Planar Embeddings of Inverse Limit Spaces of Unimodal Maps

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March 28, 2001

Abstract

For an arbitrary C^r unimodal map f, its inverse limit space X is embedded in a planar region W in such a way that the induced homeomorphism $\hat{f} : X \to X$ conjugates to a Lipschitz map $g : A \subset W \to A$. Furthermore g can be extended, in a C^r manner, to the rest of W. In the course of the proof a characterization of the end-points of X in terms of symbolic dynamics is given.

1 Introduction

One of the interesting features of inverse limit spaces is that they often model attractors of dynamical systems. For example, the attracting set of Smale's horseshoe is homeomorphic to the inverse limit space of a *full* unimodal map, i.e. a unimodal map that is topologically conjugate to $f(x) = 1 - 2x^2$ on $[-1, 1]$.

Let $f : [-1, 1] \to [-1, 1]$ be a unimodal map with critical point c. The interesting dynamics take place on the *dynamical core* $[f^2(c), f(c)]$. We consider the class $\mathcal{X}$ of inverse limit spaces with a single unimodal bonding map $f|[f^2(c), f(c)]$. For $X = ([f^2(c), f(c)], f) \in \mathcal{X}$ the induced homeomorphism is denoted by $\hat{f}$. An old result of Bing [5] states that each $X \in \mathcal{X}$ can be embedded in $\mathbb{R}^2$. But this result gives little insight what the embedded space, say $A = \varphi(X)$, looks like, neither is any smoothness of $g = \varphi \circ \hat{f} \circ \varphi^{-1}$, i.e. the homeomorphism acting on A, implemented. There are a few results in this direction. Barge and Holte showed that if f has a periodic critical point, then $(X, \hat{f})$ is topologically conjugate to (A, g), where $g : W \subset \mathbb{R}^2 \to W$ is a *generalized horseshoe map*. See [3, 12] for the details. In [7], Brucks and Diamond used symbolic dynamics to describe A in detail for a certain class of unimodal maps. We will extend this result with respect to both the class of unimodal maps and the smoothness properties of g.

Theorem 1 *Let f be a C^r unimodal map for some $r \geq 0$. Then there exists a compact set A contained in a topological disk $W \subset \mathbb{R}^2$ and a map $g : W \to W$ such that*

- *$g(A) = A$ and (A, g) is topologically conjugate to $(X, \hat{f})$.*

*Supported by the Göran Gustafsson Foundation
1991 Mathematics Subject Classification. 58H20, 58F03, 58F13
Keywords and phrases. inverse limit space, unimodal map, attractor, end-point

- *g is a Lipschitz homeomorphism on A and a C^r diffeomorphism on* $(\mathrm{int\,W}) \setminus \mathrm{A}$.
- *A attracts an open neighbourhood of itself.*

The map g has many similarities with the Hénon map $H_{a,b} : (x, y) \mapsto (1 - ax^2 + y, bx)$. g is constructed such that it reverses orientation, as does the Hénon map for $b > 0$. The attractor A roughly looks like a Hénon attractor. It can be shown that A is the closure of the unstable manifold $W^u(p)$ of the unique fixed point $p \in A$. It has a hyperbolically stable foliation, roughly transversal to the leaves of $W^u(p)$, and an unstable foliation, roughly parallel to the leaves of $W^u(p)$.

A major difference with the Hénon map, and the reason why g is not C^r on A, seems to be the occurrence of "wandering arcs". By this we mean arcs J whose iterates $g^n(J)$ lie roughly in the unstable direction, are not attracted to a periodic orbit, but which nonetheless tend to zero in length: $\mathrm{diam}(g^n(J)) \to 0$. This corresponds to the interval case. An interval map f has a *wandering interval* J if $f^n|J$ is homeomorphic for each n and $f^n(J)$ does not converge to a periodic orbit. As was shown in [15], no C^2 interval map (satisfying a non-flatness condition) admits wandering intervals. Also in a C^1 setting, it seems hard to combine the existence of a wandering interval J with a (non-uniform) expanding structure off the orbit of J (cf. [6]).

To our knowledge, a smooth embedding of X has been constructed only in a few cases. In the *generalized horseshoe* case, there exist stable periodic orbits which attract the arc J. So there is no problem in that case. For the full unimodal map, Misiurewicz [16] established a C^∞ embedding in $\mathbb{R}^3$. C^1 embeddings in $\mathbb{R}^2$ were constructed by Barge [2] and Szczechla [18] (using the assumption that the fixed point -1 of f is neutral). Interesting in this context is the result of Gambaudo et al. [10], showing that there exists a Hénon map with an attractor of Feigenbaum-Coullet-Tresser type. We don't know if the global attracting set of their example is homeomorphic to the inverse limit space of a Feigenbaum map.

At first sight, an inverse limit space $X \in \mathcal{X}$ resembles locally a Cantor set of arcs. This is not true in general. There are in fact many phenomena that make X inhomogeneous, one of them being the occurrence of *end-points*, see Section 2 for a definition. Already in the generalized horseshoe case, X has end-points, see e.g. [4]. We give a complete symbolic characterization (Proposition 2) which shows that:

Proposition 1 (cf. [4]) *If c is recurrent (i.e. $c \in \omega(c)$), but c is not periodic, then there are uncountably many end-points. The end-points of X lie dense in X if and only if $orb(c)$ is dense in* $[f^2c), f(c)]$.

Together with the results of Barge and Martin [4] this shows that X has either a finite or an uncountable number of end-points. Hénon maps can be expected to be more complicated than the spaces $X \in \mathcal{X}$. Still we hope that this approach may be useful for the study of the topology of Hénon attractors. (For a symbolic treatment of Lozi attractors, see [13, 14].)

Although $g|A$ is not C^r, the embedding is sufficiently nice to preserve some measure theoretical properties. In particular Theorem 1 enables one to construct planar versions of some of the pathologies in the theory of unimodal maps.

Corollary 1 *There exist a map $g : W \subset \mathbb{R}^2 \to W$ which is C^∞ except on its attracting set A, where g is only Lipschitz. Furthermore g satisfies one of the following sets of properties:*

1. *g admits no physical measure. More precisely, there are continuous test-functions $\chi : W \to \mathbb{R}$ such that $\frac{1}{n} \sum_{i=1}^{n} \chi \circ g^i(z)$ does not converge for Lebesgue-a.e. $z \in W$*
2. *g admits a* wild attractor *A_0. By this we mean a Cantor set $A_0 \subset A$ such that $A_0 \supset \omega_g(z)$ for Lebesgue-a.e. $z \in W$, while for a residual set of points $z \in W$, $\omega_g(z) = A$.*

The paper is organized as follows: In Section 2 we give a symbolic description (kneading theory) of the inverse limit space. Also the results concerning end-points of the inverse limit space are proved. In Section 3 we start constructing the embedding by showing that X is homeomorphic to some space $Y/\sim$, where $Y \subset \mathbb{R}^2$ has the structure of a Cantor set of lines of variable length and $\sim$ is some equivalence relation. In Section 4 some estimates are given and in Section 5 we complete the construction by bending the space $Y/\sim$ to a genuine subset of $\mathbb{R}^2$. In that section we also prove Corollary 1.

