha_n(e) - \alpha_n(\tilde{e})$ is always a multiple of N. This proves Case II.

Case III: We will use the same arguments as for Case II. Since $r = \rho(ak) - ak$, r is a cutting time and hence $\vartheta(r)$ is odd. We started with

$$\begin{cases} e &= \overline{\nu_{r+1} \dots \nu_k' \nu_1 \dots \nu_N}, \\ \tilde{e} &= \overline{\nu_{r+1} \dots \nu_k' \nu_1 \dots \nu_N} \nu_{N-k+1} \dots \nu_N'. \end{cases}$$

Since N is the smallest period of ν, $\alpha_1(e) = iN+1$ for some $i \geq 1$. If $i > 1$, then $\nu_1 \dots \nu_N \nu_{r+1} \dots \nu_k' \nu_1 \dots \nu_N = \nu_1 \dots \nu_{k-r} \nu_1 \dots \nu_N \nu_1 \dots \nu_N$, so $\rho(k-r) > N$. Note that $\vartheta(k-r) = \vartheta(k) - \vartheta(r)$ is odd. Obviously, $k - r < N/2$ and there is always a cutting time S between $N/2$ and N. By Lemma 5b, $\rho(k-r) \leq S$, giving a contradiction. Therefore $\alpha_1(e) = N+1$.

Now for $\tilde{e}$, let $\alpha := \alpha_1(\tilde{e}) = \tau_R(\tilde{e})$, so $\vartheta(\alpha - 1)$ is even. We have

$$\begin{aligned} \vartheta(\nu_{N-k+1} \dots \nu_N') &= \vartheta(\nu_1 \dots \nu_N') - \vartheta(N-k) \\ &= \vartheta(N) \pm 1 - (a-1)\vartheta(k) - \vartheta(r) \quad \text{is even.} \end{aligned}$$

If $\alpha = k+1$, then $\nu_{N-k+1} \dots \nu_N' = \nu_1 \dots \nu_k$. Therefore $\rho(N-k) = N$ whence $\rho(N-k) - (N-k) = k$ is a cutting time by Lemma 4. But $\vartheta(k) = \vartheta(\alpha - 1)$ cannot be both odd and even. If $1 < \alpha \leq k$, then since $\nu_{N-\alpha+1} \dots \nu_N' = \nu_1 \dots \nu_\alpha$ and $\vartheta(\alpha)$ is even, $\vartheta(N-\alpha)$ is also even. But then $\vartheta(N) = \vartheta(N - \alpha) + \vartheta(\nu_{N-\alpha+1} \dots \nu_N')$ is odd, contradicting our convention (1).

If $k+1 < \alpha \leq k+N$, then by Lemma 5 applied to co-cutting time N, $\vartheta(\nu_{N+k-\alpha+1} \dots \nu_N)$ is odd. Hence $\vartheta(\nu_{N+k-\alpha+1} \dots \nu_N \nu_{N-k+1} \dots \nu_N') = \vartheta(\nu_{N+k-\alpha+1} \dots \nu_N) + \vartheta(\nu_{N-k+1} \dots \nu_N')$ is odd, and $\alpha \neq \tau_R(\tilde{e})$. If $\alpha > k+N$, then we repeat the above arguments on $\beta = \alpha - iN$ such that $k+1 \leq \beta \leq k+N$.

The remaining possibility is $\alpha_1(\tilde{e}) = 1$, so

$$\begin{cases} Re &= \overline{\nu_{r+1}\dots\nu'_k\nu_1\dots\nu_N}\nu_{r+1}\dots\nu_k\nu_1\dots\nu_N \\ &= \overline{\nu_{r+1}\dots\nu'_k\nu_1\dots\nu_N}\nu_{r+1}\dots\nu'_N\nu_{N-k+1}\dots\nu_N, \\ R\tilde{e} &= \nu_{r+1}\dots\nu'_k\nu_1\dots\nu_N\nu_{N-k+1}\dots\nu_N. \end{cases}$$

Take n minimal such that $R^n e$ ends with $\nu_1\dots\nu_k$. We claim:

There is no $1 < m < n$ such that $\alpha_m(e) \geq k+1$ or $\alpha_m(\tilde{e}) \geq k+1$.

