On the structure of isentropes of polynomial maps

Henk Bruin and Sebastian van Strien

July 19, 2013

Abstract

The structure of isentropes (*i.e.,* level sets of constant topological entropy) including the monotonicity of entropy, has been studied for polynomial interval maps since the 1980s. We show that isentropes of multimodal polynomial families need not be locally connected and that entropy does in general not depend monotonically on a single critical value.

2000 Mathematics Subject Classification. 37E05 37B40 58F13 58F14

1 Introduction and Statement of Results.

Topological entropy h_{top} was introduced by Adler et al. [1], and the for many purposes more practical approach using ε-n-separated/spanning sets was developped by Bowen [3] and Dinaburg [6]. For continuous interval maps $f : I \to I$, Misiurewicz & Szlenk [17] showed that $h_{top}(f)$ represents the exponential growth rate of the number of periodic points[1], the number of laps or the variation:

$$\begin{aligned} h_{top}(f) &= \lim_{n\to\infty} \frac{1}{n} \log \#\{x : f^n(x) = x\} \\ &= \lim_{n\to\infty} \frac{1}{n} \log \ell(f^n) \\ &= \lim_{n\to\infty} \frac{1}{n} \log \mathrm{Var}(f^n). \end{aligned} \tag{1}$$

Here the *lapnumber* $\ell(f^n)$ denotes the number of maximal intervals J such that $f^n|_J$ is monotone, and the *variation* is

$$\mathrm{Var}(f) = \sup_{k\in\mathbb{N}} \sup_{x_0<\cdots<x_k} \sum_{j=1}^{k} |f(x_j) - f(x_{j-1})|.$$

[1] where it is assumed that for every n, every separate branch of f^n contains a bounded number of n-periodic points. That this assumption is necessary emerges *e.g.* from [12].

The question of how $h_{top}(f)$ depends on the map f is a major theme within interval dynamics since numerical observations made in the 1970's. The simplest question of this type is whether for the logistic family $f_a(x) = ax(1-x)$, the entropy $h_{top}(f_a)$ is increasing in a. In the early 80's, this was proved by Douady & Hubbard [7, 8], Milnor & Thurston [19] and Sullivan, see [16]. Another elegant proof was given by Tsujii [26]. It is worth emphasizing that all known monotonicity proofs rely in some way on complex analysis, and so does the recent proof of van Strien & Rempe for the sine family [23]. Milnor & Tresser [20] (see also [18]) proved that entropy is also monotone for the family of *cubic* polynomials, and Radulescu [22] proved the same thing for a certain two-dimensional *slice of quartic* polynomials. In this case, the parameter space is two-dimensional, and then (as for all higher dimensional spaces) monotonicity of entropy means that the *isentropes* (*i.e.,* sets of constant entropy) are connected.

For polynomials of *arbitrary degree* $d = b+1$ (*i.e., b*-modal polynomials) the following monotonicity statement was proved recently in [4]: Fix $\epsilon \in \{-1, 1\}$ and define

$$\mathcal{P}^b_\epsilon = \left\{ f : [-1,1] \to [-1,1] : \begin{array}{l} f \text{ is a polynomial of } \deg(f) = b+1, \\ f(-1) = \epsilon,\ f(1) \in \{-1,1\} \text{ and } f \text{ has} \\ b \text{ distinct critical points in } (-1,1) \end{array} \right\}.$$

Theorem (Milnor's Monotonicity Conjecture, see [4])**.** *For each $h \geq 0$, the isentrope*

$$L_h = \{f \in \mathcal{P}^b_\epsilon : h_{top}(f) = h\}$$

is connected.

The main purpose of this paper is to present two theorems which show that isentropes of $\mathcal{P}^b_\epsilon$ are nevertheless complicated sets:

Theorem 1.1 (Non-monotonicity w.r.t. natural parameters)**.** *Let $f_v \in \mathcal{P}^b_\epsilon$ denote the polynomial map with critical values $v = (v_1, \dots, v_b)$. For $b \geq 2$, there are fixed values of $v_2, \dots, v_b$ such that the map*

$$v_1 \mapsto h_{top}(f_v)$$

is not monotone.

Theorem 1.2 (Non-local connectivity)**.** *For any $b \geq 4$, there is a dense set $H \subset [0, \log(b-1)]$ such that for each $h \in H$, the isentrope L_h of $\mathcal{P}^b_\epsilon$ is not locally connected.*

The proof of these theorems will be given in Sections 2 and 3.

The method used in the proof of the last theorem is related to a result by Friedman & Tresser [10] which states that the zero isentrope of bimodal circle maps is not locally connected. This is one aspect of a description of the boundary of chaos behavior for bimodal circle maps; for a discussion of a more extensive program, see *e.g.* [11, 20].

Questions: For which values of entropy are the isentropes not locally connected? The values $h = \log s$ for which our proof works are for algebraic s. Are there non-algebraic values of s such that the isentrope of level $\log s$ not locally connected? Is it possible to construct non-local connectivity when $h \in (\log(b-1), \log(b+1))$. Are there values h so that there exists a dense subset of $g \in L_h$ so that L_h is non-locally connected at g?

For other questions, see [25].

1.1 Some questions about other multimodal interval maps

Before giving the proof of the main theorems of this paper, let us turn to other models of b-modal families. We will consider two such families, namely the tent maps and the stunted sawtooth maps, see Figure 1. We will always scale them such that $f : [-1, 1] \to [-1, 1]$, $f(-1) = \epsilon \in \{-1, 1\}$, $f(1) \in \{-1, 1\}$, and there are b critical points $c_i = -1 + 2i/(b + 1)$, $i = 0, \dots, b + 1$ so by default $c_0 = -1$ and $c_{b+1} = 1$ are the boundary points. The critical values $v_i = f(c_i) \in [-1, 1]$ with $v_0 = \epsilon$, $v_{b+1} = (-1)^{b+1}\epsilon$ will satisfy

$$(v_i - v_{i-1}) \cdot \epsilon \begin{cases} < 0 & \text{if } i \text{ is odd;} \\ > 0 & \text{if } i \text{ is even.} \end{cases} \tag{2}$$

- A b-modal tent-map f is obtained by choosing b critical values v_i satisfying (2), and then defining

$$f(x) = \begin{cases} v_i & \text{if } x = c_i; \\ \text{by linear interpolation} & \text{if } x \notin \{c_0, \dots, c_{b+1}\}. \end{cases}$$

  The class of b-modal tent-maps is denoted by $\mathcal{T}^b_\epsilon$. If $f \in \mathcal{T}^b_\epsilon$ has constant slope $f' = \pm s$, then $h_{top}(f) = \max\{\log s, 0\}$. This follows immediately from the variation interpretation of entropy in (1). By the same token, $|f'| \geq s$ implies $h_{top}(f) \geq \log s$. A well-known result by Milnor & Thurston [19] (preceded by Parry [21] in the context of β-transformations) is that every b-modal interval map g of entropy $h_{top}(g) = \log s > 0$ is semi-conjugate to some $f \in \mathcal{T}^{b'}_\epsilon$ with constant slope $\pm s$ for some $b' \leq b$. However, the dynamics of f may be strictly less complicated, because maps in $\mathcal{T}^b_\epsilon$ allow only periodic intervals of limited types. Hence the semi-conjugacy connecting $f \in \mathcal{T}^b_\epsilon$ and $g \in \mathcal{P}^b_\epsilon$ can collapse (pre)periodic intervals of g to (pre)periodic points of f.

