Errata and Addenda to

Topological and Ergodic Theory of Symbolic Dynamics

Henk Bruin

Page xvii: Lemańczyk should be spelled with cz, not with zc.

Page 10: For shift-commuting Cantor systems (cellular automata) that are not continuous, so without a sliding block code given by the Curtis-Hedlund-Lyndon Theorem, some results are presented in F. Blanchard, E. Formenti, P. Kůrka, *Cellular automata in the Cantor, Besicovitch, and Weyl topological spaces,* Complex Systems **11** (1997), no. 2, 107–123.

Page 39: Power entropy also appears in the literature under the name of **polynomial entropy**. Results exist (among others) by Cassaigne for subshifts, by Marco for Hamiltonian systems, and by Roth et al. for general dynamical systems. This latter paper (as an analog of the existence of horseshoes for systems with positive topological entropy) presents the notion of one-sided ℓ-horseshoe as sufficient condition to get $\log \ell - 1$ as lower bound of the polynomial entropy. For interval maps, this condition becomes close to a necessary condition too.

J. Cassaigne. *Complexité et facteurs spéciaux.* Bull. Belg. Math. Soc. Simon Stevin **4** (1997), no. 1, 67–88.

J.-P. Marco, *Polynomial entropies and integrable Hamiltonian systems.* Regul. Chaotic Dyn. **18** (2013), no. 6, 623–655.

S. Roth, S. Roth, L. Snoha, *Rigidity and flexibility of polynomial entropy.* Preprint 2021. arXiv:2107.13695v1

See also the discussion in Petersen's class notes (arXiv:1607.02425).

Page 39: The notion of amorphic complexity from Fuhrman, Gröger & Jäger was also discussed by Karl Petersen (class notes, published as preprint arXiv:1607.02425). Later, Baake, Gähler & Gohlke ("Orbit separation dimension as complexity measure of primitive inflation tilings", arXiv:2311.03541), make the case that amorhpic complexity is a misnomer, because the only systems for which the notion is useful (i.e., with finite separation numbers) are those with a pure point spectrum, so more quasi-crystal than amorphic. Instead, they suggest the term *orbit separation dimension.*

Page 82: Regarding Theorem 3.72: The references B. Li, T. Sahlsten, T. Samuel, W. Steiner, *Denseness of intermediate β-shifts of finite-type.* Proc. Amer. Math. Soc. **147** (2019), no. 5, 2045–2055 and

B. Li, T. Sahlsten, T. Samuel, *Intermediate β-shifts of finite type.* Discrete Contin. Dyn. Syst. **36** (2016), no. 1, 32300344 are useful here as well.

Page 96: In the displayed formula at line 3 "containing c" should be "containing c_n".

Page 121: In Theorem 3.120 the assumption that P doesn't contain superfluous elements is missing. That is, we need to assume that $X_P \neq X_{P\setminus\{p\}}$ for each $p \in P$. Otherwise, Exemple 3.118, i.e., $P = 2\mathbb{N}+1$, would be a counter-example; since $X_P = X_{\{1\}}$ in this case, P is effectively a finite set.

Page 131: Useful references for Example 3.135 are S. Saiki, H. Takahasi, J.A. Yorke, *Piecewise linear maps with heterogeneous chaos,* Nonlinearity 34 (2021) 5744-5761. arXiv:1903.05770 and H. Takahasi, K. Yamamoto, *Heterochaos baker maps and the Dyck system: maximal entropy measures and a mechanism for the breakdown of entropy approachability,* Preprint 2022 arXiv:2209.04905

Page 136: The period doubling substitution and the resulting Toeplitz sequence of Example 4.9 already appears in the book by Gottschalk & Hedlund [284].

Page 159: Four lines below the last displayed formula, in the formula $\chi^n \circ \tilde{\chi}$, the accent is on the wrong substitution: it should be $\tilde{\chi}^n \circ \chi$ that forces occurrences of 20 to appear at least 3^{n+1} apart.

Page 162: There are too many typos in the proof of Lemma 4.43 to makes the details comprehensible. So let me give a corrected version. First of all, for this result it is assumed that the substitutions are taken from a finite collection.

