# Equilibrium States for S-unimodal Maps

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**Abstract**

For S-unimodal maps $f$, we study equilibrium states maximizing the free energies $F_t(\mu) := h(\mu) - t\int \log|f'|d\mu$ and the pressure function $P(t) := \sup_\mu F_t(\mu)$. It is shown that if $f$ is uniformly hyperbolic on periodic orbits, then $P(t)$ is analytic for $t \approx 1$. On the other hand, examples are given where no equilibrium states exist, where equilibrium states are not unique and where the notions of equilibrium state for $t = 1$ and of observable measure do not coincide.

## 1 Introduction

If $T : M \to M$ is an Axiom A diffeomorphism with a unique basic set and if $\psi = \log J^uT$ denotes the logarithm of the Jacobian determinant of $T$ in the unstable directions, then a unique equilibrium state $\mu$ of $T$ relative to the function $-\psi$ exists (a so called SBR measure (for Sinai-Bowen-Ruelle)). It is distinguished among all $T$-invariant measures by the property of being *observable* in the sense that $\frac{1}{n}\sum_{k=0}^{n-1}\delta_{T^kx} \to \mu$ weakly for all $x \in M$ except a set of points of Riemannian volume zero. An excellent source for this and related results is Bowen's book [2].

In nonhyperbolic situations the relation between observable measures and equilibrium states is less clear. In the case of piecewise $C^2$-maps $f : [0,1] \to [0,1]$ Ledrappier [17] proved that an ergodic $f$-invariant measure with positive *Lyapunov exponent* $\lambda(\mu) := \int \log|f'|\,d\mu$ and *Kolmogorov-Sinai entropy* $h(\mu)$ is absolutely continuous with respect to Lebesgue measure if and only if $h(\mu) - \lambda(\mu) = 0$. Under the additional assumption $\inf|f'| > 0$ Takahashi [32] characterized $h(\mu) - \lambda(\mu)$ by a variational principle where the quantity to be maximized depends on Lebesgue measure. These and similar results, although they share some of the flavour of corresponding results in hyperbolic situations, neither guarantee the existence of measures maximizing $F_1(\mu)$ nor do they shed any light on the properties of measures with $\lambda(\mu) = 0$. Also the assumption $\inf|f'| > 0$ excludes some prime examples as logistic maps from consideration. Therefore it is the aim of this paper to explore to which extent the conceptual framework of the theory of equilibrium states developed for hyperbolic systems carries over to general one-dimensional unimodal maps with negative Schwarzian derivative.

In uniformly hyperbolic situations the existence of equilibrium states is usually guaranteed by the upper semicontinuity of the function $\mu \mapsto F_t(\mu) := h(\mu) - t\,\lambda(\mu)$. In the present setting it turns out that although there are some classes of maps (including the Fibonacci maps) for which this function is upper semicontinuous, this seems to be a rather unusual property among S-unimodal maps, because $\mu \mapsto \lambda(\mu)$ is never lower semicontinuous for Collet-Eckmann maps (Proposition 2.8). Instead we formulate a weaker property called the *stability* of $F_t$ in Definition 2.10 which still guarantees

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existence of equilibrium states for $F_t$ (*i.e.* of invariant measures maximizing $F_t$), but which is shared by many unimodal maps, including Collet-Eckmann maps as we shall see in Section 5. In Section 3 we bound the Lyapunov exponents of measures by the infimum of Lyapunov exponents along periodic orbits: $\lambda(\mu) \geq \lambda_{per}$. Recently Nowicki and Sands [27] showed that $\lambda_{per} > 0$ is equivalent to the Collet-Eckmann condition. In this section we also discuss the Markov extension of $f$. It is shown that an invariant measure $\mu$ for $f$ is *liftable* to an invariant measure on the Markov extension if and only if $\lambda(\mu) > 0$. In Section 4 we prove the analyticity of the pressure function $P(t) = \sup_\mu F_t(\mu)$ if $\lambda_{per} \neq 0$ (*i.e.* if $f$ is uniformly hyperbolic on periodic orbits). In Section 5 we show stability of $F_t$ for a large class of functions but also give an example of a logistic map that is not stable and has no equilibrium state. In the last section we assume that $f$ has a zero dimensional attractor or that $F_1$ is stable and show that each observable measure is an equilibrium state for $F_1$.

Our main results can be summarized briefly as:

- If $\lambda_{per} \neq 0$ and $t \approx 1$, then $F_t$ is stable, the pressure function is analytic at $t$, and $f$ has a unique equilibrium state for $F_t$. The equilibrium state for $t = 1$ is at the same time the only observable measure. (Theorems 3.4, 4.1, 5.1)
- If $\lambda_{per} = 0$, then none of the above properties needs to be satisfied (Examples 3.13, 4.2, 5.4).

Related results for a class of complex polynomials were obtained by Makarov and Smirnov [21] and by Smirnov [31].

The results on Collet-Eckmann maps require rather elaborate proofs and we have to rely heavily on results from especially [16]. For some details only the unpublished reference [29] is available. Therefore we add an appendix in which some of these details are given.

## 2 Continuity Properties of the Free Energy

We consider unimodal interval maps $f : I := [0,1] \to I$ with negative Schwarzian derivative and critical point $c$. More precisely: $f$ is $C^3$ on $I \setminus \{c\}$ with $Sf := f'''/f' - \frac{3}{2}(f''/f')^2 \leq 0$, and there are a real number $\ell > 1$ and a continuous, strictly positive function $M : I \to \mathbb{R}$ such that $|f'(x)| = M(x) \cdot |x-c|^{\ell-1}$ for all $x \in I$. $\ell$ is called the *critical order* of $f$.

Let $\mathcal{M}$ be the set of all $f$-invariant probability measures. $\mathcal{M}$ is a compact set in the weak topology. For $\mu \in \mathcal{M}$ denote by $h(\mu)$ the entropy of $\mu$ and by $\lambda(\mu) := \mu(\psi) := \int \psi \, d\mu$ the Lyapunov exponent of $\mu$ where $\psi = \log|f'|$. As $\psi \leq \log \sup_x |f'(x)| < \infty$, the Lyapunov exponent is well defined. Even more holds: Although $\psi$ has a logarithmic singularity in $c$, it follows from Proposition 3.1 below that $\sup_{\mu\in\mathcal{M}} |\lambda(\mu)| \leq \sup_{\mu\in\mathcal{M}} \int |\psi| \, d\mu < \infty$.

**2.1 Definition**

a) For each $\mu \in \mathcal{M}$ and $t \in \mathbb{R}$ we define the *free energy* $F_t(\mu) := h(\mu) - t\lambda(\mu)$.

b) The function $P(t) := \sup_{\nu\in\mathcal{M}} F_t(\nu)$ is called the *pressure function*.

c) A measure $\mu$ is an ***equilibrium state*** for $F_t$ if $F_t(\mu) = P(t)$. The set of equilibrium states is denoted by $\mathcal{E}(t)$.

**2.2 Remark**

a) The pressure function $t \mapsto P(t)$ is convex and Lipschitz continuous, because it is the supremum of affine funtions $t \mapsto F_t(\nu)$ with uniformly bounded derivatives.

b) Let $\nu = \int \nu^x \, d\nu(x)$ be the ergodic decomposition of $\nu$ into ergodic measures $\nu^x$. As the entropy is an affine upper semicontinuous function of the measure, we have $\int h(\nu^x)\, d\nu(x) = h(\nu)$, and for the Lyapunov exponent the analogous identity is obviously satisfied. Therefore $P(t) = \sup\{F_t(\nu) : \nu \in \mathcal{M}, \nu \text{ ergodic}\}$.

Existence of equilibrium states is *e.g.* guaranteed by the following elementary lemma:

**2.3 Lemma** *If $F_t$ is upper semicontinuous on $\mathcal{M}$, then $\mathcal{E}(t)$ is nonempty and compact convex.*

In our setting, the entropy function $\mu \mapsto h(\mu)$ is upper semicontinuous. This is true, because the partition of $I$ into the two monotonicity intervals of $f$ is a generator "modulo homtervals", and under our smoothness assumptions such homtervals are attracted to periodic orbits, see *e.g.* [24]. The Lyapunov exponent $\mu \mapsto \lambda(\mu)$ is in general upper but not lower semicontinuous. Therefore if $t \le 0$, then $F_t$ is obviously upper semicontinuous. However, we are mostly interested in the cases $t > 0$. In order to explore for which maps $F_t$ actually is upper semicontinuous, we make the following definition:

**2.4 Definition** $\mathcal{M}$ is *uniformly integrating* if $\lim_{\varepsilon \to 0} \sup_{\mu \in \mathcal{M}} \int_{c-\varepsilon}^{c+\varepsilon} |\psi| d\mu = 0$.

**2.5 Lemma** *$\mathcal{M}$ is uniformly integrating if and only if $\mu \mapsto \lambda(\mu)$ is lower semicontinuous.*

*Proof:* Suppose that $\mathcal{M}$ is uniformly integrating. Let $\delta > 0$ be arbitrary, and let $U$ be a neighbourhood of $c$ so small that $\int_U \psi d\nu > -\delta$ for every $\nu \in \mathcal{M}$. Let $\psi_L = \max(-L, \psi)$ and take $L$ so large that $\psi = \psi_L$ outside of $U$. Since $\psi_L$ is continuous, $\mu \to \mu(\psi_L)$ is continuous. Let $(\mu_n) \subset \mathcal{M}$, $\mu_n \to \mu$ weakly. Then

$$\liminf_{n \to \infty} \mu_n(\psi) \ge \liminf_{n \to \infty} \mu_n(\psi_L) - \delta = \mu(\psi_L) - \delta \ge \mu(\psi) - \delta.$$

Since $\delta$ is arbitrary, also $\liminf_n \mu_n(\psi) \ge \mu(\psi)$. For the reverse implication, assume by contradiction that $\mathcal{M}$ is not uniformly integrating. Then there exists $\delta > 0$ such that for every $L \in \mathbb{N}$ there exists a measure $\mu_L \in \mathcal{M}$ such that $\mu_L(\psi - \psi_L) < -\delta$. Assume without loss of generality that $\mu_L \to \mu$ weakly. Then for any $L_0$ and all $L > L_0$,

$$\mu_L(\psi) = \mu_L(\psi_L) + \mu_L(\psi - \psi_L) < \mu_L(\psi_{L_0}) - \delta.$$

Therefore $\liminf_L \mu_L(\psi) \le \mu(\psi_{L_0}) - \delta$ for all $L_0 > 0$, whence $\liminf_L \mu_L(\psi) \le \mu(\psi) - \delta$, *i.e.* $\mu \mapsto \mu(\psi)$ is not lower semicontinuous. □

**2.6 Corollary** *If $\mathcal{M}$ is uniformly integrating, then $F_t$ is upper semicontinuous.*

In order to check for specific maps whether $\mathcal{M}$ is uniformly integrating we need some more notation. For $x \in I$, write $x_i = f^i(x)$; in particular $c_i = f^i(c)$. If $x \ne c$, denote the *symmetric point* by $\bar{x}$, *i.e.* $\bar{x} \ne x$ and $f(x) = f(\bar{x})$. We define the *cutting times* $(S_n)$ and *closest precritical points* $(z_n)$ as follows:

$$S_0 = 1 \text{ and } z_0 = f^{-1}(c) \cap (0, c),$$

and for $n \ge 1$

$$S_n = \min\{k > S_{n-1} : f^{-k}(c) \cap (z_{n-1}, c) \ne \emptyset\} \text{ and } z_n = f^{-S_n}(c) \cap (z_{n-1}, c).$$

In particular $[z_n, c]$ and $[c, \bar{z}_n]$ are the maximal central intervals on which $f^{S_n+1}$ is monotone. (In the sequel we will often use intervals with $z_n$ or $\bar{z}_n$ as boundary points. Although we try to be precise whether these endpoint belong to the intervals in question, it does not really matter from a measure theoretical viewpoint. Indeed, when $c$ is not periodic, $\mu(\cup_{n \in \mathbb{Z}} f^n(c)) = 0$ for every $\mu \in \mathcal{M}$.) It is not hard to show that $S_n - S_{n-1}$ is again a cutting time which we call $S_{Q(n)}$, see [7] or [24, Section II 3b]. The map $Q : \mathbb{N} \to \mathbb{N}$ is called the *kneading map*.

**2.7 Lemma**

*a) If $\mathcal{M}$ is uniformly integrating, then* $\lim_n -\frac{1}{S_n}\log|z_n - c| = 0$.