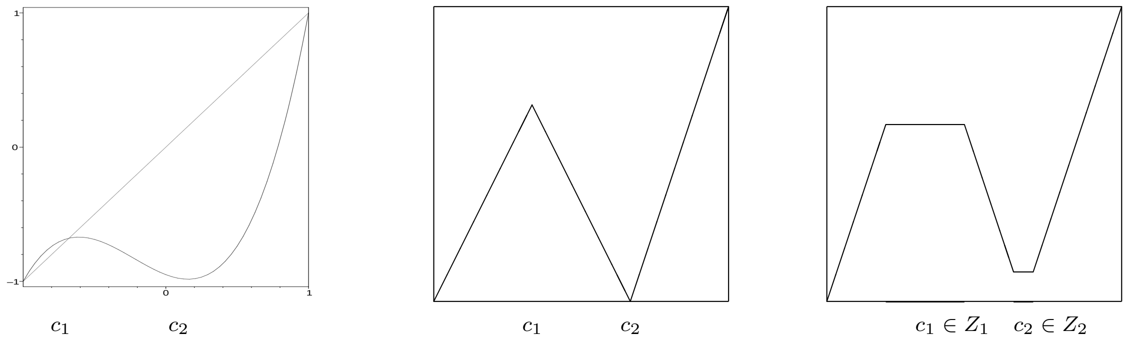

Figure 1: Bimodal maps with $\epsilon = -1$: polynomial, tent and stunted sawtooth.

- A b-modal stunted sawtooth map is obtained from a fixed b-modal sawtooth maps as follows. First constructed an unstunted sawtooth map (or full b-modal tent-map) $S_0 : [-1, 1] \to [-1, 1]$ by setting $S_0(c_i) = \epsilon$ or $-\epsilon$ if i is odd or even respectively, and defining S_0 in between by linear interpolation. Then choose new critical values v_i satisfying (2), and find (for $i = 1, \dots, b$) the maximal closed intervals $Z_i \ni c_i$ such that $(S_0(x) - v_i) \cdot (-1)^i \leq 0$ for all $x \in Z_i$. Finally, define the *stunted sawtooth map* as

$$f(x) = \begin{cases} v_i & \text{if } x \in Z_i \text{ for some } i \in \{1, \dots, b\}; \\ S_0(x) & \text{otherwise.} \end{cases}$$

  The class of b-modal stunted sawtooth maps is denoted by $\mathcal{S}^b_\epsilon$. This class of maps was heavily used in the proof of Milnor's conjecture, see [4].

It is again natural to parametrize maps in $\mathcal{T}^b_\epsilon$ and $\mathcal{S}^b_\epsilon$ by their critical values $v = (v_1, \dots, v_b)$.

Proposition 1.1. *Let $f_v \in \mathcal{S}^b_\epsilon$ denote the stunted sawtooth map with critical values $v = (v_1, \dots, v_b)$. For each $i \in \{1, \dots, b\}$, the map*

$$v_i \mapsto h_{top}(f_v)$$

is increasing/decreasing if i is odd/even. All isentropes of $\mathcal{S}^b_\epsilon$ are connected and locally connected.

Proof. See Section 2. □

Surprisingly, the corresponding questions seem to be unanswered for the space $\mathcal{T}^n_\epsilon$:

Question: Are the isentropes of $\mathcal{T}^b_\epsilon$ connected? Are they all locally connected?

2 Non-monotonicity w.r.t. single critical values

Let $\zeta_i = \epsilon\cdot(-1)^i\cdot v_i$, so increasing ζ_i always means decreasing the width of the plateau Z_i. Using these coordinates, we can reformulate the first statement of Proposition 1.1 as

> The map $\zeta = (\zeta_1, \dots, \zeta_d) \to h_{top}(T_\zeta)$ is non-decreasing in each coordinate.

Proof of Proposition 1.1. Increasing a parameter ζ_i makes a plateau narrower, and affects none of the orbits that never enter Z_i. Therefore only new orbits are created, and none destroyed. Hence entropy is non-decreasing in each ζ_i.

Now to prove local connectedness of isentrope L_s, let $x, y \in L_s$ be two arbitrary points; they subtend a parallelepiped P of dimension $\leq b$ with faces perpendicular to the coordinate axes. That is, x and y are vertices of P, and we denote the vertices at which all parameters ζ_i are minimal and maximal by l and u respectively. Foliate P by one-dimensional strands γ, connecting l to u, such that all parameters vary monotonically along γ. Then for each such γ, there is a unique compact arc (or singleton) J_γ at which the entropy is constant s. The union $\cup_\gamma J_\gamma = P \cap L_s$ and due to continuity of entropy, it is connected. Hence between any two $x, y \in L_s$ there is an arc connecting x and y within L_s with diameter $\leq d(x, y)$ in sum-metric d. This proves local connectivity. $\square$

The analogous monotonicity statement is false for $\mathcal{P}^b$ for $b \geq 2$. The below proof demonstrates that parametrizing the cubic family by its critical values v_1, v_2 results in non-monotonicity in (each of) its parameters separately.

Proof of Theorem 1.1. The 'anchored' cubic map in $\mathcal{P}^3_{-1}$ form a two-parameter family

$$f_{\alpha,\beta}(x) = \alpha x^3 + \beta x^2 + (1-\alpha)x - \beta,$$

where $0 \leq \alpha \leq 4$ and $|\beta| \leq 2\sqrt{\alpha} - \alpha$. Thus $f_{\alpha,\beta}(\pm 1) = \pm 1$ and the critical points are $c_{1,2} = \frac{1}{3\alpha}(-\beta \pm \sqrt{\beta^2 - 3\alpha(1-\alpha)})$. It is possible to parametrize the family $\mathcal{P}^b$ by critical values v_1, v_2, see [16, Theorem II.4.1], but the formulas are unpleasant. For our purposes it suffices to verify that parameter α depends monotonically on v_1 when v_2 is kept constant (close to -1).