Proof. First, recalling N from the definition of primitivity, we can define

$$K_1 := \max\{|\chi_{n-N+1} \circ \cdots \circ \chi_n(a)| : n \geq N, a \in \mathcal{A}_n\}$$

and

$$K_2 := \min\{|\chi_{n-N+1} \circ \cdots \circ \chi_n(a)| : n \geq N, a \in \mathcal{A}_n\} > 0.$$

Write $K = K_1/K_2$ and let A^{n-N} be the matrix associated to $\chi_1 \circ \cdots \circ \chi_{n-N}$. Then, for all $n \leq N$ and $a, b \in \mathcal{A}_n$:

$$\begin{aligned}\frac{|\chi_1 \circ \cdots \circ \chi_n(a)|}{|\chi_1 \circ \cdots \circ \chi_n(b)|} &= \frac{|\chi_1 \circ \cdots \circ \chi_{n-N}(\chi_{n-N+1} \circ \cdots \circ \chi_n(a))|}{|\chi_1 \circ \cdots \circ \chi_{n-N}(\chi_{n-N+1} \circ \cdots \circ \chi_n(b))|} \\ &\leq \frac{\|A^{n-N}(K_1\mathbf{1})\|}{\|A^{n-N}(K_2\mathbf{1})\|} \leq K.\end{aligned}$$

Let $u \in \mathcal{L}(X_\rho)$ such that $|u| \geq \min_{a \in \mathcal{A}_N}\{\chi_1 \circ \cdots \circ \chi_N(a)\}$ and $N' > N$ be minimal such that $|u| \leq \min_{a \in \mathcal{A}_{N'}} |\chi_1 \circ \cdots \circ \chi_{N'}(a)|$. In particular, $|u| > \min_{a \in \mathcal{A}_{N'-1}} |\chi_1 \circ \cdots \circ \chi_{N'-1}(a)|$ and every appearance of u is inside some word $\chi_1 \circ \cdots \circ \chi_{N'}(ab)$, $ab \in \mathcal{A}_{N'}^2$. Let w be a return word to u, see Definition 4.2. Since each word ab in $\rho^{N'}$ appears with a gap $\leq D$, we have

$$\begin{aligned}
|w| &\leq D \max_{c \in \mathcal{A}_{N'}} \{|\chi_1 \circ \cdots \circ \chi_{N'}(c)|\} \\
&\leq D\, K \min_{c \in \mathcal{A}_{N'}} \{|\chi_1 \circ \cdots \circ \chi_{N'}(c)|\} \\
&\leq D\, K \max_{c \in \mathcal{A}_{N'-1}} \{|\chi_1 \circ \cdots \circ \chi_{N'-1}(c)|\} \cdot \min_{c \in \mathcal{A}_{N'}} \{|\chi_{N'}(c)|\} \\
&\leq D\, K^2 \min_{c \in \mathcal{A}_{N'-1}} \{|\chi_1 \circ \cdots \circ \chi_{N'-1}(c)|\} \cdot \min_{c \in \mathcal{A}_{N'-1}} \{|\chi_{N'}(c)|\} \\
&\leq D\, K^2 \min\{|\chi_n(c)| : c \in \mathcal{A}_n, n \in \mathbb{N}\}|u|.
\end{aligned}$$

Since the lengths of the return words give the gaps between appearances of u, linear recurrence follows with constant

$$L = DK^2 \max_{n \geq N} \min_{c \in \mathcal{A}_n} |\chi_n(c)|.$$

□

Page 185: The closure bar is missing over $\{\sigma^n(x) : n \geq 0\}$ in Definition 4.84.

Page 190: Lemma 4.88 is wrong. For example, the Feigenbaum shift based on Toeplitz sequence $\rho = \rho_{\text{feig}}$ has an asymptotic pair $(0\rho, 1\rho)$, and asymptotic pairs are not compatible with uniform rigidity. The proof goes wrong in only applying to the cylinder sets containing ρ itself; but this suffices to show that ρ is uniformly recurrent, and because $X_{\boldsymbol{q}} = \overline{\text{orb}(\rho)}$, minimality of $(X_{\boldsymbol{q}}, \sigma)$ still holds.

Page 197: The result in the second bullet point requires that $\mathcal{B}$ contains infinitely many coprimes.

