

Upper and lower bounds for the correlation function via inducing with general return times

Henk Bruin and Dalia Terhesiu

May 22, 2018

Abstract

For non-uniformly expanding maps inducing with a general return time to Gibbs Markov maps, we provide sufficient conditions for obtaining higher order asymptotics for the correlation function in the infinite measure setting. Along the way, we show that these conditions are sufficient to recover previous results on sharp mixing rates in the finite measure setting for non-Markov maps, but for a larger class of observables. The results are illustrated by (finite and infinite measure preserving) non-Markov intervals maps with an indifferent fixed point.

Keywords. rates of mixing, renewal theory, inducing, Markov towers

1 Introduction

For infinite measure preserving systems (X, f, μ) , first order asymptotics of the correlation function $\rho_n(v, w) = \int_X vw \circ f^n d\mu$ for suitable observables v, w were obtained in [7, 12] via the method of operator renewal theory. The method and techniques in [7, 12] rely on the existence of some $Y \subset X$ such that the *first return* map $f^\tau : Y \rightarrow Y$ (with *first return time* $\tau(y) = \inf\{n \geq 1 : f^n y \in Y\}$) satisfies several good properties. In particular, given that μ_τ is the f^τ -invariant measure, it is essential that $\mu_\tau(y \in Y : \tau(y) > n) = \ell(n)n^{-\beta}$, where ℓ is a slowly varying function³ and $\beta \in (0, 1]$. The strong requirement on the asymptotic behavior of $\mu_\tau(\tau > n)$ originates in the works [4, 2] which provide first order asymptotics for scalar renewal sequences.

As clarified in [12], in the infinite measure setting, higher order asymptotics of the correlation function $\rho_n(v, w)$ (for suitable observables v, w) can be obtained exploiting higher order expansion of the tail probability $\mu_\tau(\tau > n)$, which can be estimated when the invariant density of the

Faculty of Mathematics, University of Vienna, Oskar Morgensternplatz 1, 1090 Vienna, Austria; e-mail: henk.bruin@univie.ac.at

Faculty of Mathematics, University of Vienna, Oskar Morgensternplatz 1, 1090 Vienna, Austria; e-mail: daliaterhesiu@gmail.com

Mathematics Subject Classification (2010): Primary 37A25; Secondary 37A40, 37A50, 37D25

³We recall that a measurable function $\ell : (0, \infty) \rightarrow (0, \infty)$ is slowly varying if $\lim_{x \rightarrow \infty} \ell(\lambda x)/\ell(x) = 1$ for all $\lambda > 0$. Examples of slowly varying functions are the asymptotically constant functions and the logarithm.

first return map f^τ is smooth enough. The work [12] obtains results on higher order asymptotics of $\rho_n(v, w)$ associated with infinite measure systems that induce with *first return* to Gibbs Markov maps (see Section 3 for details); in particular, the results in [12] (and later [16] which improves on [12]) apply to the LSV family of maps considered in [11]. A return map with smooth invariant density may be obtained by inducing with a *general return*, rather than a *first return* time. More precisely, even when the first return map is not Gibbs Markov, it might happen that there exists some *general* return time $\varphi : Y \rightarrow \mathbb{N}$ of f to Y , such that $f^\varphi = (f^\tau)^\rho$ is Gibbs Markov for some *reinduce time* $\rho : Y \rightarrow \mathbb{N}$. A good example is the class of non-Markov maps with indifferent fixed points studied in [20, 21]. Higher order asymptotics of the correlation function ρ_n for the infinite measure preserving systems studied in [20, 21] has not been addressed yet. It is mainly the question of higher order asymptotics of ρ_n for *infinite* measure preserving systems that induce with a *general return* time to Gibbs Markov maps that we answer in this paper. While focusing on this problem we also obtain some new results for *finite* measure preserving systems that induce with, again, a *general return* time to Gibbs Markov maps. The method of proof builds on [12] (in the infinite case) and on the works [15, 5] (in the finite case), which develop operator renewal theory (via first return inducing) for dynamical systems.

Let (X, μ) be a measure space (finite or infinite), and $f : X \rightarrow X$ a conservative, ergodic measure preserving map. Fix $Y \subset X$ with $\mu(Y) \in (0, \infty)$ and let τ be the first return of f to Y . Let $L : L^1(\mu) \rightarrow L^1(\mu)$ denote the transfer operator for f and define

$$T_n v := 1_Y L^n(1_Y v), \quad n \geq 0, \quad R_n v := 1_Y L^n(1_{\{\tau=n\}} v), \quad n \geq 1.$$

Thus T_n corresponds to general returns to Y and R_n corresponds to first returns to Y . The sequence of operators $T_n = \sum_{j=1}^n R_j T_{n-j}$ generalizes the notion of scalar renewal sequences (for details on the latter we refer to [3, 1] and references therein).

Operator renewal sequences via inducing with respect to the *first return* time were introduced in [15] to study lower bounds for the correlation function $\rho_n(v, w)$ (for v, w supported on Y) associated with finite measure preserving systems. This technique was later refined in [5, 8]. In particular, under suitable assumptions on the first return map f^τ , preserving a measure μ_τ , and requiring that $\mu_\tau(y \in Y : \tau(y) > n) = O(n^{-\beta})$, $\beta > 2$, [15, Theorem 1] provides higher order expansions of T_n , while [5, Theorem 1] shows that [15, Theorem 1] holds for $\beta > 1$ (see also Subsection 6.1 where we recall the latter mentioned result in a particular setting). An immediate consequence of these results is that the upper bound $|\int_X v w \circ f^n d\mu - \int_X v d\mu \int_X w d\mu| = O(n^{-(\beta-1)})$ (for appropriate observables v, w supported on Y) is sharp in the sense that there exists a lower bounds of the same order.

The work [12] developed a theory of renewal operator sequences for dynamical systems with infinite measure, generalizing the results of [4, 2] to the operator case. This work obtains first and higher order asymptotics for the n -th iterate L^n of the transfer operator associated with f . In particular, under suitable assumptions on the first return map f^τ (including the assumption that $\mu_\tau(\tau > n) = \ell(n)n^{-\beta}$, where ℓ is a slowly varying function and μ_τ is f^τ invariant)

it is shown in [12] that for $\beta \in (1/2, 1)$, $\lim_{n \rightarrow \infty} \ell(n)n^{1-\beta}L^n v = \frac{\sin \pi \beta}{\pi} \int v d\mu$, uniformly on Y and pointwise on X , for appropriate observables v . Obviously, this type of result implies that $\lim_{n \rightarrow \infty} \ell(n)n^{1-\beta} \rho_n(v, w) = \frac{\sin \pi \beta}{\pi} \int v d\mu \int w d\mu$, for suitable observables v, w and $\beta \in (1/2, 1)$. For results for $\beta \leq 1/2$ under stronger tail assumptions we refer to [7].

An important question is whether operator renewal type results/arguments can be exploited for $(Y, F = f^\varphi)$, when φ is a *general* return time of f to Y in the sense that $f^\varphi = (f^\tau)^\rho$, where $\rho : Y \rightarrow \mathbb{N}$ is some *reinduce time*. Assume that (Y, f^φ) is Gibbs Markov preserving a measure μ_0 . A Young tower over f^φ can be constructed (see [18]) and the first return map on the base of the tower is isomorphic to (Y, f^φ, μ_0) (see Section 3, which recalls this in detail).

In this work we provide sufficient conditions to answer the above question when the general return map f^φ is Gibbs Markov. In short, we formulate a tail condition on $\mu_0(\rho > k)$ that allows us to work with a decomposition (as in [6, 8]) of the transfer operator on the Young tower over (Y, μ_0, f^φ) . Our main result in the finite case Theorem 4.2 provides upper and lower bounds for the correlation function $\rho_n(v, w)$ provided that the tails $\mu_0(\varphi > n)$ and $\mu_0(\rho > k)$ are of the right form (see (H0) a) and (H1) in Section 2). More importantly, Theorem 4.2 provides upper and lower bounds of the correlation function for observables v, w supported on the whole space X (so not just on Y). To deal with observables supported on the whole space, we introduce weighted norms, with weights inverse proportional to the entrance time to Y (see Section 4).

Our main result in the infinite case Theorem 4.3 provides higher order asymptotics of the correlation function for observables supported on the whole space. This result is obtained assuming higher order expansion of $\mu_0(\varphi > n)$ (see (H0) b) in Section 2) and again, assuming that the tail $\mu_0(\rho > k)$ satisfies (H1) in Section 2. To deal with observables supported on the whole space, we use the same type of weighted norms used in the finite case (see Section 4).

We illustrate the use of the main results in the setting of non-Markov interval maps with indifferent fixed points, in particular the class of maps studied in [20, 21] (see Section 9). Below, we recall briefly the previous results on the correlation function in this non-Markov setting.

In the finite measure non-Markov setting (as in [20, 21]), upper bounds for observables supported on the whole space X have been obtained in [14]. Although not written up yet, the method in [14] can also be used to obtain lower bounds for the decay of correlations of observables supported on Y and, most probably, can be extended to deal with observables supported on the whole space. We also mention that in the same setting, the work [10] provides upper and lower bounds for the decay of correlations of observables supported on Y . In both works [14, 10] the results are obtained by exploiting operator renewal type results/arguments as developed in [15, 5] via inducing with first return times.

In the infinite setting of non-Markov maps $f : X \rightarrow X$ (as in [20, 21]), first order asymptotics of $L^n v$, for some appropriate v supported on a compact subset of $X' := X \setminus \{\text{indifferent fixed points}\}$ has been established in [12]. This result immediately implies first order asymptotics of $\rho_n(v, w)$, again for v, w supported on X' . Again, the underlying scheme relies on inducing with first return

times. The detailed results are recalled in Section 9.1.

In the setting of finite measure preserving non-Markov interval maps with indifferent fixed points, Theorems 4.2 gives upper and lower bounds for the decay of correlation of observables supported on X . In the infinite measure setting, Theorem 4.3 gives higher order asymptotics of $\rho_n(v, w)$ for v, w supported on X . In checking the required assumptions of the abstract results (*i.e.*, (H0) and (H1)) for typical examples in the class considered in [20, 21], we obtain an excellent estimate on $\mu_\tau(\tau > n)$. In the infinite measure case, the estimate on $\mu_\tau(\tau > n)$ enables us to extend the results of [12, 16] on the higher order asymptotics of L^n to the typical examples studied here; we refer to Section 9.3 for details.

Notation: We will use $a_n = O(b_n)$ and (to make proofs more readable) also $a_n \ll b_n$ to mean that there is a uniform constant C such that $a_n \leq Cb_n$.

2 The induced map and main assumptions

Given $f : X \rightarrow X$, we require that there exists $Y \subset X$ and a *general* (not necessarily first) return time $\varphi : Y \rightarrow \mathbb{N}$ such that the return map $F := f^\varphi : Y \rightarrow Y$, preserving the measure μ_0 , is a Gibbs Markov map as recalled below. For convenience we rescale such that $\mu_0(Y) = 1$.

We assume that F has a Markov partition $\alpha = \{a\}$ such that $\varphi|_a$ is constant on each partition element, and $F : a \rightarrow Y$ is a bijection mod μ_0 . Let $p = \log \frac{d\mu}{d\mu \circ F}$ be the corresponding potential. We assume that there is $\theta \in (0, 1)$ and $C_1 > 0$ such that

$$e^{p(y)} \leq C_1 \mu_0(a), \quad |e^{p(y)} - e^{p(y')}| \leq C_1 \mu_0(a) \theta^{s(y, y')} \quad \text{for all } y, y' \in a, a \in \alpha, \quad (2.1)$$

where $s(y_1, y_2) = \min\{n \geq 0 : F^n y_1 \text{ and } F^n y_2 \text{ belong to different elements of } \alpha\}$ is the *separation time*. We also assume that $\inf_{a \in \alpha} \mu_0(Fa) > 0$ (big image property).

In addition to the Gibbs Markov property above, we assume that the following holds:

- (H0)** **a)** Finite measure case: $\mu_0(y \in Y : \varphi(y) > n) = O(n^{-\beta})$ for $\beta > 1$.
 b) Infinite measure case: $\mu_0(y \in Y : \varphi(y) > n) = cn^{-\beta} + H(n)$ for $\beta \in (1/2, 1)$, some $c > 0$ and function H such that $H(n) = O(n^{-2\beta})$.

The following dynamical assumption will be verified for the class of maps described in Section 9 and will play an important role in the proofs of the main results.

- (H1)** Let $\tau : Y \rightarrow \mathbb{N}$ be the first return time to Y , and τ_k the k -th return time to Y , *i.e.*, $\tau_0 = 0$, $\tau_{k+1}(y) = \tau_k(y) + \tau(f^{\tau_k(y)}(y))$. Let ρ be the *reinduce time* for the general return, *i.e.*, $\varphi(y) = \tau_{\rho(y)}(y)$. Write $\{\varphi > n\} := \{y \in Y : \varphi(y) > n\}$. We assume that there exists $C > 0$ such that

$$\int_{\{\varphi > n\}} \rho(y) d\mu_0 \leq C \mu_0(\varphi > n)$$

for all $n \geq 0$.

Remark 2.1. *The first return time τ may be defined on a larger set than where the general return time φ is defined, but the difference in domains has measure zero, so we will ignore it.*

In order to have the norms in (4.1) below well defined, we need another mild condition on the inducing scheme.

(H2) Either $f^i(a) \subset Y$ or $f^i(a) \cap Y = \emptyset$ for all $a \in \alpha$, $0 \leq i < \varphi(a)$,

This condition certainly holds for the examples studied in Section 9 (which provides the required details).

3 The tower over the map $F = f^\varphi$

The tower Δ is the disjoint union of sets $(\{\varphi = j\}, i)$, $j \geq 1$, $0 \leq i < j$ with tower map

$$T_\Delta(y, i) = \begin{cases} (y, i+1) & \text{if } 0 \leq i < \varphi(y) - 1, \\ (Fy, 0) & \text{if } i = \varphi(y) - 1. \end{cases}$$

This map preserves the measure μ_Δ defined as $\mu_\Delta(A, i) = \mu_0(A)$ for every measurable set A , with $A \subset \{\varphi = j\}$ and $0 \leq i < j$.

Let $Y_i = \{(y, i) : \varphi(y) > i\}$ be the i -th level of the tower, so $Y = Y_0$ is the base. The restriction $\mu_\Delta|_Y = \mu_0$ is invariant under T_Δ^φ , which is the first return map to the base.