For $\beta = 2\sqrt{\alpha} - \alpha$, the rightmost critical point maps to -1, and we will use this parameter choice for the moment. That is, $c_1 = -\frac{1}{3}(1 + \frac{1}{\sqrt{\alpha}})$, $c_2 = 1 - \frac{1}{\sqrt{\alpha}}$, and these critical points are increasing in α. The first critical value

$$v_1 = f_{\alpha, 2\sqrt{\alpha}-\alpha}(c_1) = \frac{32}{27}\alpha - \frac{48}{27}\sqrt{\alpha} - \frac{1}{9} - \frac{4}{27\sqrt{\alpha}}$$

is increasing in α as well.

In fact, numerics show that $\frac{d}{d\alpha} f_{\alpha,\beta}(x) > 0$ for all $1 \le \alpha \le 4$ and $x \in (-1, c_1]$. It follows that for every two values of v_1, say $v_1(\alpha) < v_1(\alpha')$ for $1 \le \alpha < \alpha' \le 4$ and $x \in (0, c_1(\alpha))$, we have $f_{\alpha, 2\sqrt{\alpha}-\alpha}(x) < f_{\alpha', 2\sqrt{\alpha'}-\alpha'}(x)$.

It can also be verified that the parameter α such that $f_{\alpha,2\sqrt{\alpha}-1}(c_1) > c_2 > f^2_{\alpha,2\sqrt{\alpha}-1}(c_1) = c_1$ lies within the interval $[3.668,\ 3.670]$. For these values of α, $v_1 = f_{\alpha,2\sqrt{\alpha}-\alpha}(c_1) \approx 0.75$ and the left-most preimage $u \in f^{-1}_{\alpha,2\sqrt{\alpha}-\alpha}(c_2) \approx -0.72$. Also c_1 is still attracted to a period 2 orbit, and the entropy $h_{top}(f_{\alpha,2\sqrt{\alpha}-\alpha}) = \log 1+\sqrt{2}$. We find numerically that for $x \approx -0.72$,

$$\frac{d}{d\alpha} f_{\alpha,2\sqrt{\alpha}-\alpha}(x) > 2 > \frac{1}{8} > \frac{d}{d\alpha}(c_2(\alpha)) = \frac{1}{2\alpha\sqrt{\alpha}}. \tag{3}$$

Now we fix some large n, v_1^* close to 0.75 and v_2^* close to -1 such that $v_2^* = f_{\alpha,\beta}(c_2) < f^2_{\alpha,\beta}(c_2), < \cdots < f^{n-1}_{\alpha,\beta}(c_2) < c_1 < f^n_{\alpha,\beta}(c_2) = c_2$. That is, from now on α, β and $c_{1,2}$ depend on v_1 and v_2, where $v_2 = v_2^*$ is fixed and v_1 is the remaining variable such that $f^n_{\alpha,\beta}(c_2) = c_2$ for $v_1 = v_1^*$ and $v_2 = v_2^*$. We will write $f_{\alpha,\beta} = g_{v_1,v_2}$. Since v_2^* can be taken arbitrarily close to -1 (at the price of taking n large), we can assume that α is still an increasing function of v_1, and the estimates leading up to (3) remain valid.

This means that $\frac{d}{dv_1}(g_{v_1,v_2^*}(x)) > 0$ for every fixed $x \in (0, c_1)$, and for values $x \approx g^{-1}_{v_1,v_2^*}(c_2)$, we have $\frac{d}{dv_1}(g_{v_1,v_2^*}(x)) > \frac{d}{dv_1}(c_2)$. Therefore, $\frac{d}{dv_1} g^k_{v_1,v_2^*}(c_2) > 0$ for $1 < k < n$ and $\frac{d}{dv_1} g^n_{v_1,v_2^*}(c_2) > \frac{d}{dv_1}(c_2)$.

Again, for $v_1 = v_1^*$, the second critical point c_2 is (super)attracting of period n. Let us denote by $q(v_1)$ the continuation of this (attracting) n-periodic. If U is a small neighborhood of c_2, $f^{n+1}_{v_1,v_2^*}|_U$ has two branches for $v_1 \ge v_2^*$ and four branches for $v_1 < v_1^*$. If $v_1 > v_1^*$ is so large that $q(v_1)$ disappears in a saddle-node bifurcation, then $h_{top}(g_{v_1,v_2^*})$ decreases compared to $h_{top}(g_{v_1^*,v_2^*})$. However, for $v_1 \approx 1$, *i.e.*, g_{v_1,v_2^*} close to the cubic Chebyshev polynomial, the entropy is close to $\log 3$, and hence $v_1 \mapsto h_{top}(g_{v_1,v_2^*})$ cannot be monotone. $\square$

3 Isentropes which are non-locally connected

In this section we shall prove the existence of isentropes which are non-locally connected in the space $\mathcal{P}^b_\epsilon$ for any $b \ge 3$ and $\epsilon = \pm 1$: we will show that there exists a dense set of $h \in [0, \log(b-1)]$ for which $L_h \subset \mathcal{P}^b_\epsilon$ is non-locally connected.

Definition 3.1. *Given an interval map g, we say that an interval F is a* fundamental domain *in the basin of an attracting orbit of period N if* *(i) F and $g^N(F)$ have precisely one point in common, and*

(ii) $\partial g^N(F) = g^N(\partial F)$.

This notion agrees with the usual notion in the monotone case, but the assumption (ii) is added to deal with the case that $g_t^N|J_t$ is not monotone.

Theorem 3.1. *Assume that $s = e^h \in [0, b-1]$ and let $T \in \mathcal{T}_\epsilon^{b-2}$ be a map with slope $\pm s$ with $b-2$ periodic turning points. Let $\hat{q}$ be an N-periodic point of T so that $\cup_{n\geq 0} T^{-n}(\hat{q})$ is dense in $[-1,1]$ and so that the orbit of $\hat{q}$ does not contain a turning point of T. Then there exists $\delta > 0$ such that if $L_h \subset \mathcal{P}_\epsilon^b$ contains a map g with the properties:*

1. *g has a periodic point q^* of saddle-node type of period N;*
2. *two adjacent critical points $c_\eta, c_{\eta+1}$ are both contained in a fundamental domain F in the immediate basin of q^* and $|c_\eta - c_{\eta+1}| < \delta|F|$;*
3. *g is semi-conjugate to T where the semi-conjugacy only collapses basins of periodic attractors.*
4. *each other critical point of g is in the basin of a hyperbolic periodic attractor;*
5. *arbitrarily close to g, there exist maps $\tilde{g}$ in $\mathcal{P}_\epsilon^b$ which do not have a parabolic periodic point of period $\leq N$,*

then L_h is not locally connected at g.

