Chapter 1

Existence of acips for multimodal maps

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Abstract: Let $f : I \to I$ be a C^2 map with negative Schwarzian derivative and finitely many non-flat critical points. We generalize an existence proof of Nowicki & van Strien [10] of invariant probability measures to multimodal maps. More precisely, we show that if either one of the following two summability conditions hold

$$\sum_{c \in \mathrm{Crit}} \sum_{n=1}^{\infty} \left(\frac{|f^n(c) - \tilde{c}|^{\ell(\tilde{c}) - \ell(c)}}{|Df^n(f(c))|} \right)^{1/\ell(c)} < \infty$$

(with $\tilde{c}$ the critical point closest to $f^n(c)$), or

$$\sum_{c \in \mathrm{Crit}} \sum_{n=1}^{\infty} |Df^n(f(c))|^{-1/\ell_{\max}} < \infty \ ,$$

then f admits an acip μ with $\mu \in L^\tau$ for any $\tau < \ell_{\max}/(\ell_{\max} - 1)$. Here $\ell(c)$ is the order of the critical point c and $\ell_{\max} = \max_{c \in \mathrm{Crit}} \ell(c)$. In particular, the second condition holds for multimodal Collet-Eckmann maps and both conditions generalize the condition in [10] for unimodal maps. Polynomial growth rates of $D_n(c) = Df^n(f(c))$ are sufficient for acips to exist.

[1] This paper is dedicated to Floris Takens.

1.1 Introduction

Let $f : [0,1] \to [0,1]$ be a C^2 map with negative Schwarzian derivative and a single critical point c of order ℓ. A well known result of [10] states that if

$$\sum |Df^n(f(c))|^{-1/\ell} < \infty, \tag{1.1}$$

then f has an *acip*, i.e., an invariant probability measure which is absolutely continuous with respect to Lebesgue measure. In this paper we want to extend the result to multimodal maps, i.e. maps where the critical set Crit consists of at least 2, but at most finitely many points.

One issue of importance are the orders $\ell(c)$ of the critical points. If all these orders are the same (and finite), then the multimodal case turns out to bring no new difficulties. The condition is the direct analog of (1.1), and somewhat weaker than the condition used in [4], in which however also decay of correlations is addressed. Different critical points with different critical orders bring some new phenomena. For example, the forward Collet-Eckmann condition: there exist $C > 0$, $\lambda > 1$ such that

$$|Df^n(f(c))| \geq C\lambda^n \tag{1.2}$$

no longer implies the backward Collet-Eckmann condition: there exists $C > 0$, $\lambda > 1$ such that

$$Df^n(z) \geq C\lambda^n \text{ whenever } f^n(z) \in \text{Crit}. \tag{1.3}$$

Examples are given in [6] and more explicitly [5]. If all critical points have the same order, then (1.2) implies (1.3), see [7]. Let us also remark that (1.2) is invariant under topological conjugacy within the class of S-unimodal maps, see [9]. For multimodal maps, this is not clear, see [11] and the remark in [9, page 35].

We have two approaches to deal with the effects of different critical orders. One consists of including in the summability condition a factor concerning *slow recurrence*. Slow recurrence was used in the Benedicks-Carleson approach [1] of Collet-Eckmann maps in the unimodal setting. For the other approach we assume that each critical point satisfies the summability condition with the maximal critical order $\ell_{\max} = \max\{\ell(c), c \in \text{Crit}\}$.

We remark that there are maps with acip, but not satisfying the summability condition, see [3]. The same will be true of course in the multimodal setting. If $\ell(c) = \infty$ for some $c \in \text{Crit}$, then no acip needs to exist. This depends largely on the precise form of the flatness, as is shown by Benedicks & Misiurewicz [2], and in more generality by Thunberg [12]. It remains an open question whether the summability condition (1.1), or any multimodal analog, are topological invariants.

1.2 The Multimodal Summability Condition

Let $f : [0,1] \to [0,1]$ a C^2 mapping with negative Schwarzian derivative, i.e. $|f'|^{-\frac{1}{2}}$ is a convex function. We assume that f has a finite set Crit of critical points, all of which are non-flat. This means that for each $c \in$ Crit there exists $\ell = \ell(c) < \infty$ such that $\lim_{x\to c} |f(x) - f(c)|/|x-c|^\ell$ exists and is different from 0.

For each critical point c, let $D_n(c) = |Df^n(f(c))|$, and

$$\delta_n(c) = \left(\frac{|f^n(c) - \tilde{c}|^{\ell(\tilde{c})-\ell(c)}}{D_n(c)} \right)^{1/\ell(c)},$$

where $\tilde{c}$ is the critical point closest to $f^n(c)$. Obviously $\delta_n(c) = D_n(c)^{-1/\ell}$ when all critical points have the same order ℓ.

Let m denote Lebesgue measure. We are interested in finding an *acip*, i.e. an invariant probability measure μ which is absolutely continuous with respect to m.

Theorem 1. Let f be a multimodal map with negative Schwarzian derivative and finitely many non-flat critical points. If

$$(a) \qquad \sum_{c\in \mathrm{Crit}} \sum_{n=1}^{\infty} \delta_n(c) < \infty,$$

or

$$(b) \qquad \sum_{c\in \mathrm{Crit}} \sum_{n=1}^{\infty} D_n(c)^{-1/\ell_{\max}} < \infty,$$

then f admits an acip μ, and $\mu \in L^\tau$ for any $\tau < \ell_{\max}/(\ell_{\max} - 1)$.

While there are multimodal Collet-Eckmann maps that do not satisfy (a) (cf. [5]), every non-flat multimodal Collet-Eckmann map satisfies (b). Most likely, there are maps satisfying (a) but not (b). If all $\ell(c)$ are the same, both conditions are of course identical.