We extend the function φ to the tower as

$$\varphi_\Delta(y, i) := \varphi(y) - i. \quad (3.1)$$

Define $\pi : \Delta \mapsto X$ by $\pi(y, i) := f^i(y)$. Then $\mu_X = \mu_\Delta \circ \pi^{-1}$ is f -invariant, and μ_X is related to the F -invariant measure μ_0 by the usual formula

$$\mu_X(A) := \sum_{j=1}^{\infty} \sum_{i=0}^{j-1} \mu_0(f^{-i} \cap \{\varphi = j\}) = \sum_{j=0}^{\infty} \mu_0(f^{-j} A \cap \{\varphi > j\}).$$

Regardless of whether $\bar{\varphi} := \int_Y \varphi d\mu_0$ is finite (in which case we can normalize μ_X) or not, μ_0 is absolutely continuous w.r.t. μ_X .

Let v_X, w_X be observables supported on the original space X ; they lift to observables on the tower which we will denote by $v_\Delta = v_X \circ \pi$ and $w_\Delta = w_X \circ \pi$. Then

$$\int_X v_X w_X \circ f^n d\mu_X = \int_\Delta v_\Delta w_\Delta \circ T_\Delta^n d\mu_\Delta. \quad (3.2)$$

To justify (3.2), use the duality formula $\int_\Delta \pi^* v_X w_\Delta d\mu_\Delta = \int_X v_X \hat{\pi} w_\Delta d\mu_X$, where $\pi^* v_X = v_X \circ \pi$ and $\hat{\pi} w_\Delta = w_\Delta \circ \pi^{-1}$. To compute $\int_X v_X w_X \circ f^n d\mu_X$, it therefore suffices to estimate the correlation function on the tower.

4 Results for the map f under the assumptions of Section 2

Throughout we assume that f and $F = f^\varphi$ satisfy the assumptions of Section 2. In particular, we assume that F is Gibbs Markov and that the relevant forms of (H0) and (H1) hold.

We restrict to the following class of observables. Let

$$\tau^*(x) := 1 + \min\{i \geq 0 : f^i(x) \in Y\}.$$

Recall that $s(x, x')$ is the separation time of points $x, x' \in Y$ and let $\theta \in (0, 1)$ be such that (2.1) holds. Let $v_X : X \rightarrow \mathbb{R}$. For $\varepsilon > 0$ we define the weighted norm $\|\cdot\|_\theta^*$ as follows:

$$\begin{cases} \|v_X\|_\infty^* := \sup_{x \in X} |v_X(x)| \tau^*(x)^{1+\varepsilon}, \\ |v_X|_\theta^* = \sup_{a \in \alpha} \sup_{0 \leq i < \varphi(a)} \sup_{x, x' \in a} \frac{(\tau^* \circ f^i(a))^{1+\varepsilon}}{\theta^{s(x, x')}} |v_X \circ f^i(x) - v_X \circ f^i(x')|, \end{cases} \quad (4.1)$$

and $\|v_X\|_\theta^* = \|v_X\|_\infty^* + |v_X|_\theta^*$. Note that by (H2), τ^* is constant on $f^i(a)$, so the factor $\tau^* \circ f^i(a)$ in (4.1) is well-defined.

Remark 4.1. *If v_X is supported on Y , then the weighted norms $\|\cdot\|_\infty^*$ and $\|\cdot\|_\theta^*$ coincide with $\|\cdot\|_{L^\infty(\mu_0)}$ and $\|\cdot\|_\theta$ with $\|v\|_\theta = \|v\|_{L^\infty(\mu_0)} + \text{Lip}(v)$, where $\text{Lip}(v)$ is the Lipschitz constant of v w.r.t. the distance $d_\theta(x, x') := \theta^{s(x, x')}$.*

The main results in the present set-up are stated below.

Theorem 4.2 (finite measure). *Assume (H0) a) and (H1). Suppose that $v_X, w_X : X \rightarrow \mathbb{R}$ are such that $\|v_X\|_\theta^* < \infty$ and $\|w_X\|_\infty^* < \infty$. Let $d_\mu := \frac{1}{\varphi} d\mu_X$. Then*

$$\begin{aligned} \int_X v_X w_X \circ f^n d\mu - \int_X v_X d\mu \int_X w_X d\mu &= \frac{1}{\varphi} \sum_{j=n+1}^{\infty} \mu_0(\varphi > j) \int_X v_X d\mu \int_X w_X d\mu \\ &\quad + O(\|v_X\|_\theta^* \cdot \|w_X\|_\infty^* \cdot d_n), \end{aligned}$$

where

$$d_n := \begin{cases} n^{-\beta} & \text{if } \beta > 2; \\ n^{-2} \log n & \text{if } \beta = 2; \\ n^{-(2\beta-2)} & \text{if } \beta < 2. \end{cases}$$

Theorem 4.3 (infinite measure). *Assume (H0) b) and (H1). Suppose that $v_X, w_X : X \rightarrow \mathbb{R}$ are such that $\|v_X\|_\theta^* < \infty$ and $\|w_X\|_\infty^* < \infty$. Let $q = \max\{j \geq 0 : 2(j+1)\beta > 2j+1\}$. Then there exist real constants⁴ d_0, \dots, d_{q-1} such that*

$$\begin{aligned} \int_X v_X w_X \circ f^n d\mu_X &= (d_0 n^{\beta-1} + \dots + d_{q-1} n^{q(\beta-1)}) \int_X v_X d\mu_X \int_X w_X d\mu_X \\ &\quad + O(\|v_X\|_\theta^* \cdot \|w_X\|_\infty^* \cdot d_n), \end{aligned}$$

where $d_n = n^{-(\beta-1/2)}$.

⁴The constants d_0, \dots, d_{q-1} depend only on β and the constant c appearing in (H0) b). For their precise form we refer to [12, Theorem 9.1] and [16, Theorem 1.1].

Remark 4.4. *Instead of (H0) b), one can assume a stronger tail expansion of the form used in [16, Theorem 1.1] and as such obtain an improved error term in Theorem 4.3. This is just an exercise, which can be solved by the argument used in the proof of Theorem 4.3.*

Remark 4.5. *The novelty of Theorem 4.2 lies in the fact that the observables are supported on the whole space, and of course the fact that this result can be obtained by inducing with general return times. Theorem 4.3 is new, even for observables supported on Y .*

5 Transfer operators on the tower

Let L_Δ be the transfer operator associated with the tower map T_Δ and potential

$$p_\Delta(y, i) := \begin{cases} 0 & \text{if } i < \varphi(y) - 1, \\ p(y) & \text{if } i = \varphi(y) - 1. \end{cases}$$

Given that L is the transfer operator associated with (X, f, μ_X) , we have $L_\Delta \pi^* v_X = \pi^* L v_X$.

Recall that $Y = Y_0$ is the base of the tower Δ and that $F = f^\varphi$ preserves the measure μ_0 . Choose $\theta \in (0, 1)$ such that (2.1) holds and put $d_\theta(x, x') := \theta^{s(x, x')}$, where $s(x, x')$ is the separation time. Let $\mathcal{B}_\theta(Y)$ be the Banach space of d_θ -Lipschitz functions $v : Y \rightarrow \mathbb{R}$ with norm $\|v\|_\theta = \|v\|_{L^\infty(\mu_0)} + \text{Lip}(v)$, where $\text{Lip}(v)$ is the Lipschitz constant of v w.r.t. d_θ .

Let $R^* : L^1(\mu) \rightarrow L^1(\mu)$ be the transfer operator associated with $F = f^\varphi$. Under the assumption that F is Gibbs Markov, it is known that (see, for instance, [15, Section 5]):

- (P1)** **a)** The space $\mathcal{B}_\theta(Y)$ contains constant functions and $\mathcal{B}_\theta(Y) \subset L^\infty(\mu_0)$.
 b) 1 is a simple eigenvalue for R^* , isolated in the spectrum of R^* .

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\bar{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$. Given $z \in \bar{\mathbb{D}}$, we define $R^*(z) : L^1(Y) \rightarrow L^1(Y)$ to be the operator $R^*(z)v := R^*(z^\varphi v)$. By (P1) b), 1 is an isolated eigenvalue in the spectrum of $R^*(1)$. In addition, we know that (see, for instance, [15, Section 5])

- (P2)** For $z \in \bar{\mathbb{D}} \setminus \{1\}$, the spectrum of $R^*(z)$ does not contain 1.

Note that φ is the first return time of T_Δ to the base Y . Define the following transfer operators that describe the general resp. the first return to the base Y :

$$T_n^* v := 1_Y L_\Delta^n(1_Y v), \quad n \geq 0, \quad R_n^* v := 1_Y L_\Delta^n(1_{\{\varphi=n\}} v) = R^*(1_{\{\varphi=n\}} v), \quad n \geq 1.$$

By, for instance, [15, Lemma 8] there is $C > 0$ such that

- (P3)** $\|R^*(1_a v)\|_\theta \leq C \mu_0(a) \|1_a v\|_\theta$ for all $a \in \alpha$ and hence $\|R_n^*\|_\theta = O(\mu_0(\varphi = n))$.

As recalled below, $(I - R^*(z))^{-1}$ can be used to understand the asymptotics of the transfer operator of the Markov tower over the (general) induced map $F = f^\varphi$. First, it is easy to see that $R^*(z)v = R^*(z^\varphi v) = \sum_{n \geq 1} (R_n^* v) z^n$. By (P1) and (P2), when viewed as a family of operators acting on

$\mathcal{B}_\theta(Y) \subset L^\infty(\mu_0)$, the function $(I - R^*(z))^{-1}$ is bounded and continuous on $\bar{\mathbb{D}} \setminus \{1\}$. By (P3), R_n^* is bounded, so $z \mapsto (I - R^*(z))^{-1}$ is analytic on \mathbb{D} .

Since φ is a first return time of T_Δ to the base Y , we have the renewal equation on the tower $T_n^* = \sum_{j=1}^n R_j^* T_{n-j}^*$. For $z \in \bar{\mathbb{D}}$, define $T^*(z) := \sum_{n \geq 0} T_n^* z^n$ and recall $R^*(z) = \sum_{n \geq 1} R_n^* z^n$. Since $(I - R^*(z))^{-1}$ is well defined on $\bar{\mathbb{D}} \setminus \{1\}$, we have the following equation on $\bar{\mathbb{D}} \setminus \{1\}$:

$$T^*(z) = (I - R^*(z))^{-1}. \quad (5.1)$$

Under both forms of (H0) (*i.e.*, finite and infinite measure preserving) and the rest of the assumptions in Section 2, the asymptotic behavior of T_n^* is well understood ([15, 5, 12]); we also refer to Subsection 6.1 where we recall these results.

Since $v_\Delta = v_X \circ \pi$ is in general not supported on Y_0 , equation (5.1) cannot be used as such to obtain information on the asymptotic behaviour of the correlation function on the tower given by $\int_\Delta v_\Delta w_\Delta \circ T_\Delta^n d\mu_\Delta = \int_\Delta L_\Delta^n v_\Delta w_\Delta d\mu_\Delta$. Hence, one cannot just use (5.1) and (3.2) to estimate the correlation function $\int_X v_X w_\Delta \circ f^n d\mu_X$ for the map $f : X \rightarrow X$. However, the operators A_n and B_n defined below can be used to deal with precisely this problem.

Following [8, Section 2.1.1], to understand the behaviour of L_Δ^n via the behaviour of T_n^* , we need to define several operators that describe the action of L_Δ^n outside Y . Recall from (3.1) that $\varphi_\Delta(y, i) = \varphi(y) - i$ and define the operators associated with the end resp. beginning of an orbit on the tower as

$$A_n v := L_\Delta^n (1_{\{\varphi > n\}} v), \quad n \geq 0, \quad B_n v := \begin{cases} 1_Y L_\Delta^n (1_{\{\varphi_\Delta = n\}} \setminus Y v), & n \geq 1, \\ 1_Y v, & n = 0. \end{cases}$$

The operator associated with orbits that do not see the base of the tower is:

$$C_n v := L_\Delta^n (1_{\{\varphi_\Delta > n\}} \setminus Y v), \quad n \geq 0.$$

As noticed in [8, Section 2.1.1], the following equation describes the relationship between $T_n^* = 1_Y L_\Delta^n 1_Y$ and L_Δ^n .

$$L_\Delta^n = \sum_{n_1 + n_2 + n_3 = n} A_{n_1} T_{n_2}^* B_{n_3} + C_n. \quad (5.2)$$

6 Proofs of the main results: previous and new ingredients

6.1 Previous ingredients

As already mentioned, T_Δ^φ is the first return map for T_Δ to the base $Y_0 = Y$. Thus, previous results on renewal theory, in particular [5, Theorem 1] (under (H0) a)-finite measure case) and [12, Theorem 9.1] (under (H0) b)-infinite measure case), apply to T_n^* . We start by recalling these results, as relevant to the present setting. Let P denote the spectral projection corresponding to the eigenvalue 1 for $R^*(1)$. So, we can write $Pv(y) \equiv \int_Y v d\mu_0$.

Lemma 6.1. [5, Theorem 1] Assume that F is Gibbs Markov and that (H0) a) holds. Then

$$T_n^* = \frac{P}{\bar{\varphi}} + \frac{1}{\bar{\varphi}^2} \sum_{k=n+1}^{\infty} \sum_{j=k+1}^{\infty} PR_j P + E_n,$$

where E_n is an operator on $\mathcal{B}_\theta(Y)$ satisfying

$$\|E_n\|_\theta \ll \begin{cases} n^{-\beta} & \text{if } \beta > 2; \\ n^{-2} \log n & \text{if } \beta = 2; \\ n^{-(2\beta-2)} & \text{if } \beta < 2. \end{cases}$$

Lemma 6.2. [12, Theorem 9.1] Assume that F is Gibbs Markov and that (H0) b) holds. Let $q = \max\{j \geq 0 : 2(j+1)\beta > 2j+1\}$. Then there exist real constants d_0, \dots, d_{q-1} (depending only on the constants and parameters involved in (H0) b))⁵ such that

$$T_n^* = (d_0 n^{\beta-1} + \dots + d_{q-1} n^{q(\beta-1)})P + D_n, \quad n \geq 1,$$

where D_n is an operator on $\mathcal{B}_\theta(Y)$ satisfying $\|D_n\|_\theta = O(n^{-(\beta-1/2)})$.

6.2 New ingredients: estimates related to A_n, B_n, C_n under (H1)

In the results stated below we use the norm $\|\cdot\|_\theta^*$ and $\|\cdot\|_\infty^*$ as defined in (4.1). Recall that $v_\Delta = v_X \circ \pi$ and $w_\Delta = w_X \circ \pi$. The proofs of the following results are postponed to Subsection 6.4.