We note that since $c_\eta, c_{\eta+1}$ are the only two critical points in the basin of q^*, there exists a fixed point q of g^N so that the component of the basin of q^* containing q^* is equal to $(q, q^*]$, see Figure 2.

The topological entropy of a map $g \in \mathcal{P}_\epsilon^b$ as in this theorem, is the same as that of polynomial map $\hat{g}$ for which $c_\eta = c_{\eta+1}$ and which therefore has a degenerate critical point at $c_\eta = c_{\eta+1}$; such a map $\hat{g}$ has $b-2$ turning points and therefore at most topological entropy $\log(b-1)$. Since this is the only mechanism which we are aware of for obtaining non-local connectivity of isentropes, this explains the reason that we have to assume that $h \in [0, \log(b-1)]$ and can only assert that there are non-locally connected isentropes in $\mathcal{P}_\epsilon^b$ when $b \geq 3$.

Remark: The assumption that $\cup_{n\geq 0} T^{-n}(\hat{q})$ is dense in $[-1,1]$ implies that if T is renormalizable, say $U \supset T^m(U)$ for some non-degenerate interval U, then $\hat{q} \in \text{orb}(U)$. If the maximal such $m = p \cdot 2^r$ for some odd integer $p \geq 3$, then $h_{top}(T) \geq 2^{-r} \log \lambda_p$ where λ_p is the largest solution of $x^p - 2x^{p-2} = 1$ (see [2, Theorem 4.4.17]). Hence to realize small values of h, the integers N and k in condition C2 below must be multiples of 2^r for some large r.

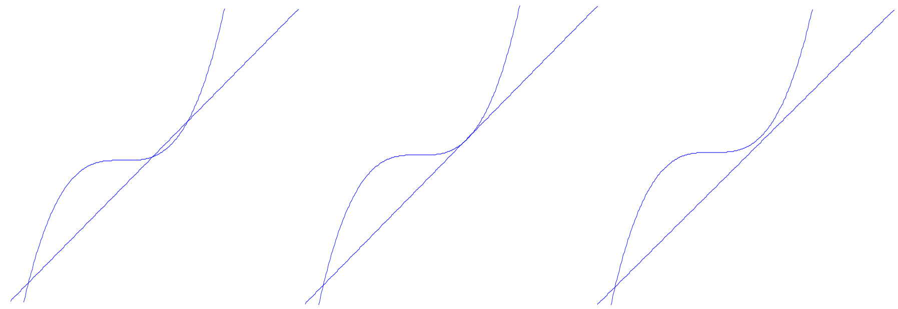

Figure 2: The graph of $\tilde{g}^N$ for $\tilde{g}$ near g. In the middle figure, g^N has a hyperbolic fixed point q and a saddle-node fixed point q^*.

Since the proof for $b = 3$ (with $c_\eta = c_1$) is the same as the general case, we present here only that case. For simplicity we shall also assume that $\epsilon = -1$.

3.1 Construction of a map g satisfying the assumptions of Theorem 3.1

To make the discussion rather explicit we will first show that maps satisfying the assumptions in Theorem 3.1 exist. To do this, choose a unimodal tent map (in the general case we choose a map with constant slope with $b-2$ turning points and replace $\log 2$ below by $\log(b-1)$). Let $h \in (0, \log 2)$ be so that there exist an integer R, $s > 1$ and a map $T \in \mathcal{T}^3_\epsilon$ with slopes $\pm s$ (where $s = \exp(h)$) and turning point $c = 0$ for which $T^R(c) = c$. Note that the set of such h's is dense in $[0, \log 2]$. For later use, pick any periodic point $\hat{q}$ of T so that $\cup_{n \geq 0} T^{-1}(\hat{q})$ is dense and so that the orbit of $\hat{q}$ does not contain a turning point of T. Let N be its period. Given $\hat{q}$, choose an integer k, an interval neighbourhood $\hat{J} \ni \hat{q}$ and a point $\hat{x}$, so that

C1 $T^N|\hat{J}$ is monotone,

C2 $\hat{x} \in T^N(\hat{J})$ and there exists a neighbourhood $\hat{V} \ni \hat{x}$ so that T^k maps $\hat{V}$ monotonically onto $\hat{J}$.

Since $\cup_{n \geq 0} T^{-1}(\hat{q})$ is dense, k, $\hat{J}$ and $\hat{x}$ always exist. We may assume that $T^N|\hat{J}$ is orientation preserving (otherwise replace N by $2N$) and in order to be definite we assume that $\hat{q}$ lies to the left of the turning point of T and that $\hat{x} > \hat{q}$ (if $\hat{x} < \hat{q}$ then we replace sup by inf in the proof below).

Consider the space $\overline{\mathcal{P}}^b_\epsilon$ of real degree four polynomials with three critical points $-1 \leq c_1 \leq c_2 < c_3 < 1$ and so that $f(\pm 1) = \epsilon = -1$. Note that $\overline{\mathcal{P}}^b_\epsilon$ can be parametrized by the *critical values* $v_i = f(c_i)$ of f and so corresponds to the space

$$\{(v_1, v_2, v_3) \in \mathbb{R}^3; -1 \leq v_2 \leq v_1 \leq 1, v_1 > -1 \text{ and } v_2 < v_3 \leq 1\},$$

see [16, Section II.4] and also [5]. Note that $\mathcal{P}^b_\epsilon$ consists of the polynomials in $\overline{\mathcal{P}}^b_\epsilon$ for which $v_2 < v_1$ holds; by *singular* polynomials we mean those in $\overline{\mathcal{P}}^b_\epsilon \setminus \mathcal{P}^b_\epsilon$. The next three steps involve singular maps; in Step 4., we construct the required non-singular map from them.