*b) If* $\sum_n -\frac{1}{S_n}\log|z_n - c| < \infty$, *then $\mathcal{M}$ is uniformly integrating.*

*Proof:* First note that by nonflatness of $f$, $f'(x) = \mathcal{O}(\ell)|x-c|^{\ell-1}$, where $\ell < \infty$ is the critical order. We can assume that $f$ has no stable or neutral periodic orbit, because otherwise there is a neighbourhood $U \ni c$ such that $\mu(U) = 0$ for all $\mu \in \mathcal{M}$ (If $c$ itself is periodic, then $\mathcal{M} \setminus \{\delta_{orb(c)}\}$ is uniformly integrating.) In absence of stable or neutral periodic orbits, $f^{S_n}(z_n, c) = f^{S_n}(c, \bar{z}_n)$ contains either $(z_n, c)$ or $(c, \bar{z}_n)$ for each $n \in \mathbb{N}$. Therefore there exists an $S_n$-periodic point $p_n \in (z_n, \bar{z}_n)$. If $\mu_n$ is the equidistribution on orb$(p_n)$, then $\int_{z_n}^{\bar{z}_n}\psi d\mu_n \le \mathcal{O}(\ell)\frac{1}{S_n}\log|p_n - c| \le \mathcal{O}(\ell)\frac{1}{S_n}\log|z_n - c|$. Therefore $\mathcal{M}$ can only be uniformly integrating if $\lim_n \frac{1}{S_n}\log|z_n - c| = 0$. This proves a). For the second statement note that by definition of closest precritical point, $f^i(z_n), f^i(z_{n+1}) \notin (z_n, \bar{z}_n)$ and $f^i(z_n, \bar{z}_n) \not\ni c$ for $0 < i < S_n$. Therefore $f^i(z_n, z_{n+1}) \cap ((z_n, z_{n+1}] \cup [\bar{z}_{n+1}, \bar{z}_n)) = \emptyset$ for all $0 < i < S_n$, so that any measure $\mu \in \mathcal{M}$ assigns mass $\le \frac{1}{S_n}$ to the set $(z_n, z_{n+1}] \cup [\bar{z}_{n+1}, \bar{z}_n)$. It follows that

$$0 \ge \int_{z_k}^{\bar{z}_k}\psi d\mu \ge \mathcal{O}(\ell)\sum_{n\ge k}\log|z_{n+1} - c|\cdot\mu((z_n, z_{n+1}] \cup [\bar{z}_{n+1}, \bar{z}_n)) \ge \mathcal{O}(\ell)\sum_{n\ge k}\frac{1}{S_n}\log|z_{n+1} - c|.$$

As $S_{n+1} = S_n + S_{Q(n)} \le 2S_n$, our assumption gives that $\sup_{\mu\in\mathcal{M}}\int_{z_n}^{\bar{z}_n}|\psi|d\mu \to 0$ as $n \to \infty$. □

**2.8 Proposition** *If $f$ is a Collet-Eckmann map,* i.e. *if* $\liminf_n \frac{1}{n}\log|(f^n)'(c_1)| > 0$, *then $\mu \mapsto \lambda(\mu)$ is not lower semicontinuous.*

*Proof:* If $f$ is Collet-Eckmann, then $\liminf_n -\frac{1}{S_n}\log|z_n - c| > 0$, see [26, Corollary 21]. □

In view of the preceding discussion and in view of the well-known strong hyperbolicity properties of Collet-Eckmann maps we are thus led to the conclusion that upper semicontinuity of the free energy is not a reasonable assumption for the investigation of equilibrium states of unimodal maps. Nevertheless we want to mention a prominent example of a unimodal map for which $F_t$ is indeed upper semicontinuous.

**2.9 Proposition** *For S-unimodal Fibonacci maps $\mathcal{M}$ is uniformly integrating.*

*Proof:* Let $f_\ell$ be an S-unimodal map of critical order $\ell$ with Fibonacci dynamics. Topologically, such a map is characterized by the property that the cutting times $S_n$ coincide with the Fibonacci numbers $1, 2, 3, 5, 8, \ldots$. Fibonacci maps and their metrical properties have been studied in several papers; the properties that we need are taken from [20, 15, 5]. ¿From those papers we can derive the following asymptotic behaviour:

$$\frac{|c_{S_{n+1}} - c|}{|c_{S_n} - c|}\begin{cases} \ge C \text{ for some } C \in (0,1) & \text{if } \ell > 2, \\ \sim 2^{-\frac{n}{3}} & \text{if } \ell = 2, \\ \sim \kappa^{(S_n^r)} \text{ for some } \kappa, r \in (0,1) & \text{if } 1 < \ell < 2. \end{cases}$$

Note that $S_n \sim \gamma^n$, where $\gamma = \frac{1+\sqrt{5}}{2}$ is the golden ratio. Since for Fibonacci maps $c_{S_n} \in (z_{n-2}, z_{n-1}) \cup (\bar{z}_{n-1}, \bar{z}_{n-2})$, we obtain the following:

- If $\ell > 2$, then $-\frac{1}{S_n}\log|z_n - c| = \mathcal{O}(1)\frac{n}{S_n}\log\frac{1}{C}$.
- If $\ell = 2$, then $-\frac{1}{S_n}\log|z_n - c| = -\frac{1}{S_n}\log\left[\mathcal{O}(1)2^{-\sum_{i=1}^{n}\frac{i}{3}}\right] = \mathcal{O}(1)\frac{n^2}{S_n}$.

- If $1 < \ell < 2$, then $-\frac{1}{S_n} \log |z_n - c| = \mathcal{O}(1)\gamma^{-n} \sum_{i=1}^n \gamma^{ri} \log \frac{1}{\kappa} = \mathcal{O}(1)\gamma^{(r-1)n}$.

In all three cases $-\frac{1}{S_n} \log |z_n - c|$ is summable, whence $\mathcal{M}$ is uniformly integrating for $\ell > 1$. For the Fibonacci tent map $\mathcal{M}$ is also uniformly integrating, because then $\psi$ is just a constant. □

In order to guarantee existence of equilibrium states it is sufficient that $F_t$ is upper semicontinuous on those parts of $\mathcal{M}$ where equilibrium states might be located. This leads us to the following

**2.10 Definition** The free energy $F_t$ is called *stable* if for every weakly converging sequence of measures $\mu_n \in \mathcal{M}$ such that $F_t(\mu_n) \to P(t)$ holds: $\mu := \lim_n \mu_n \in \mathcal{E}(t)$.

The next two lemmas already indicate that stability should be an appropriate notion in the present context. Let $\tilde{\mathcal{E}}(t) = \{\mu \in \mathcal{M} :$ there exists $(\mu_n) \subset \mathcal{M}, \mu_n \to \mu$ and $F_t(\mu_n) \to P(t)\}$.

**2.11 Lemma** *$\tilde{\mathcal{E}}(t)$ is nonempty and compact convex. Moreover, the following statements are equivalent:*

*1) $\mathcal{E}(t) = \tilde{\mathcal{E}}(t)$.*

*2) $F_t$ is stable.*

*3) $F_t$ is upper semicontinuous on $\tilde{\mathcal{E}}(t)$.*

*Proof:* By definition of $P(t)$ there exists $(\mu_n)$ such that $F_t(\mu_n) \to P(t)$. Since $\mathcal{M}$ is compact there exists a subsequence $\mu_{n_i} \to \mu \in \tilde{\mathcal{E}}(t)$, whence $\tilde{\mathcal{E}}(t) \neq \emptyset$. Suppose $(\mu_n) \subset \tilde{\mathcal{E}}(t)$ and $\mu_n \to \mu$. Let $\mu_{n,k}$ be such that $\mu_{n,k} \to \mu_n$ and $F_t(\mu_{n,k}) \to P(t)$ as $k \to \infty$. Then for a sufficiently rapidly increasing sequence $(k_n)$ also $\mu_{n,k_n} \to \mu$. Hence $\tilde{\mathcal{E}}(t)$ is closed and therefore compact. The convexity is immediate. Now for the equivalent statements, we first note that $\mathcal{E}(t) \subset \tilde{\mathcal{E}}(t)$ by definition.

1) ⇒ 3) If $\tilde{\mathcal{E}}(t) = \mathcal{E}(t)$ and $\mu_n \to \mu \in \tilde{\mathcal{E}}(t)$, then $P(t) = F_t(\mu) \geq \limsup_n F_t(\mu_n)$. Hence $F_t$ is upper semicontinuous on $\tilde{\mathcal{E}}(t)$.

3) ⇒ 2) If $F_t$ is upper semicontinuous on $\tilde{\mathcal{E}}(t)$, then for every $\mu_n \to \mu$ such that $\lim_n F_t(\mu_n) = P(t)$, we have $\mu \in \tilde{\mathcal{E}}(t)$ and hence $P(t) = \limsup_n F_t(\mu_n) \leq F_t(\mu) \leq P(t)$. Therefore $\mu \in \mathcal{E}(t)$.

2) ⇒ 1) If $\mu_n \to \mu \in \tilde{\mathcal{E}}(t)$ and $F_t(\mu_n) \to P(t)$, then stability implies that $\mu \in \mathcal{E}(t)$. □

**2.12 Lemma** *If $F_t$ is stable and $\mu_n \to \mu$ for measures $\mu_n \in \mathcal{E}(t_n)$ and some sequence $t_n \to t$, then $\mu \in \mathcal{E}(t)$.*

*Proof:* Clearly $P(t) - F_t(\mu_n) = [P(t) - P(t_n)] + [P(t_n) - (h(\mu_n) - t_n\lambda(\mu_n))] + (t - t_n)\lambda(\mu_n)$. The first term tends to 0 by continuity of the pressure function, the second term equals 0 because $\mu_n \in \mathcal{E}(t_n)$, and the third term tends to 0 because $\lambda(\mu_n)$ is bounded (see Proposition 3.1 below). Hence $|P(t) - F_t(\mu_n)| \to 0$. As $F_t$ is stable, $\mu_n \to \mathcal{E}(t)$. □

In Section 5 we prove stability for various classes of maps, among them Collet-Eckmann maps.

# 3 Lyapunov Exponents and Liftability Properties

In this section we first give a lower bound for the Lyapunov exponent of measures in $\mathcal{M}$ by means of the Lyapunov exponents along periodic orbits. Later we characterize measures with positive Lyapunov exponent by their liftability to the Markov extension.

For an $n$-periodic point $p$ let $\lambda_p = \frac{1}{n} \log |(f^n)'(p)|$. Let

$$\lambda_{per} = \inf\{\lambda_p : p \text{ is periodic}\}.$$

**3.1 Proposition** *For every $\mu \in \mathcal{M}$, $\lambda(\mu) \geq \lambda_{per}$.*

For the proof we need some more notation. For each $x \in I$ and $n \in \mathbb{N}$, denote the maximal monotonicity interval of $f^n$ containing $x$ by $Z_n[x]$. Let

$$r_n(x) = d(f^n(x), \partial f^n Z_n[x]). \tag{1}$$

*Proof:* If $\mu$ is the equidistribution on a periodic orbit (stable or not), then clearly $\lambda(\mu) \geq \lambda_{per}$. Therefore assume from now on that $\mu$ is not such an equidistribution. Without loss of generality we may assume that $\mu$ is ergodic. We divide the proof into two cases.

First assume that $\mu(c-\varepsilon, c+\varepsilon) > 0$ for all $\varepsilon > 0$. Let $x$ be a typical point of $\mu$ in the sense that $\lambda(\mu) = \lim_n \frac{1}{n}\sum_{i=0}^{n-1}\psi(x_i)$ and for each $\varepsilon > 0$, $\mathrm{orb}(x) \cap (c-\varepsilon, c+\varepsilon) \neq \emptyset$. By Birkhoff's ergodic theorem this holds $\mu$-a.e. Let $n_0 = 0$ and $n_{i+1} = \min\{n > n_i : x_n \in (x_{n_i}, \bar{x}_{n_i})\}$. In particular

$$x_{n_i} \in Y_{n_{i+1}-n_i} := \{y \in I : f^j(y) \notin (y, \bar{y}) \text{ for } j < n_{i+1} - n_i \text{ and } f^{n_{i+1}-n_i}(y) \in (y, \bar{y})\},$$

cf. [6, 25]. It is not hard to show that for the boundary points of any component of $Y_{n_{i+1}-n_i}$, say $p_i$ and $q_i$, holds: $p_i$ and $\bar{q}_i$ are $(n_{i+1}-n_i)$-periodic. Hence $|(f^{n_{i+1}-n_i})'(p_i)| \geq e^{\lambda_{per}(n_{i+1}-n_i)}$ and $|(f^{n_{i+1}-n_i})'(q_i)| \geq Ce^{\lambda_{per}(n_{i+1}-n_i)}$, where $C$ depends only on the nonsymmetry of $f$. By the Minimum Principle for maps with negative Schwarzian derivative (see [24, Lemma II.6.1]), also $|(f^{n_{i+1}-n_i})'(x_{n_i})| \geq Ce^{\lambda_{per}(n_{i+1}-n_i)}$. By the chain rule $|(f^{n_j})'(x)| \geq C^j e^{\lambda_{per} n_j}$. Because $x_{n_i} \to c$, whence $n_{i+1} - n_i \to \infty$ as $i \to \infty$, $\lim_j \frac{1}{n_j}\log|(f^{n_j})'(x)| \geq \lambda_{per}$. As $\lambda(\mu) = \lim_n \frac{1}{n}\sum_{i=0}^{n-1}\psi(x_i)$, also $\lambda(\mu) \geq \lambda_{per}$.

Now assume that $\mu(c-\varepsilon, c+\varepsilon) = 0$ for some $\varepsilon > 0$. Then for every $x \in \mathrm{supp}(\mu)$, $\mathrm{orb}(x) \cap (c-\varepsilon, c+\varepsilon) = \emptyset$ and by the Contraction Principle (see [19] or [24, page 305]), there exists $\delta > 0$ such that $r_n(x) \geq \delta$ for all $n \in \mathbb{N}$. Let $U$ be an interval, not containing a neutral or stable periodic point, such that $|U| < \frac{\delta}{2}$ and $\mu(U) > 0$. Let $x$ be a $\mu$-typical point in the sense that $\lambda(\mu) = \lim_n \frac{1}{n}\sum_{i=0}^{n-1}\psi(x_i)$ and $x$ visits $U$ infinitely often. Let $n_1 < n_2 < \dots$ be the numbers $n$ such that $x_n \in U$. Because $r_{n_i} > \delta$, there exist periodic points $p_i \in U$ of (not necessarily prime) period $n_i - n_1$. Furthermore, by the Koebe Principle [24, Theorem IV 1.2], there exists $K_\delta > 0$ such that

$$|(f^{n_{i+1}-n_1})'(x_{n_1})| \geq \frac{1}{K_\delta}|(f^{n_{i+1}-n_1})'(p_i)| \geq \frac{1}{K_\delta}e^{\lambda_{per}(n_i-n_1)}.$$

Hence $\lambda(\mu) = \lim_n \frac{1}{n}\sum_{i=0}^{n-1}\psi(x_i) \geq \lim_j \frac{1}{n_j}[(n_j - n_1)\lambda_{per} - \log K_\delta + \log|(f^{n_1})'(x)|] = \lambda_{per}$. This concludes the proof. □

**3.2 Remark** It was shown in [27] that the condition $\lambda_{per} > 0$ is equivalent to the Collet-Eckmann condition. If $\lambda_{per} < 0$, then $f$ has a unique stable periodic orbit. The case $\lambda_{per} = 0$ comprises maps with neutral periodic orbits as well as other nonhyperbolic maps.