We will follow the proof of [10], see also [8, Chapter V 4], in the sense that we give the main arguments and refer to [8] for some of the technical details, but in order to prove that (b) is a sufficient condition, we also need some results from [5]. At several places we need a version of Koebe distortion estimates, namely the Koebe Principle, the one-sided Koebe Principle and the expansion of the cross ratio. The exact formulation and proof can be found in [8, Section IV.1]

Proof: We start with the two main claims. There exists $K_0 > 0$ such that for each measurable set A and $n \geq 0$

$$m(f^{-n}(A)) \leq K_0 m(A)^{1/\ell_{\max}}, \tag{1.4}$$

where $\ell_{\max} = \max\{\ell(c); c \in \text{Crit}\}$. The invariant measure μ is constructed as the weak limit of Cesaro means $\mu_n(A) = \frac{1}{n}\sum_{i=0}^{n-1} m(f^{-i}(A))$. Claim (1.4) implies that $\mu(A) \leq K_0 m(A)^{1/\ell_{\max}}$, so absolute continuity follows, and a fortiori, the Radon-Nikodým derivative $\frac{d\mu}{dm}$ is an $L^\tau(m)$-function for any $1 \leq \tau < \ell_{\max}/(\ell_{\max}-1)$, see [8, page 378].

Let $B(c,\varepsilon)$ be the ε-neighbourhood of $c \in \text{Crit}$. Then there exists $K_1 > 0$ such that

$$m(f^{-n}(B(c,\varepsilon))) \leq K_1\varepsilon, \tag{1.5}$$

for all $c \in \text{Crit}$, $\varepsilon > 0$ and $n \geq 0$.

Proposition 1.2.1. The claim made in equation (1.5) implies the claim made in equation (1.4).

For the proof we need some notation and a lemma. Let

$$E_n(c,\varepsilon) = f^{-n}(B(c,\varepsilon)).$$

Lemma 1.2.1. There exists $K_2 > 0$ such that for any interval I for which $f^n|I$ is monotone, $m(f^n(I)) \leq \varepsilon$ and one of the boundary points of I is a critical point of f^n, we have

$$I \subset E_i(c, K_2(\frac{\varepsilon}{D_{n-i-1}(c)})^{1/\ell(c)}), \tag{1.6}$$

for some $0 \leq i \leq n$ and $c \in \text{Crit}$. (If $i = n$, then put $D_{n-i-1}(c) = 1$.)

Proof: Let $a \in \partial I$ be such that f^n has a critical point at a. Then $f^n(a) = f^j(\tilde{c})$ for some $\tilde{c} \in \text{Crit}$ and $j \leq n$. If $|f^j(\tilde{c}) - Crit| < 3\varepsilon$, then (1.6) holds for the $c \in \text{Crit}$ closest to $f^j(\tilde{c})$, $K_2 = 4$ and $i = n$. Assume therefore that $|f^j(\tilde{c}) - \text{Crit}| > 3\varepsilon$. Let B be the component of $f^{-n}((f^j(\tilde{c})-3\varepsilon, f^j(\tilde{c})+3\varepsilon))$ containing I. Also let

$$s = \max\{m \leq n; f^m(B) \cap Crit \neq \emptyset\}.$$

say $f^s(B) \ni c$. As $a \in \partial I$, $n-j \leq s < n$. Let C be the convex hull of c and $f^s(I)$. By construction, f^{n-s-1} maps a neighbourhood of $f^{s+1}(B)$ monotonically onto $(f^j(\tilde{c}) - 3\varepsilon, f^j(\tilde{c}) + 3\varepsilon)$. By the one-sided Koebe Principle

$$Df^{n-s-1}(c) \leq \mathcal{O}(1)\frac{|f^{n-s}(C)|}{|f(C)|} \leq \mathcal{O}(1)\frac{\varepsilon}{|f(C)|}.$$

Hence non-flatness gives $I \subset E_{n-s}(c, (\mathcal{O}(1)\varepsilon/D_{n-s-1}(c))^{1/\ell(c)})$. ■

Now we can prove Proposition 1.2.1

Proof: Let A be any measurable set; we only need to consider the case when $\varepsilon := m(A) > 0$. Let $f^n : (a^-, a^+) = I \to f^n(I)$ be a branch of f^n, and let $I^\pm \subset I$ be the maximal intervals such that $a^\pm \in \partial I^\pm$ and $|f^n(I^\pm)| \le \varepsilon$. By the Minimum Principle (see [8, page 154]), $m(f^{-n}(A) \cap I) \le m(I^- \cup I^+)$. Lemma 1.2.1 yields

$$I^\pm \subset E_{i^\pm}\left(c^\pm, K_2(\frac{\varepsilon}{D_{n-i^\pm-1}(c^\pm)})^{1/\ell(c^\pm)}\right),$$

where $c^\pm = f^{i^\pm}(a^\pm)$. By Claim(1.5), $|I^\pm| \le K_1 K_2(\varepsilon / D_{n-i^\pm-1}(c^\pm))^{1/\ell(c^\pm)}$. Thus summing over all branches I, we obtain

$$m(f^{-n}(A)) \le \sum_{c \in \mathrm{Crit}} \sum_{i=0}^{n-1} K_1 K_2 \left(\frac{\varepsilon}{D_{n-i^\pm-1}(c^\pm)}\right)^{1/\ell(c^\pm)}.$$

■

1.3 The Proof of Theorem 1, Part (a)

Division into Three Cases: It suffices to prove Claim (1.5). We will subdivide the components I of $E_n(c, \varepsilon)$ into three classes. Given $\sigma > 2\varepsilon$, let $I \subset I' \subset I''$ be the components of $E_n(c, \varepsilon)$, $E_n(c, 2\varepsilon)$ and $E_n(c, \sigma)$ respectively.