Step 1. Construction of a family of singular maps with the appropriate topological entropy. For any singular map $f \in \Pi := \overline{\mathcal{P}}_\epsilon^b \setminus \mathcal{P}_\epsilon^b$ one has $c_1 = c_2$ and therefore f is unimodal. Since v_3 can take any value in $(-1,1]$ this forms a full family of unimodal maps and therefore we can pick $g_0 \in \overline{\mathcal{P}}_\epsilon^b \setminus \mathcal{P}_\epsilon^b$ so that g_0 is semi-conjugate to T and so that $g_0^R(c_3(g_0)) = c_3(g_0)$. Since the semi-conjugacy only collapses the basin of periodic attractor of g_0, we have $h_{top}(g_0) = h$. Next define $\hat{\Gamma} = \{g \in \overline{\mathcal{P}}_\epsilon^b \setminus \mathcal{P}_\epsilon^b \, : \, g^R(c_3(g)) = c_3(g)$ and Γ the component of $\hat{\Gamma}$ which contains g_0. Note that Γ is a real algebraic variety of (real) dimension one. Moreover, by transversality, see [9] and also [24], Γ is a one-dimensional manifold. More specifically, take $g_0 \in \Pi$ with the additional property $c_1(g_0) = c_2(g_0)$ is not a point of the cycle $g_0^R(c_3(g_0)) = c_3(g_0)$ (maps for which this condition fails correspond to isolated points in Γ). According to [15, Theorem 2], one can introduce coordinates for all $g \in \Pi$ near g_0 such that one of these parameters is the multiplier ρ of the periodic orbit $c_3(g)$. Thus the set Γ can be defined as the subset of Π for which $\rho = 0$, which is obviously a one-dimensional manifold. As g varies in Γ the critical points $c_1(g) = c_2(g)$ move from the left endpoint of $[-1,1]$ to $c_3(g)$, see also [5]. More precisely, one can take a one-parameter family of maps f_t, $t \in (0,1)$ parametrizing Γ, so that as $t \downarrow 0$, $c_1(t) = c_2(t) \to -1$ and as $t \uparrow 1$ the distance of $c_1(t) = c_2(t)$ to $c_3(t)$ tends to zero. For each $t \in (0,1)$ the map f_t is semi-conjugate to T by a semi-conjugacy H_t which only collapses the basin of the periodic attractors of f_t. (In order to be definite we choose normalisations so that $x \mapsto H_t(x)$ is increasing). Hence

$$f \in \Gamma \implies h_{top}(f) = h.$$

Step 2. Construction of a singular map $f_{t'}$ within this family with a periodic point of saddle-node type containing $c_1(t') = c_2(t')$ in its immediate basin. For each $t \in (0,1)$, the image $H_t(c_1(t))$ of the inflection point $c_1(t) = c_2(t)$ lies to the left of the turning point of T. Moreover, $t \mapsto H_t(c_1(t))$ is continuous because the kneading invariant of $c_1(t)$ with respect to the partition $[-1, c_3(t)]$ and $[c_3(t), 1]$ depends continuously on t except when $c_1(t)$ is eventually mapped to $c_3(t)$. However, even in that case $t \mapsto H_t(c_1(t))$ is continuous since $c_3(t)$ is a periodic attractor and hence a neighbourhood of $c_3(t)$ is mapped to the turning point of T.

As mentioned above, as $t \in (0,1)$ varies from 0 to 1, we have that $c_1(t) = c_2(t)$ varies from the left endpoint of $[-1,1]$ to $c_3(t)$. In particular, by continuity of $t \mapsto H_t(c_1(t))$, and since $\hat{q}$ is to the left of the turning point of T, the set A of parameters t so that $H_t(c_1(t)) = \hat{q}$ is the N-periodic point of T from above, is non-empty. Since f_t is semi-conjugate to T, for each $t \in A$, there exists an interval $J_t \subset f_t^N(J_t)$ so that $f_t^N|J_t$ is monotone and orientation preserving and $H_t(J_t) = \hat{J}$ is the interval associated to T from above. Let U_t be the maximal interval in J_t so that $H_t(U_t) = \hat{q}$. Since $T^N(\hat{q}) = \hat{q}$ and f_t is semi-conjugate to T we get $f_t^N(U_t) = U_t$. Since $\hat{q}$ is contained in the interior of $\hat{J}$, the interval U_t is compactly contained in J_t. Moreover, since $c_1(t) = c_2(t) \in U_t$, the interval U_t is non-degenerate. Therefore and since $f_t^N|U_t$ is monotone, both endpoints of U_t are fixed points under f_t^N and $c_1(t) = c_2(t)$ are in the basin of a periodic attractor within U_t. Define $t' = \sup A$. This means that $f_{t'}$ has an N-periodic parabolic point q^* so that $c_1(t') = c_2(t')$ are in its basin. (Otherwise $c_1(t') = c_2(t')$ would be in the basin of a hyperbolic periodic attractor of period N contradicting that $t' = \sup A$.) Let $q(t'), q^*(t')$ be the boundary points of $U_{t'}$, where $q^*(t')$ is the parabolic point. Then both of these points have period N. Since $f_{t'}$ has negative Schwarzian derivative and therefore each attractor of $f_{t'}$ has a critical point in its immediate basin (see Singer's Theorem [16, Theorem II 6.1.]), it follows that $q(t')$ is hyperbolic repelling. As t varies from 0 to 1, $H_t(c_1(t))$ varies from -1 to the turning point of T. That $t' = \sup A$ implies that $H_t(c_1(t)) > \hat{q}$ for $t > t'$. Since $H_{t'}(q(t')) = \hat{q}$ it follows that $q(t') < c_1(t') = c_2(t')$. Since $f_t^N|J_t$ is monotone increasing we therefore get $q(t') < c_1(t') = c_2(t') < q^*(t')$.

Furthermore, again because $f_{t'}$ is semi-conjugate to T, there exists a point $x(t') \in f_{t'}^N(J_{t'})$ so that $H_{t'}(x(t')) = \hat{x}$ and so that there exists an interval $V_{t'} \ni x(t')$ so that $f_{t'}^k$ maps $V_{t'}$ diffeomorphically onto $J_{t'}$, where k and $\hat{x}$ are associated to T as above. Since $t' = \sup A$,

1. for each $t \in [t', 1)$ the map $f_t^N|J_t$ has a hyperbolic repelling fixed point $q(t) \in J_t$;
2. for any $t \in (t', 1)$ and any $y \in J_t$ with $y > q(t)$ one has $f_t^N(y) > y$;
3. $f_{t'}^N$ has a parabolic fixed point $q^*(t') \in J_{t'}$;
4. J_t contains $q(t') < c_1(t') = c_2(t') < q^*(t')$ for each t near t'.

This shows that $f_{t'}$ has a saddle-node periodic point of the required type.