The following result is due to Ledrappier [17].

**3.3 Proposition** *If $\lambda(\mu) > 0$, then $\mu \in \mathcal{E}(1)$ if and only if $\mu$ is absolutely continuous with respect to Lebesgue measure.*

Combining Propositions 3.1 and 3.3 we obtain already parts of

**3.4 Theorem**

*a) If $\lambda_{per} < 0$ then $\mathcal{E}(t) = \{\delta_{orb(p)}\}$ for $t \approx 1$* (i.e. *for $t$ in a neighbourhood of* 1)*. Here $\delta_{orb(p)}$ denotes the equidistribution on the (unique) stable periodic orbit of $f$.*

b) *If $\lambda_{per} = 0$ and $f$ has a neutral periodic point $p$, then $\mathcal{E}(t) = \{\delta_{orb(p)}\}$ for $t \geq 1$.*
*If $\lambda_{per} = 0$ and $f$ is infinitely renormalizable, then $\mathcal{E}(t) = \{\mu_{\omega(c)}\}$ for $t \geq 1$, where $\mu_{\omega(c)}$ is the unique probability measures supported by $\omega(c)$.*

c) *If $\lambda_{per} > 0$, then $\mathcal{E}(1)$ consists of a unique absolutely continuous measure.*

In Theorem 5.1 we show that if $\lambda_{per} > 0$, then $\mathcal{E}(t)$ consists of a single measure not only if $t = 1$ but also for $t \approx 1$.

*Proof of Theorem 3.4:* The last statement follows directly from Propositions 3.1 and 3.3 if one observes the ergodicity of Lebesgue measure in this case, see [1]. If $\lambda_{per} < 0$, and if $X$ denotes the complement of the basin of the stable periodic orbit, then $X$ is a hyperbolic repelling Cantor set and $m(X) = 0$. Proposition 3.3 shows that $X$ cannot support an equilibrium state for $t = 1$. By continuity of $\psi|_X$ and the fact that $\delta_{orb(p)}$ is an isolated point in $\mathcal{M}$ it follows that $\mathcal{E}(t) = \{\delta_{orb(p)}\}$ also for $t \approx 1$. In the case that $p$ is a neutral periodic point, $X$ is not a hyperbolic repelling set. Indeed, $X$ contains $p$. Therefore we cannot conclude that $\lambda(\mu) > 0$, and Proposition 3.3 does not apply. However Corollary 3.8 below shows that $\mathcal{E}(t) = \{\delta_{orb(p)}\}$ for $t = 1$ and the case $t > 1$ is an immediate consequence. Finally, if $f$ is infinitely renormalizable, there exists no absolutely continuous invariant probability measure. In fact, $\omega(x) \subset \omega(c)$ $m$-a.e. Hence Proposition 3.3 shows that $\mu \in \mathcal{E}(t)$ for $t \geq 1$ only if $\lambda(\mu) = 0$. Let $\mu_{\omega(c)} \in \mathcal{M}$ be the unique measure supported by $\omega(c)$. It follows from Lemma 3.10 below that $\lambda(\mu_{\omega(c)}) = h(\mu_{\omega(c)}) = 0$. Hence $\mu_{\omega(c)} \in \mathcal{E}(t)$. Now if $\mu \in \mathcal{M}$, $\mu \neq \mu_{\omega(c)}$, then there exists $\varepsilon > 0$ such that $\operatorname{supp}(\mu) \cap (c - \varepsilon, c + \varepsilon) = \emptyset$. However, any forward invariant set which does not contain $c$ or a stable or neutral periodic point in its closure is hyperbolic repelling, see *e.g.* [22]. Combining this with the proof of Proposition 3.1 it follows $\lambda(\mu) > 0$. Therefore $\mu \notin \mathcal{E}(t)$, which concludes the proof. □

In order to study measures with Lyapunov exponent zero, we need the notions of *Markov extension* and *liftability*. To each unimodal map $f : I \to I$, we can construct the *Markov extension* or *tower* $\hat{I}$ as follows: Let $D_1 = I$, and for $n \geq 1$,

$$D_{n+1} = \begin{cases} f(D_n) & \text{if } c \notin D_n, \\ [c_{n+1}, c_1] & \text{if } c \in D_n. \end{cases}$$

An inductive argument shows that for each $n$, $c_n$ is a boundary point of $D_n$. Note also that $D_n \ni c$ if and only if $n$ is a cutting time. (See Section 2 for the definition of the cutting times $S_k$.) Let $\hat{I} = \sqcup_{n \geq 0} D_n$ denote the union of disjoint copies of the intervals $D_n$ (henceforth also denoted by $D_n$), and define the action $\hat{f}$ on $\hat{I}$ as follows: If $x \in D_1$, then

$$\hat{f}(x) = f(x) \in \begin{cases} D_2 & \text{if } x \in [c, c_1], \\ D_1 & \text{if } x \notin [c, c_1]. \end{cases}$$

If $x \in D_n$ for some $n \geq 2$, then

$$\hat{f}(x) = f(x) \in \begin{cases} D_{n+1} & \text{if } c \notin [c_n, x), \\ D_{1+S_{Q(k)}} & \text{if } c \in [c_n, x) \text{ and } n = S_k \, . \end{cases}$$

The dynamical system $(\hat{f}, \hat{I})$ satisfies a Markov property in the sense that whenever $\hat{f}^n(x) \in D_k$ for some $n, k \in \mathbb{N}$, there exists an interval $J \ni x$ such that $\hat{f}^n$ maps $J$ monotonically onto $D_k$. Denote the natural projection from $\hat{I}$ to $I$ by $\pi$. Clearly $\pi \circ \hat{f} = f \circ \pi$.

It would be consequent to give any object attached to the tower a $\hat{}$-accent, but in order not to overload the notation, we write only measures and maps on $\hat{I}$ with a $\hat{}$-accent. Points or sets in the tower are written with a $\hat{}$-accent only if confusion can arise.

For later use we denote by $D^+_{S_k} := [c, c_{S_k}] = D_{S_k} \cap \hat{f}^{-1} D_{S_k+1}$ that part of $D_{S_k}$ that "climbs up" in the tower under the action of $\hat{f}$ and by $D^-_{S_k} := (c, c_{S_{Q(k)}}] = D_{S_k} \cap \hat{f}^{-1} D_{S_{Q(k)}+1}$ that part of $D_{S_k}$ that "jumps down".

**3.5 Remark** If $\mu \in \mathcal{M}$, then we can construct a measure $\hat{\mu}$ as follows: Let $\hat{\mu}_1$ be the measure $\mu$ lifted to the level $D_1$ and set $\hat{\mu}_n = \frac{1}{n}\sum_{i=0}^{n-1} \hat{\mu}_1 \circ \hat{f}^{-i}$. Clearly $\mu = \hat{\mu}_n \circ \pi^{-1}$ for each $n$. As was shown in [12], $\hat{\mu}_n$ converges vaguely. We call the limit measure $\hat{\mu}$. If $\mu$ is ergodic, then $\hat{\mu}$ is either a probability measure on $\hat{I}$, in which case we call $\mu$ *liftable*, or it is identically 0 on $\hat{I}$. In this case the mass "has escaped to infinity".

**3.6 Theorem** *Let $\mu$ be an ergodic invariant probability measure. Then $\mu$ is liftable if and only if $\mu$ has a positive Lyapunov exponent.*

*Proof:* The "if" part is proven in [12], using a construction from [17]. For the "only if" part let us start proving that the equidistribution on a stable or neutral periodic orbit is nonliftable. Indeed, let $p$ be the *central* point of this periodic orbit, *i.e.* $orb(p) \cap (p, \bar{p}) = \emptyset$. Because $f$ has negative Schwarzian derivative, the immediate basin of $p$ contains $c$, but no preimage of $c$. Therefore the intervals $f^n([c,p])$ are never cut in the construction of the Markov extension. This means that $\hat{f}^n \circ i([c,p]) \subset D_n$ for all $n$, where $i : I \to D_1$ is the inclusion map. (In particular, if $c$ itself is periodic, $D_n$ degenerates to $\{c_n\}$ for $n$ sufficiently large.) It follows that $orb(p)$ cannot support a liftable measure.

Let us assume that $\mu$ is liftable, $\hat{\mu}$ being the lifted measure. We will show that $\lambda(\mu) > 0$. Let $n \in \mathbb{N}$ be such that $\hat{\mu}(D_n) > 0$ and let $J \subset \mathrm{cl}\, J \subset \mathrm{int}\, D_n$ be an interval such that $\hat{\mu}(J) > 0$. Since $\mu$ is not the equidistribution on the orbit of a stable or neutral periodic point $p$, $\pi(\mathrm{cl}\, J)$ can be chosen disjoint from $\mathrm{orb}(p)$. Moreover we can chose $J$ such that $orb(\partial J) \cap J = \emptyset$. Let $\hat{F} : J \to J$ be the first return map to $J$. By our conditions on $J$ each branch $\hat{F} : J_i \to J$ of $\hat{F}$ is onto, and by the Markov property of $\hat{f}$, $\hat{F}|_{J_i}$ is extendable monotonically to a branch that covers $D_n$. Clearly each branch of $\hat{F}$, say $\hat{F}|_{J_i} = \hat{f}^s|_{J_i}$, contains an $s$-periodic point $q$. Due to a result by Martens, de Melo and van Strien [23] and also [24, Theorem IV B'], there exists $\varepsilon > 0$ such that the $|(\hat{f}^s)'(q)| > 1 + \varepsilon$, independently of the branch. If $J$ is sufficiently small, the Koebe Principle yields that $|\hat{F}'(x)| > 1 + \frac{\varepsilon}{2}$ for all $x \in J$. Clearly $\frac{\hat{\mu}}{\hat{\mu}(J)}$ is an $\hat{F}$-invariant probability measure on $J$. Let $J_i$, $i \in \mathbb{N}$, be the branch-domains of $\hat{F}$, and let $s_i$ be such that $\hat{F}|_{J_i} = \hat{f}^{s_i}|_{J_i}$. Since $\hat{\mu}$ can be written as $\hat{\mu}(\varphi) = \sum_i \sum_{j=0}^{s_i-1} \int_{J_i} \varphi \circ \hat{f}^j d\hat{\mu}$, we get

$$\begin{aligned} \lambda(\hat{\mu}) &= \sum_i \sum_{j=0}^{s_i-1} \int_{J_i} \log|\hat{f}'| \circ \hat{f}^j d\hat{\mu} \\ &= \sum_i \int_{J_i} \log|(\hat{f}^{s_i})'| d\hat{\mu} = \sum_i \int_{J_i} \log|\hat{F}'| d\hat{\mu} \\ &\geq \sum_i \hat{\mu}(J_i) \log(1 + \frac{\varepsilon}{2}) = \hat{\mu}(J) \log(1 + \frac{\varepsilon}{2}) > 0. \end{aligned}$$

Because $f'(\pi(x)) = \hat{f}'(x)$ for all $x \in \hat{I}$, this concludes the proof. □

**3.7 Corollary** *Let $\mu \in \mathcal{M}$ be ergodic.*

*a) If $\mu$ is nonliftable, then $F_t(\mu) = 0$ for all $t$.*

*b) If $\mu$ is liftable, then $F_1(\mu) \leq 0$ and $F_t(\mu) \leq (1-t)\lambda(\mu)$ with equality if and only if $\mu \ll m$.*

*In particular, $\mu \in \mathcal{E}(1)$ if and only if $\mu \ll m$ or $\mu$ is nonliftable.*

**3.8 Corollary** *If $f$ is an S-unimodal map with a neutral or stable periodic point $p$ and $\mu \in \mathcal{M}$, then either $\lambda(\mu) > 0$ or $\mu$ is the equidistribution on orb($p$).*

*Proof:* We can argue for each ergodic component of $\mu$ separately. Let therefore $\mu$ be ergodic. In view of Theorem 3.6 it suffices to prove that $\mu$ is either liftable or equal to the equidistribution on orb($p$). Assume that the latter is not the case, and take $\varepsilon > 0$ arbitrary. Let $B$ be the immediate basin of orb($p$) and let $U$ be a neighbourhood of $B$ so small that $\mu(U) < \varepsilon$. Since $c \in B$ and $\partial D_n \subset \text{orb}(c)$ for each $n > 1$, $\inf\{|D_n \setminus B| : D_n \not\subset B\} > 0$. Therefore there exists $N$ such that if $D_n \ni c$, the component of $D_n \setminus \{c\}$ that jumps down in the tower, jumps down to some level in $\sqcup_{m \le N} D_m$. Because $\partial B$ consists of (at least one-sided) repelling (pre-)periodic points, there exists $M$ such that whenever $x \in D_n \setminus \pi^{-1}(U)$, $\hat{f}^m(x)$ jumps down in the tower for some $m \le M$. Because $\mu(U) < \varepsilon$, also $\hat{\mu}_n(\sqcup_{j > M+N} D_j) < \varepsilon$ for each $\hat{\mu}_n$ as defined in Remark 3.5. Hence, as $\varepsilon$ was arbitrary, $\mu$ is liftable. □

For $x \in I$, let $V(x)$ denote the set of weak accumulation points of $(\frac{1}{n}\sum_{i=1}^n \delta_{x_i})_n$. Let conv denote the closure of the convex hull.