- $I \in \mathcal{R}_n$, the *regular* case, if f^n has no critical point in I''.
- $I \in \mathcal{S}_n$, the *sliding* case, if f^n has a critical point in I'' but not in I'.
- $I \in \mathcal{T}_n$, the *transport* case, if f^n contains a critical point in I'.

We treat these three cases separately.

The Regular Case: Let $I \subset \mathcal{R}_n(c)$, then $f^n(I'')$ is an interval of length 2σ and it contains a 2-scaled neighbourhood of $f^n(I)$. By the Koebe Principle, there exists a constant $K_\mathcal{R}$ such that

$$\frac{\sigma}{|I''|} \le \frac{|f^n(I'')|}{|I''|} \le K_\mathcal{R} \frac{|f^n(I)|}{|I|} \le K_\mathcal{R} \frac{\varepsilon}{|I|}.$$

We choose $K_\mathcal{R}$ so that this holds for all $c \in \mathrm{Crit}$. Since the intervals I'' corresponding to the $I \in \mathcal{R}_n$ are pairwise disjoint,

$$\sum_{I \in \mathcal{R}} |I| \le K_\mathcal{R} \sum_{c \in Crit} \frac{\varepsilon}{\sigma} \sum_{I \in \mathcal{R}_n(c)} |I''| \le K_\mathcal{R} \frac{\varepsilon}{\sigma}. \tag{1.7}$$

The Inductive Scheme: Having fixed the constant $K_{\mathcal{R}}$ let us explain the inductive structure of the proof of Claim (1.5). The induction hypothesis is

$$(IH) \qquad m(f^{-k}(B(c,\varepsilon))) \le \frac{3K_{\mathcal{R}}}{\sigma}\varepsilon$$

for all $\varepsilon > 0$, $c \in \mathrm{Crit}$ and $k < n$. This inequality is true for $n = 2$, because for σ sufficiently small, there are only regular intervals I. Assuming (IH) for all $k < n$, we proceed to prove (IH) for n for the sliding and transport case.

The Function $\nu(\sigma)$: Let $\nu(\sigma) = \min\{k \ge 1; f^k(\tilde{c}) \in \cup_c B(c,\sigma)$ for some $\tilde{c} \in \mathrm{Crit}\}$. The summability condition implies that that $f^n(\mathrm{Crit}) \cap \mathrm{Crit} = \emptyset$ for all $n \ge 1$ and therefore $\nu(\sigma) \to \infty$ as $\sigma \to 0$. (If f is a Misiurewicz map, then $\nu(\sigma) = \infty$ for σ sufficiently small. This case is harmless; it implies that all components I belong to $\mathcal{R}_n(c)$.)

The Sliding Case: Let $I \in \mathcal{S}_n(c)$. Let $T \supset I$ be the maximal interval on which f^n is monotone. We construct a sequence $n = n_0 > n_1 > \dots n_s \ge 0$ and intervals $T^s \supset \dots T^1 \supset T^0 = T$ as follows.

Let $T_0 = f^{n_0}(T^0) = f^n(T)$. Because $I \in \mathcal{S}_n$, $T_0 \supset I_0 := f^n(I) = B(c,\varepsilon)$. Let R_0 and A_0 be the components of $T_0 \setminus I_0$ such that $|R_0| \ge |A_0|$. By non-flatness, $|R_0| \ge \mathcal{O}(1)|I_0|$. Let us write $T^0 = (\alpha_0, \alpha_{-1})$ where α_0 is such that $f^{n_0}(\alpha_0) \in \partial A_0$. (Throughout the proof, (a,b) is the interval with boundary points a and b, also if $b < a$.)

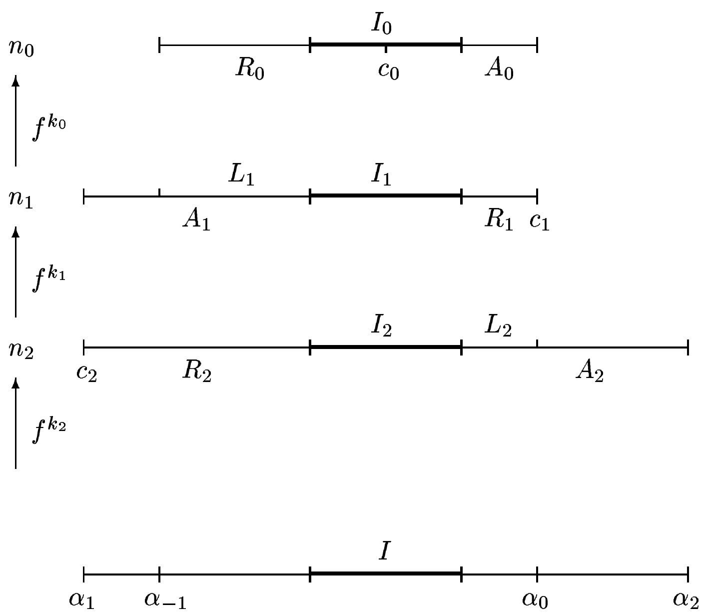

Figure 1, Construction for the Sliding Case.

