Invariant measures exist without a growth condition

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Abstract

Given a non-flat S-unimodal interval map $f$, we show that there exists $C$ which only depends on the order of the critical point $c$ such that if $|Df^n(f(c))| \geq C$ for all $n$ sufficiently large, then $f$ admits an absolutely continuous invariant probability measure (acip). As part of the proof we show that if the quotients of successive intervals of the principal nest of $f$ are sufficiently small, then $f$ admits an acip. As a special case, any S-unimodal map with critical order $\ell < 2 + \varepsilon$ having no central returns possesses an acip. These results imply that the summability assumptions in the theorems of Nowicki & van Strien [21] and Martens & Nowicki [17] can be weakened considerably.

1 Introduction

In this paper we consider S-unimodal $C^2$ maps $f : [0,1] \rightarrow [0,1]$. We assume the unique critical point $c$ has order $\ell > 1$, i.e., for $x$ near $c$, there exists a $C^2$ diffeomorphism $\varphi$ such that $f(x) = \varphi(|x - c|^\ell)$.

Theorem 1. There exists $C = C(\ell)$ so that provided $|Df^n(f(c))| \geq C$ for all $n$ sufficiently large, $f$ admits an absolutely continuous invariant probability measure (acip).

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The problem dealt with in Theorem 1 has a long history, with contributions by amongst others [1],[22],[8],[5],[19],[20],[21]. In particular Theorem 1 shows that the well-known Collet-Eckmann condition \( |Df^n(f(c))| \leq C|\gamma|^n \) for some \( \gamma \in (0,1) \), see [5]) or the more recent summability condition \( \sum_n |Df^n(f(c))|^{-\ell} \leq \infty \), see Nowicki & van Strien [21]) are far too restrictive. No growth is needed. Recently, many people are considering weakly hyperbolic systems (in particular in dimensions 2 and larger). Perhaps our techniques indicate that one might not always need to look for growth conditions.

A key idea in our proof is to construct an induced Markov map, and analyse the non-linearities and transition probabilities of the resulting random walk. This Markov map has branches with arbitrarily small ranges. The Markov map we construct is based on the so-called principal nest, and the estimates for the transition probabilities come from a careful analysis of the geometry of this principal nest. So let us define this nested sequence of neighbourhoods of the critical point \( c \) starting with \( I_0 = (q,q) \), where \( q \in (0,1) \) is the orientation reversing fixed point of \( f \) and \( f(q) = f(q) \). Then define inductively \( I_{n+1} \) to be the central domain of the first return map to \( I_n \). To continue the induction, we need to assume that \( c \) is recurrent, \( i.e. \), \( \omega(c) \subseteq c \).

Without this assumption, \( f \) is a Misûrewicz map, and the conclusions of this paper then follow easily (or from well-known results). Write

\[
\mu_n = \frac{|I_{n+1}|}{|I_n|}.
\]

Our paper deals with the case that \( \mu_n \) is small for all large \( n \).

Before stating our result second theorem, let us first discuss \( \mu_n \). Estimating the \( \mu_n \) has been an eminent problem in one-dimensional dynamics, cf. [6,7,9,12]. More precisely, it has been asked if the starting condition [9]

\[
\forall \varepsilon > 0 \exists n_0 > 0 \mu_{n_0} < \varepsilon.
\]

holds. We speak of a central return of \( c \) to \( I_n \) if the first return \( f^n(c) \) of \( c \) into \( I_n \) belongs also to \( I_{n+1} \). If \( \ell \leq 2 \) and there are no central returns, an inductive argument ([9],[12]) shows that (1) implies

\[
\forall \varepsilon > 0 \exists n_0 > 0 \forall n \geq n_0 \mu_n < \varepsilon;
\]

(if there are central returns at times \( n(k) \) then in (2) then this only holds at all ‘non-central’ times. Lyubich [12] and Graczyk & Świątek [6], using complex methods, have established the starting conditions for quadratic maps.

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Note that prior to the results [6, 12], the starting condition was verified for quadratic maps with so-called Fibonacci combinatorics [13, 11]. For this map, it is crucial that the critical order is $\ell = 2$, because for $\ell > 2$, (1) fails: $\mu_n$ does not tend to zero. More precisely, as was shown in [11],

\[
\exists \varepsilon = \varepsilon(\ell) > 0 \exists n_0 > 0 \forall n \geq n_0 \mu_n \leq \varepsilon \text{ and } \varepsilon(\ell) \searrow 0 \text{ as } \ell \searrow 2. \tag{3}
\]

In fact, when $\ell$ is large then $\mu_n$ is close to 1 for all $n$ (for the Fibonacci map); this implies that a Fibonacci map with large critical order possesses a Cantor attractor, see [4].

Recently, Shen [23] showed, by purely real methods, that for all $C^3$ S-unimodal maps without central returns that

- (1) holds for $\ell \in (1, 2]$,
- (3) holds for $\ell > 2$ close to 2.

In this paper we will show that (3), i.e., large values of $|I_n|/|I_{n+1}|$ when $n$ is large, guarantee the existence of an $f$-invariant measure $\mu$ that is absolutely continuous with respect to Lebesgue (acip).

**Theorem 2.** There exists $\varepsilon = \varepsilon(\ell)$ such that if $|I_{n+1}| \leq \varepsilon|I_n|$ for all $n$ sufficiently large, then $f$ admits an acip.

**Remark 1.** We do not need to assume that $f$ has no central returns for this theorem to hold.

Theorem 2 extends a theorem of Martens & Nowicki [17] stating that $\sum_n \mu_n^{1/\ell} < \infty$ implies the existence of an acip. In fact, as they show, $\sum_n \mu_n^{1/\ell} < \infty$ implies the Nowicki-van Strien summability condition. Theorem 2 is strictly stronger: for example for the Fibonacci map with critical order $2 + \varepsilon$ the summability conditions fail, but our assumption holds. Theorem 2 also extends the result of Keller & Nowicki [11] for Fibonacci maps of order $2 + \eta$ to more general maps:

**Corollary 1.** There exists $\eta > 0$ such that for every $C^3$ S-unimodal map $f$ with critical order $\ell < 2 + \eta$, and with a finite number of central returns holds: If $f$ has no periodic attractor, then $f$ has an acip.
Proof of Corollary 1. This follows from Shen’s result [23] that under the above conditions, there exists \( \varepsilon = \varepsilon(\ell) \) such that \( |I_{n+1}| \leq \varepsilon |I_n| \) for \( n \) sufficiently large and that \( \varepsilon \to 0 \) as \( \eta \to 0 \).

In [3], conditions (reminiscent of Fibonacci combinatorics) are given under which \( f \) has an acip, irrespective the critical order as long as \( \ell < \infty \). One can interpret Corollary 1 as a proof that the only mechanism for unimodal maps with critical order \( \ell < 2 + \eta \) not to have an acip, is by (deep) central returns, either of almost restrictive interval type (cf. [10]) or of almost saddle node type (cf. [2]).

2 Preliminaries and structure of the proof

Let us start making precise the condition on \( f \). It is a \( C^3 \) unimodal map with negative Schwarzian derivative such that \( f^2(c) < c < f(c) \) and \( f^3(c) \geq f^2(c) \). Hence we can rescale \( f \) such that \( f^2(c) = 0 \) and \( f(c) = 1 \). The critical order \( \ell \in (1, \infty) \), the critical point is recurrent but not periodic.

Let us first show that Theorem 2 implies our first theorem:

Proof of Theorem 1. Let \( k(n) \) be the minimal integer for which \( f^{k(n)}(c) \in I_n \). Then \( I_{n+1} \) is the pullback of \( I_n \) by \( f^{k(n)} \). By real bounds, [18], there exists \( \delta > 0 \) (which does not depend on \( n \)) and a neighbourhood \( T \) of \( f(I_{n+1}) \), such that \( f^{k(n)-1} \) maps \( T \) diffeomorphically onto a \( \delta \)-scaled neighbourhood of \( I_n \). Hence

\[
|Df^{k(n)}(f(c))| = |Df(f^{k(n)}(c))| \cdot |Df^{k(n)-1}(f(c))| \\
\leq \ell |I_n|^{\ell-1} \cdot K \frac{|f^{k(n)}(I_{n+1})|}{|f(I_{n+1})|} \\
\leq \ell |I_n|^{\ell-1} \cdot K \frac{|I_n|}{|I_{n+1}|^\ell} \leq \ell K \frac{|I_n|}{|I_{n+1}|} \cdot |I_{n+1}|^\ell,
\]

where we have used the non-flatness of \( f \) and Koebe. Therefore, one obtains that \( |I_{n+1}|/|I_n| \) is small provided \( |Df^{k(n)}(f(c))| \) is large.