2 The Symbolic Description

Let $f : I = [-1,1] \to I$ be a unimodal map such that $f(1) = f(-1) = -1$ and $f(x) > x$ for $x \in (-1,c]$. Let $\mathrm{orb}(c) = \{c, c_1, c_2, \dots\}$ be the critical orbit. Throughout the paper we assume that f has no wandering intervals and no periodic attractor and in particular that c is not periodic. (Inverse limit spaces of unimodal maps with periodic critical points have been treated in [3, 12].) It follows that $c_2 < c < c_1$. We will restrict f to the core $[c_2, c_1]$. Denote the left and right branches of f as $f_L : [c_2, c] \to [c_3, c_1]$ and $f_R : [c, c_1] \to [c_2, c_1]$. Let $\hat{x}$ be its *symmetric point* of x, i.e. the point such that $\{x, \hat{x}\} = f^{-1} \circ f(x)$. The same notation will be used for subsets of the interval. Let ν be the *kneading invariant* of f, i.e. $\nu = e_1 e_2 e_3 \dots$, where $e_i = 0$ if $c_i < c$ and $e_i = 1$ if $c_i > c$. Let

$$\vartheta_n = (-1)^{\#\{1 \le i \le n; e_i = 1\}}.$$

By convention $\vartheta_0 = 1$. It follows that f^n has a local maximum (minimum) in c when $\vartheta_{n-1} = 1$ ($\vartheta_{n-1} = -1$). Also $|f^n - c|$ has a local maximum (minimum) in c when $\vartheta_n = -1$ ($\vartheta_n = 1$). The inverse limit space is

$$X = \{(\dots, x_{-3}, x_{-2}, x_{-1}, x_0); x_i \in I \text{ and } x_i = f(x_{i-1})\},$$

equipped with the metric $d(x,y) = \sum_{i \le 0} 2^i |x_i - y_i|$. This space is compact and connected, i.e. X is a *continuum*. Since we restrict f to the core $[c_2, c_1]$, the arc-component with end-point $(\dots -1, -1, -1)$ is neglected, except when f is the full unimodal map. Let $\hat{f} : X \to X$, defined as

$$\hat{f}((\dots, x_{-2}, x_{-1}, x_0)) = (\dots, x_{-2}, x_{-1}, x_0, f(x_0)),$$

be the *induced homeomorphism* on X. The projections $x \mapsto x_j$ will be denoted by π_j. For each $x \in X$, let the *itinerary* $\nu(x) \in \{0, *, 1\}^{\mathbb{Z}}$ be defined as, for $i \le 0$,

$$\nu_i(x) = \begin{cases} 0 & \text{if } x_i < c, \\ * & \text{if } x_i = c, \\ 1 & \text{if } x_i > c. \end{cases}$$

and for $i > 0$

$$\nu_i(x) = \begin{cases} 0 & \text{if } f^i(x_0) < c, \\ * & \text{if } f^i(x_0) = c, \\ 1 & \text{if } f^i(x_0) > c. \end{cases}$$

Clearly $x_i = y_i$ implies $\nu_j(x) = \nu_j(y)$ for all $j \ge i$. Since c is not periodic, $\nu(x)$ can contain at most one $*$. We say that $x, y \in X$ have the same *tail* if there exists N such that $\nu_n(y) = \nu_n(x)$ for all $n \le N$. Let

$$T_N(x) = \{y \in X; \nu_n(y) = \nu_n(x) \text{ for all } n \le N\}.$$

Lemma 1 *For each $x \in X$ and $N \le 0$, T_N is homeomorphic to an open, closed or half-open interval.*

Proof: $T_N(x)$ can be parameterized by its N-th coordinate. □

Abbreviate $T(x) = T_{-1}(x)$. The set of points having the same tail as x is $T_*(x) = \cup_{N \le -1} T_N(x)$. For $x \in X$ such that $x_i \neq c$ for all $i < 0$, let

$$\tau_L(x) = \sup\{n \ge 1; \nu_{-(n-1)}(x)\nu_{-(n-2)}(x) \dots \nu_{-1}(x) = e_1 e_2 \dots e_{n-1} \text{ and } \vartheta_{n-1} = -1\}$$

and

$$\tau_R(x) = \sup\{n \ge 1; \nu_{-(n-1)}(x)\nu_{-(n-2)}(x) \dots \nu_{-1}(x) = e_1 e_2 \dots e_{n-1} \text{ and } \vartheta_{n-1} = 1\}$$

In general $\tau_L(x)$ and/or $\tau_R(x)$ could be infinite. For completeness, if f is the full unimodal map and $x = (\dots 0, 0, 0)$, set $\tau_L(x) = 1$ and $\tau_R(x) = \infty$.

Lemma 2 *If $\tau_L(x), \tau_R(x) < \infty$ and $x_i \neq c$ for all $i < 0$, then $\pi_0(T(x)) = (f^{\tau_L(x)}(c), f^{\tau_R(x)}(c))$. If $x, y \in X$ are such that $\nu_i(x) = \nu_i(y)$ for all $i < 0$ except for $i = -\tau_R(x) = -\tau_R(y)$ (or $i = -\tau_L(x) = -\tau_L(y)$), then $T(x)$ and $T(y)$ have a common boundary point.*

Proof: Suppose that $n = \tau_R(x)$. Assume without loss of generality that $x_{-n} < c$. By definition of $\tau_R(x)$, x_{-n} is contained in a neighbourhood of c, namely the set of points whose itinerary starts with $e_1 \dots e_{n-1}$. As $\vartheta_{n-1} = 1$, $x_0 \le f^n(c)$. Suppose $y_0 = \sup \pi_0(T(x)) < f^n(c)$. Then there exists $y = (\dots y_{-2}, y_{-1}, y_0)$ such that $\nu_i(x) = \nu_i(y)$ for all $i < 0$, with at most one exception. In particular $y_{-n} \in [x_{-n}, c)$. Next take $U = (\dots, U_{-2}, U_{-1}, U_0)$, $X \supset U \ni y$, where $U_{-n} = (x_{-n}, c)$, $U_i = f(U_{i-1})$ for $0 \ge i > -n$ and for $i < -n$, U_i is the component of $f^{-1}(U_{i+1})$ having x_i as boundary point.

There exists $m < -n$ such that $c_1 \in U_{m+1}$, because otherwise $y \in \text{int (U)} \subset \text{T(x)}$ contradicting the definition of y_0. But if $c_1 \in U_{m+1}$, then $c \in U_m$ and $\nu_m(x) \dots \nu_{-1}(x) = e_1 \dots e_{m-1}$. Furthermore f^{m-n} has a local maximum in c, so $\vartheta_{m-n-1} = 1$. As $x_{-n} < c_{m-n} < c$, also $\vartheta_{m-1} = 1$. This contradicts the maximality of $n = \tau_R(x)$. The proof for $\tau_L(x)$ is the same.

Now for the last statement, suppose that $\{\nu_i(x)\}_{i<0}$ and $\{\nu_i(y)\}_{i<0}$ only differ at entry $-\tau_R(x) = -\tau_R(y) = -\tau_R$. Combining the above proof for x and y, we get the common boundary point $z \in X$, where $z_{-\tau_R} = c$, and $\{\nu_i(z)\}_{i<0} = \{\nu_i(x)\}_{i<0}$ except for the entry $-\tau_R$. □

Lemma 3 *If $x \in X$, then*

$$\sup \pi_0(T(x)) = \inf\{c_n; \nu_{-n+1}(x) \dots \nu_{-1}(x) = e_1 \dots e_{n-1} \text{ and } \vartheta_{n-1} = 1\} \tag{1}$$

and

$$\inf \pi_0(T(x)) = \sup\{c_n; \nu_{-n+1}(x) \dots \nu_{-1}(x) = e_1 \dots e_{n-1} \text{ and } \vartheta_{n-1} = -1\} \tag{2}$$

If additionally $\tau_L(x)$ and $\tau_R(x)$ are both infinite and $x_i \neq c$ for all $i < 0$, then $T_*(x) = T(x)$.