First take m is minimal such that $\alpha := \alpha_m(e) > k$. If $\alpha = k+1$, then $m = n$ by definition of n. If $k+1 < \alpha \leq N+k-r$, then by Lemma 5b applied to co-cutting time N, $\vartheta(\nu_{N+k-\alpha+1}\dots\nu'_N)$ is odd. From this it follows that $\vartheta(\nu_{N+k-\alpha+1}\dots\nu'_N\nu_{N-k+1}\dots\nu_N)$ is even, and by Lemma 3, $Re \lhd R^{m+1}e$. Therefore (3) shows that $e \not\lhd R^{m+1}e$, a contradiction.

If $N+k-r < \alpha \leq 2N+k-r$, then by Lemma 5b applied to co-cutting time N we get $\vartheta(\nu_{2N+k-r-\alpha+1}\dots\nu_N)$ is odd. Since $\vartheta(r)$ is odd, also $\vartheta(\nu_{r+1}\dots\nu'_N) = \vartheta(N)\pm 1-\vartheta(r)$ is even and as shown before, $\vartheta(\nu_{N-k+1}\dots\nu_N)$ is odd. Therefore $\vartheta(\nu_{2N+k-r-\alpha+1}\dots\nu_N\nu_{r+1}\dots\nu'_N\nu_{N-k+1}\dots\nu_N)$ is even, and we apply Lemma 3 to conclude that $Re \lhd R^{m+1}e$ and hence $e \not\lhd R^{m+1}e$. Finally, if $\alpha > 2N+k-r$, then $\alpha = iN+k-r+1$ for some $i \geq 2$. However, since r is a cutting time, say $r = S_i < N/2$, and since there is a next cutting time $S_{j+1} < N$, we obtain that $\nu_{S_j+1}\dots\nu_{S_{j+1}} = \nu_1\dots\nu'_{S_{j+1}-S_j} \neq \nu_1\dots\nu_{S_{j+1}-S_j}$. This shows that $\alpha \neq iN+k-r+1$, so $\alpha > 2N+k-r$ is also impossible.

Now for $\tilde{e}$, assume that m is minimal such that $\alpha := \alpha_m(\tilde{e}) > k$. If $\alpha = k+1$, then $R^m\tilde{e}$ and $R^m e$ both end with $\nu_1\dots\nu_k$, so $m = n$. If $k+1 < \alpha \leq k+N$, then the above argument shows that $\vartheta(\nu_{N+k-\alpha+1}\dots\nu_N\nu_{N-k+1}\dots\nu_N)$ is odd, and so $m < 0$. If finally, $\alpha > N+k$, then $\alpha = iN+k+1$ for some $i \geq 1$. But then $R^m\tilde{e}$ should end with $\nu_1\dots\nu_k$, so $m = n$. This proves the claim.

Therefore $\alpha_m(e) = \alpha_m(\tilde{e})$ for $1 < m \leq n$ and

$$\begin{cases} R^n e &= \overline{\nu_{r+1}\dots\nu'_k\nu_1\dots\nu_N}\nu_{r+1}\dots\nu_k\nu_1\dots\nu_{N-k}\nu_1\dots\nu_k, \\ R^n\tilde{e} &= \overline{\nu_{r+1}\dots\nu'_k\nu_1\dots\nu_N}\nu_1\dots\nu_k. \end{cases}$$