It is possible that \( f \) is renormalizable. In that case \( k(n) \) is equal to the period \( p \) of this renormalization for all \( n \) large and \( I_n \) shrinks to the largest periodic renormalization interval \( J \) (and so \( |I_{n+1}|/|I_n| \to 1 \). Then use the same argument for the renormalization: repeat the construction of
the principal nest for $f^p|J$. Assume $f$ is $s$ times renormalizable and $J_s$ is its $s$-th renormalization interval with period $p_s$. Intervals $I_n$ associated to its $(s-1)$-th renormalization shrink to the $s$-th renormalization interval $J_s$, and therefore $|Df^{p_s}(f(c))| \leq \ell K \frac{|I_n|}{|I_{n+1}|} \leq 2\ell K$ for $n$ sufficiently large. But since $p_s \geq 2^s$, this and the assumption of Theorem 1 imply that $s$ must be bounded, and so $f$ can only be finitely often renormalizable. Then consider instead of $f$ its last renormalization $f^s|J_s$. Since the above inequality gives that $|I_n|/|I_{n+1}|$ is large for all $n$ large (and in particular $|I_n| \to 0$ as $n \to \infty$), we can apply Theorem 2 and obtain an invariant measure.

So it suffices to prove Theorem 2. The boundary points of each $I_n$ are nice in the sense of Martens [16], which means that $f^i(\partial I_n) \notin I_n$ for all $i > 0$. In fact, $f^i(\partial I_n) \notin I_{n-1}$. This allows the following priori estimates:

**Lemma 1.** If $J \subset I_n$ is a component of the domain of the first return map to $I_n$ for some $n > 0$, say $f^s|J$ is this return, then there exists an interval $T \supset f(J)$ such that $f^{-1}(T) \subset I_n$ and such that $f^{s+1}|T$ is a diffeomorphism onto $I_{n-1}$.

**Proof of Lemma 1.** See Martens [16] or Section V.1 in [18].

The idea is now to construct a Markov induced map $G$ over $f$ with the intervals $I_n$ as countable set of ranges: $G$ is defined on a countable collection of intervals $J_i$, $G|J_i = f^{s_i}|J_i$ is a diffeomorphism and $G(J_i) = I_n$ for some $n$. We then will construct a $G$-invariant measure $\nu \ll \text{Leb}$, and estimate $\nu(I_n)$:

**Proposition 1.** Assume that $\mu_n \leq \varepsilon$ for all $n \geq n_0$. If $\varepsilon$ is sufficiently small, then the induced transformation $G$ admits an a.cip $\nu$. Moreover, there exists $C_0 = C_0(f)$ such that $\nu(I_n) \leq C_0\sqrt{|I_n|}$ for all $n$.

**Corollary 2.** Under the above conditions, $f$ admits no Cantor attractor.

**Proof of Corollary 2.** This follows easily, for example, from the observation that any Cantor attractor has zero Lebesgue measure (see [15]), and, disregarding $c$, is invariant by $G$. Hence $G$ cannot carry an a.cip if a Cantor attractor is present.

It should be noted that the distortion of the branches of $G$ is in general not bounded; this comes from the fact that if $G|J = f^s|J$ is such a branch
and $G(J) = I_n$, then this branch need not be extendible, i.e., if $T \supseteq J$ is the maximal interval on which $f^*$ of monotone, then $f^*(T)$ need not contain a definite scaled neighbourhood of $I_n$. In particular, $d\nu(x)/dx$ can not be expected to be bounded on any of the sets $I_n \setminus I_{n+1}$. However, we will still be able to derive the following result:

**Theorem 3.** There exists $\varepsilon = \varepsilon(\ell)$ such that if $|I_{n+1}| \leq \varepsilon|I_n|$ for all $n \geq n_0$, then $\sum s\nu(J_i) < \infty$.

Once this is obtained, the proof of the main theorem is straight forward.

**Proof of Theorem 2.** This follows by a standard pull-back construction. Given the $G$-invariant measure $\nu$, define $\mu$ by

$$
\mu(A) = \sum_{i} \sum_{j=0}^{s_i-1} \nu(f^{-j}(A) \cap J_i).
$$

As $f$ is non-singular with respect to Lebesgue, $\mu$ is absolutely continuous, and the $f$-invariance of $\mu$ is a standard exercise. The finiteness of $\mu$ follows directly from Theorem 3.

**Comments on constants:** In the following, $\ell$ is fixed, $\varepsilon_i$ denotes constants depending only on $\varepsilon$ which are small provided that $\varepsilon$ is. Constants $\rho_i$ depend only on $\ell$. Constants $C_i$ depend only on $f$. The numbers $n_0 \in \mathbb{N}$ and $\lambda \in (0,1)$, which are defined in Section 4, also depend on $f$. For local use (i.e., within a proof), $B$ and $C = C(f)$ will denote a constant, which might vary within equations.

## 3 Construction of induced maps $G_n$ and $G$

Let $G_0$ be the first return map to $I_0$. Then $G_0$ has a finite number of branches, the central branch is the branch with the largest return time, and each non-central branch maps diffeomorphically onto $I_0$.

In this section we shall construct a sequence of maps $G_n$: $\cup_i J_i^{n+1} \rightarrow I_0$ inductively such that

1. $\cup_i J_i^{n+1}$ is a finite union and for $n \geq 1$, $G_n = G_{n-1}$ outside $I_n$;
2. The central branch $J_0^{n+1} = I_{n+1}$ and $G_n|I_{n+1}$ is the first return map to $I_n$;
3. for each $i \neq 0$, there exists $b_i \leq n$ such that such that $G_n : J_i^{n+1} \to I_{b_i}$ is a diffeomorphism;

4. the outermost branch maps onto $I_0$; more precisely, $J_i^{n+1} \subset I_n$ and $\partial J_i^{n+1} \cap \partial I_n \neq \emptyset$ imply $G_n(J_i^{n+1}) = I_0$ (and the external point of such an interval $J_i^{n+1}$ maps to the fixed point $q$);

5. $G_n(x) = f^s(x)$ implies that $f(x), \ldots, f^{s-1}(x) \notin I_n$;

By definition $G_0$ satisfies the above statements, so let us assume that by induction $G_n$ exists with the above properties, and construct $G_{n+1}$.

Set $G_{n+1}(x) = G_n(x)$ for $x \notin I_{n+1}$. Let $k_n \in \mathbb{N} := \{1, 2, 3, \ldots \}$ be minimal so that $G_n^{k_n}(c) \in I_{n+1}$. This means that $k_n = 1$ if the return to $I_n$ is central. Define $K_0 = I_{n+1}$, $K_n = I_{n+2}$ and, for $0 \leq j \leq k_n - 1$, let $K_j$ be the component of $\text{dom}(G_j^{n+1})$ which contains $c$. Next define on $K_j \setminus K_j^{n+1}$

$$G_{n+1}(x) = \begin{cases} G_j^{n+1}(x) & \text{if } G_j^{n+1}(x) \in I_{n+1} \\ G_j^{n+2}(x) & \text{otherwise.} \end{cases}$$

$G_{n+1}|I_{n+2} = G_n^{k_n}|I_{n+2}$ is the first return map to $I_{n+1}$. Properties (1) and (2) hold by construction for $G_{n+1}$. Property (3) holds because if $G_j^{n+1}(x) \in I_{n+1}$ for some $x \in I_{n+1} \setminus I_{n+2}$ then $G_{n+1}(J_i^{n+1}) = I_{n+1}$ for the corresponding domain $J_i^{n+1} \ni x$ and if $G_j^{n+1}(x) \notin I_{n+1}$ then by the induction assumption $G_{n+1}(J_i^{n+1})$ is equal to some domain $I_b$, $b \leq n$, because then $G_{n+1}(x) = G_j^{n+2}(x)$. Property (4) holds immediately because $\partial I_n$ is mapped by $G_n$ into $\partial I_0$. In order to show Property (5) holds, take $x \in K_j \setminus K_j^{n+1}$ and let $y = G_j(x)$. Note that $G_j^{n+1}|K_j^{n+1}$ is inside a component of $\text{dom}(G_n)$ and that all iterates $f(K_j), \ldots, G_j^{n+1}(K_j) \ni y$ are outside $I_{n+1}$. Since $G_j^{n+1}(x) = G_n(y)$ we get by induction that (5) holds for $G_{n+1}$ (using that it holds for $G_n$ and $y$ instead of $x$).