Step 3. Construction of a sequence of singular maps f_{t_n} with $t_n \downarrow t'$ and for which $c_1(t_n) = c_2(t_n)$ are attracting periodic orbits. Since $T^k\colon \hat{V} \to \hat{J}$ is monotone, there exists a sequence of periodic points $\hat{x}_n \in \hat{J}$, so that

$$\hat{x}_n, T^N(\hat{x}_n), \ldots, T^{(n-1)N}(\hat{x}_n) \in \hat{J}\ ,\ T^{nN}(\hat{x}_n) \in \hat{V} \text{ and } T^{nN+k}(\hat{x}_n) = \hat{x}_n.$$

Of course $\hat{x}_n \downarrow \hat{x}$ as $n \to \infty$. It follows that there exists $t_n \in (t', 1)$ so that $H_{t_n}(c_1(t_n)) = H_{t_n}(c_2(t_n)) = \hat{x}_n$ and so that $f_{t_n}^{nN+k}(c_1(t_n)) = c_1(t_n)$.

We claim that $t_n \downarrow t'$. Indeed, n iterates of $c_1(t_n)$ under the monotone map $f_{t_n}^N|J_{t_n}$ stay in J_{t_n}. It follows that there exist $0 < k_n < n$ so that $|f_{t_n}^{k_n N}(c_1(t_n)) - f_{t_n}^{(k_n+1)N}(c_1(t_n))| \to 0$ as $n \to \infty$. Since $q(t)$ is a repelling N-periodic point for each $t \in (t', 1)$, the critical point $c_1(t_n)$ cannot converge to $q(t)$ as $n \to \infty$. Therefore $|f_{t_n}^{k_n N}(c_1(t_n)) - f_{t_n}^{(k_n+1)N}(c_1(t_n))| \to 0$ as $n \to \infty$ (in other words, $f_{t_n}^N|J_{t_n}$ has 'almost a fixed point' when n is large). Since $f_t^N|J_t$ has only one fixed point as $t > t'$ (namely $q(t)$), it follows that $t_n \to t'$.

Again there exists a sequence of parameter intervals $A_n \ni t_n$ so that $c_1(t_n) = c_2(t_n)$ belongs to the basin of an periodic attractor of period $nN + k$. The parameter intervals $A_n = [t_n^-, t_n^+]$ are disjoint and converge to t'. Here $t_n^\pm$ are parameters at which $f_{t_m^\pm}^{nN+k}$ has a parabolic fixed point (this holds because for each $t \in [t_n^-, t_n^+]$, the map f_t^{nN+k} is monotone on the component of the basin of the periodic attractor containing $c_1(t_n) = c_2(t_n)$).

Step 4. Construction of a non-singular map which satisfies the assumptions of Theorem 3.1. The singular map $f_{t'}$ has a parabolic N-periodic point and there exists a codimension one algebraic set $\Sigma(t') \ni f_{t'}$ of maps in $\overline{\mathcal{P}}_\epsilon^b$ which also have a parabolic periodic point of period N. Similarly, there exist codimension one algebraic sets $\Sigma(t_n^\pm)$ containing $f_{t_n^\pm}$ with a parabolic periodic point of period $nN+k$.

Side Remark: Using the rigidity results contained in [14] and [4], one can prove that $\Sigma(t')$ and $\Sigma(t_n^\pm)$ are codimension-one manifolds of $\mathcal{P}_\epsilon^b$ and that $\Sigma(t_n^\pm)$ converges to $\Sigma(t')$. So for maps corresponding to the region between $\Sigma(t_n^-)$ and $\Sigma(t_n^+)$ the critical points c_1, c_2 are both contained in the same component of the immediate basin of a periodic point of period $nN + k$. So the situation is somewhat analogous to that of rational Arnol'd tongues for circle diffeomorphism (except that the regions between $\Sigma(t_n^-)$ and $\Sigma(t_n^+)$ do not touch). The map g we are looking for lies in the set $\Sigma(t')$.

In this paper, rather than using the property that $\Sigma(t')$ is a manifold, to show that the required g exists, it will be sufficient to use much softer arguments. Indeed, take $t^- < t' < t^+$ so that $f_{t^-}^N$ has a hyperbolic attracting fixed near $q^*(t')$ and $f_{t^+}^N$ has no fixed point near $q^*(t')$. Moreover, let $g_t \in \mathcal{P}_\epsilon^b$, $t \in [t^-, t^+]$ be a family which depends continuously on t and so that $d(f_t, g_t) < \delta'$ for each $t \in [t^-, t^+]$ (where $d(f, g)$ is the Euclidean distance of the coefficients of the two polynomials). Provided $\delta' > 0$ is sufficiently small, there exists $t_p \in [t^-, t^+]$ so that $g := g_{t_p}$ has a parabolic N-periodic point, so that g_t has no parabolic periodic point of period $\leq N$ for $t \in (t^-, t^+]$ and so that g satisfies the properties required in the assumption of Theorem 3.1.

3.2 Proof of Theorems 3.1 and 1.2

Before proving Theorems 3.1 and 1.2, we state and prove some lemmas.

Let f_{t_n} be the singular maps constructed in Step 3. above. Take $f_n \in \mathcal{P}_\epsilon^b$ sufficiently close to f_{t_n} so that f_n has a periodic attractor of $nN + k$ which still attracts the two critical points c_1 and c_2 and so that, moreover, c_3 is still in the basin of a periodic attractor of period R.

Set

$$W_n := \{\varphi \in \mathcal{P}^b_\epsilon \ : \ \varphi \text{ is hyperbolic and partially conjugate to } f_n\}, \tag{4}$$

where we say that φ and f_n are *partially conjugate* if there exists an orientation preserving homeomorphism h (depending on φ and f_n) so that $h \circ f_n = \varphi \circ h$ holds outside the basin the periodic attractors of f_n.

Lemma 3.1. *The set W_n is homeomorphic to an open three-dimensional ball.*

Proof. This lemma is special case of what is proved in [4, Theorem 2.2]. Note that the sets W_n are pairwise disjoint since each these sets correspond to maps with periodic attractors of different periods, namely $nN + k$. □

As mentioned above, the boundary of W_n consists of smooth codimension one surfaces in $\mathcal{P}^b_\epsilon$, but we shall not use or prove this fact here.

Let n be any integer. In the next lemma we will show that entropy is oscillating along any path of maps g_t which connects a map in W_n (from (4)) with g where g is the map from Theorem 3.1.

Lemma 3.2. *Take h, T and N as in the assumptions of Theorem 3.1. Then there exists $\delta_0 > 0$, so that for any $g \in \mathcal{P}^b_\epsilon$ which satisfies conditions (1)-(5) from that theorem with $\delta \in (0, \delta_0)$, there exist $h_0 > h_1 > h_2 > \cdots > h$ and $\tau_0 > 0$ so that for* any *unfolding family $(g_t)_{t\in[0,1]}$ for which $d(g_t, g) < \tau_0$ for each $t \in [0, 1]$ and with $g_{t_n} \in W_n$ for some $t_n \in (0, 1]$ there exist parameters $t_n > s_n > t_{n+1} > s_{n+1} > \cdots > 0$ so that for each $m \geq n$ the following holds:*

(A) $g_{t_m} \in W_m$ and therefore $h_{top}(g_{t_m}) = h$;

(B) $h_{top}(g_{s_m}) > h_m > h$.