Recall that $\vartheta(\nu_{N-k+1}\dots\nu_N)$ is odd, and $\vartheta(k)$ is even. Therefore $R\tilde{e}$ and $R^n\tilde{e}$ differ at an odd number of entries, so n is even. Since $R^n e$ ends with $\nu_{k+1}\dots\nu'_N\nu_1\dots\nu_k$ and $\vartheta(\nu_{k+1}\dots\nu'_N\nu_1\dots\nu_k)$ is odd, $\alpha_n(e) \leq N$. If $\alpha_n(e) > k+1$, then by Lemma 5b, $\vartheta(\nu_{N-\alpha+1}\dots\nu'_N)$ is odd, so $\vartheta(\nu_{N-\alpha+1}\dots\nu'_N\nu_1\dots\nu_k)$ is also odd. This would imply that $\alpha_n(e) \neq \tau_R(R^n e)$. Therefore $\alpha_n(e) = \tau_R(R^n e) = k+1$. The check that $\alpha_n(\tilde{e}) = \tau_R(R^n\tilde{e}) = N+k+1$ relies on the same arguments showing hat $\alpha_1(e) = N+1$. This gives

$$\begin{aligned} R^{n+1}e &= \overline{\nu_{r+1}\dots\nu'_k\nu_1\dots\nu_N}\nu_{r+1}\dots\nu_k\nu_1\dots\nu'_{N-k}\nu_1\dots\nu_k \\ &= \overline{\nu_{r+1}\dots\nu'_k\nu_1\dots\nu_N}\nu_{r+1}\dots\nu_N\nu_1\dots\nu_k \\ &= e\ \nu_{r+1}\dots\nu_N\nu_1\dots\nu_k \end{aligned}$$

and

$$\begin{aligned}
R^{n+1}\tilde{e} &= \overline{\nu_{r+1}\dots\nu'_k\nu_1\dots\nu_N}\nu_{r+1}\dots\nu_k\nu_1\dots\nu_N\nu_1\dots\nu_k \\
&= \overline{\nu_{r+1}\dots\nu'_k\nu_1\dots\nu_N}\nu_{r+1}\dots\nu_k\nu_1\dots\nu_r\nu_{r+1}\dots\nu_N\nu_1\dots\nu_k \\
&= \overline{\nu_{r+1}\dots\nu'_k\nu_1\dots\nu_N}\nu_{r+1}\dots\nu'_N\nu_{r+1}\dots\nu_N\nu_1\dots\nu_k \\
&= \tilde{e}\ \nu_{r+1}\dots\nu_N\nu_1\dots\nu_k.
\end{aligned}$$

Hence $R^{n+1}e$ and $R^{n+1}\tilde{e}$ are left-shifted copies of e and $\tilde{e}$. Since $n+1$ is odd and $\vartheta(\nu_{r+1}\dots\nu_N\nu_1\dots\nu_k)$ is odd, we can repeat the argument and find that $\underline{d}_n(e,\tilde{e}) \to \infty$ while $\alpha_n(e) - \alpha_n(\tilde{e})$ is always a multiple of N. This proves Case III. □

Proposition 2. *Case II and Case III of Theorem 1 are mutually exclusive.*

Proof. The idea of the proof is to find subwords of $\tilde{e}$ for Case II and e for Case III that cannot be simultaneously majorized, in the parity-lexicographical order, by ν. Hence at most one of Case II and Case III is admissible.

Let k be as in Case II, i.e., $k = S_b$ is the last cutting time before N and $N-k =: S_i$ is again a cutting time. Let l serve the role of k in Case III. Then $r = N - al$ is also a cutting time, say $r =: S_j$. We distinguish two cases.

1. First assume that $k < l$, so $j < i$. If in Case II,

$$\overline{\nu_1\dots\nu'_{N-k}\nu_{N-k+1}\dots\nu_k}\nu_1\dots\nu'_{N-k}\nu_{N-k+1}\dots\nu'_N$$

is admissible, then

$$\begin{aligned}
\nu_{N-k-r+1}\dots\nu'_{N-k}\nu_{N-k+1}\dots\nu_k\nu_1\dots\nu'_{N-k} &= \\
\nu_1\dots\nu_r\nu_{S_i+1}\dots\nu_{S_b}\nu_1\dots\nu'_{S_i} &\preceq \nu. \qquad (6)
\end{aligned}$$