Proof: We prove statement (1). Statement (2) can be proven in the same way. We abbreviate $a = \inf\{c_n; \nu_{-n+1}(x) \dots \nu_{-1}(x) = e_1 \dots e_{n-1} \text{ and } \vartheta_{n-1} = 1\}$. Recall that if $\vartheta_{n-1} = 1$, $c_n = \max\{f^n(y); \nu_{-n+1}(y) \dots \nu_{-1}(y) = e_1 \dots e_{n-1}\}$. This proves that $\sup \pi_0(Tx) \le a$. Next assume that $a > \sup \pi_0(Tx)$. Take $y \in X$ such that $y_0 = \frac{1}{2}(a + \sup \pi(Tx)) < a$ and $\nu_i(y) = \nu_i(x)$ for all $i < 0$. This point is well defined. Indeed, if y_i is found, then both $f_L^{-1}(y_i)$ and $f_R^{-1}(y_i)$ exist, provided $y_i < c_1$. But if $y_i \ge c_1$ for some $i < 0$, then , knowing that $y_0 > \sup \pi_0(Tx)$, we obtain that $y_0 = f^{-i}(y_i) \ge c_{-i+1}$. Furthermore $\nu_i(y) \dots \nu_{-1}(y) = e_1 \dots e_{-i+1}$, yielding $y_0 \ge a$, a contradiction.

For the third statement, suppose $\tau_L(x) = \tau_R(x) = \infty$. Assume by contradiction that there exists $y \in T_*(x)$ with $m := \min\{i < 0; \nu_i(x) \neq \nu_i(y)\} > -\infty$. Assume also, without loss of generality, that $x_m < c < y_m$. Take $n < m$ such that $\nu_{n+1}(x) \dots \nu_{-1}(x) = e_1 \dots e_{-n-1}$ and $\vartheta_{m-n-1} = 1$. This means that f^{m-n} maps $[x_n, c]$ into $(0, c)$ and $f^{m-n}(c)$ is a local maximum of f^{m-n}. As $y_m > c$, there must be a k, $n \le k < m$, such that $\nu_k(x) \ne \nu_k(y)$, contradicting minimality of m. Therefore $T_*(x) = T(x)$. □

Inverse limit spaces X of unimodal maps are known to be chainable. This means that for every $\varepsilon > 0$, there exists a finite open cover $\{U_i\}_{i=1}^N$ of X such that $\operatorname{diam}(U_i) < \varepsilon$ and $U_i \cap U_j = \emptyset$ whenever $|i - j| > 1$ and $\{i, j\} \ne \{1, N\}$. In particular X contains no *triods*, i.e. homeomorphic copies of the letter Y. Therefore the following definition makes sense: $x \in X$ is called an *end-point* of X if for each pair of subcontinua $A, B \subset X$ such that $x \in A \cap B$ either $A \subset B$ or $B \subset A$.

Proposition 2 *Let $x \in X$ be such that $x_i \ne c$ for all $i < 0$. Then $x \in X$ is an end-point of $T(x)$ if and only if $\tau_R(x) = \infty$ and $x_0 = \sup \pi_0(T(x))$ (or $\tau_L(x) = \infty$ and $x_0 = \inf \pi_0(T(x))$). Furthermore, x is an end-point of X in this case.*

Note that if $x_i = c$ for some unique $i < 0$, then x is an end-point if and only if $\hat{f}^i(x)$ is an end-point. We can apply the Proposition 2 (and Lemma 3) to $\hat{f}^i(x)$ in this case. For the proof of Proposition 2 we need the following result of Barge and Martin, [4, Theorem 1.4].

Proposition 3 *The point $x = (\dots, x_{-2}, x_{-1}, x_0)$ is an end-point of X if and only if for every $i \le 0$, every subinterval $J_i = [a_i, b_i] \ni x_i$ and $\varepsilon > 0$, there exists an integer $N > 0$ such that if $J_{i-N} \ni x_{i-N}$ and $f^N(J_{i-N}) \subset J_i$, either $f^N(J_{i-N}) \not\supset [a_i + \varepsilon, b_i - \varepsilon]$ or x_{i-N} does not separate $f^{-N}([a_i, a_i + \varepsilon]) \cap J_{i-N}$ from $f^{-N}([b_i - \varepsilon, b_i]) \cap J_{i-N}$.*

Proof of Proposition 2: If $x_0 \in \operatorname{int} \pi_0(\mathrm{T(x)})$, then x is an interior point of $\operatorname{cl} \mathrm{T(x)}$ and therefore no end-point. If $x_0 = \sup \pi_0(Tx)$ then either $\tau_R(x) < \infty$ and $x_0 = f^{\tau_R(x)}(c)$, which we excluded by assumption, or $\tau_R(x) = \infty$. Therefore let us assume that $\tau_R(x) = \infty$ and $x_0 = \sup \pi_0(T(x))$. Take an integer $i \le 0$, an interval $J_i = [a_i, b_i] \ni x_i$ and $\varepsilon > 0$ arbitrary. We assume that $\#\{i \le j < 0; \nu_j(x) = 1\}$ is even. (The other possibility is proven similarly.) If $b_i - \varepsilon \le x_i$, then Proposition 3 holds by default. Assume therefore that $x_i < b_i - \varepsilon$. By Lemma 3, $x_0 = \inf\{c_n; \nu_{-n+1}(x) \dots \nu_{-1}(x) = e_1 \dots e_{n-1}$ and $\vartheta_{n-1} = 1\}$. Therefore we can find $N \in \mathbb{N}$ such that $\nu_{i-N+1}(x)\nu_{i-N+2}(x) \dots \nu_{-1}(x) = e_1 e_2 \dots e_{N-i-1}$, $\vartheta_{N-i-1} = 1$ (and also $\vartheta_{N-1} = 1$), and $x_i < f^N(c) < b_i - \varepsilon$. Let $J_{i-N} \supset (x_{i-N}, \hat{x}_{i-N})$ be the maximal interval such that $f^N(J_{i-N}) \subset J_i$. As $f^N(c) \in J_i$, $c \in J_{i-N}$ and by maximality $\hat{J}_{i-N} = J_{i-N}$. Therefore either $f^N(J_{i-N}) \not\supset [a_i + \varepsilon, b_i - \varepsilon]$ or x_{i-N} does not separate $f^{-N}([a_i, a_i + \varepsilon]) \cap J_{i-N}$ from $f^{-N}([b_i - \varepsilon, b_i]) \cap J_{i-N}$. It follows that x is an end-point of X and therefore of $T(x)$ too. The same argument applies if $\tau_L(x) = \infty$ and $x_0 = \inf \pi_0(T(x))$. □

Corollary 2 *If f is not a full unimodal map, then X has end-points if and only if c is recurrent. The end-points of X lie dense if and only if c has a dense orbit in the dynamical core.*

Proof: We exclude the full unimodal map since its inverse limit space is the Knaster continuum, which has an end-point associated to the orientation preserving fixed point. In general this fixed point is not contained in the dynamical core.

If c is periodic, say $f^N(c) = c$, the end-points of X are $(\dots c_1, c_2 \dots c_N, c_1 \dots c_N)$ and its shifts, see [4]. We therefore assume that c is recurrent but not periodic. Let n_1 be arbitrary and let $U_1 \ni c$ be the interval $U_1 = \{x \in I; \nu_i(x) = e_i$ for $1 \le i \le n_1\}$. As c is recurrent, there exists n_2 such that $f^{n_2}(c) \in U_1$. Take $U_2 = \{x \in I; \nu_i(x) = e_i$ for $1 \le i \le n_1 + n_2\}$. Clearly $U_2 \ni c$. Continuing this way we get a nested sequence of intervals U_i, and a sequence of integers n_i. Write

$N_i = n_1 + \ldots + n_i$ and let $\{a_i\}_{i<0}$ be the sequence in $\{0,1\}^{\mathbb{Z}_-}$ defined as $a_i = e_{N_k+i}$ whenever $N_k > -i$. It is not hard to show that there exists a point $x \in X$ such that $\ldots\nu_{-2}(x)\nu_{-1}(x) = a$ and $x_0 = \lim_k f^{N_k}(c)$. By construction, $\tau_R(x) = \infty$ or $\tau_L(x) = \infty$. By the previous proposition, x must be an end-point.