We continue inductively as follows, see Figure 1: Given $T^i = (\alpha_i, \alpha_{i-1}) \supset I$ and $n_i > 0$, let

- n_{i+1} be such that $f^{n_{i+1}}(\alpha_i) \in \text{Crit}$,
- $k_i = n_i - n_{i+1}$,
- T^{i+1} be the maximal interval of the form $T^{i+1} = (\alpha_i, \alpha_{i+1})$ such that $f^{n_{i+1}}$ is monotone on T^{i+1}. So $T^{i+1} \supset T^i$ and they have the boundary point α_i in common.
- $T_{i+1} = f^{n_{i+1}}(T^{i+1})$ and $I_{i+1} = f^{n_{i+1}}(I)$.
- R_{i+1} is the component of $T_{i+1} \setminus I_{i+1}$ with the point $c_{i+1} := f^{n_{i+1}}(\alpha_i) \in$ Crit as boundary point. Let A_{i+1} be the other component. (It follows that $f^{k_i}(R_{i+1}) = A_i$.)
- L_{i+1} be the subinterval of A_{i+1} adjacent to I_{i+1} such that $f^{k_i}(L_{i+1}) = R_i$.

The construction ends at n_s when either $n_s = 0$ or when $|A_s| \geq |R_s|$. We write ℓ_i for the critical order of c_i.

Proposition 1.3.1. There exist $K_4 \geq 1$ such that

$$|I_s| \leq |f^n(I)| \cdot \prod_{i=0}^{s-1} K_4 \delta_{k_i}(c_{i+1}),$$

where $c_{i+1} \in \mathrm{Crit}$ is the critical point in ∂R_{i+1}.

Proof: The expansion of cross-ratios,

$$\frac{|f^{k_i}(L_{i+1})|}{|L_{i+1}|}\frac{|f^{k_i}(R_{i+1})|}{|R_{i+1}|} \le \frac{|f^{k_i}(I_{i+1})|}{|I_{i+1}|}\frac{|f^{k_i}(T_{i+1})|}{|T_{i+1}|},$$

in our case gives rise to

$$|I_{i+1}| \le |I_i|\frac{|R_{i+1}|}{|R_i|}\frac{|A_i \cup I_i \cup R_i|}{|L_{i+1} \cup I_{i+1} \cup R_{i+1}|}\frac{|L_{i+1}|}{|A_i|}.$$

Using this for $i = 1, \ldots, s-1$ we obtain

$$\begin{aligned}|I_s| \le |I_1|\frac{|A_1 \cup I_1 \cup R_1|}{|A_1|} &\times \prod_{i=1}^{s-1}\frac{|R_{i+1}|}{|R_i|} \\ &\times \prod_{i=2}^{s-1}\frac{|A_i \cup I_i \cup R_i|}{|L_i \cup I_i \cup R_i|}\frac{|L_i|}{|A_i|} \times \frac{|L_s|}{|L_s \cup I_s \cup R_s|}.\end{aligned} \tag{1.8}$$

The last factor is clearly less than 1, and the second last factor is less than 1 because of the inequality $\frac{l(a+w)}{a(l+w)} \le 1$ for all $w > 0$ and $a > l > 0$.

The factors $|R_{i+1}|/|R_i|$ can be estimated as follows:

$$\begin{aligned}\frac{|R_{i+1}|}{|R_i|} &\le \mathcal{O}(1)\frac{|f(R_{i+1})|^{1/\ell_{i+1}}}{|R_i|} \qquad \text{(non-flatness)} \\ &\le \mathcal{O}(1)\frac{1}{|R_i|}\left(\frac{|f(A_i)|}{D_{k_i}(c_{i+1})}\right)^{1/\ell_{i+1}} \qquad \text{(one-sided Koebe)} \\ &\le \mathcal{O}(1)\frac{|A_i|^{\ell_i/\ell_{i+1}}}{|R_i|}\frac{1}{D_{k_i}(c_{i+1})^{1/\ell_{i+1}}} \qquad \text{(non-flatness)} \\ &\le O(1)|R_i|^{(\ell_i/\ell_{i+1})-1}\frac{1}{D_{k_i}(c_{i+1})^{1/\ell_{i+1}}} \qquad (|A_i| < |R_i|) \qquad (1.9) \\ &\le \mathcal{O}(1)\,\delta_{k_i}(c_{i+1}). \qquad (|R_i| \approx |f^{k_i}(c_{i+1}) - c_i|)\end{aligned}$$

It remains to estimate

$$\begin{aligned}|I_1|\frac{|A_1 \cup I_1 \cup R_1|}{|A_1|} &\le |I_1|\frac{|L_1 \cup I_1 \cup R_1|}{|L_1|} \\ &\le |I_0|\frac{|R_1|}{|A_0|}\frac{|A_0 \cup I_0 \cup R_0|}{|R_0|} \qquad \text{(cross ratio)} \\ &\le \mathcal{O}(1)|I_0|\frac{|R_1|}{|A_0|} \qquad (\,|I_0| \le |A_0| \le |R_0|\,)\end{aligned}$$

$$
\begin{aligned}
&\le \mathcal{O}(1)|I_0|\frac{|f(R_1)|^{1/\ell_1}}{|A_0|} \quad \text{(non-flatness)}\\
&\le \mathcal{O}(1)\frac{|I_0|}{|A_0|}\left(\frac{|A_0|}{D_{k_0-1}(c_1)}\right)^{1/\ell_1} \quad \text{(one-sided Koebe)}\\
&\le \mathcal{O}(1)\frac{|I_0|}{|A_0|^{1-1/\ell_1}}\left(\frac{|f^{k_0}(c_1)-c_0|^{\ell_0-1}}{D_{k_0}(c_1)}\right)^{1/\ell_1} \quad \text{(non-flatness)}\\
&\le \mathcal{O}(1)\,|I_0|\cdot\delta_{k_0}(c_1). \quad (|A_0| \approx |f^{k_0}(c_1)-c_0|)
\end{aligned}
$$