Page 197: In the statement of Proposition 4.109, the factor $\log 2$ is missing. It should read

$$h_{top}(X_{\mathcal{B}}^{her}, \sigma) = h_{top}(X_{\mathcal{B}}^{adm}, \sigma) = (\log 2)\overline{d}(F_{\mathcal{B}}) = (\log 2)\delta(F_{\mathcal{B}}).$$

Page 198: In Example 4.110 a typo in the formula of $F_{\mathcal{B}}$. It should be $F_{\mathcal{B}} = \{n \in \mathbb{Z} : \mu(|n|) \neq 0\}$.

Page 201: In line 7-8, the factors are missing in the product $\prod_{j=1}^{k}$. It should be $\boldsymbol{a}^{\prod_{j=1}^{k} b_j}(\underline{0})$ and $\boldsymbol{a}^{\prod_{j=1}^{k} b_j}(\eta)$.

Page 232: Addition to Theorem 5.25: J. Shallit (Numeration systems, linear recurrences, and regular sets. Inform. and Comput. **113** (1994)

331—347) showed that if the G_n's satisfy a linear recurrence (cf. (8.1)) then the corresponding shift is linearly recurrent.

Page 233: The book by Ian Putnam, *Cantor Minimal Systems,* Amer. Math. Soc. University Lecture series **70** (2018) should have been mentioned for this section.

Page 233: The book by Ian Putnam, *Cantor Minimal Systems,* Amer. Math. Soc. University Lecture series **70** (2018) should have been mentioned for this section.

Page 244: In Theorem 5.44 the asusmption of finite rank is important; there are non-expansive infinite rank Bratteli-Vershik systems that are not odometers, see Gjerde & Johansen, Ergod. Th. & Dynam. Sys. **20** (2000) 1687–1710.

Page 253: On line 17, "go through $\hat{u}_k$ exactly once" is better explained as "return to the minimal path from v_0 to $\hat{u}_k$ at least once every q_{k+1} iterates".

Page 260: The caption of Figure 5.15 should read: The Bratteli diagram for $S_k = S_{k-1} + S_{\max\{k-3,0\}}$.

Page 307: In footnote 23, $\mu(A^2)$ should be $\mu(A)^2$.

Page 312: In item (3) $\Rightarrow$ (4), some μ's need to be ν's: the first term in the above expression tends to $\mu(A)\mu(B)\nu(C)\nu(D)$.

Page 330: In Proposition 6.122 it should be $(r_n(x)+\rho_n(\boldsymbol{t}(x_n))\alpha \bmod 1$ and $r_n(x)\alpha \bmod 1$ instead of $\|(r_n(x)+\rho_n(\boldsymbol{t}(x_n))\alpha\|$ and $\|r_n(x)\alpha\|$.

Page 383: An extra remark to the Calkin-Wilf function $f(x) = \frac{1}{2\lfloor x\rfloor - x+1}$: The second iterate of this map preserves $(0,1)$; it is conjugate to the dyadic odometer and therefore uniquely ergodic. However, the f^2-invariant measure is not equivalent to Lebesgue. Details can be found in G. Iommi, M. Ponce, *Odometers, backward continued fractions and counting rationals,* Preprint 2023, arXiv:2310.08329.

Page 395: Added to the history of the Lagrange spectrum: The notion was first mention by Markov in 1880 (Sur les formes quadratiques binaires indéfinies, Math. Ann. **17** (1880), no. 3, 379–399). Cusick proved (The connection between the Lagrange and Markoff spectra. Duke Math. J. **42** (1975), no. 3, 507–517) in 1975 that the Lagrange spectrum is closed.

Page 397: Some typos in the proof of Lemma 8.58. In line 19 of page 397 $a_j = \frac{1}{j}\mathbf{1}_E$ should be $a_j = \mathbf{1}_E$. On line 25, th denominator t^2 should be t and $\overline{d}$ should be $\underline{d}$, and on line 2 of page 398, $\underline{d}$ should $\overline{d}$. The proof of this lemma can also be used to prove in general that the logarith averages are squeezed in between normal averages:

$$\begin{aligned}\liminf_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} a_k &\leq \liminf_{N\to\infty} \frac{1}{\log N}\sum_{k=1}^{N} \frac{1}{k} a_k \\ &\leq \limsup_{N\to\infty} \frac{1}{\log N}\sum_{k=1}^{N} \frac{1}{k} a_k \leq \limsup_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} a_k.\end{aligned}$$

under mild conditions on the sequence $(a_k)_{k\geq 1}$.

Version of March 23, 2024.