Errata and Addenda to
Topological and Ergodic Theory of Symbolic Dynamics

Henk Bruin

I plan to maintain a list of typos, errors and useful additions, including relevant references.

Page xvii: Lemańczyk should be spelled with cz, not cz.

Page 10: For shift-commuting Cantor systems (cellular automata) that are not continuous, so without a sliding block code given by the Curtis-Hedlund-Lyndon Theorem, some results are presented in F. Blanchard, E. Formenti, P. Kůrka, *Cellular automata in the Cantor, Besicovitch, and Weyl topological spaces,* Complex Systems **11** (1997), no. 2, 107–123.

Page 39: Power entropy also appears in the literature under the name of **polynomial entropy**. Results exist (among others) by Cassaigne for subshifts, by Marco for Hamiltonian systems, and by Roth et al. for general dynamical systems. This latter paper (as an analog of the existence of horseshoes for systems with positive topological entropy) presents the notion of one-sided ℓ-horseshoe as sufficient condition to get $\log \ell - 1$ as lower bound of the polynomial entropy. For interval maps, this condition becomes close to a necessary condition too.

J. Cassaigne. *Complexité et facteurs spéciaux.* Bull. Belg. Math. Soc. Simon Stevin **4** (1997), no. 1, 67–88.

J.-P. Marco, *Polynomial entropies and integrable Hamiltonian systems.* Regul. Chaotic Dyn. **18** (2013), no. 6, 623–655.

S. Roth, S. Roth, L. Snoha, *Rigidity and flexibility of polynomial entropy.* Preprint 2021. arXiv:2107.13695v1

Page 70: Theorem 3.48 misquotes the result of Climenhaga & Thompson. The quantity q_ℓ must be replace with

$$E_\ell = \#\{w \in \mathcal{L}(X_{\mathcal{C}}) : w \text{ is a prefix or suffix of some } c \in \mathcal{C}\}.$$

Also (X, σ) should have been $(X_{\mathcal{C}}, \sigma)$ (twice).

Page 82: Regarding Theorem 3.72: The references B. Li, T. Sahlsten, T. Samuel, W. Steiner, *Denseness of intermediate β-shifts of finite-type.* Proc. Amer. Math. Soc. **147** (2019), no. 5, 2045–2055 and B. Li, T. Sahlsten, T. Samuel, *Intermediate β-shifts of finite type.*

Discrete Contin. Dyn. Syst. **36** (2016), no. 1, 32300344 are useful here as well.

Page 96: In the displayed formula at line 3 "containing c" should be "containing c_n".

Page 111: In the proof of Theorem 3.103, in understanding wy the map $c \mapsto (\theta_n)_{n\geq 1}$ is monotone, it would help to bear the follwing extra paragraph in mind:

At the heart of Douady & Hubbard's theory on the Mandelbrot set is the Riemann mapping $\psi : \mathbb{D} \to \hat{\mathbb{C}} \setminus \mathcal{M}$ by which we can define *external parameter rays* $R(\vartheta) = \psi(\{re^{2\pi i\vartheta} : 0 < r < 1\})$. The external parameter rays foliate $\hat{\mathbb{C}} \setminus \mathcal{M}$ and for Lebesgue-a.e. $\vartheta \in \mathbb{S}^1$ the ray $R(\vartheta)$ lands at a single point

$$c(\vartheta) = \lim_{r\to 1} \psi(re^{2\pi i\vartheta}) \in \partial\mathcal{M} \text{ and } c(\vartheta) = L_{c(\vartheta)}(\vartheta). \tag{0.1}$$

That is, the external parameter ray $R(\vartheta)$ lands at the same point $c(\vartheta)$ as the dynamic external ray $R_{c(\vartheta)}(\vartheta)$ for the Julia set of $\mathfrak{f}_{c(\theta)}$, and this landing point is the critical value of $\mathfrak{f}_{c(\theta)}$.

Page 131: Useful references for Example 3.135 are S. Saiki, H. Takahasi, J.A. Yorke, *Piecewise linear maps with heterogeneous chaos,* Nonlinearity 34 (2021) 5744-5761. arXiv:1903.05770 and H. Takahasi, K. Yamamoto, *Heterochaos baker maps and the Dyck system: maximal entropy measures and a mechanism for the breakdown of entropy approachability,* Preprint 2022 arXiv:2209.04905

Page 136: The period doubling substitution and the resulting Toeplitz sequence of Example 4.9 already appears in the book by Gottschalk & Hedlund [284].