Lemma 6.3. Assume that (H1) holds. Let $v_X, w_X : X \rightarrow \mathbb{R}$ such that $\|v_X\|_\infty^* < \infty$ and $\|w_X\|_{L^\infty(\mu_X)} < \infty$. Then there exists $C > 0$ such that for any $n \geq 0$,

$$\left| \int_\Delta C_n v_\Delta w_\Delta d\mu_\Delta \right| \leq C \mu_0(\varphi > n) \|v_X\|_\infty^* \|w_X\|_{L^\infty(\mu_X)}.$$

Lemma 6.4. Assume that (H1) holds. Let $v_Y : Y \rightarrow \mathbb{R}$ and $w_X : X \rightarrow \mathbb{R}$ be such that $\|v_Y\|_{L^\infty(\mu_0)} < \infty$ and $\|w_X\|_\infty^* < \infty$. Then there exists $C > 0$ such that for any $n \geq 0$,

$$\left| \int_\Delta \sum_{j \geq n} A_j v_Y w_\Delta d\mu_\Delta \right| \leq C \mu_0(\varphi > n) \|v_Y\|_{L^\infty(\mu_0)} \|w_X\|_\infty^*.$$

For the statement below we note that by definition, $B_n v_\Delta$ is a function supported on the base Y of the tower Δ , so $\|B_n v_\Delta\|_\theta$ makes sense.

Lemma 6.5. Assume that (H1) holds. Let $v_X : X \rightarrow \mathbb{R}$ such that $\|v_X\|_\theta^* < \infty$. Then there exists $C > 0$ such that for any $n \geq 0$,

$$\sum_{j \geq n} \|B_j v_\Delta\|_\theta \leq C \mu_0(\varphi > n) \|v_X\|_\theta^*.$$

Remark 6.6. Continuing from Remark 4.1, we note that under the assumption that v_X, w_X are supported on Y , all the statements in this subsection simplify since the weighted norms $\|\cdot\|_\theta^*$ and $\|\cdot\|_\infty^*$ coincide with $\|\cdot\|_\theta$ and $\|\cdot\|_{L^\infty(\mu_0)}$. Moreover, under this assumption, the proofs in this subsection are simplified, although the assumption (H1) is still required. In Subsection 6.4 we point out such a simplification for the proof of Lemma 6.3: see Remark 6.8.

⁵For the precise form of these constants we refer to [12, Theorem 9.1] and [16, Theorem 1.1].

6.3 Proofs of Theorem 4.2 and Theorem 4.3

From the statement of Theorem 4.2, recall that $d\mu = \frac{1}{\bar{\varphi}} d\mu_X$ is the normalized f -invariant measure (when $\bar{\varphi} < \infty$). By (3.2) and the definition of μ , in order to estimate the correlation function for observables on the space X , in the *finite* measure case it suffices to estimate

$$\int_X v_X w_X \circ f^n d\mu = \frac{1}{\bar{\varphi}} \int_{\Delta} v_{\Delta} w_{\Delta} \circ T_{\Delta}^n d\mu_{\Delta} = \frac{1}{\bar{\varphi}} \int_{\Delta} L_{\Delta}^n v_{\Delta} w_{\Delta} d\mu_{\Delta}. \quad (6.1)$$

Similarly, due to (3.2) in the *infinite* measure case we estimate

$$\int_X v_X w_X \circ f^n d\mu_X = \int_{\Delta} L_{\Delta}^n v_{\Delta} w_{\Delta} d\mu_{\Delta}. \quad (6.2)$$

By equation (5.2) and Lemma 6.3,

$$\int_{\Delta} L_{\Delta}^n v_{\Delta} w_{\Delta} d\mu_{\Delta} = \int_{\Delta} \sum_{n_1+n_2+n_3=n} A_{n_1} T_{n_2}^* B_{n_3} v_{\Delta} w_{\Delta} d\mu_{\Delta} + O(\mu_0(\varphi > n) \|v_X\|_{\infty}^* \|w_X\|_{L^{\infty}(\mu_X)}). \quad (6.3)$$

In order to take advantage of the full force of Lemmas 6.1 and 6.2 along with the estimates related to A_n, B_n in Subsection 6.2 we define

$$A(z) := \sum_{n \geq 0} A_n z^n, \quad B(z) := \sum_{n \geq 0} B_n z^n, \quad z \in \mathbb{D}$$

and recall that when viewed as a family of operators acting on $\mathcal{B}_{\theta}(Y) \subset L^{\infty}(\mu_0)$, $T^*(z) = \sum_{n \geq 0} T_n^* z^n$ is well defined on \mathbb{D} (in fact on $\bar{\mathbb{D}}$, but we do not use this information below). By definition, the operator sequences $A_n : L^{\infty}(\mu_0) \rightarrow L^1(\mu_{\Delta})$, $B_n : L^{\infty}(\mu_{\Delta}) \rightarrow L^{\infty}(\mu_0)$, $T_n^* : L^{\infty}(\mu_0) \rightarrow L^{\infty}(\mu_0)$ are bounded. As a consequence, $A(z)T^*(z)B(z)$, $z \in \mathbb{D}$, is well defined as a family of operators from $L^{\infty}(\mu_{\Delta})$ to $L^1(\mu_{\Delta})$. Given this we can view

$$\int_{\Delta} G_n v_{\Delta} w_{\Delta} d\mu_{\Delta} := \int_{\Delta} \sum_{n_1+n_2+n_3=n} A_{n_1} T_{n_2}^* B_{n_3} v_{\Delta} w_{\Delta} d\mu_{\Delta}$$

as the n -th coefficient of $\int_{\Delta} G(z) v_{\Delta} w_{\Delta} d\mu_{\Delta} := \int_{\Delta} A(z) T^*(z) B(z) v_{\Delta} w_{\Delta} d\mu_{\Delta}$, $z \in \mathbb{D}$.

In the sequel we also use the following statement on $A(1)$ and $B(1)$ that does not require (H1); the statement on $B(1)$ below relies on the fact that $B(1)v_{\Delta}$ is a function supported on Y .

Lemma 6.7. *We have*

$$\begin{cases} \int_{\Delta} A(1) 1_Y w_{\Delta} d\mu_{\Delta} = \int_{\Delta} w_{\Delta} d\mu_{\Delta} = \int_X w_X d\mu_X, \\ \int_{\Delta} B(1) v_{\Delta} d\mu_{\Delta} = \int_Y B(1) v_{\Delta} d\mu_0 = \int_{\Delta} v_{\Delta} d\mu_{\Delta} = \int_X v_X d\mu_X. \end{cases}$$

6.3.1 Finite measure case

Let $E_n : \mathcal{B}_{\theta}(Y) \rightarrow \mathcal{B}_{\theta}(Y)$ be as in the statement of Lemma 6.1 and put $E(z) = \sum_n E_n z^n$, $z \in \mathbb{D}$.

By Lemma 6.1,

$$\int_{\Delta} G(z) v_{\Delta} w_{\Delta} d\mu_{\Delta} = I_{main}(z)(v_{\Delta}, w_{\Delta}) + I_E(z)(v_{\Delta}, w_{\Delta}) \quad (6.4)$$

for

$$\begin{cases} I_{main}(z)(v_\Delta, w_\Delta) := \\ \frac{1}{1-z} \int_\Delta A(z) \left(\frac{P}{\varphi} + (1-z) \sum_{n=0}^{\infty} \left(\frac{1}{\varphi^2} \sum_{k=n+1}^{\infty} \sum_{j=k+1}^{\infty} PR_j P \right) z^n \right) B(z) v_\Delta w_\Delta d\mu_\Delta, \\ I_E(z)(v_\Delta, w_\Delta) := \int_\Delta A(z) E(z) B(z) v_\Delta w_\Delta d\mu_\Delta. \end{cases}$$

By the above, in order to estimate $\int_\Delta G_n v_\Delta w_\Delta d\mu_\Delta$, we need to estimate the coefficients of the functions $I_{main}(z)(v_\Delta, w_\Delta)$ and $I_E(z)(v_\Delta, w_\Delta)$, $z \in \mathbb{D}$ for appropriate v_Δ, w_Δ (equivalently for appropriate v_X, w_X). Let

$$V(z)(v_\Delta, w_\Delta) := \int_\Delta A(1) \left(\sum_{n=0}^{\infty} \left(\frac{1}{\varphi^2} \sum_{k=n+1}^{\infty} \sum_{j=k+1}^{\infty} PR_j P \right) z^n \right) B(1) v_\Delta w_\Delta d\mu_\Delta \quad (6.5)$$

and note that

$$\begin{aligned} I_{main}(z)(v_\Delta, w_\Delta) - \frac{1}{1-z} \int_\Delta A(1) \frac{P}{\varphi} B(1) v_\Delta w_\Delta d\mu_\Delta - V(z)(v_\Delta, w_\Delta) \\ = I^A(z)(v_\Delta, w_\Delta) + I^B(z)(v_\Delta, w_\Delta) \end{aligned} \quad (6.6)$$

for

$$\begin{cases} I^A(z)(v_\Delta, w_\Delta) := \int_\Delta \frac{A(z)-A(1)}{1-z} \left(\frac{P}{\varphi} + (1-z) \sum_{n=0}^{\infty} \left(\frac{1}{\varphi^2} \sum_{k=n+1}^{\infty} \sum_{j=k+1}^{\infty} PR_j P \right) z^n \right) B(1) v_\Delta w_\Delta d\mu_\Delta \\ I^B(z)(v_\Delta, w_\Delta) := \int_\Delta A(1) \left(\frac{P}{\varphi} + (1-z) \sum_{n=0}^{\infty} \left(\frac{1}{\varphi^2} \sum_{k=n+1}^{\infty} \sum_{j=k+1}^{\infty} PR_j P \right) z^n \right) \frac{B(z)-B(1)}{1-z} v_\Delta w_\Delta d\mu_\Delta \end{cases}$$

Below we provide the estimates obtained in the sequel for the coefficients of the terms in (6.5) and (6.6) and as such complete

Proof of Theorem 4.2. By Lemma 6.7,

$$\frac{1}{1-z} \int_\Delta A(1) \frac{P}{\varphi} B(1) v_\Delta w_\Delta d\mu_\Delta = \frac{1}{\varphi} \int_X v_X d\mu_X \int_X w_X d\mu_X \sum_{n=0}^{\infty} z^n.$$

By Lemma 7.1, the coefficients $V_n(v_\Delta, w_\Delta)$ of $V(z)(v_\Delta, w_\Delta)$, $z \in \mathbb{D}$ are given by

$$V_n(v_\Delta, w_\Delta) = \frac{1}{\varphi^2} \sum_{k=n+1}^{\infty} \mu_0(\varphi > k) \int_X v_X d\mu_X \int_X w_X d\mu_X.$$

We continue with the estimates for the coefficients of the terms in the RHS of (6.6). Lemmas 7.4 and 7.5 together with (H0) a) imply that the coefficients of the functions $I^A(z)(v_\Delta, w_\Delta)$ and $I^B(z)(v_\Delta, w_\Delta)$, $z \in \mathbb{D}$ are $O(\|v_X\|_\theta^* \|w_X\|_\infty^* \mu_0(\varphi > n)) = O(n^{-\beta} \|v_X\|_\theta^* \|w_X\|_\infty^*)$.

It remains to estimate the coefficients $\int_\Delta \sum_{n_1+n_2+n_3=n} A_{n_1} E_{n_2} B_{n_3} v_\Delta w_\Delta d\mu_\Delta$ of the function $I_E(z)$, $z \in \mathbb{D}$. By Lemma 6.5 and (H0) a), $\|B_n v_\Delta\|_\theta = O(\mu_0(\varphi > n) \|v_X\|_\theta^*) = O(n^{-\beta} \|v_X\|_\theta^*)$. Hence, the convolution of E_n and B_n satisfies $\|\sum_{n_2+n_3=n} E_{n_2} B_{n_3} v_\Delta\|_\theta = O(\|E_n\|_\theta \cdot \|v_X\|_\theta^*)$. Next, given that v_Y is a function supported on the base Y , the definition of A_n and (H0) a) gives

$$\left| \int_\Delta A_n v_Y w_\Delta d\mu_\Delta \right| \ll \mu_0(\varphi > n) \|w_\Delta\|_{L^\infty(\mu_\Delta)} \|v_Y\|_{L^\infty(\mu_0)} \ll n^{-\beta} \|w_X\|_{L^\infty(\mu_X)} \|v_Y\|_\theta.$$

Hence, $|\int_{\Delta} \sum_{n_1+n_2+n_3=n} A_{n_1} E_{n_2} B_{n_3} v_{\Delta} w_{\Delta} d\mu_{\Delta}| = O(\|E_n\|_{\theta} \cdot \|v_X\|_{\theta}^* \cdot \|w_X\|_{L^{\infty}(\mu_X)})$.

Putting the above together and using (6.6) and (6.3),

$$\begin{aligned} \int_{\Delta} v_{\Delta} w_{\Delta} \circ T_{\Delta}^n d\mu_{\Delta} &= \frac{1}{\bar{\varphi}} \int_X v_X d\mu_X \int_X w_X d\mu_X + \frac{1}{\bar{\varphi}^2} \sum_{k=n+1}^{\infty} \mu_0(\varphi > k) \int_X v_X d\mu_X \int_X w_X d\mu_X \\ &\quad + O(\|E_n\|_{\theta} \cdot \|v_X\|_{\theta}^* \cdot \|w_X\|_{\infty}^*). \end{aligned}$$

The conclusion follows recalling that $d\mu = \frac{1}{\bar{\varphi}} d\mu_X$ and using equation (6.1). \square

Infinite measure case. Write

$$\int_{\Delta} G(z) v_{\Delta} w_{\Delta} d\mu_{\Delta} - \int_{\Delta} A(1) T^*(z) B(1) v_{\Delta} w_{\Delta} d\mu_{\Delta} = I_{\text{inf}}^A(z) + I_{\text{inf}}^B(z) \quad (6.7)$$

for

$$\begin{cases} I_{\text{inf}}^A(z)(v_{\Delta}, w_{\Delta}) := \int_{\Delta} (A(z) - A(1)) T^*(z) B(1) v_{\Delta} w_{\Delta} d\mu_{\Delta}, \\ I_{\text{inf}}^B(z)(v_{\Delta}, w_{\Delta}) := \int_{\Delta} A(1) T^*(z) (B(z) - B(1)) v_{\Delta} w_{\Delta} d\mu_{\Delta}. \end{cases}$$

Below, we provide the estimates obtained in the sequel for the coefficients of the terms in (6.7) and as such complete

