TOPOLOGICAL CONDITIONS FOR THE EXISTENCE OF INVARIANT MEASURES FOR UNIMODAL MAPS.

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Abstract. We present a class of S-unimodal maps having an invariant measure which is absolutely continuous with respect to Lebesgue measure. This measure can often be proved to be finite. We give an example of a map which has such a finite measure and for which the lim inf of the derivatives of the iterates of the map in the critical value is finite. It will be shown that all topologically conjugate non-flat S-unimodal maps have these same properties.

1. Introduction.

In some dynamical systems the topological behaviour appears to control metrical behaviour as well. A well-known example of this is Herman's work [He] on circle diffeomorphisms, where a topological condition, namely a certain value of the rotation number, guarantees the existence of an invariant probability measure which is absolutely continuous with respect to Lebesgue measure. In a unimodal setting such a measure (which we shall call acip) is known to exists for Misiurewicz maps, i.e. maps with a non-recurrent critical point and no periodic attractor. Another example is the Fibonacci map [LM]. Here a special combinatorial form of the critical orbit yields an acip if the critical point is of quadratic type. In this paper we present a new class of unimodal maps which have acips on topological grounds. We extend the Misiurewicz condition to certain recurrent maps.

The results nevertheless require a metrical set-up concerning smoothness of the map and negative Schwarzian derivative. Some (sufficient) conditions in purely metrical terms for the existence of an acip are known. Keller [K] proved that the existence of an acip is equivalent to positive Lyapunov exponent for almost all points. In fact for almost all points x, $\limsup_{n\to\infty} \frac{1}{n} \log |Df^n(x)|$ exists and has the same value. This value is positive if and only if an acip exists. A similar result was obtained by Ledrappier [Le]. The critical orbit is not mentioned here, but as Collet and Eckmann [CE] showed, exponential growth of $|Df^n(c_1)|$ guarantees the existence of an acip. Here c_1 denotes the critical value. Nowicki and Van Strien [NS] showed that one can do with much less. Namely if

$$\sum_n |Df^n(c_1)|^{-1/\ell} < \infty, \tag{1}$$

where ℓ is the order of the critical point, then f has an acip. The proof that the Fibonacci map has an acip is based on this fact.

The class of functions we present in this paper is quite large: in general maps from this class don't satisfy the above summability condition. In fact, the derivatives of the critical value don't necessarily tend to infinity. So we can state

Theorem 1. *There exists a unimodal map with an absolutely continuous invariant probability measure for which* $\liminf_n |Df^n(c_1)| < \infty$. *Every non-flat S-unimodal map which is topologically conjugate to our example has these same properties.*

The class of maps we restrict ourselves to is the set $\mathcal{G}$ of S-unimodal functions with a non-flat critical point and long branches. By this we mean that the length of branches are uniformly bounded from below: If T is a maximal interval of monotonicity of f^n, the graph of $f^n_{|T}$ is called a branch. A branch is *long* if $|f^n(T)| \geq Z$ for some fixed positive constant Z. $\mathcal{G}$ includes the Misiurewicz maps. It is well known that the Misiurewicz maps have an acip; in fact they satisfy the Collet-Eckmann condition. In our case c can be recurrent in a special way which we call saddle node type return. This means that $f^n(c)$ can only be close to c if $f^n(U) \not\ni c$, where U is a neighbourhood of c that is folded only once by f^n. These saddle node type returns are the main tool to keep the derivatives $Df^n(c_1)$ bounded, at least for certain iterates.

In order to establish the existence of the acip, we use an induced Markov map. A Markov map has an acip; this is the so-called Folklore Theorem. We need the long-branchedness to prove that our induced map satisfies the Markov properties. The definition will be given later. The technique of inducing has been used by many authors, e.g. [GJ,JS,MS1,V]. For example one can prove that Misiurewicz maps have an acip by using the first return map to a certain interval as induced map. The main difference with our approach is that we don't induce to a fixed interval, but rather to a finite set of intervals. It depends on the branch of the induced map which of these intervals is the image. Also the boundary points of each branch of the induced map are preimages of c. The advantage of this approach is that we know exactly how many iterates of f are needed to obtain the induced map and that it is easy to define the domain of each branch of the induced map. During the research for this paper, we heard that Keller and Nowicki [KN] use a similar kind of induced map in their extension of Lyubich's and Milnor's results on Fibonacci maps.