**3.9 Proposition** *If $\mu$ has only nonliftable ergodic components, then $\mu$ is contained in conv$V(c)$.*

*Proof:* We argue for each ergodic component of $\mu$ (called $\mu$ again) separately. Therefore let $\mu \in \mathcal{M}$ be ergodic nonliftable and let $X = \{x : V(x) = \{\mu\}\}$ be its typical points. We distinguish two cases:
1) There exist $x \in X$ and a subsequence $(n_j)_j$ such that for all $\varepsilon > 0$

$$\lim_{j \to \infty} \frac{1}{n_j} \text{card}\{0 \le i < n_j : r_i(x) > \varepsilon\} = 0.$$

¿From [13, Theorem 4] (see also the proof of [24, Theorem V 5.3]) we can conclude that the accumulation points of $(\frac{1}{n_j}\sum_{i=0}^{n_j-1} \delta_{f^i(x)})_j$ are contained in conv$V(c)$. But as $V(x) = \{\mu\}$, $\mu \in \text{conv}\,V(c)$.
2) In the other case there exist $\varepsilon, \delta > 0$ such that for $\mu$-a.e. $x \in X$

$$\liminf_{n \to \infty} \frac{1}{n} \text{card}\{0 \le i < n : r_i(x) > \varepsilon\} \ge \delta > 0.$$

First assume $f$ has a stable or neutral periodic point $p$. Then if $\mu = \delta_{orb(p)}$, clearly $\{\mu\} = V(c)$, while if $\mu \ne \delta_{orb(p)}$, Corollary 3.8 and Theorem 3.6 show that $\mu$ is liftable.

Therefore we can assume that $f$ has no stable or neutral periodic point and no homtervals either, see e.g. [24, Theorem II 6.2]. Let $\mathcal{Z}_n$ be the partition of $I$ into the maximal intervals of monotonicity of $f^n$. There exists $N = N(\varepsilon)$ such that $\sup\{|Z| : Z \in \mathcal{Z}_N\} < \varepsilon$. Indeed, otherwise there exist a nested sequence $(Z_n)$, $Z_n \in \mathcal{Z}_n$, such that $\cap_n Z_n$ a nondegenerate interval. This interval must be a homterval, which we excluded.

Suppose $r_n(x) \ge \varepsilon$ and let $(\zeta_1, \zeta_2) \in \mathcal{Z}_N$ be the cylinder containing $f^n(x)$. Then $f^n(Z_n[x]) \supset (\zeta_1, \zeta_2)$, where $Z_n[x] \in \mathcal{Z}_n$ is the cylinder containing $x$, and there exist $n_1 < n_2 \le N$ such that $f^{n_i}(\zeta_i) = c$. On the tower this means that for any $y \in \pi^{-1}(x)$, $\hat{f}^{n+n_2}(y) \in D_{n_2 - n_1}$. It follows that there exists $\delta' > 0$ such that for $\mu$-a.e. $x \in X$ and every $y \in \pi^{-1}(x)$,

$$\liminf_{n \to \infty} \frac{1}{n} \text{card}\{0 \le i < n : \hat{f}^i(y) \in \sqcup_{m \le N} D_m\} \ge \delta'.$$

Because of Remark 3.5 this contradicts to $\mu$ being nonliftable. □

The map $f$ has a *persistently recurrent* critical point if $r_n(c_1) \to 0$, where $r_n$ is as in (1). For example, any map for which $\lim_n Q(n) = \infty$ has a persistently recurrent critical point [4, Proposition 3.1]. In particular, $c$ is persistently recurrent if $f$ is infinitely renormalizable.

**3.10 Lemma** *If $c$ is persistently recurrent, then for every $\mu \in \mathcal{M}$ such that supp $(\mu) \subset \omega(c)$ we have $F_t(\mu) = 0$ for all $t$.*

*Proof:* It was shown in [4, Lemma 3.3] $r_n(x) \to 0$ uniformly on $\omega(c)$. Take $\mu \in \mathcal{M}$, supp $(\mu) \subset \omega(c)$ and assume by contradiction that $\mu$ is liftable. Then there exists $n \in \mathbb{N}$ and an interval $U \subset \mathrm{cl}\, U \subset \mathrm{int}\, D_n$ such that $\hat{\mu}(U) > 0$. Let $\delta = d(U, \partial D_n) > 0$. By Birkhoff's ergodic theorem card$\{i : \hat{f}^i(y) \in U\} = \infty$ $\hat{\mu}$-a.e., and by the Markov property of $\hat{f}$, $r_n(\pi(y)) \geq \delta$ whenever $\hat{f}^n(y) \in U$. Since $\mu$ is supported by $\omega(c)$, this gives a contradiction to $r_n(x) \to 0$. Hence $\mu$ is not liftable, and by Theorem 3.6 $\lambda(\mu) = 0$. It was shown in [1] that the topological entropy $h_{top}(f|_{\omega(c)}) = 0$. Therefore $h(\mu) = 0$ and also $F_t(\mu) = 0$ for all $t$. □

The following Corollary applies in particular to infinitely renormalizable maps and to Fibonacci-like maps, *i.e.* maps whose kneading map is $Q(k) = \max(0, k - d)$.

**3.11 Corollary** *Let $f$ be any map with a persistently recurrent critical point and such that $f|_{\omega(c)}$ is uniquely ergodic with invariant measure $\nu_c$. Then*

*a) $\mathcal{E}(1)$ contains one or two extremal elequilibrium states, namely $\nu_c$ and (possibly) a liftable, absolutely continuous one.*

*b) $\mathcal{E}(t) = \{\nu_c\}$ for $t > 1$.*

**3.12 Example [Infinitely renormalizable maps]**

As infinitely renormalizable maps are uniquely ergodic on $\omega(c)$ and do not have absolutely continuous invariant measures, $\mathcal{E}(t) = \mathcal{E}(1) = \{\nu_c\}$ for $t \geq 1$ and $V(x) = \mathcal{E}(1)$ for Lebesgue-a.e. $x$.

For $t < 1$ the situation is more complicated: If $f$ is a map with Feigenbaum dynamics, then $F_t(\mu) = -t\lambda(\mu)$ for all $\mu \in \mathcal{M}$ because $f$ has entropy zero. Therefore $\mathcal{E}(t) = \{\nu_c\}$ again. If $f$ is an infinitely renormalizable map different from a Feigenbaum map, then $f$ has an ergodic invariant measure $\mu$ of positive entropy such that $F_t(\mu) = h(\mu) - t\lambda(\mu) > 0$ for $t < \frac{h(\mu)}{\lambda(\mu)}$. (Observe that $t < \frac{h(\mu)}{\lambda(\mu)} =: HD(\mu)$ is the Hausdorff dimension of $\mu$, see [9].) In particular, $P(t) > 0$ for $t < dD := \sup\{HD(\mu) : \mu \text{ ergodic}, \lambda(\mu) > 0\}$, the dynamical dimension of $\mu$. If $dD < 1$ and $t \in (dD, 1)$, then $F_t(\mu) < 0$ for all $\mu$ with $\lambda(\mu) > 0$ and hence $\mathcal{E}(t) = \{\nu_c\}$.

**3.13 Example [Fibonacci maps]**

Recall from [20] that if $f$ is a Fibonacci map, then $f|_{\omega(c)}$ is uniquely ergodic. It is known from [20] and [15] that there is $\ell_0 > 2$ such that Fibonacci maps with critical order $1 < \ell < \ell_0$ have a unique absolutely continuous invariant probability measure $\mu_1$. It is obviously the only observable measure. Therefore these maps have two extremal equilibrium states for $F_1$, the observable $\mu_1$ and $\nu_c$ which is not observable. ¿From Corollary 3.11 it follows that $\mathcal{E}(t) = \{\nu_c\}$ if $t > 1$. For $t < 1$ the situation is more complicated: As $F_t(\mu_1) = F_1(\mu_1) + (1-t)\lambda(\mu_1) = (1-t)\lambda(\mu_1) > 0$ by Corollary 3.7b), the set $\mathcal{E}(t)$ contains only liftable measures by Corollary 3.7a), and $\mathcal{E}(t) \neq \emptyset$ by Proposition 2.9.

In [5] it was proved that there is $\ell_1 > \ell_0$ such that Fibonacci maps with critical order $\ell > \ell_1$ have no absolutely continuous invariant probability measure. Hence $\mathcal{E}(t) = \mathcal{E}(1) = \{\nu_c\}$ for $t \geq 1$ and $V(x) = \mathcal{E}(1)$ for Lebesgue-a.e. $x$ in this case. For $t < 1$ we can argue as in the previous example, because $f$ has positive entropy. Hence $P(t) > 0$ for $t < dD$, and if $dD < 1$ and $t \in (dD, 1)$, then $F_t(\mu) < 0$ for all $\mu$ with $\lambda(\mu) > 0$ and $\mathcal{E}(t) = \{\nu_c\}$.

## 4 Properties of the Pressure Function

The main result of this section is the following theorem which shows that if $\lambda_{per} \neq 0$ then $P(t)$ behaves for $t \approx 1$ just as it does for uniformly hyperbolic systems.

**4.1 Theorem**

*a) If $\lambda_{per} < 0$, then $P(t) = -t\lambda_{per}$ for $t \approx 1$.*

*b) If $\lambda_{per} = 0$, then $P(t) \geq 0$ for $t < 1$ and $P(t) = 0$ for $t \geq 1$.*

*c) If $\lambda_{per} > 0$ and if $f$ satisfies some additional regularity assumptions formulated in Remark 4.4 below, then $t \mapsto P(t)$ is analytic for $t \approx 1$, $P(1) = 0$ and $P'(1) = -\lambda(\mu_1)$ where $\mu_1 \in \mathcal{M}$ is the unique absolutely continuous invariant measure.*

**4.2 Example** If $f$ is a Fibonacci map with critical order $1 < \ell < \ell_0$ and absolutely continuous ergodic invariant probability measure $\mu_1$ (see Example 3.13), then $P(t) = 0$ for $t \geq 1$ and $P(t) \geq F_t(\mu_1) = h(\mu_1) - t\,\lambda(\mu_1) = (1-t)\lambda(\mu_1)$ for $0 < t < 1$. Here $\lambda(\mu_1) > 0$ because of Proposition 3.9. In particular, $P(t)$ is not analytic at $t = 1$.

*Proof of the Theorem:* If $\lambda_{per} < 0$, then $\mathcal{E}(t)$ consists of the equidistribution on the unique attracting periodic orbit for $t \approx 1$ by Theorem 3.4. The result is immediate.

If $\lambda_{per} = 0$, then there exists a sequence $(p_n)$ of periodic orbits such that $F_t(p_n) \to 0$. Therefore $P(t) \geq 0$. Because $h(\mu) \leq \lambda(\mu)$, also $F_t(\mu) \leq 0$ for $t \geq 1$.

The main ingredient of the proof for the case $\lambda_{per} > 0$ consists of showing that $\exp P(t)$ equals the spectral radius $r_t$ of a suitable transfer operator. This is a standard technique in uniformly hyperbolic situations but needs some more care here. In the next section the identity $P(t) = \log r_t$ also plays an essential role when we prove stability of $F_t$ for Collet-Eckmann maps.

For a weight function $\hat{\Phi} : \hat{I} \to \mathbb{R}$ denote by $\mathcal{L}_{\hat{\Phi}}$ the transfer operator for $\hat{f}$ acting on functions $\hat{g} : \hat{I} \to \mathbb{C}$, namely $\mathcal{L}_{\hat{\Phi}}\hat{g}(x) = \sum_{y \in \hat{f}^{-1}x} \hat{\Phi}(y)\hat{g}(y)$. In [16] the operators $\mathcal{L}_t := \mathcal{L}_{|\hat{f}'|^{-t}}$ for $t \approx 1$ were studied. It was proved that there is a suitable Banach space of functions $\hat{g} : \hat{I} \to \mathbb{C}$, call it $B$ now[1], such that each $\mathcal{L}_t$, $t \approx 1$, acts as a quasicompact operator on $B$ and such that these operators depend analytically on $t$. If $f$ is a nonrenormalizable Collet-Eckmann map, then all $\mathcal{L}_t$ for $t \approx 1$ have a simple positive leading eigenvalue $r_t$ which is separated from the rest of the spectrum. By analytic perturbation theory the eigenvalues $r_t$ and the corresponding eigenprojections $\mathcal{P}_t$ depend analytically on $t$. In the appendix we show that there are Borel measures $\hat{m}_t$ on $\hat{I}$ and densities $\hat{h}_t : \hat{I} \to (0, \infty)$ such that $\mathcal{L}_t^* \hat{m}_t = r_t \hat{m}_t$ and $\mathcal{P}_t \hat{g} = \hat{h}_t \cdot \int \hat{g}\, d\hat{m}_t$ for all $\hat{g} \in B$. In Proposition 4.5 below we show that $P(t) = \log r_t$. Hence $P(t)$ depends analytically on $t$ and $P'(t) = (\log r_t)' = -\lambda(\hat{\mu}_t)$ where $\hat{\mu}_t = \hat{h}_t \cdot \hat{m}_t$, see [16, Section 5]. □

**4.3 Remark** Observe that $\mathcal{L}_t^* \hat{m}_t = r_t \hat{m}_t$ is equivalent to $r_t|\hat{f}'|^t = \frac{d(\hat{m}_t \circ \hat{f})}{d\hat{m}_t}$ "locally", *i.e.* $r_t|\hat{f}'|^t$ is the derivative of $\hat{f}$ with respect to the measure $\hat{m}_t$. As $\hat{f}'(x) = \hat{f}'(y)$ if $\pi(x) = \pi(y)$ (*i.e.* if $x$ and $y$ are in the same "fiber" of $\hat{I}$), it is not hard to show that a measure $m_t$ on $I$ can be unambiguously defined by setting $m_t(U) = \hat{m}_t(\hat{U})$ if $\hat{U} \subseteq D_n$ for some $n$ and $\pi(\hat{U}) = U$, see (13) of the appendix.

In fact, $m_1$ is just Lebesgue measure on $I$. $m_t$ is often called the $t$-conformal measure for $f$. Note also that $m_t$ has no atoms, see (12) of the appendix.