The induced map $G$ is defined as follows: for each $n \geq 0$, each component of the domain $J$ of $G_n$ other than the central one $I_{n+1}$ becomes a component of the domain of $G$, and $G|J = G_n|J$.

For later use, we compute by induction that if $x \in I_n \setminus I_{n+1}$, and $G(x) = f^s(x)$, then

$$s \leq t_0 \cdot (k_0 + 1) \cdots (k_{n-2} + 1) \cdot (k_{n-1} + 1),$$

where $t_0 = \min\{i > 0 \mid f^i(c) \in I_0\}$. 

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4 Distortion properties of the induced map

Suppose \( \varphi : T \to \varphi(T) \) is a \( C^1 \) map. Let us define

\[
\text{Dist}(\varphi) := \text{Dist}(\varphi, T) := \sup_{x, y \in T} \log \frac{\varphi'(x)}{\varphi'(y)}.
\]

Let us say a diffeomorphism \( h : J \to h(J) \) belongs to the distortion class \( \mathcal{F}_p^C \) if it can be written as

\[
Q \circ \varphi_q \circ Q \circ \varphi_{q-1} \circ \cdots \circ Q \circ \varphi_1,
\]

with \( q \leq p \), where \( Q(x) = |x|^t \) and \( \text{Dist}(\varphi_j) \leq C \) for all \( 1 \leq j \leq q \).

Let us fix a large positive integer \( n_0 \) such that \( |I_n| \leq \varepsilon |I_{n-1}| \) for all \( n \geq n_0 \), and such that \( f|I_{n_0} \) can be written as \( x \mapsto \varphi(|x|^t) \) with \( \text{Dist}(\varphi) \leq 1/4 \). By Lemma 1, it follows that for each \( n \geq n_0 \), if \( J \) is a return domain to \( I_n \), and \( f^n|J \) is the return, then \( f^n|J \) can be written as \( x \mapsto \varphi(|x|^t) \) with \( \text{Dist}(\varphi) \leq 1/2 \) provided \( \varepsilon \) is sufficiently small.

According to Mañé [14], the map \( G \), restricted to the set of points which stay outside \( I_{n_0+1} \) is a hyperbolic (uniformly expanding) system. Thus, there exists \( C_1 = C_1(f) > 0 \) and \( \lambda = \lambda(f) \in (0, 1) \) with the following property. For any \( k \in \mathbb{N} \)

1. if \( x \) is a point such that \( G^i(x) \) are defined and \( G^i(x) \not\in I_{n_0+1} \) for any \( 0 \leq i \leq k-1 \), then

\[
|(G^k)'(x)| \geq \frac{1}{C_1 \lambda^k};
\]

2. if \( J \) is an interval such that \( G^k|J \) is defined, and \( G^i(J) \cap I_{n_0+1} = \emptyset \) for all \( 0 \leq i \leq k-1 \), then

\[
\text{Dist}(G^k|J) \leq \log C_1.
\]

We will use the notation \( \alpha(y) = n \) if \( y \in I_n \setminus I_{n+1} \).

**Proposition 2.** Let \( m \geq 1 \), and let \( G^i : J \to I_m \) be an onto branch of \( G^i \). There exists \( C_2 = C_2(f) \) such that the following hold:

- Suppose that \( \alpha(G^{i-1}J) > m \). Let \( n > m \) and \( 1 \leq k \leq i \) be maximal such that

\[
n = \alpha(G^{i-k}J) > \alpha(G^{i-k+1}J) > \cdots > \alpha(G^{i-1}J) > m.
\]
Then $G^i|J$ can be written as $\psi \circ \varphi$ such that

$$\text{Dist}(\psi) \leq \log C_2 \text{ and } \varphi \in \mathcal{F}^1_{2(n-m+1)}.$$ 

- If $\alpha(G^{i-1}(J)) \leq m$ then $G^i|J$ can be written as $\psi \circ \varphi$ such that

$$\text{Dist}(\psi) \leq \log C_2 \text{ and } \varphi \in \mathcal{F}^1_2.$$ 

**Proof.** Let $r$ denote the maximum of $\alpha(G^j(J))$ for $0 \leq j \leq i - 1$. Let $C = C(f)$ be a big constant. We shall prove by induction on $r$ the following stronger statement: $G^i|J$ can be written as $\psi \circ H \circ Q \circ \varphi_1$ with

$$\text{Dist}(\psi) \leq \log C, \ H \in \mathcal{F}^1_{2(n-m)+1} \text{ and } \text{Dist}(\varphi_1) < 1/2.$$ 

If $r \leq n_0$, then the distortion of $G^i|J$ is bounded by $\log C_1(f)$ as we remarked above. Hence the statement is true for $C > C_1$. So let us consider the case $r > n_0$.

For $0 \leq j \leq i - 1$, let $T_j$ denote the domain of $G$ which contains $G^j(J)$. For simplicity of notation, write $\alpha_j = \alpha(G^j(J))$. By definition of $n$, we have $\alpha_{i-k-1} \leq \alpha_{i-k} = n$. Note that $G^j|J$ extends to a diffeomorphism onto $I_{\alpha_j}$ for all $1 \leq j \leq i$.

**Case 1.** $n \leq n_0$. Then $\alpha_j \leq n_0$ for all $i - k \leq j \leq i - 1$, and so $\text{Dist}(G^k|G^{i-k}(J)) \leq \log C_1$. If $G(T_{i-k-1}) \cap I_{n-1}$, then $\text{Dist}(G^{i-k}|J)$ is bounded by the Koebe principle, and thus we are done. If $G(T_{i-k-1}) \subset I_n$, then $T_{i-k-1}$ is a return domain to $I_n$. Since $n \geq m \geq 1$, this return domain is well inside $I_n$, which implies that $G^{i-k-1}|J$ has bounded distortion. Since $n \leq n_0$, the distortion of $G|T_{i-k-1}$ has bounded distortion as well, and so the proposition is true for some universal constant $C$ (which depending on the a priori real bounds).

**Case 2.** $n > n_0$. Then similarly as above, we can show that $G^{i-k}|J$ can be written as $\varphi_2 \circ h_1$, with $\text{Dist}(\varphi_2) \leq 1/2$ and $h_1 \in \mathcal{F}^{1/2}_1$. If $k = 0$, then the proposition follows. Assume $k \geq 1$. Let $J' = G_{n-1}(G^{i-k}(J))$, and let $s \in \mathbb{N}$ be such that $G = G_n = G^s_{n-1}$ on $G^{i-k}(J)$. Since $\alpha_{i-k+1} < \alpha_{i-k}$, it follows from our construction that $G^j(J') \cap I_n = \emptyset$ for all $0 \leq j < s$. The same is true for $s \leq j \leq s - 1 + k - 1$ by definition of $n$. Thus

$$\max_{j=0}^{s-1+k-1} \alpha(G^j(J')) \leq n - 1 \leq r - 1.$$ 


Applying the induction hypothesis to the map $G^{k-1} \circ G^{s-1}|J' = G^{k-1} \circ G^{s-1}_{n-1}|J'$, we see that the map can be written as $\psi \circ h \circ Q \circ \varphi$ with $\text{Dist}(\psi) < C$, and $\text{Dist}(\varphi) \leq 1/2$, and $h \in \mathcal{F}_{2(n-m)-1}$. The map $G_{n-1}|G^{i-k}(J)$ is a restriction of the first return map to $I_{n-1}$, which is of the form $\varphi_3 \circ Q$ with $\text{Dist}(\varphi_3) \leq 1/2$. Therefore

$$G^i|J = G^{s-1+k-1}|J' \circ G_{n-1}|G^{i-k}(J) \circ G^{i-k}|J$$

$$= \psi \circ h \circ Q \circ (\varphi \circ \varphi_3) \circ Q \circ \varphi_2 \circ h_1.$$