Page 159: Four lines below the last displayed formula, in the formula $\chi^n \circ \tilde{\chi}$, the accent is on the wrong substitution: it should be $\tilde{\chi}^n \circ \chi$ that forces occurrences of 20 to appear at least 3^{n+1} apart.

Page 160: There are too many typos in the proof of Lemma 4.43 to makes the details comprehensible. So let me give a corrected version.

Proof. First, recalling N from the definition of primitivity, we can define

$$K_1 := \max\{|\chi_{n-N+1} \circ \cdots \circ \chi_n(a)| : n \geq N, a \in \mathcal{A}_n\}$$

and

$$K_2 := \min\{|\chi_{n-N+1} \circ \cdots \circ \chi_n(a)| : n \geq N, a \in \mathcal{A}_n\} > 0.$$

Write $K = K_1/K_2$ and let A^{n-N} be the matrix associated to $\chi_1 \circ \cdots \circ \chi_{n-N}$. Then, for all $n \le N$ and $a, b \in \mathcal{A}_n$:

$$\begin{aligned}\frac{|\chi_1 \circ \cdots \circ \chi_n(a)|}{|\chi_1 \circ \cdots \circ \chi_n(b)|} &= \frac{|\chi_1 \circ \cdots \circ \chi_{n-N}(\chi_{n-N+1} \circ \cdots \circ \chi_n(a))|}{|\chi_1 \circ \cdots \circ \chi_{n-N}(\chi_{n-N+1} \circ \cdots \circ \chi_n(b))|} \\ &\le \frac{\|A^{n-N}(K_1\mathbf{1})\|}{\|A^{n-N}(K_2\mathbf{1})\|} \le K.\end{aligned}$$

Let $u \in \mathcal{L}(X_\rho)$ such that $|u| \ge \min_{a\in\mathcal{A}_N}\{\chi_1 \circ \cdots \circ \chi_N(a)\}$ and $N' > N$ be minimal such that $|u| \le \min_{a\in\mathcal{A}_{N'}} |\chi_1 \circ \cdots \circ \chi_{N'}(a)|$. In particular, $|u| \ge \min_{a\in\mathcal{A}_{N'-1}} |\chi_1 \circ \cdots \circ \chi_{N'-1}(a)|$ and every appearance of u is inside some word $\chi_1 \circ \cdots \circ \chi_{N'}(ab)$, $ab \in \mathcal{A}^2_{N'}$. Let w be a return word to u, see Definition 4.2. Since each word ab in $\rho^{N'}$ appears with a gap $\le D$, we have

$$\begin{aligned}|w| &\le D \max_{c\in\mathcal{A}_{N'}} \{|\chi_1 \circ \cdots \circ \chi_{N'}(c)|\} \\ &\le D\,K \min_{c\in\mathcal{A}_{N'}} \{|\chi_1 \circ \cdots \circ \chi_{N'}(c)|\} \\ &\le D\,K \max_{c\in\mathcal{A}_{N'-1}} \{|\chi_1 \circ \cdots \circ \chi_{N'-1}(c)|\} \cdot \min_{c\in\mathcal{A}_{N'}} \{|\chi_{N'}(c)|\} \\ &\le D\,K^2 \min_{c\in\mathcal{A}_{N'-1}} \{|\chi_1 \circ \cdots \circ \chi_{N'-1}(c)|\} \cdot \min_{c\in\mathcal{A}_{N'-1}} \{|\chi_{N'}(c)|\} \\ &\le D\,K^2 \min\{|\chi_n(c)| : c \in \mathcal{A}_n, n \in \mathbb{N}\}|u|.\end{aligned}$$

Since the lengths of the return words give the gaps between appearances of u, linear recurrence follows with constant

$$L = DK^2 \max_{n\ge N} \min_{c\in\mathcal{A}_n} |\chi_n(c)|.$$

□

Page 185: The closure bar is missing over $\{\sigma^n(x) : n \ge 0\}$ in Definition 4.84.

Page 197: The result in the second bullet point requires that $\mathcal{B}$ contains infinitely many coprimes.