[1] In [16] this space was called $BV_{\hat{w}}$. We point out that $B$ is a space of everywhere defined functions and not just a space of equivalence classes as in many treatments of transfer operators for piecewise monotonic maps.

**4.4 Remark** The results in [16] were in fact proved under weak additional regularity assumptions on $f$. These assumptions are satisfied *e.g.* if $f$ is a polynomial map or if $f(x) = a(1 - |2x-1|^\ell)$ for some real $\ell > 1$.

**4.5 Proposition** *If $f$ is a Collet-Eckmann map satisfying the additional regularity assumptions mentioned in Remark 4.4, then $P(t) = \log r_t$ where, as above, $r_t$ is the leading eigenvalue of the transfer operator $\mathcal{L}_{|\hat{f}'|^{-t}}$.*

*Proof:* We restrict to the case of nonrenormalizable maps, from which the result for the general case can be deduced by passing to a suitable renormalization of $f$.

Recall the construction of the Markov extension $(\hat{I}, \hat{f})$ from Section 3 where $\hat{I} = \sqcup_{n\geq 0} D_n$ is a disjoint union of intervals $D_n$. We are going to study a first return map $\hat{F}$ on a subset $J \subseteq \hat{I}$ defined as follows: For $k \geq 0$ let $A_k := D^+_{S_k} \cap \hat{f}^{-S_{Q(k+1)}} D^-_{S_{k+1}}$. $A_k$ is that subinterval of $D_{S_k}$ that climbs up in the tower under the next $S_{Q(k+1)} = S_{k+1} - S_k$ iterates of $\hat{f}$ and then jumps down to level $D_{1+S_{Q(k+1)}}$ in the next step. In fact, $\pi(A_k)$ is the interval $(z_{Q(k+1)}, c]$ or $[c, \bar{z}_{Q(k+1)})$ which is contained in $[c, c_{S_k}]$, and $z_n$ is as defined in Section 2.

Let $J = \cup_{k\geq 0} A_k$. If $x \in A_k$ is not a preimage of $c$, then there is some $j \geq Q(k+1)$ such that $\pi(x) \in (z_j, z_{j+1})$, and it is not hard to check that $\hat{f}^i(x) \notin J$ for $i = 1, \dots, S_j - 1$ and that $\hat{f}^{S_j}(x) \in A_j \subset J$. Hence $\hat{F}(x) = \hat{f}^{S_j}(x)$. Finally note that $\hat{I} \setminus \cup_{n\geq 0} \hat{f}^{-n} J$ consists only of preimages of $c$. In particular, this set supports no finite $\hat{f}$-invariant measure. Analogously as for $\hat{f}$ we define transfer operators $\tilde{\mathcal{L}}_{\hat{\Phi}}$ for $\hat{F}$ acting on functions $\hat{g} : J \to \mathbb{C}$ by $\tilde{\mathcal{L}}_{\hat{\Phi}} \hat{g}(x) = \sum_{y \in \hat{F}^{-1}x} \hat{\Phi}(y)\hat{g}(y)$ where $\hat{\Phi} : J \to \mathbb{R}$ is a weight function.

Define $\tilde{S} : J \to \mathbb{N}$, $\tilde{S}(x) = S_k$ if $\pi(x) \in [z_k, z_{k+1}) \cup (\bar{z}_{k+1}, \bar{z}_k]$. As $r_t |\hat{f}'|^t$ is the derivative of $\hat{f}$ with respect to the measure $\hat{m}_t$, the function $r_t^{\tilde{S}} |\hat{F}'|^t$ is the derivative of $\hat{F}$ with respect to $\hat{m}_t$, and as $\hat{F}$ leaves $\hat{\nu}_t = \frac{1}{\hat{\mu}(J)} \hat{h}_t \hat{m}_t|_J$ invariant, we have $\tilde{\mathcal{L}}_{(r_t^{\tilde{S}} |\hat{F}'|^t)^{-1}} (\hat{h}_t|_J) = \hat{h}_t|_J$. In terms of the weight function

$$\hat{\Phi}_t := \left( r_t^{\tilde{S}} \cdot |\hat{F}'|^t \cdot \frac{\hat{h}_t \circ \hat{F}}{\hat{h}_t} \right)^{-1}$$

this can be rewritten as $\tilde{\mathcal{L}}_{\hat{\Phi}_t} 1 = 1$. In particular, $0 \leq \hat{\Phi}_t(x) \leq 1$ for all $x$.

Suppose now that $\mu \in \mathcal{M}$ is ergodic. As $\lambda(\mu) \geq \lambda_{per} > 0$, there is a unique $\hat{f}$-invariant ergodic lift $\hat{\mu}$ of $\mu$ to $\hat{I}$, see Remark 3.5. As $\hat{\mu}(\cup_{n\geq 0} \hat{f}^{-n} J) = \hat{\mu}(\hat{I}) = 1$, it follows that $\hat{\mu}(J) > 0$. Let $\hat{\nu} = \frac{1}{\hat{\mu}(J)} \cdot \hat{\mu}|_J$. As $\hat{F}$ is the first return map on $J$, it leaves $\hat{\nu}$ invariant and $\frac{1}{\hat{\mu}(J)}$ is the expected return time to $J$ under $\hat{\nu}$. Hence

$$\begin{aligned} \sum_{k\geq 0} \hat{\nu}(A_k) \cdot S_k &= \sum_{k\geq 0} \hat{\nu}(\hat{F}^{-1} A_k) \cdot S_k = \sum_{k\geq 0} \hat{\nu} \circ \pi^{-1}([z_k, z_{k+1}) \cup (\bar{z}_{k+1}, \bar{z}_k]) \cdot S_k \\ &= \int \tilde{S}\, d\hat{\nu} = \frac{1}{\hat{\mu}(J)} < \infty\ . \end{aligned} \tag{2}$$

Denote the partition of $J$ into the sets $A_k$ by $\alpha$. As $S_k \geq k$ for all $k$, it follows that

$$H_{\hat{\nu}}(\alpha) = -\sum_{k\geq 0} \hat{\nu}(A_k) \log \hat{\nu}(A_k) < \infty$$

by taking the sum over those $k$ with $\hat{\nu}(A_k) \leq e^{-k}$ and $\hat{\nu}(A_k) > e^{-k}$ separately. As $\alpha$ is a generating partition for the dynamical system $(J, \hat{F})$ we can define the entropy of $\hat{\nu}$ under $\hat{F}$ to be $h_{\hat{F}}(\hat{\nu}) =$

$-\log\hat\Phi_{\hat\nu}\,d\hat\nu$ where $\hat\Phi_{\hat\nu}=\sum_{k\geq 0}1_{A_k}\cdot\hat\nu(A_k|\sigma(\cup_{n\geq 1}\hat F^{-n}\alpha))$. Observe that $\sum_{y\in\hat F^{-1}x}\hat\Phi_{\hat\nu}(y)=1$ for $\hat\nu$-a.e. $x\in\hat I$, and we may choose a version of $\hat\Phi_{\hat\nu}$ that satisfies this identity for all $x\in J$. In other words, $\tilde{\mathcal{L}}_{\hat\Phi_{\hat\nu}}1=1$, and it is an easy exercise to show that $\int\tilde{\mathcal{L}}_{\hat\Phi_{\hat\nu}}\hat g\,d\hat\nu=\int\hat g\,d\hat\nu$ for each $\hat g:J\to\mathbb{R}^+$. This is essentially the same setting as that studied by Ledrappier in [18], except that there shift spaces over finite alphabets are considered. In our situation the finiteness of the alphabet is replaced by the condition $H_{\hat\nu}(\alpha)<\infty$ verified above.

Our next aim is to show that

$$t\int\log|\hat F'|\,d\hat\nu=-\int\log\hat\Phi_t\,d\hat\nu-\int\tilde S\,d\hat\nu\cdot\log r_t\ . \tag{3}$$

To achieve this we must show that $\log\frac{\hat h_t}{\hat h_t\circ\hat F}$ is $\hat\nu$-integrable with integral 0. In view of [14, Lemma 2] it suffices to show that $\left(\log\frac{\hat h_t}{\hat h_t\circ\hat F}\right)^+$ is $\hat\nu$-integrable. But as

$$\log\frac{\hat h_t}{\hat h_t\circ\hat F}=\log\left(\hat\Phi_t\cdot r_t^{\tilde S}\cdot|\hat F'|^t\right)\leq\tilde S\cdot\log\left(r_t\cdot\sup_x|\hat f'(x)|^t\right)\ ,$$

this follows from $\int\tilde S\,d\hat\nu=\frac{1}{\hat\mu(J)}<\infty$.

Now we can follow the arguments in [18]:

$$\begin{aligned}
F_t(\mu) &= h_f(\mu)-t\lambda_f(\mu)=h_{\hat f}(\hat\mu)-t\lambda_{\hat f}(\hat\mu)=\hat\mu(J)\cdot\left(h_{\hat F}(\hat\nu)-t\lambda_{\hat F}(\hat\nu)\right)\\
&= \hat\mu(J)\cdot\left(\int(-\log\hat\Phi_{\hat\nu}-t\log|\hat F'|)\,d\hat\nu\right)=\hat\mu(J)\cdot\left(\int(-\log\hat\Phi_{\hat\nu}+\log\hat\Phi_t+\tilde S\log r_t)\,d\hat\nu\right)\\
&= \hat\mu(J)\cdot\int\log\frac{\hat\Phi_t}{\hat\Phi_{\hat\nu}}\,d\hat\nu+\log r_t=\hat\mu(J)\cdot\int\tilde{\mathcal{L}}_{\hat\Phi_{\hat\nu}}\left(\log\frac{\hat\Phi_t}{\hat\Phi_{\hat\nu}}\right)d\hat\nu+\log r_t\\
&\leq \hat\mu(J)\cdot\int\tilde{\mathcal{L}}_{\hat\Phi_{\hat\nu}}\left(\frac{\hat\Phi_t}{\hat\Phi_{\hat\nu}}-1\right)d\hat\nu+\log r_t\leq\hat\mu(J)\cdot\left(\int\log\tilde{\mathcal{L}}_{\hat\Phi_t}1\,d\hat\nu-\int\log\tilde{\mathcal{L}}_{\hat\Phi_{\hat\nu}}1\,d\hat\nu\right)+\log r_t\\
&= \log r_t
\end{aligned}$$

as $\tilde{\mathcal{L}}_{\hat\Phi_t}1=\tilde{\mathcal{L}}_{\hat\Phi_{\hat\nu}}1=1$. Equality occurs if and only if $\hat\Phi_t=\hat\Phi_{\hat\nu}$ $\hat\nu$-a.s. But as in [18] this is the case for the measure $\hat\mu=\hat\mu_t=\hat h_t\hat m_t$. Hence $P(t)=\sup\{F_t(\mu):\mu\in\mathcal{M},\mu\text{ ergodic}\}=\log r_t$. □

The last part of the proof yields at the same time

**4.6 Corollary** *If $\lambda_{per}>0$ and if $\mu_t=\hat\mu_t\circ\pi^{-1}$ denotes the invariant measure constructed in the appendix, then $\mu_t\in\mathcal{E}(t)$.*

# 5 Stability

In this section we will show that when $\lambda_{per}\neq 0$ (*i.e.* $f$ is either Collet-Eckmann or has a stable periodic orbit), $F_t$ is stable for $t\approx 1$. It is shown that stability also occurs for certain nonhyperbolic maps. At the end of this section we give an example where stability is violated.

**5.1 Theorem**

*a) If $\lambda_{per}<0$, then $F_t$ is stable for all $t$.*

*b) If $\lambda_{per} = 0$ and if $f$ either has a neutral periodic point or is infinitely renormalizable, then $F_t$ is stable for all $t$.*

*c) If $\lambda_{per} > 0$,* i.e. *if $f$ is a Collet-Eckmann map, then $F_t$ is stable for $t \approx 1$, and $\mathcal{E}(t)$ consists of the single measure $\mu_t$ constructed in the appendix.*

In Theorem 3.4 we already characterized $\mathcal{E}(t)$ when $f$ has a stable or neutral periodic orbit, and when $f$ is infinitely renormalizable.

**5.2 Remark** We conjecture that a more general statement than b) is true, namely that $F_t$ is stable for all $t$ if $f$ has a persistently recurrent critical point. For Fibonacci maps *e.g.* this is true, see Proposition 2.9 and Corollary 2.6.

*Proof of Theorem 5.1:* Suppose that $f$ has a stable or neutral periodic point $p$. Let $X$ be the complement of the basin of orb$(p)$. Since $f$ has negative Schwarzian derivative, there exists $\varepsilon > 0$ such that $(c-\varepsilon, c+\varepsilon) \cap X = \emptyset$. In particular, $\mu(c-\varepsilon, c+\varepsilon) = 0$ for all $\mu \in \mathcal{M}$. Therefore $\mu \mapsto \lambda(\mu)$ is continuous, and it follows from Lemma 2.11 that $F_t$ is stable.