Remark: Because there are uncountably many ways to construct such a sequence U_i as can easily be seen, there are uncountably many end-points.

Conversely, if $x \in X$ were an end-point, and $\tau_R(x) = \infty$ say, there must be integers m and n such that

$$\nu_{-n-m}(x)\ldots\nu_{-n}(x)\ldots\nu_{-1}(x) = e_1\ldots e_{n+m-1} = e_1\ldots e_m e_1\ldots e_{n-1}.$$

This means that $f^m(c)$ is close to c and because n can be arbitrarily large, $c \in \omega(c)$.

Now for the second statement, assume that orb(c) is dense. Let $U \subset X$ be an arbitrary open set and let $a_{-n}\ldots a_n \in \{0,1\}^{2n+1}$ be such that $x \in U$ whenever $\nu_{-n}(x)\ldots\nu_n(x) = a_{-n}\ldots a_n$. Because orb$(c)$ is dense, there exists n_1 such that $e_{n_1-2n}\ldots e_{n_1} = a_{-n}\ldots a_n$. Starting with this number n_1, construct an end-point $x \in X$ by the above method. Then $\hat{f}^{-n-1}(x)$ is an end-point which is contained in U.

Conversely, if there exists an interval $J \subset I$ such that orb$(c) \cap J = \emptyset$, then by Proposition 3, there can be no end-point x of X such that $\pi_i(x) \in \mathrm{int}\, J$ for any $i \in \mathbb{Z}$. □

3 Representation in the Plane

Let $\lambda \in (0,1)$ be some number to be specified in Section 4. Let C be the middle $1-2\lambda$ Cantor set embedded in the (vertical) interval $[-1,1]$. The *bridges*, i.e. the closed intervals creating the Cantor set, will be coded with finite strings of zeroes and ones, according to the following parity-lexicographical order: The string $e_n e_{n+1}\ldots e_{-1} < e'_n e'_{n+1}\ldots e'_{-1}$ if $e_{k+1}e_{k+2}\ldots a_{-1} = e'_{k+1}e'_{k+2}\ldots e'_{-1}$ and either $e_k < e'_k$ and $e_{k+1}e_{k+2}\ldots e_{-1}$ contains even number of zeroes, or $e_k > e'_k$ and $e_{k+1}e_{k+2}\ldots e_1$ contains odd number of zeroes. See figure 1.

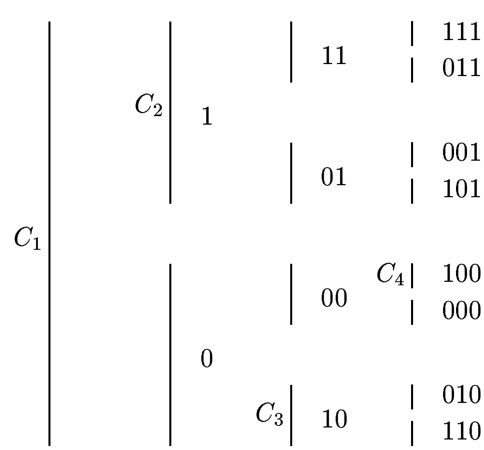

Figure 1: Construction of the Cantor set C for $\nu = 100\ldots$

By this process, the points of C will be coded by an infinite string in $\{0,1\}^{\mathbb{Z}_-}$. Note that the strings corresponding to boundary points of components of $\mathbb{R}\setminus C$ have tails consisting of only ones.

It follows that the arc component through the fixed point $(\dots p, p, p)$ will be the only part of the embedded inverse limit space that is accessible from the complement.

Define the *special bridge* C_n to be the bridge whose code equals $e_1 \dots e_{n-1}$. In particular $C_1 = [-1, 1]$. Note that $|C_n| = 2\lambda^{n-1}$ for each n.

To each $x \in X$ we can attach a point $\kappa(x) \in C$ such that $\{\nu_i(x)\}_{i<0}$ is precisely the code of $\kappa(x)$. If $x_i = c$ (i.e. $\nu_i(x) = *$) for some $i < 0$, then we attach two points to x: $\kappa_0(x)$ and $\kappa_1(x)$. The codes of these points coincide with $\{\nu_i(x)\}_{i<0}$, except for the entry i, which is 0 and 1 respectively.

Let $\pi_I : I \times C \to I$ and $\pi_C : I \times C \to C$ be the natural projections. Define $\psi : X \to I \times C$ as

$$\psi(x) = \begin{cases} (x_0, \kappa(x)) & \text{if } x_i \neq c \text{ for all } i < 0 \\ (x_0, \kappa_0(x)) \cup (x_0, \kappa_1(x)) & \text{if } x_i = c \text{ for some } i < 0 \end{cases}$$

Let $Y = \psi(X)$ and let $F : Y \to Y$ be defined as

$$F(x, y) = \begin{cases} (f(x), -1 - \lambda(y - 1)) & \text{if } x < c, \\ (f(x), 1 + \lambda(y - 1)) & \text{if } x > c, \\ \{(c_1, -1 - \lambda(y - 1)), (c_1, 1 + \lambda(y - 1))\} & \text{if } x = c. \end{cases} \qquad (3)$$

Lemma 4 *The diagram*

$$\begin{array}{ccc} X & \xrightarrow{\hat{f}} & X \\ \psi \downarrow & & \downarrow \psi \\ Y & \xrightarrow{F} & Y \end{array}$$

commutes.

The proof is straightforward. The next step is to identify points in Y. Let $a \sim b$ whenever $\{a, b\} \subset \psi(x)$ for some $x \in X$. If $a \neq b \sim a$, then $\pi_I(a) = \pi_I(b) = c_n$ for some $n > 0$, and the codes $\pi_C(a)$ and $\pi_C(b)$ differ at entry $-n$ only. In this case we denote b as $\tilde{a}$ (or a as $\tilde{b}$).

Proposition 4 *The map $\psi : X \to Y/\sim$ is a homeomorphism.*

Proof: Let us first verify that $Y/\sim$ is Hausdorff. Let $\pi_I : Y \to I$ and $\pi_C : Y \to C$ be the standard projections. Take $x \not\sim y \in Y/\sim$. If $|\pi_I(x) - \pi_I(y)| = d_I > 0$, then $\{z; |\pi_I(z) - \pi_I(x)| < \delta_I/3\}$ and $\{z; |\pi_I(z) - \pi_I(z)| < \delta_I/3\}$ are disjoint open neighbourhoods of x and y respectively. If $d_I = 0$, then $d_C = \min\{|\pi_C(x) - \pi_C(y)|, |\pi_C(x) - \pi_C(\tilde{y})|\} > 0$. Now $\{z; |\pi_C(z) - \pi_C(x)|$ or $|\pi_C(z) - \pi_C(\tilde{x})| < d_C/3\}$ and $\{z; |\pi_C(z) - \pi_C(y)|$ or $|\pi_C(z) - \pi_C(\tilde{y})| < d_C/3\}$ are disjoint open neighbourhoods of x and y respectively.

Next we prove that ψ is continuous. Pick $U \subset Y/\sim$ open, then it suffices to prove that for every $x \in U$, $\psi^{-1}(x)$ is an interior point of $\psi^{-1}(U)$. Take a neighbourhood $V \subset U$ of x. Hence $\pi_I(V)$ contains an neighbourhood of $\pi_I(x)$. Because f is continuous, the sets $\pi_i \circ \psi^{-1}(V)$ are neighbourhoods of $\pi_i \circ \psi^{-1}(x)$ for all $i \leq 0$. If it was not for the identifications in $Y/\sim$, ψ is clearly continuous. For each $y \in U$ let $V_y \subset U$ be a neighbourhood of $\tilde{y}$, if $\tilde{y}$ exists, and $V_y = \emptyset$ otherwise. Since U is open, this is possible. Then $\psi^{-1}(V \cup \cup_{y \in U} V_y) \subset \psi^{-1}(U)$ is a neighbourhood of $\psi^{-1}(x)$.