Note that $\text{Dist}(\varphi \circ \varphi_3) < 1$, and the induction step is completed. \hfill \Box

We will need another proposition to treat the case $m = 0$. By taking $C_2$ larger if necessary, we prove:

**Proposition 3.** Consider any branch $G^i|J$. Let $n = \max_{j=0}^{i-1} \alpha(G^j|J)$. Then $G^i|J$ can be written $\psi \circ H$ with

$$\text{Dist}(\psi) \leq \log C_2 \text{ and } H \in \mathcal{F}_{2n}^1.$$

**Proof.** First note that if $G^i(J) \subset I_1$, then the assertion follows immediately from the previous proposition. So we shall assume $G^i(J) = I_0$. Let us prove by induction that $G^i|J$ can be written as $\psi \circ H \circ Q \circ \varphi$, where $\psi$ is an iterate of $G|I_0 \setminus I_{n+1}$, and $H \in \mathcal{F}_{n-1}^1$, and $\text{Dist}(\varphi) < 1/2$.

If $n \leq n_0$, then the claim is clearly true. Assume $n > n_0$. Let $0 \leq p < i$ be the largest such that $\alpha_p = n$. Using similar argument as in the proof of the previous proposition, the map $G^p|J$ can be written as $\varphi_0 \circ h$, where $\text{Dist}(\varphi_0) < 1/2$, and $h \in \mathcal{F}_{1/2}^1$. Note that $\alpha(G^{p-1}) \leq \alpha(G^p|J)$ by the maximality of $\alpha(G^p|J)$. Let $s$ be the positive integer such that

$$G^p|J = G_{\alpha_p}|G^p|J = G_{\alpha_p-1}|G^p|J,$$

and let $J' = G_{\alpha_p-1}(G^p|J)$. It follows from the construction of $G$ and the maximality of $\alpha_p$ that $\alpha(G^j(J)) \leq n-1$ for all $0 \leq j \leq s-2+(i-p)$. By the induction hypothesis, we can decompose the map $G^{i-p+s-1}|J'$ as $\psi_1 \circ H_1 \circ Q \circ \varphi_1$ such that $\psi_1$ is an iterate of $G|I_0 \setminus I_{n+1}$, and $H_1 \in \mathcal{F}_{n-2}^1$. The map $G_{n-1}|G^p|J$ is a restriction of the first return map to $I_{n-1}$, and thus it can be written as $\varphi \circ Q$ with $\text{Dist}(\varphi) < 1/2$. Combining all these facts, we decompose

$$G^i|J = \psi_1 \circ \{H_1 \circ [Q \circ (\varphi_1 \circ \varphi_0)]\} \circ h,$$

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as required. This completes the proof of the induction step. □

We are going to use the following lemma many times.

**Lemma 2.** If $h : J \to I$ is a diffeomorphism in $\mathcal{F}_p^1$, and $A \subset J$ is a measurable set, then

$$
\frac{1}{(\ell e)^p} \frac{\text{Leb}(h(A))}{|I|} \leq \frac{\text{Leb}(A)}{|J|} \leq e^p \left( \frac{\text{Leb}(h(A))}{|I|} \right)^{1/p}.
$$

**(Proof.** First we note that for any interval $T \subset \mathbb{R} \setminus \{0\}$ and any measurable set $A \subset T$, we have

$$
\frac{\text{Leb}(A)}{|T|} \leq \left( \frac{\text{Leb}(Q(A))}{|Q(T)|} \right)^{1/t}.
$$

To see this, note that for a fixed $\text{Leb}(Q(A))$, the left hand side takes its maximum in the case that $A$ is an interval adjacent to the endpoint of $\partial T$ which is closer to 0.

It suffices to prove the two inequalities in case $p = 1$. So let us consider the case $h = Q \circ \varphi$ with $\text{Dist}(\varphi) \leq 1$. For any $A \subset J$, we have

$$
\frac{\text{Leb}(A)}{|J|} \leq e \frac{\text{Leb}(\varphi(A))}{|\varphi(J)|} \leq e \left( \frac{\text{Leb}(h(A))}{|h(I)|} \right)^{1/t}.
$$

This proves the second inequality of (5). On the other hand,

$$
\frac{\text{Leb}(\varphi(A))}{|\varphi(J)|} = 1 - \frac{\text{Leb}(\varphi(J \setminus A))}{|\varphi(J)|}
\geq 1 - \left( \frac{\text{Leb}(h(J \setminus h(A)))}{|h(J)|} \right)^{1/t}
= 1 - \left( 1 - \frac{\text{Leb}(h(A))}{|I|} \right)^{1/t}
\geq 1 - \frac{1 \text{Leb}(h(A))}{\ell |I|},
$$

and thus

$$
\frac{\text{Leb}(A)}{|J|} \geq \frac{1}{e} \frac{\text{Leb}(\varphi(A))}{|\varphi(J)|} \geq \frac{1}{e\ell} \frac{\text{Leb}(h(A))}{|I|},
$$

proving the first inequality. □
5 Outermost branches

Within $I_n$, there are two special branches which have common endpoints with $I_n$. These branches always mapped onto $I_0$ by the map $G$, and need special care in our argument. In this section, we shall prove that these branches can not be too small.

**Proposition 4.** There exist a constant $\rho_1 = \rho_1(\ell) > 0$ and a constant $C_3 = C_3(f) > 0$, such that if $J_n$ is one of the two outermost branches of $G$ in $I_n$, then

$$\frac{|J_n|}{|I_n|} \geq \frac{\rho_1^n}{C_3}.$$  

**Proof.** Let $\delta_n := |J_n||I_n|$ and $\hat{J}_{n-1}$ the outer-most branch of $I_{n-1} \setminus I_n$ for which $\hat{J}_{n-1} \supset G_{n-1}(J_n)$. Write $G_{n-1}|I_n = f^n$. Since this is a first return, one has Dist($f^{n-1}|f(I_n)$) $\leq 1$ for all $n$ sufficiently big.

*Case 1. $G_{n-1}(c) \notin \hat{J}_{n-1}$.** Then by the distortion bound for $f^{n-1}|f(I_n)$,

$$\frac{|f(a) - f(c)|}{|f(b) - f(c)|} = 1 + \frac{|f(a) - f(b)|}{|f(b) - f(c)|} \geq 1 + C\delta_{n-1},$$

where $a$ and $b$ are the end points of $J_n$ with $b$ between $a$ and $c$. Hence, using that $c$ is a critical point of order $\ell$,

$$\frac{|a - c|}{|b - c|} \geq (1 + C\delta_{n-1})^{1/\ell} \geq 1 + \frac{C\delta_{n-1}}{\ell}.$$  

Hence

$$\delta_n = \frac{|J_n|}{|I_n|} = \frac{1}{2} \frac{|a - b|}{|a - c|} \geq \frac{1}{2} \left(1 - \frac{1}{1 + C\delta_{n-1}/\ell}\right) \geq C\delta_{n-1}/\ell.$$  

By induction, $|J_n||I_n| \geq \rho_1^n/C_3$ for $\rho_1 = \rho_1(\ell) \asymp 1/\ell$.