If $f$ is infinitely renormalizable, there exist $n_i$-periodic points $p_i$ such that $p_{i+1} \in (p_i, \bar{p}_i)$ and such that $f^{n_i}([p_i, \bar{p}_i]) \subset [p_i, \bar{p}_i]$. The intervals $[p_i, \bar{p}_i]$ are called *restrictive*. Fix some $p_i$ and let $R_i = (p_i, \bar{p}_i)$. Each measure $\nu \in \mathcal{M}$ can be split into $\nu^{R_i} + \nu^{X_i}$ where $\operatorname{supp}(\nu^{X_i}) \subset X_i := I \setminus \cup_j f^{-j}(R_i)$, and $\operatorname{supp}(\nu^{R_i}) \subset \operatorname{orb}([f^{n_i}(c), f^{2n_i}(c)])$. Because $[f^{n_i}(c), f^{2n_i}(c)] \subset (p_i, \bar{p}_i)$, there exists $\delta > 0$ independent of $\nu$ such that $d(\operatorname{supp}(\nu^{R_i}), \operatorname{supp}(\nu^{X_i})) \geq \delta$. Therefore, if $(\mu_n) \subset \mathcal{M}$ and $\mu_n \to \mu$ weakly, also $\mu_n^{R_i} \to \mu^{R_i}$ and $\mu_n^{X_i} \to \mu^{X_i}$. Assume that also $F_t(\mu_n) \to P(t)$, and that $\mu^{X_i}$ is not identically 0. Then because $\psi$ is continuous on $X_i$, $\lim_n F_t(\frac{\mu_n^{X_i}}{\mu_n(X_i)}) \leq F_t(\frac{\mu^{X_i}}{\mu(X_i)}) \leq P(t)$. However, $F_t(\mu_n) = \mu_n(X_i)\, F_t(\frac{\mu_n^{X_i}}{\mu_n(X_i)}) + \mu_n(I \setminus X_i)\, F_t(\frac{\mu_n^{R_i}}{\mu_n(I \setminus X_i)})$ and $F_t(\frac{\mu_n^{R_i}}{\mu_n(I \setminus X_i)}) \leq P(t)$. Therefore also $P(t) \leq \lim_n F_t(\frac{\mu_n^{X_i}}{\mu_n(X_i)})$. As this is true for every restrictive interval, $\mu$ can be decomposed in equilibrium states on $X_i \setminus X_{i-1}$ and a measure $\mu_{\omega(c)}$ on $\omega(c)$. By Lemma 3.10 $\mu_{\omega(c)}$ is a nonliftable equilibrium state. Hence $\mu \in \mathcal{E}(t)$.

For the proof of c) we need

**5.3 Proposition** *Suppose that $f$ is a Collet-Eckmann map. For $g \in C(I)$ and $\gamma \in \mathbb{R}$ let*

$$\Sigma_n g(x) = \sum_{k=0}^{n-1} g(f^k(x)) \quad \text{and} \quad B_n = \left\{ x \in I : \frac{1}{n}\Sigma_n g(x) > \gamma \right\} .$$

*If $\nu \in \mathcal{M}$ is ergodic and satisfies $\int g\, d\nu > \gamma$ and $\lambda(\nu) > 0$, then*

$$\liminf_{n\to\infty} \frac{1}{n} \log m_t(B_n) \geq F_t(\nu) - P(t)$$

*for $t \approx 1$, where $m_t$ is the $t$-conformal measure for $f$, see Remark 4.3.*

*Proof:* We combine the large deviations approach that can be found *e.g.* in the proof of [34, Theorem 1] with some distortion estimates based on negative Schwarzian derivative.

Let $0 < \varepsilon < \int g\, d\nu - \gamma$, and for $n \in \mathbb{N}$ let

$$\mathcal{Z}_n' = \left\{ Z \in \mathcal{Z}_n : \exists x \in Z \text{ such that } \frac{1}{n}\Sigma_n g(x) > \gamma + \varepsilon \text{ and } \frac{1}{n}\log|(f^n)'(x)| < \lambda(\nu) + \varepsilon \right\}$$

where $\mathcal{Z}_n$ denotes the family of monotonicity intervals of $f^n$. By Birkhoff's ergodic theorem there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ holds

$$\nu(\cup \mathcal{Z}'_n) > \frac{7}{8} .$$

Recall that $Z_n[x]$ denotes that element of $\mathcal{Z}_n$ that contains $x$. The point $f^n(x)$ subdivides the interval $f^n Z_n[x]$ into two parts, say $L_n[x]$ and $R_n[x]$. Let $r_n^t(x) = \min\{m_t(L_n[x]), m_t(R_n[x])\}$. $r_n^t(x)$ is the "$m_t$-distance" of $f^n(x)$ to the endpoints of $f^n Z_n[x]$. In particular, $r_n^1(x) = r_n(x)$ with $r_n(x)$ as defined in equation (1), because $m_1$ is Lebesgue measure. For $n \in \mathbb{N}$ and $\delta > 0$ let

$$U_n^\delta = \{x \in I : r_n^t(x) > \delta, m_t(f^n Z_n[x]) > 3\delta\} .$$

As $\lim_{n\to\infty} \sup_{Z \in \mathcal{Z}_n} m_t(Z) = 0$ by (12), there is $n(\delta) > 0$ such that $m_t(Z) < \frac{\delta}{2}$ for each $Z \in \mathcal{Z}_{n(\delta)}$. Let $\eta = \eta(\delta) = \min\{m_1(Z) : Z \in \mathcal{Z}_{n(\delta)}\}$. Then each interval of $m_t$-length at least $\delta$ has $m_1$-length at least $\eta$. In particular, if $r_n^t(x) \geq \delta$ for some $x \in I$, then $r_n(x) > \eta$.

We describe $U_n^\delta$ in terms of the Markov extension introduced in Section 3. Denote by $i : I \to D_1 \subset \hat{I}$ the canonical embedding of $I$ into $\hat{I}$. Then $x \in U_n^\delta$ if and only if $\hat{f}^n(i(x)) \in V_\delta$ where $V_\delta$ is the set of points $y \in \hat{I}$ that belong to some $D_k \subset \hat{I}$ with $\hat{m}_t(D_k) > 3\delta$ and have at least $\hat{m}_t$-distance $\delta$ to the endpoints of this $D_k$. If $\hat{\nu}_n = \frac{1}{n}\sum_{k=0}^{n-1} \hat{\nu}_1 \circ \hat{f}^{-k}$ for $\nu \in \mathcal{M}$ is as in Remark 3.5, then $\nu(U_k^\delta) = \hat{\nu}_1(\hat{f}^{-k}(V_\delta))$ so that

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \nu(U_k^\delta) = \liminf_{n\to\infty} \hat{\nu}_n(V_\delta) \geq \hat{\nu}(V_\delta)$$

because $\hat{\nu}_n \to \hat{\nu}$ vaguely, where $\hat{\nu}$ is the $\hat{f}$-invariant lift of $\nu$ to $\hat{I}$. As $\hat{m}_t$ has no atoms, the set $\hat{I} \setminus \cup_{\delta>0} V_\delta$ contains only the countably many endpoints of intervals $D_k$ and thus cannot support any finite $\hat{f}$-invariant measure. Therefore, given $\frac{1}{2} < d < 1$, we can fix $\delta > 0$ such that $\hat{\nu}(V_\delta) > d$. It follows that there is a set $\Gamma \subseteq \mathbb{N}$ of asymptotic density at least $\frac{d}{2-d}$ such that $\nu(U_n^\delta) > \frac{d}{2}$ for all $n \in \Gamma$. Therefore, if $n \in \Gamma$ is sufficiently large, then

$$\nu(U_n^\delta \cap (\cup \mathcal{Z}'_n)) > \frac{d}{4} .$$

Because of the Koebe Principle there exists a constant $K_\eta$ such that for $Z \in \mathcal{Z}_n$ and $x \in Z \cap U_n^\delta$ holds

$$\frac{m_t(f^n(Z \cap U_n^\delta))}{m_t(Z \cap U_n^\delta)} = \frac{1}{m_t(Z \cap U_n^\delta)} \int_{Z \cap U_n^\delta} r_t^n |(f^n)'|^t \, dm_t \leq K_\eta \cdot r_t^n \, |(f^n)'(x)|^t . \tag{4}$$

Let $\mathcal{Z}''_n = \{Z \in \mathcal{Z}'_n : Z \cap U_n^\delta \neq \emptyset\}$. Then $\nu(\cup \mathcal{Z}''_n) \geq \nu(U_n^\delta \cap (\cup \mathcal{Z}'_n)) > \frac{d}{4}$ for sufficiently large $n \in \Gamma$, and the Shannon-McMillan-Breiman theorem implies that

$$\liminf_{\Gamma \ni n \to \infty} \frac{1}{n} \log \operatorname{card}(\mathcal{Z}''_n) \geq h(\nu) .$$

As $\Gamma$ has asymptotic density at least $\frac{d}{2-d}$ and as one can easily check that $\operatorname{card}\mathcal{Z}''_n \leq \operatorname{card}\mathcal{Z}''_{n+1} \leq \ldots$, it follows that

$$\liminf_{n\to\infty} \frac{1}{n} \log \operatorname{card}(\mathcal{Z}''_n) \geq \frac{d}{2-d} \cdot h(\nu) .$$

On the other hand, observing (4), each $Z \in \mathcal{Z}''_n$ satisfies

$$\begin{aligned} m_t(Z) &\geq m_t(Z \cap U_n^\delta) = m_t(f^n(Z \cap U_n^\delta)) \cdot \left(\frac{m_t(f^n(Z \cap U_n^\delta))}{m_t(Z \cap U_n^\delta)}\right)^{-1} \\ &\geq (3\delta - 2\delta) \cdot K_\eta^{-1} \cdot r_t^{-n} \, e^{-nt(\lambda(\nu)+\varepsilon)} . \end{aligned}$$

Hence,

$$\begin{aligned}
\liminf_{n\to\infty} \frac{1}{n} \log m_t(\cup \mathcal{Z}_n'') &\geq \liminf_{n\to\infty} \frac{1}{n} \log \left( \operatorname{card}(\mathcal{Z}_n'') \cdot \delta K_\eta^{-1} \cdot r_t^{-n}\, e^{-nt(\lambda(\nu)+\varepsilon)} \right) \\
&\geq \frac{d}{2-d} h(\nu) - t\lambda(\nu) - \log r_t - t\varepsilon \\
&= F_t(\nu) - \log r_t - t\varepsilon - 2\frac{1-d}{2-d} h(\nu) \ .
\end{aligned}$$

As $\varepsilon > 0$ and $\frac{1}{2} < d < 1$ were arbitrary, it remains to show that $B_n \supseteq \cup \mathcal{Z}_n''$. But this is an immediate consequence of the following observation: As $\lim_{n\to\infty} \sup\{m_1(Z) : Z \in \mathcal{Z}_n\} = 0$, we have for each $g \in C(I)$:

$$\lim_{n\to\infty} \sup_{Z\in\mathcal{Z}_n} \sup_{x,y\in Z} \left| \frac{1}{n}\Sigma_n g(x) - \frac{1}{n}\Sigma_n g(y) \right| = 0 \ . \tag{5}$$

□

*Proof of Theorem 5.1c):* If $\lambda_{per} > 0$ and if $t \approx 1$, then $f$ has a unique invariant probability measure $\mu_t$ which is absolutely continuous with respect to the conformal measure $m_t$, see the appendix. We may assume that $f$ is not renormalizable (otherwise we pass to a suitable renormalization). In order not to overload the notation, we denote $\mu_t$ simply by $\mu$ in this proof.

Fix $g \in BV \cap C(I)$ where $BV$ denotes the space of functions of bounded variation on $I$, and let $\gamma > \int g\, d\mu$. It is proved in [16, Theorem 1.2] that there is some $\alpha(\gamma) \leq 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \log m_t(B_n) \leq \alpha(\gamma)$$

and that (this is the important point) $\alpha(\gamma) < 0$ provided $\sigma_g^2 > 0$. [2] Here

$$\sigma_g^2 = \int g_0^2\, d\mu + 2\sum_{n=1}^{\infty} \int g_0 \cdot (g_0 \circ f^n)\, d\mu$$

and $g_0 = g - \int g\, d\mu$. Because of (15) of the appendix the terms of this series decrease exponentially. Observing that $\lambda(\nu) \geq \lambda_{per} > 0$ for each $\nu \in \mathcal{M}$ by Proposition 3.1, we conclude from the preceding proposition that

$$\sup\{F_t(\nu) - P(t) : \nu \in \mathcal{M},\ \nu \text{ ergodic}, \int g\, d\nu > \gamma\} \leq \alpha(\gamma) < 0 \tag{6}$$

if $\sigma_g^2 > 0$.

Suppose now that $\nu_n \in \mathcal{M}$, $\nu_n \to \nu$, and that $F_t(\nu_n) \to P(t)$. We only must show that $\nu = \mu$, because $F_t(\mu) = P(t)$ by Corollary 4.6. To this end let $J$ be any open subinterval of $I$ and suppose that $\tau_n : I \to \mathbb{R}$, $n \geq 1$, is a sequence of nonnegative trapezoidal functions converging to $\chi_J$ monotonously from below. It is obviously sufficient to show that $\int \tau_n\, d\mu = \int \tau_n\, d\nu$ for sufficiently large $n$. We accomplish this in two steps: First we prove that this equality holds for all $n$ for which $\sigma_{\tau_n}^2 > 0$, and then we verify $\sigma_{\tau_n}^2 > 0$ for sufficiently large $n$.

[2] This was stated explicitly only for $t = 1$ in [16]. However, once $m_t$ and $\mu$ are constructed, the same proof (that relies on a representation of Laplace transforms in terms of transfer operators) applies for $t \approx 1$.