To prove the other direction, suppose by contradiction that ψ^{-1} is discontinuous. Then there exist an open set $U \subset X$ such that $\psi(U)$ is not open in $Y/\sim$. Let $x \in \psi(U)$ be a non-interior point of $\psi(U)$. As $Y/\sim$ is Hausdorff, there exist for each $y \in Y/\sim \setminus \psi(U)$ a neighbourhood $V_y \ni y$ which is disjoint from some neighbourhood of x. Since ψ is continuous, $U \cup \cup_{y \in Y/\sim \setminus \psi(U)} \psi^{-1}(V_y)$

is an open cover of X. This cover however has no finite subcover, contradicting the compactness of X. □

In order to get a clearer picture we will connect the identified points with a semi-circle. Suppose y and $\tilde{y} \in Y$ are such that $\pi_I(y) = \pi_I(\tilde{y}) = c_n$, then we draw this semi-circle to the right (left) if $\vartheta_{n-1} = 1$ ($\vartheta_{n-1} = -1$).

Proposition 5 *The above semi-circles can be drawn simultaneously crossing neither each other nor Y.*

Proof: Assume $x, \tilde{x}, y, \tilde{y} \in Y$ exist and $\pi_C(x) \leq \pi_C(y) \leq \pi_C(\tilde{x})$. Let m be such that $\pi_I(x) = c_m$, so $-m$ is the only entry in which $\kappa(\pi_C(x))$ and $\kappa(\pi_C(y))$ differ. Without loss of generality we can take $\vartheta_{m-1} = 1$, so the semi-circle connecting x and $\tilde{x}$ is directed to the right. Let also $\pi_I(y) = c_n$. As $\pi_C(x) \leq \pi_C(y) \leq \pi_C(\tilde{x})$,

$$\kappa(\pi_C(x))_i = \kappa(\pi_C(y))_i \text{ for all } -m < i < 0. \tag{4}$$

Since $\vartheta_{m-1} = 1$,

$$c_m = \max\{z_0; z \in X \text{ and } \nu_{-m+1}(z) \dots \nu_{-1}(z) = e_1 \dots e_{m-1}\},$$

and therefore $c_m = \pi_I(x) \geq \pi_I(y)$. Therefore the semi-circle connecting x and $\tilde{x}$ does not cross Y. To see that this semi-circle does not cross the semi-circle between y and $\tilde{y}$, we check two possibilities:

- $\vartheta_{n-1} = -1$. Therefore the semi-circle between y and $\tilde{y}$ is directed to the left. As $c_n \leq c_m$, there is no crossing.
- $\vartheta_{n-1} = 1$. It follows from (4) and the definition of τ_R that $n \geq m$. Therefore $\pi_C(x) \leq \pi_C(\tilde{y}) \leq \pi_C(\tilde{x})$. Because also $c_n \leq c_m$, there is again no crossing.

Remark: What this proof says is that if $(x, y) \in Y$ is such that $y \in C_n$, then $x \leq c_n$ ($x \geq u_n$) if $\vartheta_{n-1} = 1$ ($\vartheta_{n-1} = -1$). □

We can extend the map F to $I \times I$, using the same definition (3), obtaining a discontinuous map $F : I \times I \to I \times I$. Clearly F maps vertical lines into vertical lines, squeezing them by a factor λ. Therefore Y is a normally hyperbolic attracting set.

4 The Choice of λ

So far the discussion has been purely topological. However, in the construction of the embedding of the inverse limit space, we want to retain as much of the original smoothness as possible. Therefore let f be C^r for some $r \geq 1$. Let $\ell > 1$ be the *critical order*, i.e. $|f(x) - f(c)| = \mathcal{O}(1)|x - c|^\ell$. We recall that f is assumed to have no wandering intervals and periodic attractors. The following lemmas show that certain points in the critical orbit cannot lie too close together.

Lemma 5 (Nowicki [17]) *If f is C^1 and has no periodic attractor, then there exists $\lambda_0 > 0$ such that for all $n \geq 1$, $|c_n - c| \geq \lambda_0^n$.*

Proof: Let $L = \max_{x\in I} |Df(x)|$ and let K be such that $|Df(x)| \leq K|x-c|^{\ell-1}$ for all $x \in I$. Here $\ell > 1$ is the critical order. Assume by contradiction that $|c_n - c| \leq \frac{1}{2}(\frac{1}{2KL^n})^{\frac{1}{\ell-1}}$. Take $x < c$ the point closest to c such that $|Df^n(x)| = \frac{1}{2}$. In particular, f^n is monotone on (x,c). We have $\frac{1}{2} = |Df^n(x)| < K|x-c|^{\ell-1}L^n$. It follows that

$$|c - x_n| \leq |c_n - c| + |c_n - x_n| \leq \frac{1}{2}(\frac{1}{2KL^n})^{\frac{1}{\ell-1}} + \int_x^c |Df^n(y)|dy \leq \frac{1}{2}|x-c| + \frac{1}{2}|x-c|.$$

This means that f^n maps (x,c) into itself and $|Df^n| \leq \frac{1}{2}$ on (c,x). This yields a periodic attractor. Hence the lemma is true for $\lambda_0 = (\frac{1}{2})^{\frac{2\ell-1}{\ell-1}}(\frac{1}{KL})^{\frac{1}{\ell-1}}$. □

Let $\rho(n) = \min\{i \geq 1; e_i \neq e_{n+i}\}$.

Lemma 6 *If f is C^3, has negative Schwarzian derivative and no periodic attractor, then there exists $\lambda_1 > 0$ with the following property: If $n \geq 1$, $k \geq 1$ are such that*

$$e_1 \dots e_{k-1} = e_{n+1} \dots e_{n+k-1},$$

then $|c_k - c_{n+k}| \geq \lambda_1^{n+k}$.

Note that because $\nu = 10\dots$, the assumption on n and k can only hold if $n \geq 2$. Note also that Lemma 6 gives another proof of Proposition 5, since the semi-circles attached to $\{c_n\} \times C_n$ are all concentric.

Proof: By assumption $\rho(n) \geq k$ and $\rho(n)$ is the smallest integer such that $c_{\rho(n)}$ and $c_{n+\rho(n)}$ lie on different sides of c. If $\rho(n) \leq 3n$, then by Lemma 5

$$|c_{n+k} - c_k| \geq \frac{|c_{n+\rho(n)} - c_{\rho(n)}|}{L^{\rho(n)-k}} \geq \frac{\lambda_0^{\rho(n)}}{L^{\rho(n)-k}} \geq (\frac{\lambda_0}{L})^{3(n+k)}.$$

The assertion holds for $\lambda_1 = (\frac{\lambda_0}{L})^3$. For this part, the assumption on the Schwarzian derivative is not necessary. If $\rho(n) > 3n$, then, as shown in e.g. [8], $\vartheta_n = 1$ and $|f^n - c|$ has a local minimum in c. The situation is as follows (see figure 2): There exists m dividing n such that $\vartheta_m = 1$, c_m is a closest return to c. Furthermore $(c_n, c) \supset (c_m, c)$.