From the acip of the induced map one can derive an absolutely continuous invariant measure for the unimodal map itself. However, for the finiteness of this measure one has to check a summability condition on domains of branches and corresponding iterates. Checking this is the hardest part of the proof. In general this summability condition doesn't hold. In Theorem 2 we show that in this case f has an absolutely continuous invariant measure which is only σ-finite, and we give an example of such a map. This example has the same flavour as Johnson's example [J]. It is not unlikely that one has to follow either Johnson's or our scenario to obtain an S-unimodal map without acip. Both cases involve that for every m, c returns at least m times in a row to the central branch domain of f^n under iteration of f^n, for some $n = n(m)$. We think that this is essential.

Conjecture. *Let f is be an S-unimodal map with critical order 2. Let $U_n \ni c$ be the maximal interval such that $f^{n-1}_{|f(U)}$ is diffeomorphic. If there exists $M \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $f^{mn}(c) \notin U_n$ for some $m \leq M$, then f has an absolutely continuous invariant probability measure.*

The main combinatorial tool is the kneading map. This map was developed by Hofbauer [H] and gives a characterization of the kneading invariant of a unimodal map. We will use it to define our induced map and to characterize combinatorially our notion of long branches and saddle node type returns. In particular we will show that long-branchedness is equivalent to a bounded kneading map. Furthermore it is a guide for the construction of the kneading invariant of our examples. We will devote a section to these ideas.

We want to thank Sebastian van Strien for his advices and the many fruitful discussions. Also we are grateful to the referee for carefully reading the manuscript and the many useful suggestions.

2. Outline of the proof.

We consider the class $\mathcal{G}$ of non-flat S-unimodal maps with long branches. That is

- $f : [0,1] \to [0,1]$ is an interval map such that $f(0) = f(1) = 0$ and 0 is hyperbolic repelling. f is at least C^3 and has negative Schwarzian derivative: $\frac{D^3 f(x)}{Df(x)} - \frac{3}{2}(\frac{D^2 f(x)}{Df(x)})^2 < 0$ wherever it is defined.
- There exists a unique critical point c. f is increasing on $[0,c]$ and decreasing on $[c,1]$.
- There exist a C^3-coordinate transformation φ such that $f(\varphi(x)) = f(c) - |x-c|^\ell$ in a neighbourhood of c. We assume that the order of the critical point $1 < \ell < \infty$. It follows that constants $0 < O_1 < O_2 < \infty$ can be found such that $\ell O_1 |x-c|^{\ell-1} \leq |Df(x)| \leq \ell O_2 |x-c|^{\ell-1}$.
- There exists a uniform constant $Z > 0$ such that if T is a maximal interval of monotonicity of f^n, $|f^n(T)| \geq Z$.

In general we assume that f has no periodic attractor. As a model we think of a full family f_α of unimodal maps. This means that f_α displays any possible combinatorial type for some value of α. The quadratic family $f_\alpha(x) = \alpha x(1-x)$ is such a family.

A probability measure μ is said to be invariant and absolutely continuous with respect to Lebesgue measure if $\mu(A) = \mu(f^{-1}(A))$ and if $|A| = 0$ implies $\mu(A) = 0$. It has been shown that every S-unimodal map without periodic attractor is ergodic with respect to Lebesgue measure (or μ) [BL2]. Other proofs can be found in [Ma1,MS1]. Moreover, μ is a Bowen-Ruelle-Sinai measure, which means that for all continuous functions φ,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) = \int \varphi d\mu$$

for Lebesgue almost all x. A consequence of this is that for any open set A,

$$\mu(A) = \lim_{n\to\infty} \frac{1}{n} \#\{i \mid 0 \leq i < n \text{ and } f^i(x) \in A\}$$

for Lebesgue almost all x.