Combining these estimates gives the result. ∎

The next step is to compare I with a component of $E_{n_s}(c_s,\varepsilon')$ for an appropriate ε'. Let $J \subset T^s$ be an interval such that $|J| = |I|$ and $G := f^{n_s}(J) \subset I_s \cup R_s$ is adjacent to c_s. Since $|A_s| > |R_s|$, the one-sided Koebe Principle gives

$$\frac{|G|}{|J|} \le \mathcal{O}(1)\frac{|f^{n_s}(I)|}{|I|}.$$

Because $|I| = |J|$, Proposition 1.3.1 gives

$$|G| \le K_{\mathcal{S}}|f^n(I)|\prod_{i=0}^{s-1} K_4\delta_{k_i}(c_{i+1}),$$

for some $K_{\mathcal{S}} > 0$. Therefore I is a subinterval of a component of the set $E_{n_s}(c_s, \varepsilon K_{\mathcal{S}}\prod_{i=0}^{s-1} K_4\delta_{k_i}(c_{i+1}))$. For every s-tuple $(k_0,\dots,k_{s-1})$, there are at most $(2\#\text{Crit})^s$ intervals I such that $f^{n_s}(I)$ slides to the same interval G. Furthermore, all the intervals T_i have size $\le \sigma$, so that $k_i \ge \nu(\sigma)$ for all $0 \le i < s$. Therefore,

$$\sum_{I\in\mathcal{S}_n(c)} |I| \le \sum_{\tilde c\in Crit} \sum_{\substack{k_j \ge \nu(\sigma)\\ \sum_j k_j = n-n_s \le n}} (2\#\text{Crit})^s |E_{n_s}(\tilde c, \varepsilon K_{\mathcal{S}}\prod_{i=0}^{s-1}\delta_{k_i}(c_{i+1}))|. \tag{1.10}$$

The Transport Case: Suppose $I \in \mathcal{T}_n(c)$. Then f^n has a critical point in I'. Let $k < n$ be maximal such that $f^k(I')$ contains a critical point, say $\tilde c$, in its closure. Let us say that $I \in \mathcal{T}_n^k(\tilde c, c)$ in this case. Clearly f^{n-k-1} maps $f^{k+1}(I')$ diffeomorphically into $B(c, 2\varepsilon)$.

Proposition 1.3.2. There exists $K_{\mathcal{T}} > 0$ such that

$$\sum_{I\in\mathcal{T}_n(c)} |I| \le \sum_{k=0}^{n-1}\sum_{I\in\mathcal{T}_n^k(\tilde c,c)} |I| \le \sum_{\tilde c\in Crit}\sum_{k=0}^{n-1} E_k(\tilde c, K_{\mathcal{T}}\varepsilon\delta_{n-k}(\tilde c)) \tag{1.11}$$

for all $c \in \text{Crit}$ and $n \geq 1$.

Proof: Clearly $f(\tilde{c}) \in f^{k+1}(I') \subset [x,y]$ where f^{n-k-1} maps $[x,y]$ monotonically onto $B(c,2\varepsilon)$. Because $B(c,2\varepsilon)$ contains an 2-scaled neighbourhood of $B(c,\varepsilon) \supset f^n(I)$, the Koebe Principle assures that $|f^n(I)|/|f^{k+1}(I)| = \mathcal{O}(1)D_{n-k-1}(\tilde{c})$. Therefore there exists K_5 such that

$$|f^k(I)| \leq K_5 \left(\frac{\varepsilon}{D_{n-k-1}(\tilde{c})}\right)^{1/\ell(\tilde{c})}.$$

Since $f^{n-k}(\tilde{c}) \in B(c,2\varepsilon)$, the Chain Rule and non-flatness give $D_{n-k}(\tilde{c}) < \mathcal{O}(\ell(c))D_{n-k-1}(\tilde{c})\varepsilon^{(\ell(c)-1)}$. Therefore there exists $K_{\mathcal{T}}$ such that

$$|f^k(I)| \leq \mathcal{O}(1)\frac{\varepsilon^{\ell(c)/\ell(\tilde{c})}}{D_{n-k}(\tilde{c})^{1/\ell(\tilde{c})}} \leq |f^n(I)|K_{\mathcal{T}}\delta_{n-k}(c).$$

Summing over all such I gives the proposition. ∎

Conclusion of the Proof: Using [8, Lemma 4.9] one can choose σ so small (and therefore $\nu(\sigma)$ so large) that for any $n \geq 1$

$$\sum_{\tilde{c}\in Crit} \sum_{\substack{k_j \geq \nu(\sigma) \\ \sum_j k_j = n - n_s \leq n}} 3K_{\mathcal{S}} \prod_{j=0}^{s-1} 2\#\text{Crit}K_4\delta_{k_j}(\tilde{c}_j) \leq 1, \tag{1.12}$$

and

$$\sum_{\tilde{c}\in Crit} \sum_{k\geq\nu(\sigma)} 3K_{\mathcal{T}}\delta_k(\tilde{c}) \leq 1. \tag{1.13}$$

By (1.10) and (1.11), for any $c \in \text{Crit}$

$$\begin{aligned}
|E_n(c,\varepsilon)| &\leq \sum_{I\in\mathcal{R}_n(c)} |I| + \sum_{I\in\mathcal{S}_n(c)} +|I| \sum_{I\in\mathcal{T}_n(c)} |I| \\
&\leq K_{\mathcal{R}}\frac{\varepsilon}{\sigma} \\
&\quad + \sum_{\tilde{c}\in Crit} \sum_{\substack{k_j \geq \nu(\sigma) \\ \sum_j k_j = n - n_s \leq n}} (2\#\text{Crit})^s |E_{n_s}(\tilde{c}, 7K_{\mathcal{S}}\varepsilon \prod_{j=0}^{s-1} K_4\delta_{k_j}(\tilde{c}_j))| \\
&\quad + \sum_{\tilde{c}\in Crit} \sum_{k=\nu(\sigma)}^{n-1} |E_k(\tilde{c}, K_{\mathcal{T}}\varepsilon\delta_{n-k}(\tilde{c}))|.
\end{aligned}$$