Remark: Provided $\tau_0 > 0$ is sufficiently small, $g_t \in \mathcal{P}^b_\epsilon$ is non-singular for each $t \in [0, 1]$.

Proof. Let us again assume that $c_\eta, c_{\eta+1} = c_1, c_2$. For $\delta > 0$ sufficiently small, the critical points $c_3, \ldots, c_b$ are still in the basins of hyperbolic periodic attractors. Let H_t be the orientation preserving semi-conjugacy between g_t and a piecewise affine map T_t with slope $\pm e^{h_{top}(g_t)}$; it maps c_k to periodic orbits for $k = 3, \ldots, b$.

Take $\hat{c}_1(t) \in (c_2(t), c_3(t))$ so that $g_t(\hat{c}_1(t)) = g_t(c_1(t))$ and define

$$G_t(x) = \begin{cases} g_t(c_1(t)) & \text{if } x \in [c_1(t), \hat{c}_1(t)]; \\ g_t(x) & \text{otherwise.} \end{cases}$$

Since the kneading invariants of G_t and T for the critical points $c_3, \ldots, c_b$ are the same, for each t there exists a semi-conjugacy $\bar{H}_t$ between G_t and T. Let $\hat{q} = \bar{H}_t(q^*)$

and let k, $\hat{J}$ and $\hat{x}$ be as in Step 1 in Section 3.1. Moreover, choose an interval $J_t \ni q^*$ so that $\bar{H}_t(J_t) = \hat{J}$ and take $x_t \in g_t^N(J_t)$ so that $H_t(x_t) = \hat{x}$. As in Step 3 of Section 3.1, there exists parameters $t_n > t_{n+1} > \cdots > 0$ so that for $m \geq n$ one has $g_{t_m}^{mN+k}(c_1(g_{t_m})) = G_{t_m}^{mN+k}(c_1(g_{t_m})) = c_1(g_{t_m})$. For each $m \geq n$, H_{t_m} semi-conjugates g_{t_m} to the same map T and $H_{t_m}([c_1(t_m), \hat{c}_1(t_m)])$ is a single $mN+k$-periodic point for T. Provided $\delta > 0$ is sufficiently small, for each $t \in [0,1]$ there exists a fundamental domain (recall Definition 3.1) $F_t \subset J_t$ containing $c_1(g_t)$ and $c_2(g_t)$. Now take $t > 0$ and $n \in \mathbb{N}$ so that $F_t, g_t^N(F_t), \dots, g_t^{(n-1)N}(F_t) \subset J_t$ have disjoint interiors and so that $x_t \in g_t^{nN}(F_t)$.

Claim: There exists $\rho_0 > 0$ so that $|g_t^{nN}(F_t)|, |F_t| \geq \rho_0$ for each $t > 0$.

Indeed, let U_0 be the maximal neighbourhood of $c_1(t)$ on which $|Dg_t(x)| < 1$, and let $U_1 \ni c_1(t)$ be a subinterval of diameter $\frac{1}{4}|U_0|$. If $F_t \subset U_1$, then since also $g_t^N(F_t)$ is adjacent to F_t (recall that it is a fundamental domain) $c_1(t)$ is contained in the basin of an attracting fixed point of $g_t^N|J_t$, contradicting assumption (c) on the family g_t. That $|g_t^{nN}(F_t)|$ is not small holds because this interval is about to "leave J_t" (in less than k steps, where k does not depend on n).

The previous claim implies that $Dg_t^{mN}|F_t$ is bounded uniformly in $t > 0$ for each $m \geq n$. Since k does not depend on t, it follows that $Dg_{t_n}^{mN+k}|F_t$ is also bounded uniformly in $t > 0$. In particular, when $\delta > 0$ is sufficiently small, $|Dg_{t_m}^{mN+k}| < 1$ on the interval connecting $c_1(g_t)$ and $c_2(g_t)$ and therefore $c_2(g_t)$ is contained in the basin of attracting periodic orbit $c_1(g_t)$ when $t = t_m$. This shows that $g_{t_m} \in W_m$ and therefore $h_{top}(g_{t_m}) = h$. In other words, the family f_t passes through W_m for each m large, which means that the 'teeth of the comb' in the h-isentrope don't decrease in height as $m \to \infty$.

There exists a sequence $s_n > s_{n+1} > \cdots > 0$ so that $t_n > s_n > t_{n+1} > s_{n+1} > \cdots > 0$ and so that $c_1(g_{s_m})$ is not in the basin of a periodic attractor of g_{s_m} for $m \geq n$. This holds because the set of parameters s for which $c_1(g_s)$ is contained in the basin of a periodic attractor of g_s is a union of disjoint intervals with disjoint closures, and therefore the complement has to be non-empty.

We need to show that $h_{top}(g_{s_m}) > h_m$ for some $h_{m-1} > h_m > h := h_{top}(G_{s_m})$. Write $Y = (c_1(t_m), \hat{c}_1(t_m))$, $Y^* = \cup_{k \geq 0} g_{s_m}^k(Y)$ and $Y^{**} = \{y \in Y^* \ : \ g_{t_m}^k(y) \notin Y \text{ for all } k \geq 0\}$. Since Y is not asymptotically periodic, Y^* is a g_{s_m}-invariant set consisting of finitely many interval, and it carries a unique measure of maximal entropy ν with $\nu(Y) > 0$, see [13]. On the other hand, the restriction $g_{s_m}|Y^{**}$ (which coincides with the restriction $G_{s_m}|Y^{**}$) has its own measure of maximal entropy $\bar{\nu}$ and $\bar{\nu}(Y) = 0$. Therefore ν and $\bar{\nu}$ are both g_{s_m}-invariant and $\nu \neq \bar{\nu}$. Since ν is the unique measure of maximal entropy, the metric entropies satisfy $h_\nu(g_{s_m}) > h_{\bar{\nu}}(g_{s_m}) = h$ and the required $h_m \in (h, h_{m-1})$ can indeed be found. The above claim gives in fact $|g_{s_m}^{mN}(Y)| \geq \rho_0$, so there exists $K = K(\rho_0) \in \mathbb{N}$ such that

$\cup_{k=mN}^{mN+K} g_{s_m}^k(Y) = Y^*$ independently of m. Therefore we can choose h_m depending only on f, m and τ_0, but not on the family g_t. $\square$

Lemma 3.3. *Let X be connected subset of $\mathbb{R}^b$. For each $x \in X$, choose $\tau(x) > 0$ and define $U = \cup_{x \in X} B_{\tau(x)}(x)$. Then U is path connected.*

Proof. Since X is connected, so is U. Since U is open, it is also path connected. $\square$

Now we are ready to prove the main theorems.