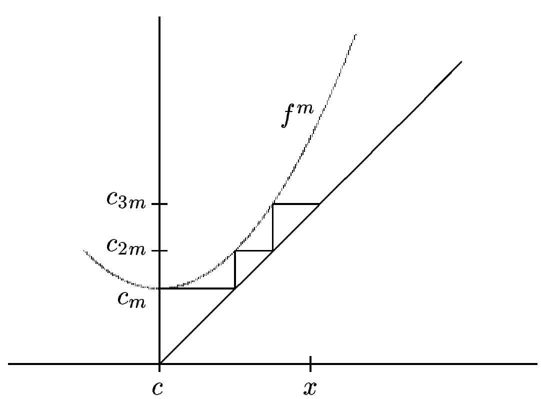

Figure 2: A close return c_m with $\vartheta_m = 1$

Therefore it suffices to show the lemma for $n = m$. The central branch of f^m is almost tangent to the diagonal. The interval (c_{2m}, c_m) will be much smaller than (c_m, c) (it may be expected that $|c_{2m} - c_m| = \mathcal{O}(|c_m - c|^\ell)$, but afterwards the intervals (c_{2m}, c_{3m}), (c_{3m}, c_{4m}), etc. can shrink no faster than geometrically, say with rate $\kappa > 0$. Take r minimal such that $rm \geq m + k$. Then, using the first part of the proof,

$$|c_{rm} - c_{(r-1)m}| \geq \kappa^{r-1} |c_{2m} - c_m| \geq \kappa^{r-1} (\frac{\lambda_0}{L})^{3m},$$

whence, for m sufficiently large, $|c_{m+k} - c_k| \geq \frac{1}{L^{rm-k}} |c_{rm} - c_{(r-1)m}| \geq \kappa^{r-1} \frac{\lambda_0^{(3m}}{L^{rm-k+3m}} \geq (\frac{\lambda_0}{L})^{4m} \geq (\frac{\lambda_0}{L})^{3(m+k)}$. Therefore the lemma is fulfilled for the same value $\lambda_1 = (\frac{\lambda_0}{L})^3$. □

Take $\lambda = \frac{1}{3}\lambda_1 < \frac{1}{3}$. Let as before C be the middle $1 - 2\lambda$ Cantor sets. The special bridges C_n (as all bridges of order n) have length $2\lambda^{n-1}$.

5 The Embedding of the Inverse Limit Space

In this section we construct a genuine embedding. Although it may be not so easy to recognize, the basic idea is to squeeze the semi-circles that were introduced in Section 3 simultaneously to points. For a function $r : B \subset \mathbb{R}^2 \to \mathbb{R}^2$ we denote the smallest Lipschitz constant as

$$\text{Lips}(r|B) = \inf\{k; |r(x) - r(y)| \leq k|x - y| \text{ for all } x, y \in B\}.$$

Let us introduce some notation. Let u_n be the middle point of the special bridge C_n. For $\alpha > 0$ and $n \geq 1$, let $C_n(\alpha) = [u_n - \frac{\alpha}{2}|C_n|, u_n + \frac{\alpha}{2}|C_n|]$, i.e. the interval concentric with C_n such that $|C_n(\alpha)| = \alpha|C_n|$. Take $\varepsilon > 0$ small and let

$$\tilde{V}_n = [c_n - \varepsilon|C_n|, c_n + \varepsilon|C_n|\,] \times C_n.$$

Then define recursively $V_1 = \tilde{V}_1$ and

$$V_{n+1} = \tilde{V}_{n+1} \cap F(V_n).$$

Therefore $\{c_n\} \times C_n \subset V_n \subset \tilde{V}_n$ for each n, and if c_n is close to c, $F(V_n) = V_{n+1}$. Let $J_n = \pi_I V_n$ and let J_n^+ be the component of $J_n \setminus \{c_n\}$ that lies to the right (left) of c_n if $\vartheta_{n-1} = 1$ ($\vartheta_{n-1} = -1$).

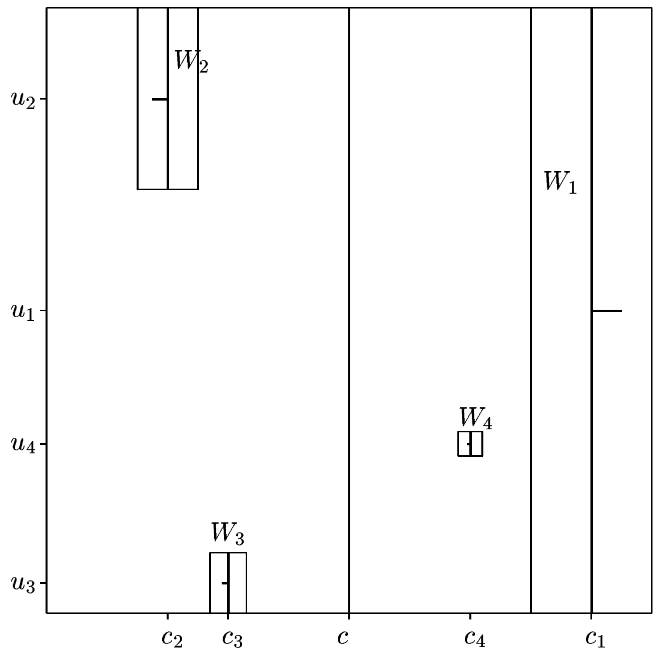

Figure 3: The sets W_i, $\{c_i\} \times C_i$ and $J_i^+ \times \{u_i\}$ for $1 \leq i \leq 4$ and $\nu = 1001\dots$

Let

$$W_n = [c_n - 2\varepsilon n^2|C_n|, c_n + 2\varepsilon n^2|C_n|\,] \times C_n,$$

and let W_n^- be the component of $W_n \setminus (\{c_n\} \times C_n)$ which lies to the left (right) if $\vartheta_{n-1} = 1$ ($\vartheta_{n-1} = -1$). Let also

$$\tilde{W}_n = [c_n - 2\varepsilon L n^2|C_n|, c_n + 2\varepsilon L n^2|C_n|\,] \times C_n(2).$$

Clearly $V_n \subset W_n \subset \tilde{W}_n$ and because $L = \max_x |Df(x)|$, also $F(W_n) \subset \tilde{W}_{n+1}$ for all $n \geq 1$. Recall that $|C_n| = 2\lambda^{n-1}$. The choice of λ and the results from Section 4 guarantee that for ε sufficiently small,

$$\tilde{W}_n \cap (\{c_m\} \times C_m) = \emptyset \text{ for all } n > m \geq 1.$$

In fact, if $n > m$ then $\tilde{W}_n \cap \tilde{W}_m \neq \emptyset$ implies $W_n \subset W_m^-$. Furthermore, $\tilde{W}_n \cap (\{c\} \times I) = \emptyset$ for all $n \geq 1$, and $\tilde{W}_n \cap \tilde{W}_{n+1} = \emptyset = F(\tilde{W}_n) \cap \tilde{W}_n$.

Let $\delta > 0$ and extend f to the interval $[-1-\delta, 1+\delta]$ in such a way that the extension is still unimodal and $f(1+\delta) = f(-1-\delta) = -1-\delta$. Let

$$W = [-1-\delta, 1+\delta] \times [1 - \frac{1}{\lambda}, \frac{1}{\lambda} - 1].$$

Extend F to W, keeping the definition (3). Then F maps W into itself, F is discontinuous on $\{c\} \times [1-\frac{1}{\lambda}, \frac{1}{\lambda}-1]$ and F maps $[-1-\delta, 1+\delta] \times \{1-\frac{1}{\lambda}\}$ in a two-to-one fashion onto $[-1-\delta, 1+\delta] \times \{0\}$. F maps the remaining part of W diffeomorphically onto its image.