*Case 2. $G_{n-1}(c) \in \hat{J}_{n-1}$.** Note that $G_{n-1}(\hat{J}_{n-1}) = I_0$ and that $G_{n-1}^2 J_n$ intersects an outermost branch $\hat{J}_0$ of $I_0$. Let $p \geq 0$ be minimal so that $G_{n-1}^{p+2}(c) \notin \hat{J}_0$. Then $|\hat{J}_0||G_{n-1}^{p+2} I_n|$ is bounded from below (by a bound which depends only on $f$), and since $G_{n-1}^{p+2}(J_n) = \hat{J}_0$, and $f|(I_0 \setminus I_1)$ is hyperbolic this implies

$$|G_{n-1}^2 J_n||G_{n-1}^2 I_n| \geq C > 0.$$
According to the distortion control on $G_{n-1}|I_{n-1}$ given by Proposition 3, this implies

$$|G_{n-1}J_n|/|G_{n-1}I_n| \geq C\rho^n > 0.$$ 

Since $I_n$ is a first return domain of $G_{n-1}$, by Lemma 1, this implies

$$|J_n|/|I_n| \geq \rho_1^n/C_3,$$

with $\rho_1 = \rho_1(\ell) \times 1/\ell$ and $C_3 = C_3(f) \times 1/C$. 

\[\square\]

6 Improved decay for deep returns

Let $x$ and $m$ be so that $G_m^m(x)$ is well-defined and $G_m^i(x) \notin I_{n+1}$ for $0 \leq i < m$. Let $T_i = T_i(x)$ be the component of $\text{dom}(G_n)$ which contains $G_m^i(x)$. Define $\alpha(y) = j$ if $y \in I_j \setminus I_{j+1}$ and $s(y) = s$ if $G(y) = f^s(y) = G_{\alpha(y)}(y)$. Let $t_n$ be the return time of $c$ to $I_n$ under $f$. Define

$$\Lambda = \{0 \leq i \leq m-2 \ ; \ \alpha(T_{i+1}) \geq \alpha(T_i)\},$$

$$N = \sum_{i \in \Lambda} [\alpha(T_{i+1}) - \alpha(T_i) + 1] \text{ and } r = \#\Lambda.$$ 

Moreover, define

$$T_0^i = \{y \in T_0 \ ; \ G_n^i(y) \in T_i \text{ for all } i \leq m - 1\}.$$ 

If $\varphi : T \to \varphi(T)$ is a homeomorphism and $J \subset T$ is a subinterval of $T$, we denote the components of $T \setminus J$ by $L$ and $R$, and write

$$Cr(T, J) := \frac{|T| \cdot |J|}{|L| \cdot |R|}$$

for the cross-ratio of $J$ in $T$.

**Lemma 3.** Assume that $\alpha(T_i) \geq n_0$ for all $i = 0, \ldots, m-2$, then for $\varepsilon_1 \ll \varepsilon^{1/\ell}$

- $Cr(T_0, T_0^i) \leq \varepsilon_1^N$ if $r \geq 1$;

- for each interval $J \subset G_n(T_{m-1})$ with $J \ni G_n^m(x)$, and $J' := \{y \in T_0^i \ ; \ G_n^m(y) \in J\}$ we have

$$Cr(T_0, J') \leq \varepsilon_1^N \cdot Cr(G_n(T_{m-1}), J)$$

(even if $r = 0$).
Proof of Lemma 3. For $0 \leq j \leq m - 2$, write
\[
Cr(I_{a(T_j)}, G_n^j T_0^i) \leq Cr(T_j, G_n^j T_0^i) \\
\leq Cr(G_n T_j, G_n^{j+1} T_0^i) \\
\leq Cr(I_{a(T_{j+1})}, G_n^{j+1} T_0^i).
\]
Here the first and third inequality hold by inclusion of intervals, and the second inequality because $f$ has negative Schwarzian derivative. Note that $G_n T_j \supset I_{a(T_j)}$. If $j \in \Lambda$ then one gets improved inequalities: if
\[
G_n T_j \supset I_{a(T_j)-1} \supset I_{a(T_{j+1})-1},
\]
then in the third inequality one gets an additional factor $\varepsilon_1^{[\alpha(T_{j+1})-\alpha(T_j)+1]}$, while if $G_n T_j = I_{a(T_j)} \supset I_{a(T_{j+1})}$ then in the first inequality one gets a factor $\varepsilon_1$ (because then $G_n$ is a first return and so a composition of $x^t$ and a map which extends diffeomorphically to $I_{a(T_j)-1}$) and in the third we get an additional factor
\[
\varepsilon_1^{[\alpha(T_{j+1})-\alpha(T_j)]}.
\]
To prove the second assertion of the lemma one proceeds in the same way. Note that all this holds, provided $\alpha(T_j) \geq n_0$ for each $j \in \Lambda$ where $n_0$ is chosen so that $|I_{n+1}|/|I_n| < \varepsilon$ for $n \geq n_0$.

Let $k_n$ be as in Section 3.

**Corollary 3.** There exists $C_4 = C_4(f) > 1$ and $\varepsilon_2 \approx \varepsilon_1^{1/t}$ with the following property.

1. If $\alpha(G_n^i(I_{n+2})) \geq n_0$ for all $0 \leq i \leq k_n$, then
\[
\frac{|I_{n+2}|}{|I_{n+1}|} \leq C_4 \varepsilon_2^{k_n}.
\]
2. If $\alpha(G_n^i(I_{n+2})) \leq n_0$ for some $1 \leq i \leq k_n$, then
\[
\frac{|I_{n+2}|}{|I_{n+1}|} \leq C_4 \varepsilon_2^{n-n_0}.
\]

**Proof.** (1) Let $x = G_n(c)$ and $m = k_n - 1$, and let $T_i, \Lambda, N$ be defined as above. Write $n' = \alpha(G_n^{k_n-1}(c))$. Note that $\alpha(G_n(c)) = n$. Then
\[
\sum_{i=0}^{m-1} [\alpha(T_{i+1}) - \alpha(T_i)] = n' - n.
\]
Thus
\[
N - r = \sum_{i \in \Lambda} [\alpha(T_{i+1}) - \alpha(T_i)]
\]
\[
= n' - n + \sum_{i \not\in \Lambda} [\alpha(T_i) - \alpha(T_{i+1})]
\]
\[
\geq n' - n + m - r,
\]
which implies
\[
N \geq n' - n + m. \quad (6)
\]

Let \( J = G^k_n(I_{n+2}) \). Then
\[
Cr(G_n(T_{m-1}), J) \leq Cr(I_{n'}, I_{n+1}) \leq 3\varepsilon_1^{n-1-n'}.
\]

Applying the last part of the previous lemma, we obtain
\[
Cr(T_0, G_n(I_{n+2})) \leq 3\varepsilon_1^n \varepsilon_1^{n-1-n'} \leq 3\varepsilon_1^{m+1} = 3\varepsilon_1^k,
\]
which implies this corollary.

(2) Let \( p < k_n \) be the largest integer for which \( \alpha(G^p_n(I_{n+2})) \leq n_0 \). Let
\( \hat{\Lambda} = \{p \leq i \leq k_n - 2 : i \in \Lambda \} \), and let \( \hat{N} = \sum_{i \in \hat{\Lambda}} [\alpha(T_{i+1}) - \alpha(T_i) + 1] \). Then we can show similarly
\[
\hat{N} \geq n' - n_0 + k_n - p \geq n' - n_0,
\]
and
\[
Cr(T_p, G^p_n(I_{n+2})) \leq \varepsilon_1^n Cr(I_{n'}, I_{n+1}) \leq \varepsilon_1^{n-n_0},
\]
which implies the statement. \( \square \)

7 Improved decay in general

Let \( I_{n+1} = K^0 \supset K^1 \supset \cdots \supset K^{k_n} = I_{n+2} \) be the domains of \( G^j_n \) as in Section 3.
Lemma 4. Assume
\[ K^i \supseteq K^{i+1} = K^{i+2} = \cdots = K^{i+m} \supseteq K^{i+m+1}. \]

Then there exists \( C_5 = C_5(f) > 0 \) and \( \rho_2 = \rho_2(\ell) \in (0, 1) \) such that (provided \( n \) is sufficiently large)
\[ \frac{|K^{i+1}|}{|K^i|} \leq (1 - \rho_2^n / C_5)^m. \tag{7} \]

Proof of Lemma 4. By construction, \( G^{i+1}K^i \) contains the outermost domain of some interval \( I_j \), with \( j \leq n \), while \( G^{i+1}K^{i+1} \subset I_j \) is not contained in that outermost domain. By Proposition 4, this outermost domain is at least \( \rho_i^2 / C_3 (\geq \rho_i^n / C_3) \) times as long as \( |I_j| \). By Propositions 2 and 3, the map \( G^i|G_nK^i = G^i|G_nK^i \) can be written as \( \psi \circ H \) with
\[ \text{Dist(\psi)} \leq \log C_3 \text{ and } H \in \mathcal{F}_{2n}. \]