By the induction hypothesis (IH) this is smaller than

$$K_{\mathcal{R}}\frac{\varepsilon}{\sigma} + \sum_{\tilde{c}\in Crit} \sum_{\substack{k_j \geq \nu(\sigma) \\ \sum_j k_j = n - n_s \leq n}} 3K_{\mathcal{R}}\frac{\varepsilon K_{\mathcal{S}}}{\sigma}\prod_{j=0}^{s-1} 2\#\mathrm{Crit}K_4\delta_{k_j}(\tilde{c}_j)$$
$$+ \sum_{\tilde{c}\in Crit}\sum_{k=\nu(\sigma)}^{n-1} 3K_{\mathcal{R}}\frac{\varepsilon}{\sigma}K_{\mathcal{T}}\delta_{n-k}(\tilde{c})$$

which by formulas (1.12) and (1.13) is smaller than $3K_{\mathcal{R}}\varepsilon/\sigma$. This proves the induction. Therefore Claim (1.5) holds for $K_1 = 3K_{\mathcal{R}}/\sigma$. ■

1.4 The Proof of Theorem 1, Part (b)

The proof is an adaptation of the proof of Part (a) combined with a result from [5]. We indicate the differences.

Proof: We will again prove Claim (1.4), but we change Claim (1.5) into: there exists K_6 such that

$$m(f^{-n}(B(c,\varepsilon)) \leq K_6\varepsilon^{\ell(c)/\ell_{\max}}, \tag{1.14}$$

for all $c \in$ Crit, $\varepsilon > 0$ and $n \geq 0$. The proof that Claim (1.14) implies Claim (1.4) is the same as Proposition 1.2.1. The construction of $\mu \in L^\tau$ remains the same as well. To prove Claim (1.14), we use the same division into cases, with the addition that the sliding cases falls into two subclasses.

The regular Case: Keep the same $K_{\mathcal{R}}$ as in the proof of part (a). Obviously (1.7) gives

$$\sum_{I\in\mathcal{R}} |I| \leq K_{\mathcal{R}}\frac{\varepsilon}{\sigma} \leq \frac{K_{\mathcal{R}}}{\sigma}\varepsilon^{\ell(c)/\ell_{\max}}$$

The Inductive Scheme: We use the induction hypothesis

$$(IH') \qquad m(f^{-k}(B(c,\varepsilon))) \leq \frac{5K_{\mathcal{R}}}{\sigma}\varepsilon^{\ell(c)/\ell_{\max}}$$

The Sliding Case: Take $L > 10$ so large that $50L^{-1/\ell_{\max}} \leq 1$. Keeping the construction, we divide the case $I \in \mathcal{S}_n$ into

- $I \in \mathcal{S}_n'$, if $\ell_0 \leq \ell_s$ or $|A_0| \leq L^2\varepsilon$, and
- $I \in \mathcal{S}_n''$, if $\ell_{\max} = \ell_0 > \ell_s$ and $|A_0| > L^2\varepsilon$.
- $I \in \mathcal{S}_n'''$, if $\ell_{\max} > \ell_0 > \ell_s$ and $|A_0| > L^2\varepsilon$.

First the case $I \in \mathcal{S}_n'$:

Proposition 1.4.1. There exist $K_7 \geq 1$ such that for every $I \in \mathcal{S}'_n$ holds

$$|I_s| \leq \frac{|f^n(I)|^{\ell_0/\ell_s}}{\prod_{i=0}^{s-1} K_7 D_{k_i}(c_{i+1})^{1/\ell_s}},$$

where $c_{i+1} \in \text{Crit}$ is the critical point in ∂R_{i+1}.

Proof: The proof is the same as the proof of Proposition 1.3.1, except that we need to estimate the first and second factor in (1.8) a bit differently. Indeed, now we estimate $|R_{i+1}| \leq \mathcal{O}(1) D_{k_i}(c_{i+1})^{-1/\ell_{i+1}} |R_i|^{\ell_i/\ell_{i+1}}$, which follows immediately from (1.9). Hence

$$\begin{aligned} |R_s| &\leq |R_{s-1}|^{\ell_{s-1}/\ell_s} \frac{\mathcal{O}(1)}{D_{k_{s-1}}(c_s)^{1/\ell_s}} \\ &\leq |R_{s-2}|^{\ell_{s-2}/\ell_s} \frac{\mathcal{O}(1)}{D_{k_{s-2}}(c_{s-1})^{1/\ell_s}} \frac{\mathcal{O}(1)}{D_{k_{s-1}}(c_s)^{1/\ell_s}} \\ &\vdots \qquad \vdots \qquad \vdots \\ &\leq |R_1|^{\ell_1/\ell_s} \prod_{i=1}^{s-1} \frac{1}{K_7 D_{k_i}(c_{i+1})^{1/\ell_s}}. \end{aligned}$$

Hence

$$\prod_{i=1}^{s-1} \frac{|R_{i+1}|}{|R_i|} = \frac{|R_s|}{|R_1|} \leq |R_1|^{(\ell_1-\ell_s)/\ell_s} \prod_{i=1}^{s-1} \frac{1}{K_7 D_{k_i}(c_{i+1})^{1/\ell_s}}.$$