Proof of Theorem 4.3. By Lemma 6.2, the n -th coefficient $[\int_{\Delta} A(1) T^* B(1) v_{\Delta} w_{\Delta} d\mu_{\Delta}]_n$ of the function $\int_{\Delta} A(1) T^*(z) B(1) v_{\Delta} w_{\Delta} d\mu_{\Delta}$, $z \in \mathbb{D}$, satisfies

$$\begin{aligned} [\int_{\Delta} A(1) T^* B(1) v_{\Delta} w_{\Delta} d\mu_{\Delta}]_n &= (d_0 n^{\beta-1} + \dots + d_{q-1} n^{q(\beta-1)}) \int_{\Delta} A(1) P B(1) v_{\Delta} w_{\Delta} d\mu_{\Delta} \\ &\quad + \int_{\Delta} A(1) D_n B(1) v_{\Delta} w_{\Delta} d\mu_{\Delta}, \end{aligned}$$

where $\|D_n\|_{\theta} = O(n^{-(\beta-1/2)})$. By Lemma 6.5, $\|B(1) v_{\Delta}\|_{\theta} \leq C \|v_X\|_{\theta}^*$ for some $C > 0$. Hence,

$$\begin{aligned} |\int_{\Delta} A(1) D_n B(1) v_{\Delta} w_{\Delta} d\mu_{\Delta}| &= O(n^{-(\beta-1/2)} \|w_{\Delta}\|_{L^{\infty}(\mu_{\Delta})} \|B(1) v_{\Delta}\|_{\theta}) \\ &= O(n^{-(\beta-1/2)} \|w_X\|_{L^{\infty}(\mu_X)} \|v_X\|_{\theta}^*). \end{aligned}$$

By Lemma 6.7, $\int_{\Delta} A(1) P B(1) v_{\Delta} w_{\Delta} d\mu_{\Delta} = \int_X v_X d\mu_X \int_X w_X d\mu_X$. Putting these together,

$$\begin{aligned} [\int_{\Delta} A(1) T^*(z) B(1) v_{\Delta} w_{\Delta} d\mu_{\Delta}]_n &= (d_0 n^{\beta-1} + \dots + d_{q-1} n^{q(\beta-1)}) \int_X v_X d\mu_X \int_X w_X d\mu_X \\ &\quad + O(n^{-(\beta-1/2)} \|w_X\|_{L^{\infty}(\mu_X)} \|v_X\|_{\theta}^*). \end{aligned}$$

By Lemma 8.1 the coefficients of the functions $I_{\text{inf}}^A(z)(v_{\Delta}, w_{\Delta})$ and $I_{\text{inf}}^B(z)(v_{\Delta}, w_{\Delta})$, $z \in \mathbb{D}$, are $O(n^{-\beta} \|v_X\|_{\theta}^* \|w_X\|_{\infty}^*)$.

Putting the above together and using equations (6.7) and (6.3),

$$\begin{aligned} \int_{\Delta} v_{\Delta} w_{\Delta} \circ T_{\Delta}^n d\mu_{\Delta} &= (d_0 n^{\beta-1} + \dots + d_{q-1} n^{q(\beta-1)}) \int_X v_X d\mu_X \int_X w_X d\mu_X \\ &\quad + O(n^{-\beta} \|v_X\|_{\theta}^* \|w_X\|_{\infty}^*). \end{aligned}$$

The conclusion follows from the above equation together with equation (6.2). \square

6.4 Proofs of the lemmas in Subsections 6.2 and 6.3

Proof of Lemma 6.3. Recall that $\tau_0(y) = 0$ and $\tau_k(y) = \tau_{k-1}(y) + \tau(f^{\tau_{k-1}(y)}(y))$ is the k -th return time to Y . Compute for $y \in Y$

$$\begin{aligned}
\sum_{j=\tau_k(y)}^{\tau_{k+1}(y)-1} |v_\Delta \circ T_\Delta^j(y, 0)| &= \sum_{j=0}^{\tau_{k+1}(y)-\tau_k(y)-1} |v_\Delta \circ T_\Delta^j(T^{\tau_k(y)}(y, 0))| \\
&= \sum_{j=0}^{\tau_{k+1}(y)-\tau_k(y)-1} |v_X \circ f^j(f^{\tau_k(y)}(y))| \\
&\leq \sum_{j=0}^{\tau_{k+1}(y)-\tau_k(y)-1} \|v_X\|_\infty^* (\tau_{k+1}(y) - \tau_k(y) - j)^{-(1+\varepsilon)} \leq C_\varepsilon \|v_X\|_\infty^*, \tag{6.8}
\end{aligned}$$

where $C_\varepsilon = \sum_{j \geq 1} j^{-(1+\varepsilon)}$. Thus,

$$\begin{aligned}
\left| \int_\Delta C_n v_\Delta w_\Delta d\mu_\Delta \right| &= \left| \int_\Delta L_\Delta^n(1_{\{\varphi_\Delta > n\}} \setminus Y v_\Delta) w_\Delta d\mu_\Delta \right| \\
&\leq \int_{\{\varphi_\Delta > n\} \setminus Y} |v_\Delta| |w_\Delta \circ T_\Delta^n| d\mu_\Delta \leq \|w_\Delta\|_{L^\infty(\mu_\Delta)} \int_{\{\varphi > n\}} \sum_{j=1}^{\varphi(y)-n-1} |v_\Delta \circ T_\Delta^j| d\mu_0 \\
&\leq \|w_X\|_{L^\infty(\mu_X)} \int_{\{\varphi > n\}} \sum_{k=0}^{\rho(y)-1} \sum_{j=\tau_k(y)}^{\tau_{k+1}(y)-1} |v_\Delta \circ T_\Delta^j| d\mu_0 \\
&\leq \|w_X\|_{L^\infty(\mu_X)} C_\varepsilon \|v_X\|_\infty^* \int_{\{\varphi > n\}} \rho(y) d\mu_0 \leq CC_\varepsilon \mu_0(\varphi > n) \|v_X\|_\infty^* \|w_X\|_{L^\infty(\mu_X)},
\end{aligned}$$

where the last inequality is obtained using (H1). \square

Remark 6.8. Continuing from Remark 6.6 we note the following. If v_X is supported on Y , then the sum $\sum_{j=\tau_k(y)}^{\tau_{k+1}(y)-1} |v_\Delta \circ T_\Delta^j(y, 0)|$ reduces to single term, namely $|v_\Delta \circ T_\Delta^{\tau_k(y)}(y, 0)|$. In this case the constant C_ε appearing in (6.8) disappears, but condition (H1) is still required.

Proof of Lemma 6.4. Using (6.8) for w_Δ instead of v_Δ we find

$$\begin{aligned}
\left| \int_\Delta \sum_{j=n}^\infty A_j v_Y w_\Delta d\mu_\Delta \right| &= \left| \int_Y \sum_{j=n}^\infty 1_{\{\varphi > j\}} v_Y w_\Delta \circ T_\Delta^j d\mu_0 \right| \\
&\leq \int_{\{\varphi > n\}} |v_Y| \sum_{j=n}^{\varphi(y)-1} 1_{\{\varphi > j\}} |w_\Delta \circ T_\Delta^j| d\mu_0 \leq \|v_Y\|_{L^\infty(\mu_0)} \int_{\{\varphi > n\}} \sum_{k=0}^{\rho(y)-1} \sum_{j=\tau_k(y)}^{\tau_{k+1}(y)-1} |w_\Delta \circ T_\Delta^j| d\mu_0 \\
&\leq \|v_Y\|_{L^\infty(\mu_0)} C_\varepsilon \|w_X\|_\infty^* \int_{\{\varphi > n\}} \rho(y) d\mu_0 \leq CC_\varepsilon \mu_0(\varphi > n) \|v_Y\|_{L^\infty(\mu_0)} \|w_X\|_\infty^*
\end{aligned}$$

by (H1). This completes the proof. \square

Proof of Lemma 6.5. For the purpose of the argument below, we define weighted norms on the tower analogous to (4.1). Let $\tau_\Delta^*(x) = 1 + \min\{i \geq 0 : T_\Delta^i(x) \in \hat{Y}\}$ for $\hat{Y} = \pi^{-1}(Y)$, and define

$$\begin{cases} \|v_\Delta\|_{\Delta,\infty}^* = \sup_{x \in \Delta} |v_\Delta(x)| \tau_\Delta^*(x)^{1+\epsilon}, \\ \|v_\Delta\|_{\Delta,\theta}^* = \sup_{a \in \alpha} \sup_{0 \leq i < \varphi(a)} \sup_{x, x' \in a} \frac{\tau_\Delta^*(x, i)}{\theta^{s(x, x')}} |v_\Delta(x, i) - v_\Delta(x', i)|, \end{cases}$$

and $\|v_\Delta\|_{\Delta,\theta}^* = \|v_\Delta\|_{\Delta,\infty}^* + |v_\Delta|_{\Delta,\theta}^*$. In this way, $\|v_X\|_\theta^* = \|v_\Delta\|_{\Delta,\theta}^*$ whenever $v_\Delta = v_X \circ \pi$.

For $a \in \alpha$, $0 \leq j < \varphi(a)$, define $B_{j,a}v_\Delta := L_\Delta^j(1_{\{(y,i): y \in a, i+j=\varphi(y)\}}v_\Delta)$. The definition of p_Δ implies that for points on level i of the tower, the potential p_Δ satisfies $\sum_{j=0}^{\varphi(y)-i-1} p_\Delta \circ T_\Delta^j(y, i) = p(y)$. Writing $y = F^{-1}(x) \cap a$ and $y' = F^{-1}(x') \cap a$, we compute using (2.1)

$$\begin{aligned} \|B_{j,a}v_\Delta\|_\theta &= \sup_{x \in Y} e^{p(y)} |v_\Delta(y, \varphi(y) - j)| \\ &\quad + \sup_{x, x' \in Y} \theta^{-s(x, x')} \left| e^{p(y)} v_\Delta(y, \varphi(y) - j) - e^{p(y')} v_\Delta(y', \varphi(y') - j) \right| \\ &\leq \mu_0(a) (\tau^* \circ f^{\varphi(a)-j}(a))^{-(1+\epsilon)} \|v_\Delta\|_{\Delta,\infty}^* \\ &\quad + \sup_{x, x' \in Y} \theta^{-s(x, x')} \left(|e^{p(y)} - e^{p(y')}| (\tau^* \circ f^{\varphi(a)-j}(a))^{-(1+\epsilon)} \|v_\Delta\|_{\Delta,\infty}^* \right. \\ &\quad \left. + \mu_0(a) (\tau^* \circ f^{\varphi(a)-j}(a))^{-(1+\epsilon)} |v_\Delta|_\theta^* \right) \\ &\leq C_1 \mu_0(a) (\tau^* \circ f^{\varphi(a)-j}(a))^{-(1+\epsilon)} \|v_\Delta\|_{\Delta,\theta}^*. \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{j \geq n} \|B_j v_\Delta\|_\theta &\leq \sum_{\substack{a \in \alpha \\ \varphi(a) > n}} \sum_{j=1}^{\varphi(a)-1} \|B_{j,a} v_\Delta\|_\theta \leq \sum_{\substack{a \in \alpha \\ \varphi(a) > n}} \sum_{k=0}^{\rho(a)-1} \sum_{j=\tau_k(a)+1}^{\tau_{k+1}(a)} \|B_{j,a} v_\Delta\|_\theta \\ &\leq \sum_{\substack{a \in \alpha \\ \varphi(a) > n}} C_1 \mu_0(a) \sum_{k=0}^{\rho(a)-1} \sum_{j=\tau_k(a)+1}^{\tau_{k+1}(a)} (1 + \tau_{k+1}(a) - j)^{-(1+\epsilon)} \|v_\Delta\|_{\Delta,\theta}^* \\ &\leq C_1 C_\epsilon \|v_\Delta\|_{\Delta,\theta}^* \sum_{\substack{a \in \alpha \\ \varphi(a) > n}} \mu_0(a) \rho(a) = C_1 C_\epsilon \|v_\Delta\|_{\Delta,\theta}^* \int_{\{\varphi > n\}} \rho(y) d\mu_0 \\ &\leq C_1 C_\epsilon C \|v_\Delta\|_{\Delta,\theta}^* \mu_0(\varphi > n), \end{aligned}$$

where the last inequality was obtained using (H1). \square

Proof of Lemma 6.7. By direct computation:

$$\begin{aligned} \int_\Delta A(1) 1_Y w_\Delta d\mu_\Delta &= \int_\Delta \sum_{n \geq 0} L_\Delta^n(1_{\{\varphi > n\}}) w_\Delta d\mu_\Delta = \int_\Delta \sum_{n \geq 0} 1_{\{\varphi > n\}} w_\Delta \circ T_\Delta^n d\mu_\Delta \\ &= \int_Y \sum_{n \geq 0}^{\varphi-1} w_\Delta \circ T_\Delta^n d\mu_0 = \int_\Delta w_\Delta d\mu_\Delta. \end{aligned}$$

The statement on A follows. For the statement on B , let $v_k = 1_{\{\varphi_\Delta = k\}} v_\Delta$ for $k \geq 0$ and set $v_{k,j} = 1_{Y_j} v_k$ where Y_j is the j -th level of the tower. Then $v_{k,j}$ have disjoint supports, and for

each element $a \in \alpha$, there is only one j such that $v_{k,j}$ is supported on $f_\Delta^j(a) \subset Y_j$, namely, $k + j = \varphi(a)$. Let $u_k(y, 0) = v_k(y, j)$ and compute that

$$L_\Delta^k v_k = \sum_{j=1}^{\infty} L_\Delta^k v_{k,j} = \sum_{j=1}^{\infty} L_\Delta^k L_\Delta^j u_k = L_\Delta^\varphi u_k = R^* u_k, \quad (6.9)$$

Recall that $B(1)v_\Delta$ is supported on Y . Hence,

$$\int_Y B(1)v_\Delta d\mu_0 = \int_Y \sum_{k=0}^{\infty} L_\Delta^k v_k d\mu_0 = \int_Y \sum_{k=0}^{\infty} R^* u_k d\mu_0 = \int_Y \sum_{k=1}^{\infty} u_k d\mu_0 = \int_\Delta v_\Delta d\mu_\Delta.$$

The conclusion follows since $\int_\Delta v_\Delta d\mu_\Delta = \int_X v_X d\mu_X$. \square

7 Proofs of Lemmas used in the proof of Theorem 4.2 (finite measure case)

7.1 Estimating the coefficients of $V(z)(v_\Delta, w_\Delta)$ defined in (6.5)

Lemma 7.1. *Assume the setting of Lemma 6.1. Then the coefficients $V_n(v_\Delta, w_\Delta)$ of the function $V(z)(v_\Delta, w_\Delta)$, $z \in \mathbb{D}$ are given by*

$$V_n(v_\Delta, w_\Delta) = \frac{1}{\varphi^2} \sum_{k=n+1}^{\infty} \mu_0(\varphi > k) \int_X v_X d\mu_X \int_X w_X d\mu_X.$$

Proof. By Lemma 6.7, $\int_Y B(1)v_\Delta d\mu_0 = \int_X v_X d\mu_X$. Recalling $R_n^* v = 1_Y L_\Delta^n (1_{\{\varphi=n\}} v)$, we compute that

$$\left(\sum_{n=0}^{\infty} \left(\frac{1}{\varphi^2} \sum_{k=n+1}^{\infty} \sum_{j=k+1}^{\infty} P R_j^* P \right) z^n \right) B(1)v_\Delta = \frac{1}{\varphi^2} \int_X v_X d\mu_X \sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} \mu_0(\varphi > k) z^n.$$

Thus, using Lemma 6.7 (first, for the statement on B and at the end of the argument for A),

$$\begin{aligned} V(z)(v_\Delta, w_\Delta) &= \int_\Delta A(1) \left(\sum_{n=0}^{\infty} \left(\frac{1}{\varphi^2} \sum_{k=n+1}^{\infty} \sum_{j=k+1}^{\infty} P R_j^* P \right) z^n \right) B(1)v_\Delta w_\Delta d\mu_\Delta \\ &= \frac{1}{\varphi^2} \int_X v_X d\mu_X \sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} \mu_0(\varphi > k) z^n \int_\Delta A(1) 1_Y w_\Delta d\mu_\Delta \\ &= \frac{1}{\varphi^2} \int_X v_X d\mu_X \int_X w_X d\mu_X \sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} \mu_0(\varphi > k) z^n. \end{aligned}$$

The conclusion follows. \square

7.2 Estimating the coefficients of $I^A(z)(v_\Delta, w_\Delta)$ and $I^B(z)(v_\Delta, w_\Delta)$ defined in (6.6)

We begin with some immediate consequences of Lemmas 6.4 and 6.5.