By the left inequality of (5), this implies that
\[ \frac{|G_n(K^i \setminus K^{i+1})|}{|G_nK^i|} \geq \rho^n / C, \]
for some \( \rho = \rho(\ell) \in (0, 1) \). Since \( G_n|K^i \) is a restriction of the first return map to \( I_n \), it follows that
\[ \frac{|K^{i+1}|}{|K^i|} \leq 1 - \rho^n / C. \]

for \( n \) large. Hence, at least provided \( \frac{\log m}{n} \) is not too large, i.e., bounded by a universal constant, (7) holds (taking \( \rho_2 > 0 \) small). So we need to consider the case that \( \frac{\log m}{n} \) is large. Then \( K^{i+1} = \cdots = K^{i+m}, G_nT_n K^{i+1} \) is contained in an outermost domain, and so one of the endpoints of \( G_nT_n K^{i+1} \) is a boundary point of \( I_0 \). Using that \( K^{i+1} = \cdots = K^{i+m}, \)
\[ \frac{|G_n^{-i}K^{i+1}|}{|I_0|} \leq \lambda^m, \]
where \( I_0 \) is the outermost branch of \( I_0, C = C(f), \) and \( \lambda \in (0, 1) \) comes from the beginning of Section 4. The distortion control given by Proposition 3 gives
\[ \frac{|G_n^{-i}K^{i+1}|}{|T_n|} \leq C\lambda^{m/\ell^2}, \]

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where $T_{i+1}$ is the domain of $G_n^2$ containing $G_n^{i+1}K^{i+1}$. Since $|G_n^{i+1}(K^i \setminus K^{i+1})| \geq \rho_i^n |T_{i+1}|/C$, it follows

$$\frac{|G_n^{i+1}K^{i+1}|}{|G_n^{i+1}K^i|} \leq C^\lambda^{m/\ell^{2n}} \rho_i^n.$$

Using the distortion control given by Proposition 2 or 3, and equation (5), we obtain

$$\frac{|G_n(K^{i+1})|}{|G_n(K^i)|} \leq C e^{-n\lambda^m/\ell^n} \rho_1^n / \rho_1^{n+2n}.$$

Pulling back by the first return map $G_n|K^i$, we obtain

$$\frac{|K^{i+1}|}{|K^i|} \leq C e^{n/\ell} \lambda^m / \rho_1^{n+2n},$$

which clearly implies (7) when \( \frac{\log m}{n} \gg 4\log \ell \) and $\rho_2 \ll \ell^{-4}$. \qed

**Lemma 5.** Let $\lambda \in (0,1)$ be as in the beginning of Section 4. Let $m$ be so that $I_{n+1} = K^0 = \cdots = K^m \neq K^{m+1} \supset I_{n+2}$. Assume $m \geq 1$. Then

$$\frac{|I_{n+1}|}{|I_n|} \leq C_6 \lambda^{m/\ell^{n+1}},$$

where $C_6 = C_6(f)$ is a constant.

**Proof of Lemma 5.** Note that $G_n|I_{n+1}$ is a first return map to $I_n$, and so there exists a neighbourhood $T \ni f(c)$ such that $f^{i_{n-1}} : T \to I_{n-1}$ is a diffeomorphism and $f^{-1}(T) \subset I_n$. Therefore

$$\frac{|I_{n+1}|}{|I_n|} \leq \left( \frac{|G_n I_{n+1}|}{|I_{n-1}|} \right)^{1/\ell}.$$

If $m \geq 1$, then $G_n(I_{n+1})$ is contained in an outermost branch $J_n$ in $I_n$. Similarly as before

$$\frac{|G_n I_{n+1}|}{|J_n|} \leq C \lambda^{m/\ell^n}$$

and so

$$\frac{|I_{n+1}|}{|I_n|} \leq C \lambda^m / \ell^{n+1}.$$

\qed
Lemma 6. There exists \( \varepsilon(\ell) \) so that if \(|I_{n+1}| \leq \varepsilon|I_n|\) for all \( n \) sufficiently large, then for all \( n \) sufficiently large,

\[
\frac{|I_{n+2}|}{|I_{n+1}|} \leq \frac{1}{(k_n + 1)^4}.
\]

Proof of Lemma 6. Consider \( \alpha(G_i^t) \) for \( 1 \leq i < n \). If all these are larger than \( n_0 \) then by Corollary 3

\[
\frac{|I_{n+2}|}{|I_{n+1}|} \leq C_4 \varepsilon_2^{k_n} < \frac{1}{(k_n + 1)^4}.
\]

So assume that there exists \( 1 \leq i < n \) such that \( \alpha(G_i^t) \leq n_0 \). Then at least we have

\[
\frac{|I_{n+2}|}{|I_{n+1}|} \leq C_4 \varepsilon_2^{(n-n_0)/\ell},
\]

by the second statement of Corollary 3. This implies the lemma, unless \( k_n \geq \varepsilon_2^{-(n-n_0)/(4\ell)/C_4} \). Let \( m \) as before be so that \( I_{n+1} = K^0 = K^1 = \cdots = K^m \neq K^{m+1} \supset I_{n+2} \). Then respectively by the previous lemma and by Lemma 4,

\[
\frac{|I_{n+1}|}{|I_n|} \leq C \lambda^{m/\ell n+1} \text{ and } \frac{|I_{n+2}|}{|I_{n+1}|} \leq (1 - \rho_2^C \delta)^{k_n-m}.
\]

Case 1. \( m < k_n/2 \). According to the second inequality, we have

\[
\frac{|I_{n+2}|}{|I_{n+1}|} \leq (1 - \rho_2^C \delta)^{k_n/2} \leq \frac{1}{(k_n + 1)^4},
\]

provided we choose \( \varepsilon(\ell) \) so small that for \( \varepsilon_2 \) from Corollary 3, \( \varepsilon_2 < \rho_2^\ell \) and we take \( n \) sufficiently large. Here we have used the assumption that \( k_n \geq \varepsilon_2^{-(n-n_0)/(4\ell)/C_4} \).

Case 2. \( m \geq k_n/2 \). Then by the first inequality,

\[
C_r(I_n, I_{n+1}) \times \frac{|I_{n+1}|}{|I_n|} \leq \lambda^{k_n/2\ell n+1}.
\]

By Lemma 1, there is an interval \( T \ni f(c) \) such that \( f^{-1}(T) \subset I_{n+1} \) and such that \( f^{t_{n+1}-1} : T \to I_n \) is a diffeomorphism, where \( t_{n+1} \) is the first return time.
of $c$ to $I_{n+1}$. Since also $f^{n+1}(I_{n+2}) \subset I_{n+1}$, we obtain
\[
Cr(T, f(I_{n+2})) \leq Cr(f^{n+1-1}(T), f^{n+1}(I_{n+2}))
\leq Cr(I_n, I_{n+1})
\leq \lambda^{k_n/2l^{n+1}}.
\]
Since $f^{-1}(T) \subset I_{n+1}$, $f(I_{n+1})$ contains a component of $T \setminus f(I_{n+2})$. Thus
\[
\frac{|f(I_{n+2})|}{|f(I_{n+1})|} \leq Cr(T, f(I_{n+2})) \leq C\lambda^{k_n/2l^{n+1}}.
\]
Finally, the non-flatness of the critical point gives
\[
\frac{|I_{n+2}|}{|I_{n+1}|} \leq C\lambda^{k_n/2l^{n+2}} \leq \frac{1}{(k_n + 1)^4},
\]
provided that $\varepsilon_2 < \ell^{-4d}$ and $n$ is sufficiently large. \hfill \Box

8 The measure for the induced map

In this section we prove the existence of an acip for the induced map $G$.

**Proof of Proposition 1.** We will use the result by Straube [24] claiming that $G$ has an acip if (and only if) there exists some $\eta \in (0, 1)$ and $\delta > 0$ such that for every measurable set $A$ of measure $\text{Leb}(A) < \delta$ holds $\text{Leb}(G^{-k}(A)) < \eta|I_0|$.