For the remaining factors we find as before

$$\begin{aligned} &|I_1| \cdot |R_1|^{(\ell_1-\ell_s)/\ell_s} \cdot \frac{|A_1 \cup I_1 \cup R_1|}{|A_1|} \\ &\leq \mathcal{O}(1) \frac{|I_0|}{|A_0|} |R_1|^{\ell_1/\ell_s} \text{ (cross-ratio)} \\ &\leq \mathcal{O}(1) \frac{|I_0|}{|A_0|} |f(R_1)|^{1/\ell_s} \text{ (non-flatness)} \\ &\leq \mathcal{O}(1) \frac{|I_0|}{|A_0|} \left(\frac{|A_0|^{\ell_0}}{D_{k_0}(c_1)}\right)^{1/\ell_s} \text{ (one-sided Koebe and non-flatness)} \\ &\leq \mathcal{O}(1) |I_0|^{\ell_0/\ell_s} \frac{1}{D_{k_0}(c_1)^{1/\ell_s}}, \end{aligned}$$

where the last inequality holds because $\ell_0 \leq \ell_s$ or $|A_0| \leq L^2 |I_0|$. This proves the proposition. ■

The same sliding trick now gives an $K_{\mathcal{S}} \geq 1$ such that $\sum_{I \in \mathcal{S}'_n(c)} |I|$ is at most

$$\sum_{\tilde{c} \in Crit} \sum_{\substack{k_j \geq \nu(\sigma) \\ \sum_j k_j = n - n_s \leq n}} (2\#\mathrm{Crit})^s |E_{n_s}(\tilde{c}, \frac{\varepsilon^{\ell(c)/\ell(\tilde{c})} K_{\mathcal{S}}}{\prod_{i=0}^{s-1} D_{k_i}(c_{i+1})^{1/\ell(\tilde{c})}})|.$$

For the case $I \in \mathcal{S}''_n$, we use Theorem 1.2 and Proposition 3.1 from [5]. By the Mean Value Theorem, there exists $\xi \in I$ such that $|Df^n(\xi)| = \varepsilon/|I| \leq \varepsilon^{\ell(c)/\ell_{\max}}/|I|$. In the notation of [5], n is a critical time of type (AP) for ξ. Choose $\ell(\xi) = \ell_0 = \ell_{\max}$. Then there exists K_8 independently of ξ, such that

$$|Df^n(\xi)| \geq \prod_{i=0}^{s'-1} K_8 D_{k_i}(c_{i+1})^{1/\ell_{\max}},$$

where $0 = n_{s'} < n_{s'-1} < \ldots < n_0 = n$ are the critical times of ξ in the construction of [5]. Furthermore $k_i = n_i - n_{i+1}$ and as before $k_i \geq \nu(\sigma)$ for all i. Because there are at most $(2\#\mathrm{Crit})^{s'}$ intervals $I \in \mathcal{S}''_n$ with the same critical times $n_{s'}, n_{s'-1}, \ldots, n_0$, we find

$$\sum_{I \in \mathcal{S}''_n(c)} |I| \leq \sum_{\tilde{c} \in Crit} \sum_{\substack{k_j \geq \nu(\sigma) \\ \sum_j k_j = n}} \varepsilon^{\ell(c)/\ell_{\max}} \prod_{i=0}^{s'-1} \frac{2\#\mathrm{Crit}}{K_8 D_{k_i}(c_{i+1})^{1/\ell_{\max}}}.$$

The case $\mathcal{S}'''_n$ will be dealt with in the conclusion of the proof.

The Transport Case: Proposition 1.3.2 still ensures the existence of $K_{\mathcal{T}} \geq 1$ such that

$$\sum_{I \in \mathcal{T}_n(c)} |I| \leq \sum_{k=0}^{n-1} \sum_{I \in \mathcal{T}_n^k(\tilde{c}, c)} |I| \leq \sum_{\tilde{c} \in Crit} \sum_{k=0}^{n-1} E_k(\tilde{c}, \frac{K_{\mathcal{T}} \varepsilon^{\ell(c)/\ell(\tilde{c})}}{D_{n-k}(\tilde{c})^{1/\ell(\tilde{c})}})$$

for all $c \in$ Crit and $n \geq 1$.

Conclusion of the Proof: Choose σ so small that for any $n \geq 1$

$$\sum_{\tilde{c} \in Crit} \sum_{\substack{k_j \geq \nu(\sigma) \\ \sum_j k_j = n - n_s \leq n}} 5K_{\mathcal{S}} \prod_{j=0}^{s-1} \frac{2\#\mathrm{Crit}}{K_7 D_{k_j}(\tilde{c}_j)^{1/\ell_{\max}}} \leq 1, \tag{1.15}$$

$$\sum_{\tilde{c} \in Crit} \sum_{\substack{k_j \geq \nu(\sigma) \\ \sum_j k_j = n}} \prod_{j=0}^{s'-1} \frac{2\#\mathrm{Crit}}{K_8 D_{k_j}(\tilde{c}_j)^{1/\ell_{\max}}} \leq 1, \tag{1.16}$$

and

$$\sum_{\tilde{c}\in Crit}\sum_{k\geq\nu(\sigma)}\frac{5K_{\mathcal{T}}}{D_k(\tilde{c})^{1/\ell_{\max}}}\leq 1. \tag{1.17}$$