Page 197: In the statement of Proposition 4.109, the factor log 2 is missing. It should read

$$h_{top}(X^{her}_{\mathcal{B}}, \sigma) = h_{top}(X^{adm}_{\mathcal{B}}, \sigma) = (\log 2)\overline{d}(F_{\mathcal{B}}) = (\log 2)\delta(F_{\mathcal{B}}).$$

Page 198: In Example 4.110 a typo in the formula of $F_{\mathcal{B}}$. It should be $F_{\mathcal{B}} = \{n \in \mathbb{Z} : \mu(|n|) \ne 0\}$.

Page 201: In line 7-8, the factors are missing in the product $\prod_{j=1}^k$. It should be $\boldsymbol{a}^{\prod_{j=1}^k b_j}(\underline{0})$ and $\boldsymbol{a}^{\prod_{j=1}^k b_j}(\eta)$.

Page 266: In addition to Zeckendorf's Theorem, it implies that every natural number can be written as sum of Fibonacci numbers in a unique way, provided you don't repeat the same number. But it also says that the Fibonacci sequence is the only recursive sequence of natural numbers with this uniqueness property.

Page 228: Being a greedy representation, an enumeration scales is also called greedy numeration systems.

Page 232: Addition to Theorem 5.25: J. Shallit (Numeration systems, linear recurrences, and regular sets. Inform. and Comput. **113** (1994) 331—347) showed that if the G_n's satisfy a linear recurrence (cf. (8.1)) then the corresponding shift is linearly recurrent.

Page 233: The book by Ian Putnam, *Cantor Minimal Systems,* Amer. Math. Soc. University Lecture series **70** (2018) should have been mentioned for this section.

Page 233: Shimomura (Adv. Math. 2020) proved, using graph covers, that every homeomorphism on a zero-dimensional set has a representation as Bratteli-Vershik system.

Page 307: In footnote 23, $\mu(A^2)$ should be $\mu(A)^2$.

Page 326: We can add the remark to Corollary 6.117 that the same proof shows that every balanced shift $X \subset \mathcal{A}^{\mathbb{Z} \text{ or } \mathbb{N}}$ has a continuous eigenvalue $\mu([a])$ for each $a \in \mathcal{A}$ and the unique probability measure μ.

Page 330: In Proposition 6.122 it should be $(r_n(x) + \rho_n(\boldsymbol{t}(x_n))\alpha \bmod 1$ and $r_n(x)\alpha \bmod 1$ instead of $\|(r_n(x) + \rho_n(\boldsymbol{t}(x_n))\alpha\|$ and $\|r_n(x)\alpha\|$.

Page 395: Added to the history of the Lagrange spectrum: The notion was first mention by Markov in 1880 (Sur les formes quadratiques binaires indéfinies, Math. Ann. **17** (1880), no. 3, 379–399). Cusick proved (The connection between the Lagrange and Markoff spectra. Duke Math. J. **42** (1975), no. 3, 507–517) in 1975 that the Lagrange spectrum is closed.

Page 397: Some typos in the proof of Lemma 8.58. In line 19 of page 397 $a_j = \frac{1}{j}\mathbf{1}_E$ should be $a_j = \mathbf{1}_E$. On line 25, th denominator t^2 should be t and $\overline{d}$ should be $\underline{d}$, and on line 2 of page 398, $\underline{d}$ should $\overline{d}$. The proof of this lemma can also be used to prove in general that the logarith averages are squeezed in between normal averages:

$$\begin{aligned}\liminf_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} a_k &\leq \liminf_{N\to\infty} \frac{1}{\log N}\sum_{k=1}^{N} \frac{1}{k} a_k \\ &\leq \limsup_{N\to\infty} \frac{1}{\log N}\sum_{k=1}^{N} \frac{1}{k} a_k \leq \limsup_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} a_k.\end{aligned}$$

under mild conditions on the sequence $(a_k)_{k\geq 1}$.

Page 400: Regarding Lemma 8.59, it was shown in by Ferenczi et al. (Minimality and unique ergodicity for adic transformations.(English summary) J. Anal. Math. **109** (2009), 1-–31) that this example is uniquely ergodic if and only if $\sum_n \frac{1}{c_n}$ diverges.

Version of December 7, 2024.