For the first step we consider any $g \in BV \cap C(I)$ with $\sigma_g^2 > 0$. Let $\nu_n = \int \nu_n^x \, d\nu_n(x)$ be the ergodic decomposition of $\nu_n$ into ergodic measures $\nu_n^x$. As the entropy is an affine upper semicontinuous function of the measure, we have $\int h(\nu_n^x) \, d\nu_n(x) = h(\nu_n)$, and for the Lyapunov exponent the analogous identity is obviously satisfied. Therefore

$$\int F_t(\nu_n^x) \, d\nu_n(x) = F_t(\nu_n) \to P(t)$$

where $F_t(\nu_n^x) \leq P(t)$ for each $x$. Hence, given $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ there is $A_n \subseteq I$ such that $\nu(A_n) > 1 - \frac{\varepsilon}{\|g\|_\infty}$ and such that $F_t(\nu_n^x) - P(t) > \alpha(\gamma)$ for each $x \in A_n$. As $\sigma_g^2 > 0$, it follows from (6) that $\int g \, d\nu_n^x \leq \gamma$ for each $x \in A_n$, and we can conclude that for $n \geq n_0$

$$\int g \, d\nu_n = \int \left( \int g \, d\nu_n^x \right) d\nu_n(x) \leq \gamma + \frac{\varepsilon}{\|g\|_\infty} \cdot \|g\|_\infty = \gamma + \varepsilon \; .$$

Hence $\int g \, d\nu = \lim_{n\to\infty} \int g \, d\nu_n \leq \gamma$, and as $\gamma > \int g \, d\mu$ was arbitrary, it follows that $\int g \, d\nu \leq \int g \, d\mu$. The same consideration applied to $-g$ yields

$$\int g \, d\nu = \int g \, d\mu, \quad \text{if } \sigma_g^2 > 0.$$

We still have to show $\sigma_{g_n}^2 > 0$ for $n$ sufficiently large. Because of (15) of the appendix there are constants $C > 0$ and $\rho \in (0,1)$ such that for bounded measurable $g_1, g_2 : I \to \mathbb{R}$ with $g \in BV$ holds

$$\left| \int g_1 \cdot (g_2 \circ f^n) \, d\mu \right| \leq C \cdot \rho^n \cdot \operatorname{var}(g_1) \cdot \sup |g_2| \tag{7}$$

provided that $\int g_1 \, d\mu = \int g_2 \, d\mu = 0$.

Let $T : L_\mu^2 \to L_\mu^2$, $g \mapsto g \circ f$, and denote by $T^*$ the dual operator of $T$ on $L_\mu^2$. Then $T^* 1 = 1$ as $\mu$ is $f$-invariant, and for each $g \in BV$ with $\int g \, d\mu = 0$ holds

$$\sum_{k=0}^{\infty} \|(T^*)^k g\|_2 = \sum_{k=0}^{\infty} \left| \int g \cdot T^k (T^{*k} g) \, d\mu \right|^{\frac{1}{2}} \leq \sum_{k=0}^{\infty} \left( C \cdot \rho^k \cdot \operatorname{var}(g) \cdot \underbrace{\sup |T^{*k} g|}_{\leq \sup |g|} \right)^{\frac{1}{2}} < \infty \; .$$

Hence $g_\infty := \sum_{k=0}^{\infty} (T^{*k}) g$ exists in $L_\mu^2$, and if $\sigma_g^2 = 0$, then $g = T(T^* g_\infty) - T^* g_\infty$ $\mu$-a.e. (see [28] or [11, Lemma 9.1]).

Suppose now that $g = \chi_A - \mu(A)$ for some $A \subseteq I$, and denote $G := T^* g_\infty$. Then (see [28] for this argument)

$$e^{2\pi i G} \circ f = e^{2\pi i T G} = e^{2\pi i G} e^{2\pi i g} = e^{2\pi i G} e^{-2\pi i \mu(A)} \; ,$$

and as $(f, \mu)$ is mixing, this implies $\mu(A) = 0$ or $\mu(A) = 1$. In particular, if $A = J$ is a proper, nontrivial subinterval of $[c_2, c_1]$ (recall we assumed that $f$ is not renormalizable), then $0 < m_t(J) < 1$ and $\sigma_g^2 > 0$. Observe that in this case $\|g\|_\infty = 1$ and $\operatorname{var}(g) = 2$.

Recall that $\tau_n \geq 0$ are trapezoidal functions converging to $\chi_J$ monotonously from below. Let $g_n = \tau_n - \mu(J)$. We are going to show that $\lim_{n\to\infty} \sigma_{\tau_n}^2 = \lim_{n\to\infty} \sigma_{g_n}^2 = \sigma_g^2 > 0$. Observing that $\|g\|_\infty, \|g_n\|_\infty \leq 1$ and $\operatorname{var}(g), \operatorname{var}(g_n) \leq 2$ we find that for any $N > 0$,

$$|\sigma_g^2 - \sigma_{g_n}^2|$$

$$
\begin{aligned}
&= \left| \int (g^2 - g_n^2)\, d\mu + 2 \sum_{k=1}^{\infty} \int (g - g_n) \cdot (g \circ f^k)\, d\mu + 2 \sum_{k=1}^{\infty} \int g_n \cdot ((g - g_n) \circ f^k)\, d\mu \right| \\
&\leq 2 \int |g - g_n|\, d\mu + 4N \int |g - g_n|\, d\mu + 4 \sum_{k=N+1}^{\infty} 4C\rho^k
\end{aligned}
$$

in view of (7). As $\lim_{n\to\infty} \int |g - g_n|\, d\mu = \lim_{n\to\infty} \int |\chi_J - \tau_n|\, d\mu = 0$, it follows that $\liminf_{n\to\infty} \sigma_{g_n}^2 \geq \sigma_g^2 - \frac{16C}{1-\rho}\rho^{N+1}$, and in the limit $N \to \infty$ we obtain the desired estimate. □

**5.4 Example** We present a map which has no equilibrium state at all and for which, a fortiori, the free energy is unstable. This example comes from [8, Theorem 2]. Using the technique of almost restrictive intervals of [10], a map is constructed which has no absolutely continuous invariant probability measure. Instead $V(x) = \delta_p$ for $m$-a.e. $x$, where $p$ is the repelling orientation reversing fixed point. We briefly describe the combinatorics, and then explain why the map has the asserted properties.

Let $(\alpha_i)$ and $(\beta_i)$, $\alpha_i < \beta_i < \alpha_{i+1}$ for all $i$, be integer sequences. Our kneading map is defined as

$$
Q(n) = \begin{cases} 0 & \text{if } n < \alpha_1 \\ 2 & \text{if } n = \alpha_i \\ 1 & \text{if } \alpha_i < n \leq \beta_i \\ \beta_i & \text{if } \beta_i < n \leq \alpha_{i+1} \end{cases}
$$

It is not hard to verify that $Q$ satisfies $\{Q(n+j)\}_{j\geq 1} \succeq \{Q(Q^2(n)+j)\}_{j\geq 1}$ for all $n$ and for every choice of $(\alpha_i)$ and $\beta_i)$. According to [8], this means that $Q$ is *admissible*, *i.e.* that there exists unimodal maps, and in particular a quadratic map $f_a(x) = ax(1-x)$, with precisely this kneading map.

The next step is to show that $V(c) = \{\delta_p\}$. We have $p \in (\bar{z}_1, \bar{z}_0)$, and in general

$$
f^{S_n}(c) \in (z_{Q(n+1)-1}, z_{Q(n+1)}] \cup [\bar{z}_{Q(n+1)}, \bar{z}_{Q(n+1)-1}). \tag{8}
$$

Since $Q(n) = 1$ for $\alpha_i < n < \beta_i$, $f^j(c) \in (\bar{z}_1, \bar{z}_0)$ for all $S_{\alpha_i} < j < S_{\beta_i}$. By taking $\beta_i > 2^i \alpha_i$, we can at least guarantee that $\{\delta_p\} \in V(c)$. Because $Q(n)$ is large for $\beta_i < n < \alpha_{i+1}$, the orbit of $f^{S_n}(c)$ will follow the orbit of $c$ closely for these values of $n$. Therefore $\delta_p$ is indeed the only measure in $V(c)$.

The last ingredient is the *almost restrictive interval.* Since $f^{S_n}$ maps either $(z_{n-1}, z_n)$ or $(\bar{z}_n, \bar{z}_{n-1})$ in a monotone orientation preserving way onto itself, there exist an $S_n$-periodic orientation preserving point $p_n \in (z_{n-1}, z_n) \cup (\bar{z}_n, \bar{z}_{n-1})$. We call the interval $[p_n, \bar{p}_n]$ *almost restrictive* if $f^{S_n}([p_n, \bar{p}_n])$ is only slightly bigger than $[p_n, \bar{p}_n]$. ¿From formula (8) one can derive that this is the case if there is a large number $j_0$ such that $Q(n+j) = n$ for $1 \leq j < j_0$ and $Q(n+j_0) < n$ (see also [3, page 95]). In our construction, $f$ has an $S_{\beta_i}$-periodic almost restrictive interval whenever $\alpha_{i+1} - \beta_i$ is large. The idea behind almost restrictive intervals is that mass gathered inside $(p_n, \bar{p}_n)$ can only slowly escape from a neighbourhood of $\cup_{i=0}^{S_n-1} f^i([p_n, \bar{p}_n])$. Using cascades of these intervals, Johnson was able to construct maps for which $m$-a.e. point spends at least a $1-\varepsilon$ portion of its time ($\varepsilon > 0$ arbitrary) in neighbourhoods of orb$(c)$ of arbitrarily small $m$-measure. Such maps cannot have absolutely continuous invariant probability measures. For our purpose we state this result in the following form:

**5.5 Lemma** *There exists a recursive choice of $(\alpha_i)$ (where $\alpha_i$ depends only on the family $(f_a)_a$ and on $\alpha_j$ and $\beta_j$ for $j \leq i$; once $\alpha_i$ is fixed, $\beta_i$ can be chosen arbitrary), such that the map $f_a$ with the limit kneading map satisfies $V(x) = V(c) = \{\delta_p\}$.*

Choosing $\alpha_i$ as in this lemma, we obtain a map which has neither absolutely continuous nor nonliftable equilibrium states. Hence by Propositions 3.3 and 3.9, $\mathcal{E}(1) = \emptyset$. Because for $n = \beta_i$, $f^{S_n}([p_n, \bar{p}_n])$ is only slightly larger than $[p_n, \bar{p}_n]$, the multipliers of $p_{\beta_i}$ are bounded. Therefore, if $\delta_i$ is the equidistribution on $\mathrm{orb}(p_{\beta_i})$, $\lambda(\delta_i) \to 0$ and $F_t(\delta_i) \to 0$ as $i \to \infty$. But $p_{\beta_i}$ follows the critical orbit closely in the first $S_{\beta_i}$ iterates, whence $\delta_i \to \delta_p$ weakly. This gives a direct argument for the unstability of $F_t$ in this example.

# 6 Observable measures

We call a measure $\mu \in \mathcal{M}$ *observable*, if $\frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i x} \to \mu$ as $n \to \infty$ for a set of points $x$ of positive Lebesgue measure. In uniformly hyperbolic situations $\mu$ is observable if and only if $\mu \in \mathcal{E}(1)$. Here we collect some partial results in this direction. For maps with $\lambda_{per} \neq 0$ the claim of the following theorem is well-known.

**6.1 Theorem**

*a) If $f$ has an absolutely continuous invariant probability measure or a zero dimensional attractor (being a stable or neutral periodic orbit, a solenoidal attractor or a Cantor attractor), then $V(x) \subset \mathcal{E}(1)$ m-a.e.*

*b) If $F_1$ is stable, then $V(x) \cap \mathcal{E}(1) \neq \emptyset$ m-a.e.*

By a solenoidal attractor we mean the attractor that exists in the infinitely renormalizable case. If $f$ is finitely renormalizable, but $\omega(c)$ is a Cantor set which nonetheless attracts $m$-a.e. point (*i.e.* $\omega(x) \subset \omega(c)\, m$-a.e.), then $\omega(c)$ is called a *Cantor attractor.* It was shown in [5, 4] that certain unimodal maps have such Cantor attractors.

**6.2 Corollary** *In all cases of the preceding theorem each observable measure belongs to $\mathcal{E}(1)$.*

**6.3 Remark** If $\lambda_{per} \neq 0$ then Theorem 5.1 shows that there is a unique measure in $\mathcal{E}(1)$ which is at the same time the unique observable measure.

**6.4 Remark** The example in the previous section shows that an observable measure need not belong to $\mathcal{E}(1)$. On the other hand, Fibonacci maps with critical order $\ell < 2 + \varepsilon$, which are known to possess an absolutely continuous $\mu \in \mathcal{M}$ (see [20, 15]), provide examples of nonobservable equilibrium states in $\mathcal{E}(1)$ (cf. Corollary 3.11).

*Proof of Theorem 6.1:* If $f$ has an absolutely continuous invariant probability measure $\mu$, then by Proposition 3.3 $\mu \in \mathcal{E}(1)$, and Birkhoff's ergodic theorem implies that $\{\mu\} = V(x)$ $m$-a.e. If $f$ has an attractor $A$, then $\omega(x) \subset A$ for $m$-a.e. $x$, whence $\mathrm{supp}\,(\mu) \subset A$ for every $\mu \in V(x)$. But $A$ can only support equilibrium states. This is clear if $A$ is a stable or neutral periodic orbit or a solenoidal attractor. If $A$ is a Cantor attractor, then by [1, Lemma 11.1] $c$ must be persistently recurrent. Hence Lemma 3.10 yields $V(x) \subset \mathcal{E}(1)$.