Additionally to Theorem 1, we show that long-branchedness is not a sufficient condition for an acip to exist.

Theorem 2. *There exists a unimodal map f such that no map in $\mathcal{G}$ which is topologically conjugate to f has an absolutely continuous invariant probability measure. However, every map in $\mathcal{G}$ has a σ-finite absolutely continuous invariant measure.*

We explicitly use the fact that $\omega(c)$, the set of accumulation points of the forward orbit of c, is a nowhere dense for $f \in \mathcal{G}$.

In the sequel (x, y) is an interval irrespective the ordering of the boundary points. If $x \neq c$, $\hat{x}$ denotes the involution of x. So $\hat{x} \neq x$ and $f(\hat{x}) = f(x)$. Furthermore c_n denotes $f^n(c)$ and if $n < 0$ it should be clear from the context which preimage of c is meant. Also let $\mathcal{N} = [f^2(c), f(c)]$ and $L = \max_{x \in \mathcal{N}} |Df(x)|$.

The proof of the theorems relies on two major ideas. First we have the notion of saddle node cascades. At a saddle node bifurcation of period n the graph of f^n is tangent to the diagonal. So we have a one-sided n-periodic attractor which, due to negative Schwarzian derivative, attracts the critical point. It is obvious from the graph that $f^n(c)$ is close to c, but that for the neighbourhood U of c which is folded only once by f^n, $f^n(U) \not\ni c$. This is why we call this kind of return a saddle node type return. It is also obvious that $|Df^n(c_1)| < 1$.

Under an appropriate, but arbitrarily small perturbation the periodic point vanishes, and we are left with an almost periodic behaviour. So c, $f^n(c)$, $f^{2n}(c)$,... will still be close together. Also derivatives hardly change, so $Df^n(c_1)$ remains bounded. This will be made precise in Lemma 5.

A perturbation of a saddle node map can still be a saddle node map of a higher period. In the proof we shall define a saddle node cascade as a converging sequence of maps at saddle node bifurcations of increasing period. The limit map inherits its behaviour of each of the approximating saddle node maps. In particular, the derivatives $|Df^n(c_1)|$ of the limit map will be bounded whenever n is the period of an approximating saddle node map. This will cause $\liminf_n |Df^n(c_1)|$ to be finite.

Using techniques from Lemmas 1 and 2 of the next section, we will construct the kneading invariants of a saddle node cascade. Because we present our examples in this form, the result holds for topologically conjugate maps, provided they are in $\mathcal{G}$.

In Theorem 2 this technique is used to construct a map which doesn't satisfy the summability condition (2) below. In Lemma 3 we prove that $\omega(c)$ is nowhere dense for long-branched maps, which, according to [Ma2,HK2], leads to the existence of a σ-finite measure. Using a general argument from ergodic theory, and ergodicity of f, we show that μ is unique up to a multiplicative constant. In order to prove Theorem 1, we need to be more careful in constructing our saddle node cascade.

The other idea is the induced Markov map. We will define the inducing using the tools of the next section. In short, we show that c_{-S_j} are the closest preimages of c for a certain sequence $\{S_j\}_{j \geq 0}$. These preimages come in pairs; if c_{-S_j} is a closest preimage, so is $\hat{c}_{-S_j}$. Let $\Gamma_j = (c_{-S_{j-1}}, \hat{c}_{-S_{j-1}})$ and $A_j = int(\Gamma_j \setminus \Gamma_{j+1})$. A_j which consists of two intervals, which are the involutions of each other. $\cup_j A_j$

is a partition of the interval $\mathcal{N}$ into pairwise disjoint sets. The map F defined by $F_{|A_j} = f^{S_{j-1}}_{|A_j}$ is an induced Markov map.