First take $\varepsilon\geq\sigma/L^2$; this means that the case $\mathcal{S}_n'''$ does not occur. Then for any $c\in$ Crit,

$$\begin{aligned}
|E_n(c,\varepsilon)| &\leq \sum_{I\in\mathcal{R}_n(c)}|I|+\sum_{I\in\mathcal{S}_n'(c)}+\sum_{I\in\mathcal{S}_n''(c)}+|I|\sum_{I\in\mathcal{T}_n(c)}|I| \\
&\leq K_{\mathcal{R}}(\frac{\varepsilon}{\sigma})^{\ell(c)/\ell_{\max}} \\
&+\sum_{\tilde{c}\in Crit}\sum_{\substack{k_j\geq\nu(\sigma)\\ \sum_j k_j=n-n_s\leq n}}(2\#\text{Crit})^s|E_{n_s}(\tilde{c},\frac{K_{\mathcal{S}}\varepsilon^{\ell(c)/\ell(\tilde{c})}}{\prod_{j=0}^{s-1}K_7D_{k_j}(\tilde{c}_j)^{1/\ell(\tilde{c})}})| \\
&+\varepsilon^{\ell(c)/\ell_{\max}}\sum_{\tilde{c}\in Crit}\sum_{\substack{k_j\geq\nu(\sigma)\\ \sum_j k_j=n}}\prod_{j=0}^{s'-1}\frac{2\#\text{Crit}}{K_8D_{k_j}(\tilde{c}_j)^{1/\ell_{\max}}} \\
&+\sum_{\tilde{c}\in Crit}\sum_{k=\nu(\sigma)}^{n-1}|E_k(\tilde{c},\frac{K_{\mathcal{T}}\varepsilon^{\ell(c)/\ell(\tilde{c})}}{D_{n-k}(\tilde{c})^{1/\ell(\tilde{c})}})|.
\end{aligned}$$

By the induction hypothesis (IH') this is smaller than

$$\begin{aligned}
&\frac{K_{\mathcal{R}}}{\sigma}\varepsilon^{\ell(c)/\ell_{\max}} \\
&+\sum_{\tilde{c}\in Crit}\sum_{\substack{k_j\geq\nu(\sigma)\\ \sum_j k_j=n-n_s\leq n}}\frac{4K_{\mathcal{R}}}{\sigma}(\varepsilon K_{\mathcal{S}})^{\ell(c)/\ell_{\max}}\prod_{j=0}^{s-1}\frac{2\#\text{Crit}}{K_7D_{k_j}(\tilde{c}_j)^{1/\ell_{\max}}} \\
&+\varepsilon^{\ell(c)/\ell_{\max}}\sum_{\tilde{c}\in Crit}\sum_{\substack{k_j\geq\nu(\sigma)\\ \sum_j k_j=n}}\prod_{j=0}^{s'-1}\frac{2\#\text{Crit}}{K_8D_{k_j}(\tilde{c}_j)^{1/\ell_{\max}}} \\
&+\sum_{\tilde{c}\in Crit}\sum_{k=\nu(\sigma)}^{n-1}\frac{4K_{\mathcal{R}}}{\sigma}\varepsilon^{\ell(c)/\ell_{\max}}\frac{K_{\mathcal{T}}}{D_{n-k}(\tilde{c})^{1/\ell_{\max}}}
\end{aligned}$$

which by formulas (1.15), (1.16) and (1.17) is smaller than $\frac{4K_{\mathcal{R}}}{\sigma}\varepsilon^{\ell(c)/\ell_{\max}}$. This proves (IH') for n and all $\varepsilon\geq\sigma/L^{M+1}$ for $M=1$. We continue the induction in M, so suppose that (IH') holds for all $\varepsilon\geq\sigma/L^{M+1}$, and take $\sigma/L^{M+2}\leq\varepsilon<\sigma/L^{M+1}$. The calculations are as before, except that now case $\mathcal{S}_n'''$ can occur. Suppose therefore that $I_0=f^n(I)$ for some $I\in\mathcal{S}_n'''$. Note that $|A_0|<\sigma$, otherwise we were in the regular case. Find $N\leq M$

such that
$$\frac{|A_0|}{L^{N+2}} \leq \varepsilon = |I_0| < \frac{|A_0|}{L^{N+1}}.$$
Let $\tilde{I} \supset I$ be such that $\tilde{I}_0 := f^n(\tilde{I})$ is an interval of length $L^N\varepsilon$ centered around I_0. As $L > 10$, the Koebe Principle gives $|I|/|\tilde{I}| \leq 10|I_0|/|\tilde{I}_0|$. Let $\mathcal{S}'''_{n,N}$ be the set of all intervals $I \in \mathcal{S}'''_n$ with this value N. Then
$$\sum_{I\in\mathcal{S}'''_{n,N}} |I| \leq 10L^{-N} \sum_{I\in\mathcal{S}'''_{n,N}} |\tilde{I}| \leq 10L^{-N} m(f^{-n}(B(c, L^N\varepsilon)).$$
By (IH') applied to $L^N\varepsilon$ and since $\ell_0 = \ell(c) \leq \ell_{\max} - 1$, this is smaller than
$$10L^{-N}\frac{5K_{\mathcal{R}}}{\sigma}(L^N\varepsilon)^{\ell(c)/\ell\max} \leq \frac{50K_{\mathcal{R}}}{\sigma}L^{-N/\ell_{\max}}\varepsilon^{\ell(c)/\ell-\max}.$$
Summing over all $N \geq 1$ we find
$$\sum_{I\in\mathcal{S}'''_n} |I| \leq \frac{50K_{\mathcal{R}}}{\sigma}\sum_{N\geq 1} L^{-N/\ell_{\max}}\varepsilon^{\ell(c)/\ell_{\max}} \leq \frac{K_{\mathcal{R}}}{\sigma}\varepsilon^{\ell(c)/\ell_{\max}}.$$
by the choice of L. Thus if we add the term $\sum_{I\in\mathcal{S}'''_n}|I|$ in the earlier calculations, we still find $|E_n(c,\varepsilon)| \leq \frac{5K_{\mathcal{R}}}{\sigma}\varepsilon^{\ell(c)/\ell_{\max}}$. This proves the induction. Therefore Claim (1.14) holds for $K_6 = 5K_{\mathcal{R}}/\sigma$. ■

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