AN ALGORITHM TO COMPUTE THE TOPOLOGICAL ENTROPY OF A UNIMODAL MAP.

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We describe an algorithm, due to F. Hofbauer, to compute the topological entropy of a unimodal interval map.

In this note we want to draw attention to an algorithm, due to F. Hofbauer, to compute the topological entropy of an unimodal map. It was developed in [Hofbauer, 1979, Hofbauer, 1980] as a theoretical tool and is maybe therefore less known than we think it should. The method may be applicable to multimodal maps too, but for unimodal maps it is very easy to program.

The entropy is computed directly from the kneading invariant, as the logarithm of the spectral radius of an appropriate transition matrix (namely of the canonical Markov extension). Unlike the algorithm proposed by Block et al. [Block et al., 1989], tent maps are not used. (Recall that the tent map with slope $\pm a$ has entropy $\log a$.) Therefore one is not restricted by the large Lyapunov exponent of the tent map. However, by means of the tent family, error estimates can be made.

Let $f : I \to I$ be a unimodal map with turning point c. We assume that $f(c)$ is a maximum. Let $K(f) = e_1 e_2 e_3 \dots \in \{0,1\}^{\mathbb{N}}$ be the *kneading invariant* defined as

$$e_i = \begin{cases} 0 & \text{if } f^i(c) < c, \\ 1 & \text{if } f^i(c) \geq c. \end{cases}$$

The initial word $e_1 \dots e_n$ of the kneading invariant will be denoted by $K_n(f)$. Define the *cutting times* [Bruin, 1995] as $S_0 = 1$ and

$$S_k = \min\{n > S_{k-1} ; e_n \neq e_{n-S_{k-1}}\}.$$

Next let $M(n) = (m_{i,j})_{i,j=2}^n$ be the $n-1 \times n-1$-matrix given by

$$m_{i,j} = \begin{cases} 1 & \text{if } j = i+1 \\ 1 & \text{if } i = S_k \text{ and} \\ & \quad j = S_k - S_{k-1} + 1 \text{ for some } k, \\ 0 & \text{otherwise.} \end{cases}$$

Let $r(n)$ be the spectral radius of $M(n)$.

Theorem A. *Let h be the topological entropy of f. Then $\log r(n) \nearrow h$ and $|h - \log r(n)| \leq Ce^{-hn}$ for some uniform constant C.*

Another algorithm, solely based on the kneading invariant, or more precisely on the *kneading determinant* was proposed by Collet et al. [Collet et al., 1988]. They also conclude on an exponential rate of convergence: assuming that f is not renormalizable, the error based on $K_n(f)$ is of order $\mathcal{O}(2^{-\frac{n}{2}})$.

Proof. (Sketch) The first statement follows from [Hofbauer, 1980], see also [Collet et al., 1988]. For the second statement we remark that any unimodal map $\tilde{f}$ such that $K_n(\tilde{f}) = K_n(f)$, $M(n)$ is the same for $\tilde{f}$ and f. Furthermore, if $T_a : [0,1] \to [0,1]$ is the tent map with slope $\pm a$ ($T_a(x) = \min\{ax, a(1-x)\}$), then $a \mapsto T_a^n(\frac{1}{2})$ is an oscillating function: its intervals of monotonicity are precisely the parameter windows on which $K_{n-1}(T_a)$ is constant. It is known that

$$|\frac{d}{da}T_a^n(\frac{1}{2})| = \mathcal{O}(a^n),$$

see e.g. [Brucks & Misiurewicz, 1996]. Therefore $\{a; K_n(f) = K_n(T_a) \text{ for } i \leq n\}$ is an interval of length $\mathcal{O}(e^{-hn})$. (Here it is important to realize that if f is renormalizable of period n, f is semi-conjugate to a tent map with an n-periodic turning point. The semiconjugacy preserves the entropy.) This gives the second statement. ∎

Remarks concerning the implementation

(i) For the calculation of the spectral radius, it is probably easiest (mainly because of the sparseness of $M(n)$), to take an arbitrary unit vector $v_0 \in \mathbb{R}^{n-1}$ and iterate along the scheme $v_{i+1} = \frac{M(n)v_i}{||M(n)v_i||}$. For almost all choices of v_0, $\frac{||M(n)v_i||}{||v_i||}$ converges (exponentially) to $r(n)$.

(ii) The main difficulty seems to lie in obtaining $K(f)$ with sufficient accuracy. The number of accurate kneading coordinates is directly related to the Lyapunov exponent along the critical orbit of f. This is because in many cases $Df_a^{n-1}(c_1)$ and $\frac{d}{da}f_a^n(c)$ have the same order of magnitude. For the tent family itself, small slopes give a small Lyapunov exponent, and $K(f)$ is easily computed to large accuracy. This gain is undone again by the smaller accuracy of $r(n)$ as estimate for h, see Theorem A. This effect was observed numerically in [Block et al., 1989]. For arbitrary unimodal maps the best results seem to be obtained in the region of larger entropy.

(iii) For one-parameter families f_a of smooth unimodal maps it is well known that there are windows in parameter space for which f_a has an attracting periodic point. These windows are followed (or preceded) by countably many other windows, who together represent a period doubling cascade. Along the full range of this cascade, the topological entropy is constant. However, because the kneading invariant still changes in such parameter regions, and the algorithm uses truncated matrices, the algorithm may produce a nonconstant result here. This effect gradually disappears when n is increased.

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