Lemma 7.2. *Assume (H0) a) and (H1). Let $w_X : X \rightarrow \mathbb{R}$ such that $\|w_X\|_\infty^* < \infty$. Then*

$$\int_{\Delta} \frac{A(z) - A(1)}{1 - z} 1_Y w_{\Delta} d\mu_{\Delta} = \sum_{n \geq 0} a_n z^n,$$

where $|a_n| = O(\mu_0(\varphi > n) \|w_X\|_\infty^*)$.

Proof. Compute that

$$-\int_{\Delta} \frac{A(z) - A(1)}{1 - z} 1_Y w_{\Delta} d\mu_{\Delta} = \int_{\Delta} \sum_{n \geq 0} \sum_{j \geq n} A_j 1_Y w_{\Delta} z^n d\mu_{\Delta}.$$

The conclusion follows from the above equation together with Lemma 6.4. \square

Lemma 7.3. *Assume (H0) a) and (H1). Let $v_X : X \rightarrow \mathbb{R}$ such that $\|v_X\|_{\theta}^* < \infty$. Then*

$$\int_{\Delta} \frac{B(z) - B(1)}{1 - z} v_{\Delta} d\mu_{\Delta} = \sum_{n \geq 0} a_n z^n,$$

where $|a_n| = O(\mu(\varphi > n) \|v_X\|_{\theta}^*)$.

Proof. The conclusion follows by the argument used in the proof of Lemma 7.2, using Lemma 6.5 instead of Lemma 6.4. \square

The coefficients of $I^A(z)$ will be obtained by decomposing this term into $I^A(z) = D_A(z) + F_A(z)$, where

$$D_A(z)(v_{\Delta}, w_{\Delta}) := \int_{\Delta} \frac{A(z) - A(1)}{1 - z} \frac{P}{\bar{\varphi}} B(1) v_{\Delta} w_{\Delta} d\mu_{\Delta}$$

and

$$F_A(z)(v_{\Delta}, w_{\Delta}) := \int_{\Delta} \frac{A(z) - A(1)}{1 - z} Q(z) B(1) v_{\Delta} w_{\Delta} d\mu_{\Delta},$$

with

$$Q(z)v := (1 - z) \sum_{n=0}^{\infty} \left(\frac{1}{\bar{\varphi}^2} \sum_{k=n+1}^{\infty} \sum_{j=k+1}^{\infty} P R_j^* P \right) z^n v. \quad (7.1)$$

Lemma 7.4. *Assume the setting of Lemma 6.1. Assume that (H1) holds. Suppose that $v_X, w_X : X \rightarrow \mathbb{R}$ are such that $\|v_X\|_{L^\infty(\mu_X)} < \infty$ and $\|w_X\|_\infty^* < \infty$. Then the coefficients $D_{A,n}(v_{\Delta}, w_{\Delta})$, $F_{A,n}(v_{\Delta}, w_{\Delta})$ of the functions $D_A(z)(v_{\Delta}, w_{\Delta})$ and $F_A(z)(v_{\Delta}, w_{\Delta})$, $z \in \mathbb{D}$ satisfy*

$$\begin{cases} |D_{A,n}(v_{\Delta}, w_{\Delta})| = O(\mu_0(\varphi > n) \|v_X\|_{L^\infty(\mu_X)} \|w_X\|_\infty^*), \\ |F_{A,n}(v_{\Delta}, w_{\Delta})| = O(\mu_0(\varphi > n) \|v_X\|_{L^\infty(\mu_X)} \|w_X\|_\infty^*). \end{cases}$$

Proof. By Lemma 6.7 (the statement on B),

$$D_A(z)(v_{\Delta}, w_{\Delta}) = \frac{1}{\bar{\varphi}} \int_{\Delta} v_{\Delta} d\mu_{\Delta} \int_{\Delta} \frac{A(z) - A(1)}{1 - z} 1_Y w_{\Delta} d\mu_{\Delta}.$$

By Lemma 7.2, $\int_{\Delta} \frac{A(z)-A(1)}{1-z} 1_Y w_{\Delta} d\mu_{\Delta} = \sum_{n \geq 0} a_n z^n$ with $|a_n| = O(\mu_0(\varphi > n) \|w_X\|_{\infty}^*)$. The statement on $|D_{A,n}(v_{\Delta}, w_{\Delta})|$ follows.

Next, by definition,

$$\begin{aligned} Q(z)B(1)v_{\Delta} &= \frac{1}{\bar{\varphi}^2} \int_{\Delta} B(1)v_{\Delta} d\mu_{\Delta} \times (1-z) \sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} \mu_0(\varphi > k) z^n \\ &= \frac{1}{\bar{\varphi}^2} \int_{\Delta} v_{\Delta} d\mu_{\Delta} \times \sum_{n=0}^{\infty} \mu_0(\varphi > n) z^n. \end{aligned}$$

Thus,

$$F_A(z)(v_{\Delta}, w_{\Delta}) = \frac{1}{\bar{\varphi}^2} \int_{\Delta} v_{\Delta} d\mu_{\Delta} \times \sum_{n=0}^{\infty} \mu_0(\varphi > n) z^n \times \int_{\Delta} \frac{A(z)-A(1)}{1-z} 1_Y w_{\Delta} d\mu_{\Delta}.$$

We already know that the coefficients of $\int_{\Delta} \frac{A(z)-A(1)}{1-z} 1_Y w_{\Delta} d\mu_{\Delta}$ are $O(\mu_0(\varphi > n) \|w_X\|_{\infty}^*)$. The statement on $|F_{A,n}(v_{\Delta}, w_{\Delta})|$ follows. \square

The next result provides estimates for the coefficients of $I^B(z)(v_{\Delta}, w_{\Delta})$, $z \in \mathbb{D}$ defined in (6.6). Write $I^B(z) = D_B(z) + F_B(z)$, where given that $Q(z)$ is as defined in (7.1),

$$D_B(z) := \int_{\Delta} A(1) \frac{P}{\bar{\varphi}} \frac{B(z) - B(1)}{1-z} v_{\Delta} w_{\Delta} d\mu_{\Delta}$$

and

$$F_B(z) := \int_{\Delta} A(1) Q(z) \frac{B(z) - B(1)}{1-z} v_{\Delta} w_{\Delta} d\mu_{\Delta}.$$

Lemma 7.5. *Assume the setting of Lemma 7.4. Then the coefficients $D_{B,n}(v_{\Delta}, w_{\Delta})$, $F_{B,n}(v_{\Delta}, w_{\Delta})$ of the functions $D_B(z)(v_{\Delta}, w_{\Delta})$, $F_B(z)(v_{\Delta}, w_{\Delta})$, $z \in \mathbb{D}$ satisfy*

$$\begin{cases} |D_{B,n}(v_{\Delta}, w_{\Delta})| = O(\mu_0(\varphi > n) \|v_X\|_{\theta}^* \|w_X\|_{\infty}^*), \\ |F_{B,n}(v_{\Delta}, w_{\Delta})| = O(\mu_0(\varphi > n) \|v_X\|_{\theta}^* \|w_X\|_{\infty}^*). \end{cases}$$

Proof. The required argument is similar to the one used in the proof of Lemma 7.4 with Lemma 7.3 replacing Lemma 7.2. \square

8 Proofs of Lemmas used in the proof of Theorem 4.3 (infinite measure case)

Recall that for $z \in \mathbb{D}$, the functions $I_{\text{inf}}^A(z)(v_{\Delta}, w_{\Delta}) := \int_{\Delta} (A(z) - A(1)) T^*(z) B(1) v_{\Delta} w_{\Delta} d\mu_{\Delta}$ and $I_{\text{inf}}^B(z)(v_{\Delta}, w_{\Delta}) := \int_{\Delta} A(1) T^*(z) (B(z) - B(1)) v_{\Delta} w_{\Delta} d\mu_{\Delta}$ were defined in equation (6.7). Let $I_{\text{inf},n}^A(v_{\Delta}, w_{\Delta})$ and $I_{\text{inf},n}^B(v_{\Delta}, w_{\Delta})$ denote their n -th coefficients. The next result provides estimates for $I_{\text{inf},n}^A(v_{\Delta}, w_{\Delta})$ and $I_{\text{inf},n}^B(v_{\Delta}, w_{\Delta})$ and it was used in the proof of Theorem 4.3.

Lemma 8.1. *Assume (H0) b) and (H1). Let $v_X, w_X : X \rightarrow \mathbb{R}$ such that $\|v_X\|_\theta^*, \|w_X\|_\infty^* < \infty$. Then*

$$|I_{\text{inf},n}^A(v_\Delta, w_\Delta)| = O(n^{-\beta} \|v_X\|_\theta^* \|w_X\|_\infty^*), \quad |I_{\text{inf},n}^B(v_\Delta, w_\Delta)| = O(n^{-\beta} \|v_X\|_\theta^* \cdot \|w_X\|_\infty).$$

The proof of the above result relies on standard continuity properties of the functions $I_{\text{inf}}^A(z)$ and $I_{\text{inf}}^B(z)$, $z \in \mathbb{D}$ which we recall below.

8.1 Continuity properties of $I_{\text{inf}}^A(z)$ and $I_{\text{inf}}^B(z)$, $z \in \mathbb{D}$

First we note some standard consequences of Lemmas 6.4, 6.5 and 8.2 which give the continuity properties of some quantities involving $A(z), B(z)$, $z \in \mathbb{D}$.

Lemma 8.2. *Let $a(z)$ be a function acting on some function space \mathcal{B} with norm $\|\cdot\|$, well defined on \mathbb{D} . Suppose that its coefficients satisfy $\sum_{j>n} \|a_j\| \leq C_1 n^{-\beta}$ for $\beta > 0$ and $C_1 > 0$. Then there exists $C_2 > 0$ such that for all $h > 0$, all $u > 0$ and all $\theta \in (-\pi, \pi]$, $\|a(e^{-(u-i(\theta+h))}) - a(e^{-(u-i\theta)})\| \leq C_2 h^\beta$.*

Proof. This proof is standard. We provide it here only for completeness. Compute that

$$\|a(e^{-(u-i(\theta+h))}) - a(e^{-(u-i\theta)})\| \leq h \sum_{j \leq h^{-1}} j \|a_j\| + \sum_{j > h^{-1}} \|a_j\|. \quad (8.1)$$

By assumption, the second term is bounded by $C_1 h^\beta$. Next, let $s_n := \sum_{j>n} \|a_j\|$ and note that

$$\begin{aligned} \sum_{j \leq h^{-1}} j \|a_j\| &= \sum_{j \leq h^{-1}} j (s_{j-1} - s_j) = \sum_{j \leq h^{-1}} (j-1) s_{j-1} - \sum_{j \leq h^{-1}} j s_j + \sum_{j \leq h^{-1}} s_{j-1} \\ &\leq C_1 (h^{-1} - 1) h^\beta + C_1 h^{\beta-1} \leq 2C_1 h^{\beta-1}. \end{aligned}$$

Hence, the first term of (8.1) is bounded by $2C_1 h^\beta$, as required. \square

Lemma 8.3. *Assume (H0) (either a) or b) and (H1). Let $v_Y : Y \rightarrow \mathbb{R}$, $w_X : X \rightarrow \mathbb{R}$ such that $\|v_Y\|_{L^\infty(\mu_0)}, \|w_X\|_\infty^* < \infty$. Then there exist $C_1, C_2 > 0$ such that for all $h > 0$, all $u > 0$ and all $\theta \in (-\pi, \pi]$,*

$$\begin{aligned} \left| \int_{\Delta} (A(e^{-(u-i(\theta+h))}) - A(e^{-(u-i\theta)})) v_Y w_\Delta d\mu_\Delta \right| &\leq C_1 h^\beta \|v_Y\|_{L^\infty(\mu_0)} \|w_X\|_\infty^*, \\ \left| \int_{\Delta} (A(e^{-(u-i\theta)}) - A(1)) v_Y w_\Delta d\mu_\Delta \right| &\leq C_2 |\theta|^\beta \|v_Y\|_{L^\infty(\mu_0)} \|w_X\|_\infty^*. \end{aligned}$$

Proof. By Lemma 6.4, there exists $C > 0$ such that $\sum_{j=n}^\infty |\int_Y A_j v_Y w_\Delta d\mu_\Delta| \leq C \mu_0(\varphi > n) \|v_Y\|_{L^\infty(\mu_0)} \|w_X\|_\infty^*$, for all $n \geq 0$. The conclusion follows by Lemma 8.2. \square

Lemma 8.4. *Assume (H0) (either a) or b) and (H1). Let $w_X : X \rightarrow \mathbb{R}$ such that $\|v_X\|_\theta^* < \infty$. Then there exist $C_1, C_2 > 0$ such that for all $h > 0$, all $u \geq 0$ and all $\theta \in (-\pi, \pi]$,*

$$\begin{aligned} \|(B(e^{-(u-i(\theta+h))}) - B(e^{-(u-i\theta)})) v_\Delta\|_\theta &\leq C_1 h^\beta \|v_X\|_\theta^*, \\ \|(B(e^{-(u-i\theta)}) - B(1)) v_\Delta\| &\leq C_2 |\theta|^\beta \|v_X\|_\theta^*. \end{aligned}$$

Proof. By Lemma 6.5, there exists $C > 0$ such that $\sum_{j \geq n} \|B_j v_\Delta\| \leq C \|v_X\|_\theta^*$, for all $n \geq 0$. The conclusion follows from Lemma 8.2. \square

The following result was obtained in [12, Lemma 4.1] (see also [13, Lemma 2.4] and its proof for a different argument).