The assumptions give that there exists a constant $B$ with the following property: If $J$ is any branch of $G^k$ and $G^k(J) = I_n$, then
\[
\frac{\text{Leb}(|x \in J; G^k(x) \in I_{n+m}|)}{|J|} \leq B \frac{|I_{n+m}|}{|I_n|}.
\]
(8)
This includes trivially the branch of $G^0$, that is the identity. Note that $B$ is a distortion constant, and $B \leq 2$ for $\varepsilon \approx 0$ and $n \geq n_0$. So we can assume that $B\sqrt{\varepsilon}/(1 - \sqrt{\varepsilon}) < 1/3$. Moreover, $|I_n| \leq \varepsilon^{n-m}|I_m|$ for all $n \geq m \geq n_0$.

**Lemma 7.** If $J$ is a branch of $G^{k-1}$ such that $G^{k-1}(J) = I_{n+1}$, then
\[
\text{Leb}(|x \in J; \alpha(G^k(x)) \geq n + 1|) \leq \frac{1}{6}|J|,
\]
(9)
provided $n \geq n_0$. 

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Proof. Let $I_{n+1} = K^0 \supset K^1 \supset \cdots \supset K^{k_n} = I_{n+2}$ be as in Section 3. For each $0 \leq i \leq k_n - 1$ with $K^i \neq K^{i+1}$, there can be at most two branches of $G$, symmetric w.r.t. the critical point, which map onto $I_{n+1}$. We claim that each of these branches $P$ lies deep inside $K^i$ (if they exist). To see this, let $s \in \mathbb{N}$ be such that $G|P = f^s|P$. Then by our construction, $f^{s-1}$ maps an interval $T \ni f(c)$ onto some interval $I_j$ with $j \leq n$, and $f^{-1}(T) = K_i$. Since $f^{s-1}(f(P)) = I_{n+1}$ lies deep inside $I_j$, it follows from the Koebe principle that $f(P)$ lies deep inside $T$. The claim follows from the non-flatness of the critical point.

Let $U_{n+1}$ be the union of those domains of $G$ inside $I_{n+1} \setminus I_{n+2}$ which are mapped onto $I_{n+1}$ by $G$. Then it follows from the Koebe principle

$$\text{Leb}(\{x \in J : G^{k-1}(x) \in U_{n+1}\}) \leq \frac{1}{10} |J|.$$ 

It remains to consider branches of $J'$ of $G^k|J$ for which $G^k(J') = I_{n'}$ with $n' \leq n$. But using the remark before this lemma, we obtain an estimate for this part also, and thus we conclude the proof. \hfill \square

Write $y_{n,k} = \text{Leb}(\{x \in I_0 ; \alpha(G^k(x)) = n\})$. Take $C_0 > 6B/|I_{n_0}|$. We will show by induction that $y_{n,k} \leq C_0 \sqrt{|I_n|}$ for all $n, k \geq 0$. For $k = 0$, this is obvious, and the choice of $C_0$ assures that $y_{n,k} \leq C_0 \sqrt{|I_n|}$ for all $n < n_0$.

Now for the inductive step, assume that $y_{n,k-1} \leq C_0 \sqrt{|I_n|}$ for all $n$. Pick $n$ such that (9) holds (i.e., $n \geq n_0 + 1$), and write $y_{n,k}$ for the measure of the set $x$ such that (9) holds and $\alpha(G^k x) = n$.

Then by equations (8), (9) and induction,

$$y_{n,k} = \sum_{n' < n} y_{n,k-1}^{n'} + \sum_{n_0 \leq n' < n} y_{n,k-1}^{n'} + \sum_{n' > n} y_{n,k-1}^{n'}$$

$$\leq B|I_n| + \sum_{n' < n} C_0 B |I_{n'}| \sqrt{|I_n|} + \frac{C_0}{6} \sqrt{|I_n|} + \sum_{n' > n} C_0 \sqrt{|I_{n'}|}$$

$$\leq C_0 \sqrt{|I_n|} \left( \frac{1}{6} + \sum_{n' < n} B(\sqrt{\varepsilon})^{n-n'} + \frac{1}{6} + \sum_{n' > n} (\sqrt{\varepsilon})^{n'-n} \right)$$

$$< (\frac{1}{6} + \frac{1}{3} + \frac{1}{6} + \frac{1}{3}) C_0 \sqrt{|I_n|} = C_0 \sqrt{|I_n|}.$$ 

If an acip $\nu$ exists, then it can be written as $\nu(A) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \text{Leb}(G^{-i}A)$. 

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Therefore,
\[ \nu(I_n) \leq C_0 \sqrt{|I_n|}. \] (10)

Take \( \eta \in (0,1) \). Fix \( n_1 \) such that \( \sum_{n \geq n_1} y_{n,k} < \eta/2 \) for all \( k \geq 0 \). We need to show that we can choose \( \delta > 0 \) so that if \( A \subset I_0 \) is a set of measure \( \text{Leb}(A) < \delta \), then \( \text{Leb}(G^{-k}(A)) < \eta \) for all \( k \geq 0 \). By the choice of \( n_1 \), it suffices to show that \( \text{Leb}(G^{-k}(A)) < \eta/2, k \geq 0 \), for any \( A \subset I_0 \setminus I_{n_1} \).

Assume that \( A \subset I_n \setminus I_{n+1} \) for some \( n < n_1 \). Proposition 2 shows that any onto branch \( G^k : J \to I_n \) can be written as \( \psi \circ \varphi \) with
\[
\text{Dist}(\psi) \leq \log C_2 \text{ and } \varphi \in \mathcal{F}_{3(m-n+1)},
\]
where
\[
m = \alpha(G^i J) > \alpha(G^{i+1} J) > \cdots > \alpha(G^{k-1} J) > n
\]
for some \( i < k \). Clearly \( i \geq k - m + n - 1 \). For such a branch we have
\[
\text{Leb}(G^{-k} A \cap J) \leq C_2 B \left( \frac{|A|}{|I_n|} \right)^{1/\ell^3(m-n+1)} |J|.
\]
For fixed \( m \), the total measure of the set of points arriving to \( I_n \) in this fashion is bounded by \( \sum_{i=k-m+n-1}^{k-1} y_{m,i} \leq (m - n + 1) \cdot C_0 \cdot \sqrt{|I_n|} \). Summing over all branches \( J \) (including the ones that do have extensions and hence distortion bounded by \( C_1 \)), and all \( m \geq n \), we find
\[
\text{Leb}(G^{-k} A) \leq C_1 \frac{|A|}{|I_n|} + \sum_{m \geq n} (m - n + 1) C_0 \sqrt{|I_n|} C_2 B \left( \frac{|A|}{|I_n|} \right)^{1/\ell^3(m-n+1)}.
\]
Thus \( \text{Leb}(G^{-k} A) \leq \eta/2n_1 \) for any integer \( k \) and any \( A \subset I_n \setminus I_{n+1} \), \( n < n_1 \), with \( |A| \leq \delta \), provided \( \delta \) is sufficiently small. It follows that if \( A \subset I_0 \setminus I_{n_1} \) has sufficiently small measure, then \( \text{Leb}(G^{-k} A) < n_1 \eta/(2n_1) = \eta/2 \). This concludes the verification of Straube’s condition.

\[ \square \]

9 Summability

We finish by proving Theorem 3. This theorem follows immediately from the next lemma.
Lemma 8. The partial sum \( \sum_{j \in I_{n+1} \setminus I_{n+2}} s_j \nu(J_j) \) is exponentially small in \( n \).