Proof of Theorems 3.1. Take g and τ_0 be as in the assumption of Theorem 3.1. Suppose by contradiction that L_h is locally connected at g. Then, by definition, there exists an open set Y which is contained in a $\tau_0/2$-neighbourhood of g so that $X := L_h \cap Y$ is connected. By Lemma 3.2 for n sufficiently large, $W_n \cap X \neq \emptyset$. Pick such an integer n and let $h_n > h$ be as in Theorem 3.1. By Lemma 3.3, there exists a path connected open set $U \supset X$ which is contained in $\{\tilde{g}; h_{top}(\tilde{g}) < h_n \text{ and } d(\tilde{g}, X) < \tau_0/2\}$. So choose a path $(g_t)_{t \in [0,1]}$ in U so that $g_0 = g$ and $g_1 \in W_n \cap X$. In particular, $h_{top}(g_t) < h_n$ and $d(g_t, g) < \tau_0$ for each $t \in [0, 1]$. However, by Lemma 3.2, there exists $s_n \in (0, t_n)$ (where we can take $t_n = 1$) so that $h_{top}(g_{s_n}) > h_n$. This contradicts that $h_{top}(g_t) < h_n$ for each $t \in [0, 1]$. Thus we have shown that L_h is not locally connected at g. $\square$

Proof of Theorem 1.2. This immediately follows from Theorem 3.1. $\square$

References

[1] R. L. Adler, A. G. Konheim, M. H. McAndrew, *Topological entropy,* Trans. Amer. Math. Soc. **114** (1965), 309–319.

[2] L. Alsedá, J. Llibre, M. Misiurewicz, *Combinatorial dynamics and entropy in dimension one,* Advanced Series in Nonlinear Dynamics, World Scientific, New Jersey (1993).

[3] R. Bowen, *Entropy for group endomorphisms and homogeneous spaces,* Trans. Amer. Math. Soc. **181** (1973), 509–510.

[4] H. Bruin, S. van Strien, *Monotonicity of entropy for real multimodal maps,* Preprint 2009. arXiv:0905.3377

[5] D. Cheraghi, S. van Strien, *Monotonicity of entropy in a three parameter family of unimodal maps*, in preparation.

[6] E. I. Dinaburg, *The relation between topological entropy and metric enropy,* Dokl. Akad. Nauk SSSR., **190** (1970), 19–22, Soviet Math. Dok. **11** (1970), 13-16.

[7] A. Douady, *Topological entropy of unimodal maps: Monotonicity for quadratic polynomials*. In: Real and Complex Dynamical Systems, B. Branner and P. Hjorth, Eds. Dordrecht: Kluwer, (1995), 65–87.

[8] A. Douady, J. H. Hubbard, *Etude dynamique des polynômes quadratiques complexes*, I, (1984) & II, (1985), Publ. Mat. d'Orsay.

[9] A. Epstein, *Transversality in holomorphic dynamics*, see http://homepages.warwick.ac.uk/∼mases/Transversality.pdf.

[10] B. Friedman, C. Tresser, *Comb structure in hairy boundaries: some transition problems for circle maps,* Phys. Lett. A **117** (1986), 15–22.

[11] J.-M. Gambaudo, J. Los, C. Tresser, *A horseshoe for the doubling operator: topological dynamics for metric universality,* Phys. Lett. A **123** (1987), no. 2, 60–64.

[12] V. Kaloshin, O. Kozlovski, *A C^r unimodal map with an arbitrary fast growth of the number of periodic points,* Ergod. Th. Dynam. Sys. **32** (2012), no. 1, 159–165.

[13] F. Hofbauer, *On intrinsic ergodicity of piecewise monotonic transformations with positive entropy, II.* Israel J. of Math. **38** (1981) 107-115.

[14] O. Kozlovski, W. Shen, S. van Strien, *Rigidity for real polynomials,* Annals of Math. **165** (May 2007), 749–841.

[15] G. Levin, *Multipliers of periodic orbits in spaces of rational maps*, Ergod. Th. Dynam. Sys. **31**, 197–243.

[16] W. de Melo, S. van Strien, *One-Dimensional Dynamics,* Springer, Berlin Heidelberg New York, (1993).

[17] M. Misiurewicz, W. Szlenk, *Entropy of piecewise monotone mappings,* Studia Math. **67** (1980), no. 1, 45–63.

[18] J. Milnor, *Remarks on iterated cubic maps*, Experimental Math. **1** (1992), 5–24.

[19] J. Milnor, W. Thurston, *On iterated maps of the interval: I, II*, Lect. Notes in Math. **1342**, Springer, Berlin New York (1988), 465–563.

[20] J. Milnor, C. Tresser, *On entropy and monotonicity for real cubic maps*, Comm. Math. Phys. **209** (2000), 123–178.

[21] W. Parry, *On the β-expansion of real numbers,* Acta Math. Acad. Sci. Hungar **11** (1960), 401–416.

[22] A. Radulescu, *The connected isentropes conjecture in a space of quartic polynomials,* Discrete Contin. Dyn. Syst. **19** (2007), 139–175.

[23] L. Rempe, S. van Strien, *Density of hyperbolicity for real transcendental entire functions with real singular values*, http://arxiv.org/abs/1005.4627.

[24] S. van Strien, *Misiurewicz maps unfold transversaly in families*, Fundamenta Mathematica **163** (2000), 39–54.

[25] S. van Strien, *Milnor's Conjecture on Monotonicity of Topological Entropy: results and questions.* To appear in: Frontiers in Complex Dynamics (Celebrating John Milnor's 80th birthday).

[26] M. Tsujii, *A simple proof for monotonicity of entropy in the quadratic family,* Ergod. Th. Dynam. Sys. **20** (2000), 925–933.

Faculty of Mathematics, University of Vienna,
Oskar Morgensternplatz 1, A-1090 Vienna, Austria.
`henk.bruin@univie.ac.at`
`http://www.mat.univie.ac.at/`$\sim$`bruin`

Department of Mathematics, Imperial College,
180 Queen's Gate, London SW7 2AZ, UK.
`s.van-strien@imperial.ac.uk`
`http://www2.imperial.ac.uk/`$\sim$`svanstri/`