Lemma 8.5. *Assume that F is Gibbs Markov and (H0) b) holds. Then for all $u \geq 0$ and $\theta \in (-\pi, \pi]$, there exist $C_1, C_2 > 0$ such that $\|T^*(e^{-(u-i\theta)})\|_\theta \leq C_1 |u - i\theta|^{-\beta}$. Moreover, for all $h > 0$, all $u \geq 0$ and all $\theta \in (-\pi, \pi]$, $\|T^*(e^{-(u-i(\theta+h))}) - T^*(e^{-(u-i\theta)})\|_\theta \leq C_2 h^\beta |u - i\theta|^{-2\beta}$.*

Combining Lemmas 8.3, 8.4 and 8.5, we obtain

Corollary 8.6. *There exist positive constants C_1, C_2 such that for all $u \geq 0$ and all $\theta \in (-\pi, \pi]$, $|I_{\inf}^A(e^{-(u-i\theta)})(v_\Delta, w_\Delta)| \leq C_1 \|v_X\|_\theta^* \|w_X\|_{L^\infty(\mu_X)}$ and similarly, $|I_{\inf}^B(e^{-(u-i\theta)})(v_\Delta, w_\Delta)| \leq C_2 \|v_X\|_\theta^* \|w_X\|_{L^\infty(\mu_X)}$. Moreover, there exist positive constants C_3, C_4 such that for all $h > 0$, all $u \geq 0$ and $\theta \in (-\pi, \pi]$,*

$$\begin{aligned} |(I_{\inf}^A(e^{-(u-i(\theta+h))}) - I_{\inf}^A(e^{-(u-i\theta)}))(v_\Delta, w_\Delta)| &\leq C_3 h^\beta |u - i\theta|^{-\beta} \|v_X\|_\theta^* \|w_X\|_{L^\infty(\mu_X)} \\ |(I_{\inf}^B(e^{-(u-i(\theta+h))}) - I_{\inf}^B(e^{-(u-i\theta)}))(v_\Delta, w_\Delta)| &\leq C_4 h^\beta |u - i\theta|^{-\beta} \|v_X\|_\theta^* \|w_X\|_{L^\infty(\mu_X)} \end{aligned}$$

8.2 Proof of Lemma 8.1

The first result below will be instrumental in the proof of Lemma 8.1.

Lemma 8.7. *Let $b(z)$ be a function well defined on \mathbb{D} . Assume that there exist $C_1, C_2 > 0$ such that for any $h > 0$ and for all $\theta \in (-\pi, \pi]$, $|b(e^{-(u-i\theta)})| \leq C_1$ and $|b(e^{-(u-i(\theta+h))}) - b(e^{-(u-i\theta)})| \leq C_2 h^\beta |u - i\theta|^{-\beta}$, for $\beta \in (0, 1)$. Then the n -th coefficient b_n of $b(z)$, $z \in \mathbb{D}$ is $O(n^{-\beta})$.*

Proof. We give the standard short proof only for completeness. We estimate the coefficients of $b(z)$, $z \in \mathbb{D}$, on the circle $\Gamma = \{e^{-u} e^{i\theta} : -\pi \leq \theta < \pi\}$ with $e^{-u} = e^{-1/n}$, where $n \geq 1$. Write

$$b_n = \frac{1}{2\pi i} \int_\Gamma \frac{b(z)}{z^{n+1}} dz = \frac{e}{2\pi} \int_{-\pi}^{\pi} b(e^{-1/n} e^{i\theta}) e^{-in\theta} d\theta.$$

Note that

$$|b_n| \ll \left| \int_0^{1/n} b(e^{-1/n} e^{i\theta}) e^{-in\theta} d\theta \right| + \left| \int_{1/n}^{\pi} b(e^{-1/n} e^{i\theta}) e^{-in\theta} d\theta \right|.$$

Since $|b(e^{-1/n} e^{i\theta})| \leq C_1$ we have $|\int_0^{1/n} b(e^{-1/n} e^{i\theta}) e^{-in\theta} d\theta| \leq C_1 n^{-1}$. To estimate the second term, let $I := \int_{1/n}^{\pi} b(e^{-1/n} e^{i\theta}) e^{-in\theta} d\theta$ and note that

$$I = \int_{1/n}^{\pi} b(e^{-1/n} e^{i\theta}) e^{-in\theta} d\theta = - \int_{(1+\pi)/n}^{\pi+\pi/n} b(e^{-1/n} e^{i\theta}) e^{-in\theta} d\theta.$$

Thus,

$$2I = \int_{1/n}^{\pi} b(e^{-1/n} e^{i\theta}) e^{-in\theta} d\theta - \int_{(1+\pi)/n}^{\pi+\pi/n} b(e^{-1/n} e^{i(\theta-\pi/n)}) e^{-in\theta} d\theta = I_1 + I_2 - I_3,$$

where

$$I_1 = \int_{\pi}^{\pi+\pi/n} b(e^{-1/n} e^{i\theta}) e^{-in\theta} d\theta, \quad I_2 = \int_{1/n}^{(1+\pi)/n} b(e^{-1/n} e^{i\theta}) e^{-in\theta} d\theta,$$

$$I_3 = \int_{(1+\pi)/n}^{\pi} (b(e^{-1/n} e^{i(\theta-\pi/n)}) - b(e^{-1/n} e^{i\theta})) e^{-in\theta} d\theta.$$

Clearly, $|I_1| \ll n^{-1}$ and $|I_2| \ll n^{-1}$. By assumption, $|b(e^{-1/n} e^{i(\theta-\pi/n)}) - b(e^{-1/n} e^{i\theta})| \leq C_2 n^{-\beta} |1/n - i\theta|^{-\beta}$. Thus, $|I_3| \ll n^{-\beta}$ and the conclusion follows. \square

We can now complete

Proof of Lemma 8.1. By Corollary 8.6, $I_{\text{inf}}^A(e^{-(u-i\theta)})(v_{\Delta}, w_{\Delta})$ and $I_{\text{inf}}^B(e^{-(u-i\theta)})(v_{\Delta}, w_{\Delta})$ satisfy (in ||) the assumptions of Lemma 8.7 (where the involved constants include the product $\|v_X\|_{\theta}^* \|w_X\|_{\infty}$). The conclusion follows by applying Lemma 8.7 to $I_{\text{inf}}^A(e^{-(u-i\theta)})(v_{\Delta}, w_{\Delta})$ and $I_{\text{inf}}^B(e^{-(u-i\theta)})(v_{\Delta}, w_{\Delta})$. \square

9 Non-Markov interval maps with indifferent fixed points

The works [20, 21] studied a class of non-Markov interval maps $f : [0, 1] \rightarrow [0, 1]$, with indifferent fixed points, called AFN maps, which stands for **F**inite image, **N**on-uniformly expanding maps satisfying **A**dler's distortion condition: $f''/(f')^2$ is bounded.

9.1 Known results for f via first return inducing

For infinite measure preserving topologically mixing AFN maps $(f, [0, 1], \mu)$, with $\mu_{\tau}(\tau > n) = n^{-\beta} \ell(n)$ for $\beta \in (1/2, 1)$ and ℓ a slowly varying function, and transfer operator L , [12, Theorem 1.1] shows that $\lim_{n \rightarrow \infty} \ell(n) n^{1-\beta} L^n v = \frac{\sin \pi \beta}{\pi} \int v d\mu$, uniformly on compact subsets of $[0, 1] \setminus I_p$, where I_p is the set of indifferent fixed points, for all $v = u/h$, u is a Riemann integrable on $[0, 1]$ and $h(x) = \frac{d\mu(x)}{dx}$. In particular, [12, Theorem 1.1] holds in the setting of (9.1) below, for $v(x) = x^q$ with $q\beta \geq 1$. For the LSV family of maps studied in [11], which induce with first return to a Gibbs Markov map, the work [12] also obtains higher order asymptotics of $L^n v$, for some suitable v supported on $(0, 1]$ (we recall that such a map has a single indifferent fixed point at 0); this result of [12] implies higher order asymptotics for the correlation function $\rho_n(v, w) = \int v w \circ f^n d\mu$ associated with the LSV family of maps [11] (for the suitable v and $w \in L^{\infty}$). These results on higher order asymptotics have been improved in [16] and again, they apply to LSV maps [11]. Higher order asymptotics of $\rho_n(v, w)$ in the setting of AFN maps without Markov partition has *not* been addressed. The only obstacle in [12, 16] was that the invariant density of the induced map is BV and thus, the arguments used in [12, 16] to obtain higher order expansion of $\mu_{\tau}(\tau > n)$ (which require smoothness of the induced invariant density) do not apply⁶.

⁶Higher order expansion of $\mu(\tau > n)$ is required for results which aim to address any type of error term in the infinite measure set-up: see [12, 16, 17].

In what follows, in the process of verifying (H0) and (H1) for AFN maps, we obtain excellent estimates for $\mu_\tau(\tau > n)$. This allows one to infer that the results in [12, 16] on higher order asymptotics of L^n hold in the setting of (9.1), a typical examples in the class of AFN maps [20, 21]; we recall that Theorems 4.2 and 4.3 only address the asymptotics of the correlation function $\rho_n(v, w)$ for appropriate v, w (so a weaker result than higher order asymptotics of L^n). For details we refer to Section 9.3.

In the setting of finite measure preserving non-Markov, non-uniformly expanding interval maps $f : [0, 1] \rightarrow [0, 1]$ with a single indifferent fixed point at 0, the works [14, 10] consider a first return induced map to $Y = [z, 1]$, $z > 0$, to obtain upper/sharp mixing rates. The relevant Banach space in which renewal type arguments are developed or verified is BV . The sharp results in [10] are for observables supported on Y .

9.2 Verifying conditions (H0)-(H2)

One can verify the abstract conditions in Section 2, and hence prove Theorems 4.2 and 4.3 for the general class of AFN maps studied [20, 21]. For simplicity, we restrict here to the following example:

$$f(x) = f_{\alpha,b}(x) = x(1 + bx^\alpha) \bmod 1, \quad \alpha > 0, b \in (0, 1]. \quad (9.1)$$

We induce on the interval $Y = [e_0, 1]$, where $e_0 \in (0, 1)$ is such that $f(e_0) = 0$. The fact that the orbit of e_0 is disjoint from the interior of Y implies that $e_0 \notin f^i(a)$ for every $a \in \alpha$, $0 \leq i < \varphi(a)$, and therefore condition (H2) follows immediately.

Adler's condition fails at $x = 0$ if $\alpha \in (0, 1)$ in (9.1), but the first return map f^τ to Y is uniformly expanding and Adler's condition does hold for it. This gives a uniform bound on the distortion of $g := f^\tau$. Indeed, if $g : J \rightarrow g(J)$ is a branch of f^τ with $|g''(s)/(g'(s))^2| \leq C$, then for all $x, y \in J$,

$$\begin{aligned} \left| \log \frac{g'(y)}{g'(x)} \right| &= \left| \int_x^y \frac{d}{ds} \log g'(s) ds \right| \\ &= \int_x^y \left| \frac{g''(s)}{g'(s)} \right| ds \leq C \int_x^y |g'(s)| ds = C|g(y) - g(x)|. \end{aligned} \quad (9.2)$$

The same bound applies to iterates of g . As a consequence, the proportion of subintervals of the branch domains of g^k doesn't vary too much under the map g^k . This fact will be used throughout this section.

In general, f is not Markov, but preserves an absolutely continuous measure which is finite if and only if $\alpha \in (0, 1)$. Set $\beta = 1/\alpha$. Let $x_0 = e_0$ and for $n \geq 1$, define recursively $x_{n+1} < x_n$ so that $f(x_n) = x_{n-1}$. From [9] (in fact, sharper estimates can be found in [16, Section B]) one can establish the asymptotics

$$x_n = \frac{c^*}{(n+1)^\beta} + O\left(\frac{\log(n+1)}{(n+1)^{\beta+1}}\right) \quad \text{for some } c^* = c^*(\alpha) > 0. \quad (9.3)$$

For instance the condition $\|v_X\|_\infty^* < \infty$ can thus be written as $\sup_{x \in (0,1]} x^{-(1+\varepsilon)/\beta} |v_X(x)| < \infty$.

For each $k \geq 1$, let $e_k > e_{k-1}$ be the right-most point such that $f^{\tau_k(e_k)}(e_k) = e_0$. Then f^{τ_k} maps $[e_k, 1)$ monotonically but in general not surjectively into Y . The general return time is $\varphi(y) = \tau_k(y) + \tau \circ f^{\tau_k(y)}(y)$ for $y \in [e_k, e_{k+1})$. The map f^φ has only onto branches and thus, it is a Gibbs Markov induced map with good distortion properties derived from (9.2), and $\{y \in Y : \varphi(y) = \tau_{k+1}(y)\} = [e_k, e_{k+1})$.

Lemma 9.1. *The family of maps (9.1) satisfy condition (H0).*

Proof. For $n \geq 1$, let $A_k := \{y \in Y : \varphi(y) = \tau_{k+1}(y) > n\} \subset [e_k, e_{k+1})$. Let in the remainder of this section $\tau_k = \tau_k(1)$. We first estimate the derivatives $\lambda_k := Df^{\tau_k+1}(e_k)$ and lengths $|A_k|$.

For $j \geq 1$, let $y_j \in [e_0, e_1)$ be such that $f(y_j) = x_{j-1}$. Hence $\tau(y_j) = j$ and $f^j(y_j) = e_0$, so that $\{\tau > n\} = (e_0, y_n)$. Let σ_j be the integers such that $f^{\tau_j}(1) \in [y_{\sigma_{j+1}}, y_{\sigma_{j+1}-1})$ for $j \geq 0$, see Figure 1. Then also $f^{\tau_{k-1}}(A_k) \subset f^{\tau_{k-1}}([e_k, 1)) \subset [y_{\sigma_k}, y_{\sigma_k-1})$ for each $k \geq 0$. Using (9.2) (or

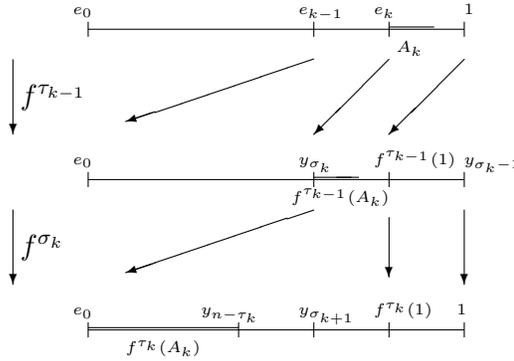


Figure 1: The points $e_{k-1}, e_k, 1$ and set A_k and their images.

the fact that all branches of f^τ are convex upwards), we find

$$\frac{|1 - e_{j+1}|}{|1 - e_j|} \leq \frac{|f^{\tau_j}(1) - y_{\sigma_{j+1}}|}{|f^{\tau_j}(1) - e_0|} \leq \frac{|y_{\sigma_{j+1}-1} - y_{\sigma_{j+1}}|}{|y_{\sigma_{j+1}} - e_0|} \frac{|f^{\tau_j}(1) - y_{\sigma_{j+1}}|}{|y_{\sigma_{j+1}-1} - y_{\sigma_{j+1}}|}.$$

Using (9.3), we can bound $\frac{|y_{\sigma_{j+1}-1} - y_{\sigma_{j+1}}|}{|y_{\sigma_{j+1}} - e_0|} \leq \min(\gamma/\sigma_{j+1}, 1/\lambda)$ for some uniform $\gamma > 0, \lambda > 1$.