For the second statement we only have to consider maps without attractor or absolutely continuous invariant probability measure. By [13, Theorem 3(b)] (see also [24, Theorem V 3.2])

$$\limsup_{n\to\infty} \frac{1}{n} \log |(f^n)'(x)| = 0 \quad m\text{-a.e.} \tag{9}$$

Because there is no attractor, the action $(\hat{I}, \hat{f})$ is recurrent $m$-a.e. in the sense that there exists $k$ and an interval $J$, $\mathrm{cl}\, J \subset \mathrm{int}\, D_k$, such that

$$\mathrm{card}\{n : \hat{f}^n(y) \in J\} = \infty \text{ for every } y \in \pi^{-1}(x) \text{ and } m\text{-a.e. } x \in I. \tag{10}$$

Choose $x$ satisfying both (9) and (10). Let $n_1 < n_2 < \ldots$ be the integers $n$ such that $\hat{f}^n(y) \in J$. By the Markov property of $(\hat{I}, \hat{f})$, $r_{n_i}(x) \geq \varepsilon$ for every $i$, where $\varepsilon := d(J, \partial D_k) > 0$. Furthermore there exist periodic points $p_i \in \pi(J)$ of (not necessarily prime) period $n_i - n_1$ that shadow the orbit $f^{n_1}(x), f^{n_1+1}(x), \ldots, f^{n_i-1}(x)$. By the Koebe Principle, there exists $K_\varepsilon \geq 1$ such that

$$\frac{1}{K_\varepsilon}|(f^{n_i-n_1})'(x_{n_1})| \leq |(f^{n_i-n_1})'(p_i)| \leq K_\varepsilon |(f^{n_i-n_1})'(x_{n_1})|.$$

Because $x$ satisfies (9) and $p_i$ is not stable, $\lim_i \frac{1}{n_i-n_1} \log|(f^{n_i-n_1})'(p_i)| = \lim_i \frac{1}{n_i} \log|(f^{n_i})'(x)| = 0$. Let $\mu$ be a weak accumulation point of $(\delta_{orb(p_i)})_i$, then by stability of $F_1$ we have $F_1(\mu) = \lim_i F_1(\delta_{orb(p_i)}) = 0$, whence $\mu \in \mathcal{E}(1)$. But as $(\delta_{orb(p_i)})_i$ and $(\frac{1}{n_i}\sum_{n=0}^{n_i-1} \delta_{f^n(x)})_i$ clearly have the same accumulation points, $V(x) \cap \mathcal{E}(1) \neq \emptyset$. □

**6.5 Remark** We do not know whether stability implies that $V(x) \subset \mathcal{E}(1)$ $m$-a.e. We want to mention, however, that Tsujii [33] introduced the following notion of weak regularity: The point $x$ is weakly regular for a mapping $f$, if the restriction of the mapping $\lambda : \nu \mapsto \lambda(\nu)$ to the closure of the set $\{\frac{1}{n}\sum_{k=0}^{n-1} \delta_{f^k x} : n > 0\}$ is continuous, and the mapping $f$ is weakly regular, if $m$-a.e. point $x$ is weakly regular for $f$. He proves among other things that a $C^2$-map $f$ is weakly regular, if all its critical values are weakly regular. In our context this means that if (9) holds for a S-unimodal map $f$ and if $f(c)$ is weakly regular, then $V(x) \subseteq \mathcal{E}(1)$ $m$-a.e. This is closely related to the fact (see [13, Theorem 4]) that if (9) holds, then $V(x) \subseteq \operatorname{conv} V(c)$ $m$-a.e.

# Appendix: Conformal and Invariant Measures for CE-Maps

In this appendix we construct the measures $m_t$ and $\mu_t$ for Collet-Eckmann maps satisfying the additional regularity assumptions mentioned in Remark 4.4, and we prove the exponential decay of correlations for $\mu_t$. We rely heavily on results and proofs in [16], in fact, what follows cannot be read independently of that paper.

Let $\hat{BV}$ denote the Banach space of functions of bounded variation on $\hat{I}$ as in [16, Sect. 2.2], and let $\hat{w} : \hat{I} \to (0, \infty]$ be the function constructed in [16, Sect. 6.2] which has exactly the singularities the invariant density for $\hat{f}$ is expected to have. Consider transfer operators $\hat{\mathcal{L}}_{\hat{\Phi}}$ acting on functions $\hat{g} : \hat{I} \to \mathbb{C}$ by $\hat{\mathcal{L}}_{\hat{\Phi}}\hat{g}(x) = \sum_{y \in \hat{f}^{-1}x} \hat{\Phi}(y)\hat{g}(y)$ where $\hat{\Phi} = \hat{\Psi}^t \cdot e^{u\hat{F}}$ with $\hat{\Psi} = (|\hat{f}'|\frac{\hat{w}\circ\hat{f}}{\hat{w}})^{-1}$ and arbitrary $\hat{F} \in \hat{BV}$. For $t \approx 1$ and $u \approx 0$ this operator acts quasicompactly on $\hat{BV}$, and if $\hat{f}$ is a nonrenormalizable Collet-Eckmann map, then it has the spectral representation $\hat{\mathcal{L}}_{\hat{\Phi}}^n = r_{t,u}^n \hat{\mathcal{P}}_{t,u} + \hat{\mathcal{L}}_{\hat{\Phi}}^n \hat{\mathcal{P}}_{t,u}^\perp$ where $r_{t,u}$ and the one-dimensional projection $\hat{\mathcal{P}}_{t,u}$ depend analytically on $t, u \in \mathbb{C}$. For $t = 1$ and $u = 0$ we have $r_{1,0} = 1$ and $\hat{\mathcal{P}}_{1,0} = \int \hat{g}\, dx \cdot \tilde{h}$ where $\hat{h} := \tilde{h}\hat{w}$ is the unique invariant probability density for $\hat{f}$, see [16, Sect. 4.2, Sect. 5 and (2.3)].

Let $u = 0$, $t \in \mathbb{R}$, $t \approx 1$. In this situation we suppress the subscript $u$. As $\hat{\mathcal{L}}_{\hat{\Phi}} \geq 0$, also $\hat{\mathcal{P}}_t \geq 0$, so $\hat{\mathcal{P}}_t(\hat{g}) = \gamma_t(\hat{g}) \cdot \tilde{h}_t$ for some positive linear functional $\gamma_t : \hat{BV} \to \mathbb{R}$ and $\tilde{h}_t = r_t^{-1}\hat{\mathcal{P}}_t\tilde{h}_t$ is normalized such that $\gamma_t(\tilde{h}_t) = 1$. As we tacitly assume that all preimages of the critical point are doubled (*cf.* [16, Sect. 3]), all $\hat{Z} \in \cup_n \hat{\mathcal{Z}}_n$ are compact, and the finitely additive measure $\tilde{m}_t$ on $\cup_n \hat{\mathcal{Z}}_n$ specified by $\tilde{m}_t(\hat{Z}) = \gamma_t(1_{\hat{Z}})$ extends to a $\sigma$-finite Borel measure on $\hat{I}$. There are constants $C > 0$, $\Theta \in (0, r_t)$ such that

$$\tilde{m}_t(\hat{Z}) = \gamma_t(1_{\hat{Z}}) = r_t^{-n}\gamma_t(\hat{\mathcal{L}}_{\hat{\Phi}}^n 1_{\hat{Z}}) < C\,(\Theta\, r_t^{-1})^n\ , \tag{11}$$

see [16, Lemma 4.1].

In order to show that $\int \hat{g}\, d\tilde{m}_t = \gamma_t(\hat{g})$ for all $\hat{g} \in \hat{BV}$ we follow [29]: Let $\hat{\alpha}_n : \hat{BV} \to \hat{BV}$ be the projection on piecewise constant functions defined in [16, Sect. 4.1]. If $\hat{g}|_{D_k} = 0$ for all except finitely many $k$, then $\hat{\alpha}_n \hat{g}$ is a finite linear combination of indicator functions $1_{\hat{Z}}$, whence $\int \hat{\alpha}_n \hat{g}\, d\tilde{m}_t = \gamma_t(\hat{\alpha}_n \hat{g})$. As $|\int \hat{\alpha}_n \hat{g}\, d\tilde{m}_t - \int \hat{g}\, d\tilde{m}_t| \leq \mathrm{var}(\hat{g}) \cdot C\,(\Theta\, r_t^{-1})^n \to 0$ by (11) and as

$$|\gamma_t(\hat{\alpha}_n \hat{g}) - \gamma_t(\hat{g})| = r_t^{-n}\, |\gamma_t(\hat{\mathcal{L}}_{\hat{\Phi}}^n \hat{\alpha}_n \hat{g} - \hat{\mathcal{L}}_{\hat{\Phi}} \hat{g})| \leq C\,(\Theta\, r_t)^{-n} \to 0$$

by [16, Lemma 4.2], it follows that $\int \hat{g}\, d\tilde{m}_t = \gamma_t(\hat{g})$. For arbitrary $\hat{g} \in \hat{BV}$ let $\hat{g}_k = \hat{g} \cdot 1_{\sqcup_{j \leq k} D_j}$. Then $\|\hat{g} - \hat{g}_k\|_{\hat{BV}} \to 0$ by definition of $\|\,.\,\|_{\hat{BV}}$ [16, Sect. 2.2] so that $\int \hat{g}\, d\tilde{m}_t = \lim_k \int \hat{g}_k\, d\tilde{m}_t = \lim_k \gamma_t(\hat{g}_k) = \gamma_t(\hat{g})$. Hence $\int \hat{\mathcal{L}}_{\hat{\Phi}} \hat{g}\, d\tilde{m}_t = \gamma_t(\hat{\mathcal{L}}_{\hat{\Phi}} \hat{g}) = r_t \gamma_t(\hat{g}) = r_t \int \hat{g}\, d\tilde{m}_t$ for all $\hat{g} \in \hat{BV}$, and as $\hat{BV}$ is dense in $L^1_{\tilde{m}_t}$ it follows that $\hat{\mathcal{L}}_{\hat{\Phi}}^* \tilde{m}_t = r_t \tilde{m}_t$.

Let $\hat{m}_t = \frac{1}{\hat{w}} \tilde{m}_t$. As $\hat{w} \cdot \hat{\mathcal{L}}_{\hat{\Phi}}(\frac{\hat{g}}{\hat{w}}) = \mathcal{L}_t(\hat{g})$ ($\mathcal{L}_t$ from Section 4 of *this* paper), it follows that

$$\mathcal{L}_t^* \hat{m}_t = r_t \hat{m}_t \quad \text{and} \quad \mathcal{L}_t \hat{h}_t = r_t \hat{h}_t$$

and, because of (11), $\hat{m}_t$ has no atoms. In particular

$$\lim_{n\to\infty} \sup_{\hat{Z} \in \hat{\mathcal{Z}}_n} \hat{m}_t(\hat{Z}) = 0\ . \tag{12}$$

Let $\hat{Z}, \hat{Z}' \in \hat{\mathcal{Z}}_n$ for some $n$. If $\pi\hat{Z} = \pi\hat{Z}'$, then $\mathcal{L}_t^n 1_{\hat{Z}} = \mathcal{L}_t^n 1_{\hat{Z}'}$ by definition of $\mathcal{L}_t$. It follows that

$$\hat{m}_t(\hat{Z}) = r_t^{-n}\, \hat{m}_t(\mathcal{L}_t^n 1_{\hat{Z}}) = r_t^{-n}\, \hat{m}_t(\mathcal{L}_t^n 1_{\hat{Z}'}) = \hat{m}_t(\hat{Z}')\ , \tag{13}$$

*i.e.* $\hat{m}_t$ is level independent and thus gives rise to a measure $m_t$ on $I$ defined by

$$m_t \circ \pi(U) = \hat{m}_t(U) \quad \text{if } U \subseteq D_n \text{ for some } n. \tag{14}$$

Let $\hat{\mu}_t = \hat{h}_t \cdot \hat{m}_t = \tilde{h} \cdot \tilde{m}_t$. Then $\hat{\mu}_t$ is $\hat{f}$-invariant, and $\mu_t := \hat{\mu}_t \circ \pi^{-1}$ is $f$-invariant. Obviously $\mu_t \ll m_t$.

Next we show that $(f, \mu_t)$ has exponentially decreasing correlations in the following sense: There are constants $C > 0$ and $\rho \in (0,1)$ such that for all bounded $g_1, g_2 : I \to \mathbb{R}$ such that $g_1 \in BV$ and $\int g_1\, dm_t = \int g_2\, dm_t = 0$ holds

$$\left| \int g_1 \cdot (g_2 \circ f^n)\, d\mu_t \right| \leq C \cdot \rho^n \cdot \mathrm{var}(g_1) \cdot \sup |g_2|\ . \tag{15}$$

The proof is essentially the same as that in [16, Appendix B] for the case $t = 1$, except that the following estimate must be used:

$$\int_{D_i} |g_2 \circ \pi|\, \hat{w}\, d\hat{m}_t \leq \sup |g_2| \cdot \tilde{m}_t(D_i) \leq \|\gamma_t\| \cdot \sup |g_2|$$

because $\tilde{m}_t(D_i) = \gamma_t(1_{D_i}) \leq \|\gamma_t\| \cdot \|1_{D_i}\|_{\hat{BV}} = \|\gamma_t\| < \infty$.

Finally let $BV_{\hat{w}} = \{\hat{w} \cdot \hat{g} : \hat{g} \in \hat{BV}\}$. $BV_{\hat{w}}$ is in a natural way isomorphic to $\hat{BV}$, and it is this space that we call $B$ in Section 4. In particular, $\frac{d\hat{\mu}_t}{d\hat{m}_t} = \tilde{h}_t\, \hat{w}_t =: \hat{h}_t \in BV_{\hat{w}}$.

We summarize the main points of this discussion:

**6.6 Theorem** *Let $f : I \to I$ be a Collet-Eckmann map satisfying the additional regularity assumptions mentioned in Remark 4.4 and denote its Markov extension by $\hat{f} : \hat{I} \to \hat{I}$. Let $t \approx 1$. Then there are a Borel measure $\hat{m}_t$ on $\hat{I}$, a density $\hat{h}_t : \hat{I} \to (0,\infty)$ and a real number $r_t > 0$ such that* *a)* $\mathcal{L}_t^* \hat{m}_t = r_t \hat{m}_t$*,* $\mathcal{L}_t \hat{h}_t = r_t \hat{h}_t$*, and* $r_t$ *depends analytically on* $t$ *for* $t \approx 1$.

*b) The spectral projection* $\mathcal{P}_t$ *associated with the eigenvalue* $r_t$ *of* $\mathcal{L}_t$ *can be represented as* $\mathcal{P}_t \hat{g} = \hat{h}_t \cdot \int \hat{g}\, d\hat{m}_t$.

*c) Denote by* $m_t$ *the measure on* $I$ *defined in (14). Then* $\mu_t := (\hat{h}_t \hat{m}_t) \circ \pi^{-1} = h_t m_t$ *is an* $f$*-invariant measure with density* $h_t(x) = \sum_{\hat{x} \in \pi^{-1} x} \hat{h}_t(\hat{x})$*. It has exponentially decreasing correlations in the sense of inequality (15).*

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