We will construct perturbations F_i of F and maps h_i such that the following diagram commutes:

$$\begin{array}{ccccccccc}
W & \xrightarrow{h_1} & W & \xrightarrow{h_2} & W & \xrightarrow{h_3} & W & \xrightarrow{h_4} & \dots \\
{\scriptstyle F}\downarrow & & {\scriptstyle F_1}\downarrow & & {\scriptstyle F_2}\downarrow & & {\scriptstyle F_3}\downarrow & & \\
W & \xrightarrow{h_1} & W & \xrightarrow{h_2} & W & \xrightarrow{h_3} & W & \xrightarrow{h_4} & \dots
\end{array}$$

We will need a sequence of mappings $\Phi_n : W \to W$ with the following properties:

- Φ_n is the identity outside $\tilde{W}_n$ and C^r outside $\{c_n\} \times C_n$.
- Φ_n maps W_n into itself.
- Φ_n maps $\{c_n\} \times C_n$ onto $J_n \times \{u_n\}$ in such a way that if $|y - u_n| < \frac{1}{2}|C_n|$,
$$\lim_{x \nearrow c_n} \Phi_n(x,y) = (c_n + \vartheta_{n-1}|J_n^+|\frac{|y-u_n|}{|C_n|}, u_n)$$
and
$$\lim_{x \searrow c_n} \Phi_n(x,y) = (c_n + \vartheta_{n-1}|J_n^+|(1 - \frac{|y-u_n|}{|C_n|}), u_n).$$
Extending Φ_n to $\{c_n\} \times C_n$ by "continuity", we get that the point (c_n, y) has two images under Φ_n, each of which has two preimages in $\{c_n\} \times C_n$. We say that Φ_n maps $\{c_n\} \times C_n$ in a *two-to-two fashion* onto $J_n^+ \times \{u_n\}$. See Figure 4.

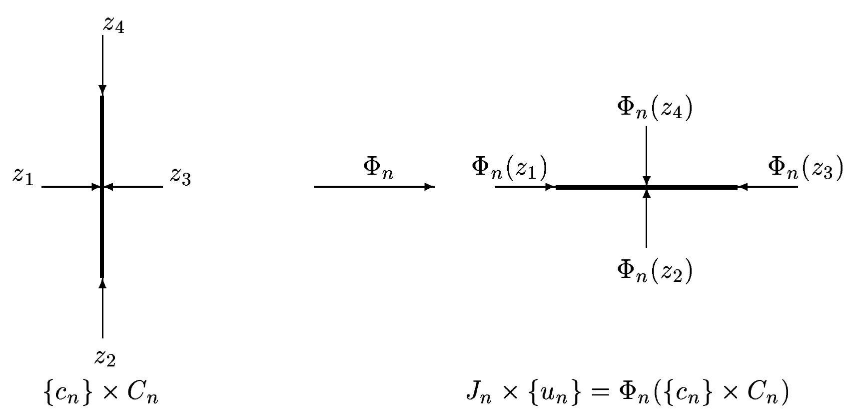

Figure 4: The two-to-two mapping $\Phi_n|\{c_n\} \times C_n$

- $\text{Lips}(\Phi_n|W_n^-) = 1 + \mathcal{O}(\frac{1}{n^2})$ and also $\text{Lips}(\Phi_n^{-1}|\Phi_n(\tilde{W}_n) \cap U^{\pm}) \le 1 + \mathcal{O}(\frac{1}{n^2})$, where $U^+ = \{(x,y) \in W; y \ge u_n\}$ and $U^- = \{(x,y) \in W; y \le u_n\}$. Since $|\pi_I(W_n)| \ge 2n^2|J_n|$, this is no problem.
- Φ_n is symmetric in the sense that $\Phi_n(x, u_n + \eta) = \Phi_n(x, u_n - \eta)$ for all x and η.

First set
$$h_1 = \Phi_1 \text{ and } F_0 = F.$$
If h_n and F_{n-1} are constructed, let
$$F_n = h_n \circ F_{n-1} \circ h_n^{-1}.$$
It is not hard to see that F_1 is a C^r diffeomorphism outside $J_1 \times \{u_1\}$. Indeed $J_1^+ \times \{u_1\}$ is mapped in a two-to-two fashion into $h_1(\{c_2\} \times C_2)$. In general F_n will be C^r, except in the arc $h_n \circ \ldots \circ h_1(J_n \times \{u_n\})$. This set is mapped in a two-to-two fashion into $h_n \circ \ldots \circ h_1(\{c_{n+1}\} \times C_{n+1})$. The action h_{n+1} will cancel this discontinuity, but induce a "smaller" discontinuity of F_{n+1} elsewhere.

Define h_{n+1} as follows:

- h_{n+1} is the identity outside $F_n \circ h_n \circ \ldots \circ h_1(\tilde{W}_n)$.
- For $(x, y) \in F_n \circ h_n \circ \ldots \circ h_1(\tilde{W}_n) \supset h_n \circ \ldots \circ h_1(W_{n+1})$, let

$$h_{n+1}(x, y) = h_n \circ \ldots \circ h_1 \circ \Phi_{n+1} \circ h_1^{-1} \circ \ldots \circ h_n^{-1}(x, y).$$

- By adjusting Φ_{n+1} if necessary, we can also assume that $h_{n+1} \circ F_n$ is C^r in the neighbourhood $h_n \circ \ldots \circ h_1(\tilde{W}_n)$ of $h_{n-1} \circ \ldots \circ h_1(J_n^+ \times \{u_n\})$, with a Lipschitz constant $\leq L + \mathcal{O}(\frac{1}{n^2})$.

Finally, let

$$h = \ldots \circ h_3 \circ h_2 \circ h_1, \quad A = h(Y)$$

and

$$G = \lim_{n \to \infty} F_n = h \circ F \circ h^{-1}.$$

This construction has the following properties:

1. $\text{Lips}(h_n | h_{n-1} \circ \ldots \circ h_1(W_n^-)) \leq \prod_{i=1}^n 1 + \mathcal{O}(\frac{1}{n^2})$. Indeed, $\text{Lips}(\Phi_n | W_n^-) \leq 1 + \mathcal{O}(\frac{1}{n^2})$, and if $W_n^- \cap W_m \neq \emptyset$ for some $m < n$, then $W_n \subset W_m^-$. This follows from the remark at the end of the proof of Proposition 5. Therefore the statement follows by induction. In particular, there exists K such that $\text{Lips}(h_n | h_{n-1} \circ \ldots \circ h_1(W_n^-)) \leq K$ for all n.
2. h_{n+1} maps $h_n \circ \ldots \circ h_1(\{c_{n+1}\} \times C_{n+1})$ in a two-to-two fashion onto $h_n \circ \ldots \circ h_1(J_{n+1}^+ \times \{u_{n+1}\})$.
3. $F_n = h_n \circ F_{n-1} \circ h_n^{-1}$ maps $h_{n-1} \circ \ldots \circ h_1(J_n \times \{u_n\})$ in a two-to-two fashion onto $h_n \circ \ldots \circ h_1(\{c_{n+1}\} \times C_{n+1})$. But then h_{n+1} maps $h_n \circ \ldots \circ h_1(\{c_{n+1}\} \times C_{n+1})$ in a two-to-two fashion onto $h_n \circ \ldots \circ h_1(J_{n+1} \times \{u_{n+1}\})$. Therefore F_{n+1} restores the continuity at $h_{n-1} \circ \ldots \circ h_1(J_n \times \{u_n\})$ that was violated by F_n. A fortiori, due to the third property of h_{n+1}, F_{n+1} is even C^r in a neighbourhood of $h_{n-1} \circ \ldots \circ h_1(J_n \times \{u_n\})$.
4. F_n maps $h_{n-2} \circ \ldots \circ h_1(J_{n-1} \times \{u_{n-1}\})$ onto $h_{n-1} \circ \ldots \circ h_1(J_n \times \{u_n\})$. Because $h_m | h_{n-2} \circ \ldots \circ h_1(J_{n-1} \times \{u_{n-1}\})$ and $h_{m+1} | h_{n-1} \circ \ldots \circ h_1(J_n \times \{u_n\})$ are the identity for $m > n$, we obtain that in the limit

$$G \circ h_{n-2} \circ \ldots \circ h_1(J_{n-1} \times \{u_{n-1}\}) = h_{n-1} \circ \ldots \circ h_1(J_n \times \{u_n\}),$$

for all $n \geq 2$. Therefore the sets $\{h_{n-2} \circ \ldots \circ h_1(J_{n-1} \times \{u_{n-1}\})\}_n$ play the role of a wandering arc. The existence of such an arc is the main difficulty in achieving smoothness for G.