We bound the second factor $\frac{|f^{\tau_j}(1) - y_{\sigma_{j+1}}|}{|y_{\sigma_{j+1}-1} - y_{\sigma_{j+1}}|} \leq 1$, except for $j = k - 1$. Taking the product over $j = 0, \dots, k - 1$ gives

$$\frac{|1 - e_k|}{|1 - e_0|} \leq \frac{|f^{\tau_{k-1}}(1) - y_{\sigma_k}|}{|y_{\sigma_{k-1}} - y_{\sigma_k}|} \prod_{j=1}^k \min\left\{\frac{\gamma}{\sigma_j}, \frac{1}{\lambda}\right\}. \quad (9.4)$$

Remark 9.2. *This shows that the reinduced scheme by itself has exponential tails.*

Boundedness of distortion of $f^{\sigma_k} : [y_{\sigma_k}, y_{\sigma_k-1}) \rightarrow [e_0, 1)$ together with (9.3) implies

$$\frac{|f^{\tau_{k-1}}(1) - y_{\sigma_k}|}{|y_{\sigma_{k-1}} - y_{\sigma_k}|} \ll \frac{|f^{\tau_k}(1) - e_0|}{|1 - e_0|} \leq \frac{c^*}{|1 - e_0|} \sigma_{k+1}^{-\beta} (1 + o(1)). \quad (9.5)$$

The boundedness of distortion of $f^{\tau_k+1} : [e_k, 1] \rightarrow [0, f^{\tau_k+1}(1)]$ combined with (9.4) and the first inequality of (9.5) leads to

$$\begin{aligned} \frac{1}{\lambda_k} &\ll \frac{1}{f'(e_0)} \frac{|1 - e_k|}{|f^{\tau_k}(1) - e_0|} \\ &\ll \frac{|f^{\tau_{k-1}}(1) - y_{\sigma_k}|}{|y_{\sigma_{k-1}} - y_{\sigma_k}|} \frac{|1 - e_0|}{|f^{\tau_k}(1) - e_0|} \prod_{j=1}^k \min\left\{\frac{\gamma}{\sigma_j}, \frac{1}{\lambda}\right\} \ll \prod_{j=1}^k \frac{1}{\max\{\sigma_j/\gamma, \lambda\}}. \end{aligned}$$

As $\tau_k = \sum_{j=1}^k \sigma_j$, the quotients τ_k/λ_k decrease exponentially and are thus summable in k .

Define $h_{\max} := \sup_{x \in [e_0, 1]} h(x)$ for the density $h = \frac{d\mu_0}{dx}$. Take n_0 so large that

$$\prod_{j=1}^{k_0} \frac{1}{\max\{\sigma_j/\gamma, \lambda\}} < \frac{1}{n^{\beta+1}} \quad \text{and} \quad c^* \sum_{k=k_0}^{\infty} \frac{h_{\max}}{\lambda_k} < \frac{1}{n^{\beta+1}} \quad (9.6)$$

for all $n \geq n_0$, where $k_0 = k_0(n) = \min\{k \geq 1 : 2\tau_{k+2} \geq n\}$.

Let g_k denote the inverse branch of $f^{\tau_k+1} : A_k \rightarrow [0, x_{n-\tau_k-1}]$. With the notation $\lambda_k = Df^{\tau_k+1}(e_k)$, $\lambda'_k = D^2 f^{\tau_k+1}(e_k)$ and $q_k = -\lambda'_k/\lambda_k^2$ (which is bounded in k because of Adler's condition), we find

$$g_k(x) - e_k = \frac{1}{\lambda_k} \left(x + \frac{q_k}{2} x^2 + O(x^3) \right).$$

Since the density h is C^2 smooth, $\mu_0(A_k) = h(e_k)(g_k(x_{n-\tau_k}) - e_k) + \frac{h'(e_k)}{2}(g_k(x_{n-\tau_k}) - e_k)^2 + O((g_k(x_{n-\tau_k}) - e_k)^3)$. Inserting the asymptotics for $g_k(x) - e_k$ and for $x_{n-\tau_k-1}$ from (9.3), gives

$$\begin{aligned} \mu_0(A_k) &= \frac{h(e_k)c^*}{\lambda_k} (n - \tau_k)^{-\beta} + \frac{(c^*)^2}{2\lambda_k} \left(h(e_k)q_k + \frac{h'(e_k)}{\lambda_k} \right) (n - \tau_k)^{-2\beta} \\ &\quad + O\left(\frac{h(e_k)}{\lambda_k} (n - \tau_k)^{-(\beta+1)} \log(n - \tau_k) \right). \end{aligned}$$

Applying this for $k < k_0$ and therefore $n > 2\tau_k$, we can use the asymptotics $(n - \tau_k)^{-\beta} \leq n^{-\beta}(1 + 2\beta\tau_k/n)$. Therefore

$$\mu_0(A_k) = n^{-\beta} \frac{h(e_k)c^*}{\lambda_k} + O\left(\frac{\tau_k}{\lambda_k} n^{-(\beta+1)} + n^{-(\beta+1)} \log(n) + n^{-2\beta} \right).$$

Set $c := c^* \sum_{k \geq 0} \frac{h(e_k)}{\lambda_k}$. Because $\sum_{k \geq 0} \frac{\tau_k}{\lambda_k} < \infty$ and $|1 - e_{k_0}| \leq |1 - e_0| \prod_{j=1}^{k_0} \frac{1}{\max\{\sigma_j/\gamma, \lambda\}}$ by (9.4) for $k = k_0$, we obtain

$$\begin{aligned} |\mu_0(\varphi > n) - cn^{-\beta}| &= \left| \sum_{k=0}^{k_0-1} \mu_0(A_k) + \mu_0([e_{k_0}, 1]) - \frac{c^*}{n^\beta} \sum_{k=1}^{\infty} \frac{h(e_k)}{\lambda_k} \right| \\ &= \frac{c^*}{n^\beta} \sum_{k=k_0}^{\infty} \frac{h(e_k)}{\lambda_k} + \prod_{j=1}^{k_0} \frac{h_{\max}}{\max\{\sigma_j/\gamma, \lambda\}} \\ &\quad + O(n^{-(\beta+1)} \log n) + O(n^{-(\beta+1)}) + O(n^{-2\beta}). \end{aligned}$$

Recalling the choice of n_0 and hence k_0 in (9.6), we conclude that $|\mu_0(\varphi > n) - cn^{-\beta}| = O(n^{-2\beta}, n^{-(\beta+1)} \log n)$, and condition (H0) follows. \square

Lemma 9.3. *For the maps given by (9.1) we have $\left| \frac{1}{\rho} \mu_0(\varphi > n) - \mu_\tau(\tau > n) \right| = O(n^{-(1+\beta)})$ and (H1) holds.*

Proof. To estimate $\left| \frac{1}{\rho} \mu_0(\varphi > n) - \mu_\tau(\tau > n) \right|$, we use Lemma A.1. Recall that $\lambda_k = Df^{\tau_k+1}(e_k)$ and $h_{\max} = \sup_{x \in [e_0, 1]} h(x)$. Take again n_0 and $k_0 = k_0(n)$ as in (9.6) and assume that $n \geq n_0$, so $n > 2\tau_k$ for all $k < k_0$.

Recall that $\{\varphi = \tau_{k+1}\} = [e_k, e_{k+1})$. The set of $y \in [e_k, e_{k+1})$ such that $n - \tau_k < \tau(f^{\tau_k}(y)) \leq n$ is $O\left(\frac{|y_n - y_{n-\tau_k}|}{|y_{\sigma_{k+1}} - e_0|}\right)$ due to the boundedness of distortion of $f^{\tau_k} : [e_k, 1] \rightarrow [e_0, f^{\tau_k}(1)]$. Using also (9.3) to estimate $|y_n - y_{n-\tau_k}|$, we obtain for the first sum in (A.1):

$$\sum_{k \geq 0} \int_{\{\varphi = \tau_{k+1}\}} 1_{\{n \geq \tau > n - \tau_k\}} \circ f^{\tau_k} d\mu_0 \leq h_{\max} \sum_{k \geq 1} |y_n - y_{n-\tau_k}| \frac{|e_{k+1} - e_k|}{|y_{\sigma_{k+1}} - e_0|}.$$

By boundedness of distortion, $\frac{|e_{k+1} - e_k|}{|y_{\sigma_{k+1}} - e_0|} \ll \frac{1}{\lambda_k}$ and since $\frac{\tau_k}{\lambda_k}$ is summable by Lemma 9.1, we conclude

$$\begin{aligned} \sum_{k \geq 0} \int_{\{\varphi = \tau_{k+1}\}} 1_{\{n \geq \tau > n - \tau_k\}} \circ f^{\tau_k} d\mu_0 &\ll h_{\max} \left(\sum_{k=1}^{k_0-1} \frac{(n - \tau_k)^{-\beta} - n^{-\beta}}{\lambda_k} + \sum_{k \geq k_0} \frac{|1 - e_0|}{\lambda_k} \right) \\ &\leq \left(h_{\max} \sum_{k=1}^{k_0-1} \frac{2\beta\tau_k}{\lambda_k} + \frac{|1 - e_0|}{c^*} \right) n^{-(\beta+1)} \ll n^{-(\beta+1)}. \end{aligned} \quad (9.7)$$

From (9.3) we derive $\frac{|f^{\tau_k}(1) - y_{\sigma_{k+1}}|}{|f^{\tau_k}(1) - e_0|} \ll \frac{1}{\sigma_{k+1}}$. Using (9.4), the second sum in (A.1) can be estimated as

$$\begin{aligned} \int_{\{\varphi > \tau_{k+1}\}} 1_{\{\tau > n\}} \circ f^{\tau_k} d\mu_0 &= \sum_{\sigma_{k+1} > n} \mu_0([e_{k+1}, 1]) \\ &\leq h_{\max} \sum_{\sigma_{k+1} > n} \frac{|f^{\tau_k}(1) - y_{\sigma_{k+1}}|}{|f^{\tau_k}(1) - e_0|} \prod_{i=1}^k \frac{1}{\max(\sigma_j/\gamma, \lambda)} \\ &\ll \frac{h_{\max}}{n} \sum_{\sigma_{k+1} > n} \prod_{j=0}^k \frac{1}{\max(\sigma_j/\gamma, \lambda)} \\ &\leq \frac{h_{\max}}{n} \prod_{j=1}^{k_0} \frac{1}{\max(\sigma_j/\gamma, \lambda)} \sum_{k \geq k_0, \sigma_{k+1} > n} \prod_{j=k_0+1}^k \frac{1}{\max(\sigma_j/\gamma, \lambda)} \end{aligned}$$

because $\sigma_{k+1} \leq n$ for $k \leq k_0$ by the definition of $k_0 = k_0(n)$ in (9.6), which also gives $\prod_{j=1}^{k_0} \frac{1}{\max(\sigma_j/\gamma, \lambda)} \leq n^{-(1+\beta)}$ for all $n \geq n_0$. Therefore the quantity of the previous displayed equation is $O(n^{-(\beta+2)})$, which is clearly negligible compared to the first term (9.7).

To check condition (H1), we continue the proof of Lemma 9.1. Boundedness of distortion of $f^{\sigma_{k+1}} : [y_{\sigma_{k+1}}, y_{\sigma_{k+1}-1}) \rightarrow [e_0, 1)$ gives $\frac{|A_k|}{|1 - e_k|} \ll \frac{|y_{n-\tau_k} - e_0|}{|y_{\sigma_{k+1}-1} - e_0|} \ll \frac{\sigma_{k+1}^\beta}{(n - \tau_k)^\beta}$. Combining this with (9.4) and (9.5) we get

$$(k+1)|A_k| \ll \frac{k+1}{(n - \tau_k)^\beta} \prod_{j=1}^k \min\left\{\frac{\gamma}{\sigma_j}, \frac{1}{\lambda}\right\} = \frac{1}{n^\beta} \left(\frac{n}{(n - \tau_k)\tau_k} \right)^\beta \frac{(k+1)\tau_k^\beta}{\prod_{j=1}^k \max\left\{\frac{\sigma_j}{\gamma}, \lambda\right\}}. \quad (9.8)$$

Since $n \leq (n - \tau_k)\tau_k$, this gives

$$\int_{\{\varphi > n\}} \rho(y) d\mu_0 \leq h_{\max} \sum_{k \geq 0} (k+1) |A_k| \ll \frac{1}{n^\beta} \sum_{k \geq 0} \frac{(k+1)\tau_k^\beta}{\prod_{j=1}^k \max\{\frac{\sigma_j}{\gamma}, \lambda\}}.$$

Recall that $\tau_k = \sum_{j=1}^k \sigma_j$, so the sum in this expression is finite. Since $n^{-\beta} = O(\mu_0(\varphi > n))$, condition (H1) follows. \square

9.3 Further results for the infinite measure setting of (9.1)

Recall that μ_τ is the absolutely continuous probability measure preserved by the first return map $f^\tau : Y \rightarrow Y$. Lemma 9.3 shows that the tails $\frac{1}{\bar{\rho}}\mu_0(\varphi > n)$ and $\mu_\tau(\tau > n)$ coincide up to $O(n^{-(1+\beta)})$. As shown in Lemma 9.1, $\mu_0(\varphi > n)$ satisfies (H0) b). Thus, $\mu_\tau(\tau > n)$ also satisfies (H0) b). Moreover, using sharper estimates of x_n (as in [16, Section B]), one obtains sharper estimates for $\mu_0(\varphi > n)$; in particular, $\mu_0(\varphi > n)$ satisfies condition (H) in [16], and by Lemma 9.3, $\mu_\tau(\tau > n)$ satisfies condition (H) in [16] as well.