Proof of Lemma 8. Let \( I_{n+1} = K^0 \supseteq \cdots \supseteq K^{k_n} = I_{n+2} \) be as in Section 3, and let \( m \geq 0 \) be minimal such that \( K^m \supseteq K^{m+1} \). Let us first comment on the induce times \( s_j \). If \( J_j \subset K^i \setminus K^{i+1} \), then \( G|J_j \) corresponds to at most \((i + 2)\) iterates of \( G_n \), and thus \( s_j \leq (i + 2)t_0(k_0 + 1) \cdots (k_{n-1} + 1) \) according to (4). For \( J_j \subset K^{m} \setminus K^{m+1} \), we need a better estimate than (4). Note that if \( m \geq 2 \), then \( G_n^2(J_j) \) is contained in one of the outermost branches in \( I_0 \) for all \( 2 \leq p \leq m - 1 \), where iterates of \( G \) corresponds to \( f^2 \), and thus we have in this case that

\[
s_j \leq 2t_0(k_0 + 1) \cdots (k_{n-1} + 1) + 2m. \tag{11}
\]

By Lemma 6, we have

\[
\sqrt{|I_{n+1}|} (k_0 + 1)(k_1 + 1) \cdots (k_{n-1} + 1) \leq C|I_{n+1}|^{1/4} \leq C\varepsilon^{n/4}. \tag{12}
\]

A direct computation shows that \( m\lambda^{m/\ell} / R^n \leq C = C(\lambda) \) for \( \lambda \in (0, 1) \) and all \( m, n \geq 0 \), provided \( R > \ell \). So, by Lemma 5, we have

\[
m|I_{n+1}|^{1/2} \leq Cm\lambda^{m/2\ell + 1}|I_n|^{1/2} \leq C\ell^m|I_n|^{1/2} \leq C(\sqrt{\ell^2})^n. \tag{13}
\]

**Sum over outermost branches:** Note that if \( J_i \) is an outermost branch in \( I_{n+1} \), then \( s_i \leq 2t_0(k_0 + 1) \cdots (k_{n-1} + 1) + 2m \) by (11). Using also the obvious estimate \( \nu(J_i) \leq \nu(I_{n+1}) \leq C_0\sqrt{|I_{n+1}|} \), we obtain

\[
s_i \nu(J_i) \leq 4C_0(t_0(k_0 + 1) \cdots (k_{n-1} + 1) + m)\sqrt{|I_{n+1}|}
\leq C(\varepsilon^{n/4} + (\sqrt{\ell^2})^n)
\]

according to (12) and (13). Since there are only two outermost branches, the term over these branches is exponentially small in \( n \) (provided that \( \varepsilon \) is sufficiently small).

**Sum over all other branches:** Note that if \( A \) is a subset of a component of \( I_{n+1} \setminus I_{n+2} \), and the distance \( d(A, \partial I_{n+1}) \geq \delta \cdot \text{diam}(A) \), then the Koebe distortion lemma gives that for every \( i \geq 0 \) and every onto branch \( G^i : J \to I_{n+1} \), we have

\[
\frac{\text{Leb}(G^{-i}(A) \cap J)}{|J|} \leq K(\delta) \frac{\text{Leb}(A)}{|I_{n+1}|},
\]

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where \(K(\delta) = 2(1+\delta)^2/\delta^2\). Hence
\[
\text{Leb}(G^{-i}A) \leq \sum_{G \subseteq I_{n+1}} K(\delta) \frac{\text{Leb}(A)}{|I_{n+1}|} |J|
+ \sum_{G \supseteq I_{n+1}} K(1/\epsilon) \frac{\text{Leb}(A)}{|I_{n+1}|} \text{Leb}(G^{-i}(I_{n+1}) \cap J),
\]
so that \(\text{Leb}(G^{-i}A)/\text{Leb}(G^{-i}I_{n+1}) \leq K(\delta)\text{Leb}(A)/|I_{n+1}|\). In particular, this implies that
\[
\nu(A) \leq K(\delta) \nu(I_{n+1}) \frac{\text{Leb}(A)}{|I_{n+1}|}.
\]
By Proposition 4, the length of each of the outermost branches is as least \(\rho_1^n/C_3\), and thus for any other branch \(J_j \subseteq I_{n+1} \setminus I_{n+2}\),
\[
d(J_j, \partial I_{n+1}) \geq \rho_1^n |J_j|/C_3,
\]
which implies
\[
\frac{\nu(J_j)}{|J_j|} \leq \frac{C \nu(I_{n+1})}{\rho_1^{2n} |I_{n+1}|} \leq \frac{C}{\rho_1^{2n} \sqrt{|I_{n+1}|}}.
\]
Therefore the sum of \(s_j \nu(J_j)\) over all branches other than the outermost ones is bounded from above by the following
\[
\frac{C}{\rho_1^{2n}} \frac{C_0}{\sqrt{|I_{n+1}|}} \sum_{J_j} s_j |J_j| = \frac{C}{\rho_1^{2n}} \frac{C_0}{\sqrt{|I_{n+1}|}} \left( \sum_{J_j \subseteq K^m \setminus K^{m+1}} + \sum_{J_j \subseteq K^{m+1} \setminus I_{n+2}} \right) s_j |J_j|
\]
(note that \(K^m = I_{n+1}\)). Let us first estimate the first part of this sum. Using (11), Corollary 3, Lemma 5 and (13), we obtain
\[
\frac{1}{\rho_1^{2n} \sqrt{|I_{n+1}|}} \sum_{J_j \subseteq K^m \setminus K^{m+1}} s_j |J_j| \leq
\]
\[
\leq \frac{2}{\rho_1^{2n}} (t_0(k_0 + 1) \cdots (k_{n-1} + 1) + m) \frac{|K^m \setminus K^{m+1}|}{\sqrt{|I_{n+1}|}}
\]
\[
\leq \frac{2}{\rho_1^{2n}} (t_0(k_0 + 1) \cdots (k_{n-1} + 1) + m) \sqrt{|I_{n+1}|}
\]
\[
\leq \frac{2}{\rho_1^{2n}} (t_0(k_0 + 1) \cdots (k_{n-1} + 1)) \sum_{k_0+\cdots+k_{n-1}} C_0^{k_0+\cdots+k_{n-1}} C_0 \lambda^{m+n+1} +
\]
\[
+ 2C \left( \frac{\sqrt{\epsilon \rho_1^2}}{\rho_1} \right)^n,
\]
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is exponentially small provided that \( \varepsilon \) is sufficiently small.

For each domain \( J_j \subset K^i \setminus K^{i+1} \) with \( i \geq m + 1 \), we have

\[
s_j \leq (i + 2) t_0 (k_0 + 1) \cdots (k_{n-1} + 1) \leq C(i + 2) \left( \frac{1}{|I_{n+1}|} \right)^{1/4}.
\]

Therefore,

\[
\frac{1}{\sqrt{|I_{n+1}|}} \sum_{J_j \subset K^{m+1} \setminus I_{n+2}} s_j |J_j| = \frac{1}{\sqrt{|I_{n+1}|}} \sum_{i=m+1}^{k_n-1} \sum_{J_j \subset K^i \setminus K^{i+1}} s_j |J_j|
\]
\[
\leq \frac{C}{|I_{n+1}|^{3/4}} \sum_{i=m+1}^{k_n-1} (i + 2) |K^i|
\]
\[
= C \cdot |I_{n+1}|^{1/4} \sum_{i=m+1}^{k_n-1} (i + 2) \frac{|K^i|}{|I_{n+1}|}.
\]

By Lemma 4 (applied repeatedly),

\[
\frac{|K^i|}{|I_{n+1}|} = \left( 1 - \frac{\rho_2^n}{C_5} \right)^{i-m},
\]

which implies

\[
\sum_{i=m+1}^{k_n-1} (i + 2) \frac{|K^i|}{|I_{n+1}|} \leq \sum_{i=m} (i + 2) (1 - \rho_2^n / C_5)^{i-m}
\]
\[
\leq (m + 2) \frac{C_5}{\rho_2^n} + \left( \frac{C_5}{\rho_2^n} \right)^2
\]
\[
\leq 2C(m + 2) \frac{1}{\rho_2^{2n}}.
\]

Thus, using again Lemma 5,

\[
\frac{1}{\sqrt{|I_{n+1}|}} \sum_{J_j \subset K^{m+1} \setminus I_{n+2}} s_j |J_j| \leq C |I_{n+1}|^{1/4} (m + 2) \frac{1}{\rho_2^{2n}}
\]
\[
\leq C \left( \frac{\varepsilon^{1/4}}{\rho_2^3} \right)^n C_6 (m + 2) \lambda^m \ell^{n+1},
\]

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and so
\[
\frac{1}{\rho_1^{2m}} \frac{1}{\sqrt{|I_{n+1}|}} \sum_{j_i \in K_{m+1} \setminus I_{n+2}} s_i |J_i| \leq C(m + 2) \lambda^{m / \ell n + 1} \left( \frac{\varepsilon}{\rho_1 \rho_2} \right)^{n/4},
\]
which is again exponentially small in $n$ provided that $\varepsilon$ is sufficiently small. This completes the proof. \hfill \Box

References


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