If in case Case III, If $\overline{\nu_{r+1}\dots\nu'_l\nu_1\dots\nu_N}$ is admissible, then

$$\nu_{N-k+1}\dots\nu_N\nu_{r+1}\dots\nu_l = \nu_1\dots\nu_{S_i}\nu_{r+1}\dots\nu_l \preceq \nu. \qquad (7)$$

Recall the kneading map $Q : \mathbb{N} \to \mathbb{N} \cup \{0\}$ from Lemma 4. By construction $Q(k) < k$ for each k. Condition (4) is in fact a way to express the parity-lexicographic order. If ν and $\tilde{\nu}$ are two 01-words (or kneading sequences) starting with 1, then for both cutting times and kneading maps Q respectively $\tilde{Q}$ can be defined. By induction, it is not hard to show that $\nu \preceq \tilde{\nu}$ if and only if $\{Q(t)\}_{t\geq 1} \geq_{lex} \{\tilde{Q}(t)\}_{t\geq 1}$. Applying this to (6) and (7), we obtain

$$\{Q(t)\}_{t=i+1}^{b} \geq_{lex} \{Q(t)\}_{t=j+1}^{b+j-i} \text{ and } \{Q(t)\}_{t=j+1}^{b} \geq_{lex} \{Q(t)\}_{t=i+1}^{b+i-j}. \qquad (8)$$

Therefore $Q(i+t) = Q(j+t)$ for $t = 1, \dots, b-i$, which is still possible, but it implies that $Q(t) < i$ for $t \leq b$. A closer look at (6) shows that to fulfill the second condition of (8), we also need $N - S_b = S_i = S_{Q(b+j-i)}$, but this is impossible.

2. Assume that $l < k$. Recall that $N = al + r$ and $r = S_j$. Let $S_{\tilde{b}}$ be the last cutting time before l. Thus $S_i, S_j \le S_{\tilde{b}} < l < k = S_b$. If, in Case II

$$\overline{\nu_1 \dots \nu'_{N-k} \nu_{N-k+1} \dots \nu_k} \nu_1 \dots \nu'_{N-k} \nu_{N-k+1} \dots \nu'_N$$

is admissible, then since $\nu_{l+1} \dots \nu_k \nu_1 \dots \nu'_{N-k} = \nu_{l+1} \dots \nu_k \nu_{k+1} \dots \nu'_N$ and $\nu_{al+1} \dots \nu'_N = \nu_1 \dots \nu_r$ we find the subword

$$\nu_{al+1} \dots \nu'_N \nu_{N-k+1} \dots \nu_k = \nu_1 \dots \nu_{S_j} \nu_{S_i+1} \dots \nu_{S_b} \preceq \nu. \tag{9}$$

If in Case III: $\overline{\nu_{r+1} \dots \nu'_l \nu_1 \dots \nu_N}$, is admissible, then

$$\nu_{N-k+1} \dots \nu_N \nu_{r+1} \dots \nu'_l = \nu_1 \dots \nu_{S_i} \nu_{S_j+1} \dots \nu'_l \preceq \nu. \tag{10}$$

Combining (9) and (10) gives

$$\{Q(t)\}_{t=i+1}^{b} \ge_{lex} \{Q(t)\}_{t=j+1}^{b+j-i} \text{ and } \{Q(t)\}_{t=j+1}^{\tilde{b}} \ge_{lex} \{Q(t)\}_{t=i+1}^{\tilde{b}+i-j}. \tag{11}$$

Therefore $Q(i+t) = Q(j+t)$ for $t = 1, \dots, \tilde{b} - j$. Combining (10) with the second part of (11), we get that $S_{Q(\tilde{b}+1)} > l - S_{\tilde{b}} \ge S_{Q(\tilde{b}+i-j+1)}$. The first part of (11), however, yields $Q(i + \tilde{b} - j + 1) \ge Q(\tilde{b} + 1)$. This contradiction completes the proof. □

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