As mentioned in Section 9.1, the only obstruction in [12, 16] to obtain higher order asymptotics of the transfer operator L^n for maps such as (9.1) uniformly on $(0, 1]$, for BV functions supported on $(0, 1]$, was the higher order expansion of $\mu_\tau(\tau > n)$. But as shown here $\mu_\tau(\tau > n)$ satisfies (H0) b) (by Lemmas 9.1 and 9.3) and hence, the required tail conditions of [12, Theorem 9.1, Theorem 11.4] and [16, Theorem 1.1, Proposition 1.6]. As a consequence, these results apply to the map (9.1).

Using the fact that [12, Theorem 11.4] and [16, Proposition 1.6] hold for the map (9.1), one also obtains [12, Corollary 9.10] and [16, Proposition 1.7], which provide error rates for the arcsine law. (It is known that arcsine laws hold for the general class of AFN maps, see [19].) As shown in [21, 19], the Darling Kac law holds for the general class of AFN maps considered in [21]. Error rates in the Darling Kac law for maps such as the one studied in [11], characterized by good higher order asymptotics of the tail of the first return time, were obtained in [17, Theorem 1.1]. Again, the only obstruction in [17] to show that [17, Theorem 1.1] applies to maps of the form (9.1), was the lack of knowledge on the higher order expansion of $\mu_\tau(\tau > n)$. Given the information on $\mu_\tau(\tau > n)$ obtained here, one obtains that [17, Theorem 1.1] applies to the setting of (9.1).

A Comparing general and first returns

In this appendix we prove a result used in Lemma 9.3, namely Lemma A.1. A consequence of Lemma 9.3, namely Corollary A.2 below, allows for a direct comparison between $\sum_{j \geq n} \frac{1}{\bar{\rho}} \mu_0(\varphi > j)$ (the leading term of the correlation decay in Theorem 4.2) and $\sum_{j \geq n} \frac{1}{\bar{\rho}} \mu_\tau(\tau > j)$ (the leading term of the correlation decay, possibly, obtained via inducing with respect to first return time); we refer to Remark A.3 for details in the setting of (9.1).

Lemma A.1. *Suppose that μ_0 and μ_τ are equivalent measures on Y that are preserved by the general return map f^φ and first return map f^τ , respectively, and $\bar{\rho} = \int_Y \rho d\mu_0 < \infty$. Then*

$$\begin{aligned} & \frac{1}{\bar{\rho}} \mu_0(\varphi > j) - \mu_\tau(\tau > j) \\ &= \frac{1}{\bar{\rho}} \sum_{k \geq 0} \left(\int_{\{\varphi = \tau_{k+1}\}} 1_{\{j \geq \tau > j - \tau_k\}} \circ f^{\tau_k} d\mu_0 - \int_{\{\varphi > \tau_{k+1}\}} 1_{\{\tau > j\}} \circ f^{\tau_k} d\mu_0 \right). \end{aligned} \quad (\text{A.1})$$

Proof. The set $\hat{Y} = \pi^{-1}(Y) = \sqcup_{k \geq 0} \hat{Y}_k = \sqcup_{k \geq 0} \{(y, \tau_k) : \rho > k\}$ can be considered as a subtower of Δ with dynamics

$$T_\Delta^\tau(y, \tau_i) = \begin{cases} (y, \tau_{i+1}) & \text{if } 0 \leq i < \rho(y) - 1, \\ (Fy, 0) & \text{if } i = \rho(y) - 1, \end{cases}$$

see Figure 2. Clearly μ_0 is the invariant measure of the return map to the base. Recall that μ_Δ is the ‘‘pushed-up’’ measure from μ_0 onto Δ . Restricted to \hat{Y} , μ_Δ is T_Δ^τ -invariant, and $\mu_\Delta(\hat{Y}) = \int_Y \rho d\mu_0 = \bar{\rho} < \infty$. The projection $\hat{\pi} : \hat{Y} \rightarrow Y$, $(y, k) \mapsto f^{\tau_k}(y)$ pushes μ_Δ down to an f^τ -

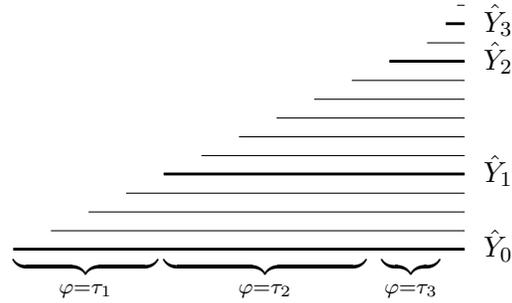


Figure 2: The tower Δ and \hat{Y} in between with the bold-face levels.

invariant measure, which when normalized has the formula $\mu_\tau(A) = \frac{1}{\bar{\rho}} \sum_{k \geq 0} \mu_0(\hat{\pi}^{-1}(A) \cap \hat{Y}_k)$. Applying this to $A = \{\tau > n\}$ and recalling that $\hat{Y}_k = \{\varphi > \tau_k\} \times \{\tau_k\}$ gives

$$\mu_\tau(\tau > n) = \frac{1}{\bar{\rho}} \sum_{k \geq 0} \int_{\{\varphi > \tau_k\}} 1_{\{\tau > n\}} \circ f^{\tau_k}(y) d\mu_0(y). \quad (\text{A.2})$$

Next we specify the set $\{\varphi > n\}$. For each $y \in Y$, pick $k = \rho(y) - 1$, so that (y, k) is at the top level of the subtower \hat{Y} , so $\varphi(y) = \tau_k(y) + \tau(f^{\tau_k}(y))$. Therefore

$$\begin{aligned} \mu_0(\varphi > n) &= \sum_{k \geq 0} \mu_0(\varphi = \tau_{k+1} \wedge \tau \circ f^{\tau_k} > n - \tau_k) \\ &= \sum_{k \geq 0} \int_{\{\varphi = \tau_{k+1}\}} 1_{\{\tau > n - \tau_k\}} \circ f^{\tau_k} d\mu_0. \end{aligned} \quad (\text{A.3})$$

Combining (A.2) and (A.3) we get:

$$\begin{aligned} & \frac{1}{\bar{\rho}} \mu_0(\varphi > j) - \mu_\tau(\tau > j) \\ &= \frac{1}{\bar{\rho}} \sum_{k \geq 0} \left(\int_{\{\varphi = \tau_{k+1}\}} \mathbf{1}_{\{\tau > j - \tau_k\}} \circ f^{\tau_k} d\mu_0 - \int_{\{\varphi > \tau_k\}} \mathbf{1}_{\{\tau > j\}} \circ f^{\tau_k} d\mu_0 \right) \\ &= \frac{1}{\bar{\rho}} \sum_{k \geq 0} \left(\int_{\{\varphi = \tau_{k+1}\}} \mathbf{1}_{\{j \geq \tau > j - \tau_k\}} \circ f^{\tau_k} d\mu_0 - \int_{\{\varphi > \tau_{k+1}\}} \mathbf{1}_{\{\tau > j\}} \circ f^{\tau_k} d\mu_0 \right), \end{aligned}$$

proving (A.1). \square

We have the following corollary in the finite measure setting, for which we recall that $\bar{\varphi} = \int_Y \varphi d\mu_0$ and $\bar{\tau} = \int_Y \tau d\mu_\tau$ are finite.

Corollary A.2. *Suppose that μ_0 and μ_τ are equivalent measures on Y that are preserved by the general return map f^φ and first return map f^τ , respectively. Then*

$$\left| \sum_{j \geq n} \frac{1}{\bar{\varphi}} \mu_0(\varphi > j) - \frac{1}{\bar{\tau}} \mu_\tau(\tau > j) \right| \leq \frac{1}{\bar{\varphi}} \sum_{k \geq 1} \int_{\{\varphi = \tau_{k+1} > n\}} \tau_k d\mu_0.$$

Remark A.3. *In the setting of (9.1), we can replace $k + 1$ with τ_k in (9.8) and obtain that $\sum_{k \geq 1} \int_{\{\varphi = \tau_{k+1} > n\}} \tau_k d\mu_0 = O(\mu_0(\varphi > n))$. This together with Corollary A.2 implies that the leading term in Theorem 4.2 applied to (9.1) matches the leading term of the correlation decay results in [10, 12]; although not exactly the same, the difference in the main terms can be absorbed in the error term.*

Proof. Observe that

$$\begin{aligned} \bar{\varphi} &= \int_Y \varphi d\mu_0 = \mu_\Delta(\Delta) = \sum_{k=0}^{\infty} \sum_{j=0}^{\tau_{k+1} - \tau_k - 1} \mu_\Delta(Y_j) = \sum_{k=0}^{\infty} \int_{\hat{Y}_k} \tau \circ f^{\tau_k} d\mu_\Delta \\ &= \int_Y \tau d\mu_\Delta \circ \pi^{-1} = \bar{\rho} \int_Y \tau d\mu_\tau = \bar{\rho} \cdot \bar{\tau}. \end{aligned}$$

Therefore the statement of the corollary is equivalent to

$$\left| \sum_{j \geq n} \frac{1}{\bar{\rho}} \mu_0(\varphi > j) - \mu_\tau(\tau > j) \right| \leq \frac{1}{\bar{\rho}} \sum_{k \geq 1} \int_{\{\varphi = \tau_{k+1} > n\}} \tau_k d\mu_0. \quad (\text{A.4})$$

Continuing from Lemma A.1, and since τ is constant on $f^{\tau_k}(\{\varphi > \tau_{k+1}\})$, there is at most one $j \geq n$ for which $\mathbf{1}_{\{\tau > j\}} \circ f^{\tau_k} = 1$. Therefore, using $\tau_{k+1} = \tau \circ f^{\tau_k} + \tau_k$, we can sum the second sum of the integrals in (A.1) over $j \geq n$ and compute

$$\begin{aligned} & \sum_{j \geq n} \sum_{k \geq 0} \int_{\{\varphi > \tau_{k+1}\}} \mathbf{1}_{\{\tau > j\}} \circ f^{\tau_k} d\mu_0 \leq \sum_{k \geq 0} \mu_0(\varphi > \tau_{k+1} > \tau_k + n) \\ & \leq \sum_{k \geq 1} \mu_0(\varphi \geq \tau_{k+1} > n) \leq \sum_{k \geq 1} k \mu_0(\varphi = \tau_{k+1} > n) = \sum_{k \geq 1} \int_{\{\varphi = \tau_{k+1} > n\}} k d\mu_0, \end{aligned}$$

which is definitely less than the first sum of integrals in (A.1), which we will estimate now.

For the first sum of integrals in (A.1), there are at most τ_k values of $j \geq n$ making the indicator function 1. Using again that $\tau_{k+1} = \tau \circ f^{\tau_k} + \tau_k$, we can sum the first sum of integrals in (A.1) over $j \geq n$ and compute

$$\begin{aligned} \sum_{j \geq n} \sum_{k \geq 0} \int_{\{\varphi = \tau_{k+1}\}} 1_{\{j \geq \tau > j - \tau_k\}} \circ f^{\tau_k} d\mu_0 &\leq \sum_{k \geq 0} \int_{\{\varphi = \tau_{k+1}\}} \tau_k 1_{\{\tau > n - \tau_k\}} \circ f^{\tau_k} d\mu_0 \\ &\leq \sum_{k \geq 1} \int_{\{\varphi = \tau_{k+1} > n\}} \tau_k d\mu_0. \end{aligned}$$

This proves (A.4) and hence the corollary. \square

References

- [1] N. H. Bingham, C. M. Goldie, J. L. Teugels. *Regular variation*. Encyclopedia of Mathematics and its Applications **27**, Cambridge University Press, Cambridge, 1987.
- [2] K. B. Erickson. Strong renewal theorems with infinite mean. *Trans. Amer. Math. Soc.* **151** (1970) 263–291.
- [3] W. Feller. *An Introduction to Probability Theory and its Applications, II*. Wiley, New York, 1966.
- [4] A. Garsia, J. Lamperti. A discrete renewal theorem with infinite mean. *Comment. Math. Helv.* **37** (1962/1963) 221–234.
- [5] S. Gouëzel. Sharp polynomial estimates for the decay of correlations. *Israel J. Math.* **139** (2004) 29–65.
- [6] S. Gouëzel. Berry-Esseen theorem and local limit theorem for non uniformly expanding maps. *Ann. Inst. H. Poincaré Probab. Statist.* **41** (2005) 997–1024.
- [7] S. Gouëzel. Correlation asymptotics from large deviations in dynamical systems with infinite measure. *Colloquium Math.* **125** (2011) 193–212.
- [8] S. Gouëzel. Vitesse de décroissance et théorèmes limites pour les applications non uniformément dilatantes. Ph. D. Thesis, Ecole Normale Supérieure, 2004.
- [9] M. Holland, Slowly mixing systems and intermittency maps. *Ergodic Theory Dynam. Systems* **25** (2005) 133–159.
- [10] H. Hu, S. Vaienti. Polynomial decay for non-uniformly expanding maps. Preprint 2013, arXiv:1307.0359

- [11] C. Liverani, B. Saussol, S. Vaienti. A probabilistic approach to intermittency. *Ergodic Theory Dynam. Systems* **19** (1999) 671–685.
- [12] I. Melbourne, D. Terhesiu. Operator renewal theory and mixing rates for dynamical systems with infinite measure. *Invent. Math.* **1** (2012) 61–110.
- [13] I. Melbourne, D. Terhesiu. First and higher order uniform ergodic theorems for dynamical systems with infinite measure. *Israel J. Math.* **194** (2013) 793–830.
- [14] I. Melbourne, D. Terhesiu. Decay of correlation for non-uniformly hyperbolic systems with general return times. *Ergodic Theory Dynam. Systems* **34** (2014) 893–918.
- [15] O. M. Sarig. Subexponential decay of correlations. *Invent. Math.* **150** (2002) 629–653.
- [16] D. Terhesiu. Improved mixing rates for infinite measure preserving transformations. *Ergodic Theory Dynam. Systems* **35** (2015) 585–614.
- [17] D. Terhesiu. Error rates in the Darling Kac law. *Studia Math.* **220** (2014), no. 2, 101–117.
- [18] L.-S. Young. Recurrence times and rates of mixing. *Israel J. Math.* **110** (1999) 153–188.
- [19] M. Thaler, R. Zweimüller. Distributional limit theorems in infinite ergodic theory. *Probab. Theory Related Fields* **135** (2006) 15–52.
- [20] R. Zweimüller. Ergodic structure and invariant densities of non-Markovian interval maps with indifferent fixed points. *Nonlinearity* **11** (1998) 1263–1276.
- [21] R. Zweimüller. Ergodic properties of infinite measure-preserving interval maps with indifferent fixed points. *Ergodic Theory Dynam. Systems* **20** (2000) 1519–1549.