We come to the main result.

Theorem 2 *The following properties hold:*

1. *h is C^r outside* $\mathrm{cl} \cup_n \{c_n\} \times C_n$, *continuous on Y and absolutely continuous on W.*
2. *G is C^r outside* $\cap_N \mathrm{cl} \cup_{n > N} h(W_n)$ *and Lipschitz homeomorphic on* $\mathrm{int}\, W$.
3. *$G|W$ is a factor of $F|W$ and $G|A$ is conjugate to $\hat{f}|X$,*
4. *The set A has zero Lebesgue measure and A is the global attractor of $h((-1,1) \times [1 - \frac{1}{\lambda}, \frac{1}{\lambda} - 1])$ in the sense that for each B, $A \subset B \subset \mathrm{cl}\, B \subset h((-1,1) \times [1 - \frac{1}{\lambda}, \frac{1}{\lambda} - 1])$, we have $A = \cap_n G^n(B)$.*

Proof: h_n is C^r outside $h_{n-1}\circ\ldots\circ h_1(\{c_n\}\times C_n)$, and the identity outside $h_{n-1}\circ\ldots\circ h_1(\tilde{W}_n)$, the area of which decreases exponentially. Therefore h is C^r outside $\mathrm{cl}\cup_n\{c_n\}\times C_n$. Moreover $\mathrm{cl}\cup_n\{c_n\}\times C_n$ has Lebesgue measure 0 and is mapped into itself. Therefore h is also absolutely continuous. The discontinuity of h_n is only caused by the fact that for $|y-u_n|\leq\frac{1}{2}|C_n|$, $\lim_{x\nearrow c_n}\Phi_n(x,y)\neq\lim_{x\searrow c_n}\Phi_n(x,y)$. Since $Y\cap W_n$ lies only on one side of $\{c_n\}\times C_n$, this discontinuity is not visible on Y. This proves 1.

By property 3., F_n is C^r outside $h(W_n)$. Moreover, for $m>n$, $F_m=F_n$, except on $h(\cup_{n'>n}\tilde{W}_{n'})$. Therefore $G=\lim_n F_n$ is C^r except in $\cap_N\mathrm{cl}\cup_{n>N}\mathrm{h}(\mathrm{W_n})$. Note also that $\mathrm{diam}(F_n(\tilde{W}_n))\to 0$ as $n\to\infty$. Let us prove that $\mathrm{Lips}(F_n|W\setminus h_n\circ\ldots\circ h_1(\tilde{W}_n))$ is uniformly bounded. If $(x,y)\notin h_n\circ\ldots\circ h_1(\tilde{W}_{n+1}\cup\tilde{W}_n)$, then F_{n+1} coincides with F_n. Therefore take $(x,y)\in h_n\circ\ldots\circ h_1(\tilde{W}_n)$. Then

$$\begin{aligned}F_{n+1}(x,y) &= h_{n+1}\circ F_n\circ h_{n+1}^{-1}(x,y)=h_{n+1}\circ F_n(x,y)\\ &= h_n\circ\ldots\circ h_1\circ\Phi_{n+1}\circ h_1^{-1}\circ\ldots\circ h_n^{-1}\circ F_n(x,y)\\ &= h_n\circ\ldots\circ h_1\circ\Phi_{n+1}\circ F\circ h_1^{-1}\circ\ldots\circ h_n^{-1}(x,y).\end{aligned}$$

Now $\mathrm{Lips}(h_n\circ\ldots\circ h_1|\Phi_{n+1}\circ F(\tilde{W}_n))\leq\prod_{i=1}^n 1+\mathcal{O}(\frac{1}{i^2})$. The discontinuity of $h_n^{-1}|h_n\circ\ldots\circ h_1(\tilde{W}_n)$ is canceled by $\Phi_{n+1}\circ F|\tilde{W}_n$, giving a Lipschitz constant $L_0=\mathcal{O}(L)$ which is independent of n. Because $\tilde{W}_n\cap\{c_m\}\times C_m=\emptyset$ for all $m<n$, $h_n\circ\ldots\circ h_1(\tilde{W}_n)$ lies always to one side of $h_{m-1}\circ\ldots\circ h_1(\{u_m\}\times J_m^+)$. Therefore $\mathrm{Lips}(h_m^{-1}|h_m\circ\ldots\circ h_1(\tilde{W}_n))\leq 1+\mathcal{O}(\frac{1}{m^2})$ and $\mathrm{Lips}(h_1^{-1}\circ\ldots\circ h_{n-1}^{-1}|h_{n-1}\circ\ldots\circ h_1(\tilde{W}_n))\leq\prod_{i=1}^{n-1}1+\mathcal{O}(\frac{1}{i^2})$. We obtain

$$\mathrm{Lips}(F_{n+1}|h_n\circ\ldots\circ h_1(\tilde{W}_n)\leq L_0(\prod_{i=1}^{\infty}1+\mathcal{O}(\frac{1}{i^2}))^2=:K,$$

which is independent of n. Now for the limit map $\mathrm{Lips}(G|W)\leq K+2$. Indeed, let $z\neq z'\in W$ be arbitrary. Take N so large that $\mathrm{diam}(F_n(\tilde{W}_n))<|z-z'|$ for all $n>N$. Then also

$$|G(z)-G(z')|\leq|G(z)-F_n(z)|+|F_n(z)-F_n(z')|+|F_n(z')-G(z')|\leq 3K|z-z'|.$$

This proves 2.

Since $h\circ G=F\circ h$, G is a factor of F by definition. Note that if $h^{-1}(x,y)$ is multi-valued (two-valued, to be precise), then $h\circ F$ maps the different images to one point again. Furthermore, $h(x,y)=h(x',y')$ only if $(x,y)\sim(x',y')$ by the equivalence relation introduced in Section 3. Therefore $G|A$ is indeed conjugate to $F|Y/\sim$, which is conjugate to $\hat{f}|X$.

For the last statement, $A=h(Y)$ has measure 0 because Y has and h is absolutely continuous. Furthermore Y attracts every point in $(-1,1)\times[1-\frac{1}{\lambda},\frac{1}{\lambda}-1]$ under iteration of F. Therefore A attracts every point of $h((-1,1)\times[1-\frac{1}{\lambda},\frac{1}{\lambda}-1])$ under iteration of G. □

Proof of Theorem 1: Take $g=G$ as in Theorem 2. Then all the assertions of Theorem 1 are satisfied except that if f is a full map. In this case A need not attract a neighbourhood of itself. For example the points $(c_1+|J_1^+|+\eta,u_1)$ (and its iterates) need not be attracted to A for any $\eta>0$. Note that $(c_1+|J_1^+|,u_1)\in A$. However, let F_1 be a C^r perturbation of F which maps both branches of $F|\{c\}\times I$ into the interior of $\{c_1\}\times I$. The rest of the construction remains the same. Then the limit map $g=G$ maps $\{c\}\times I$ into the interior of $J_1^+\times\{u_1\}$, and by continuation, $g^i(\{c\}\times I)$ lies in the interior of $J_i^+\times\{u_i\}=h(\{c_i\}\times C_i)$. Now A indeed attracts a neighbourhood of itself. □

Proof of Corollary 1: Because $F|Y$ is just a skew product of f and a vertical contraction, and because h is absolutely continuous, the assertions follow directly from the one-dimensional counterparts. For statement 1., this is a result by Hofbauer and Keller [11, Theorem 1]. (Note that also Theorems 2 and 3 of that paper have, by this technique, planar versions.) For case 2., this is the main result of [9]. □

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