By the Folklore Theorem F has an acip. The question whether $f \in \mathcal{G}$ itself has an acip reduces, and is equivalent, to the following summability condition

$$\sum_j j|A_j| < \infty. \tag{2}$$

It would be sufficient to prove that $|A_j|$ decreases exponentially, and it is not hard to show that this is the case if f is a Misiurewicz map. However the exponential decrease of $|A_j|$ persists in the sense that we can indicate regions in $\mathbb{N}$ where $|A_j|$ decreases exponentially, provided the critical orbit stays long enough outside a neighbourhood of c. This will be made precise in Lemma 6.

In general $|A_j|$ doesn't decrease exponentially for saddle node cascades. Theorem 2 clearly gives a counter-example. The idea is to alternate the Misiurewicz and the saddle node behaviour in such a way that due to the Misiurewicz behaviour (2) holds, and due to the saddle node behaviour (1) is not fulfilled. As the saddle node behaviour tends to disturb (2), we need to find a good balance between the two kinds of behaviour. The crucial estimation for this is contained in the last section.

We assume negative Schwarzian derivative because we want to apply the Koebe Principle. Let $T \supset I$ be two intervals. Then T is said to contain a δ-scaled neighbourhood of I if both components of $T \setminus I$ are longer than $\delta|I|$. Let the distortion on I of a function g be defined as $\sup_{x,y\in I} \log \frac{|Dg(x)|}{|Dg(y)|}$. With these notions one can formulate the

Koebe Principle. *For all $\delta > 0$ there exists $K > 0$ such that if $T \supset I$, $g_{|T}$ is monotone and $g(T)$ contains a δ-scaled neighbourhood of $g(I)$, then the distortion of $g_{|I}$ is bounded by K. Moreover, $K \to 0$ as $\delta \to \infty$.*

This result holds if g has negative Schwarzian derivative. For the proof we refer to [MS1]. We will use this principle in Section 4 and in the proof of Lemma 6.

3. The kneading map.

In this section we give a combinatorial characterization of long branches (Lemma 1) and saddle node type returns (Lemma 2). They enable us to describe a saddle node cascade with long branches in terms of kneading invariants. Also we introduce a notation for preimages of the critical point, which we will use for the definition of the induced map of the next section. Along our proof we have a rather combinatorial approach to these objects: we will often compare pieces of itineraries. The reader should keep in mind that if itineraries coincide for a long piece, the corresponding iterates of the critical point must be close together.

Notice f admits no wandering intervals, as was proved by Guckenheimer [G] in a S-unimodal setting and by others [BL1,Ly1,MMS,MS2] under more general hypotheses. A wandering interval is an interval I such that $f^n_{|I}$ is monotone for every I and $\omega(I)$ is not an attracting periodic orbit. A corollary of this is the following: For all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f^n(I)| > \delta \tag{3}$$

whenever $|I| > \varepsilon$ and $f^n_{|I}$ is monotone. The proof can be found in for example [MS1].

The kneading map is a tool developed by Hofbauer [H] by which the kneading invariant of a unimodal map can be described. Let the *itinerary* of x be the infinite sequence

$$\nu(x) = e_1(x)e_2(x)e_3(x)...$$

of zeroes and ones such that $e_i = 1$ if $f^i(x) > c$ and $e_i = 0$ if $f^i(x) < c$. We are not interested in $e_0(x)$, the position of x itself. This means that x and $\hat{x}$ have the same itinerary. Also we neglect precritical points. The word *kneading invariant* is reserved for the itinerary $\nu = \nu(c)$ of the critical point. We disregard the case that c is periodic. There is a unique way to split this kneading invariant into *basic* blocks:

$$\nu = 1\Delta_1\Delta_2\Delta_3...,$$

where each block $\Delta_j = e_{i+1}e_{i+2}...e_{i+k}$ coincides with $e_1e_2...e_{k-1}e'_k$. Here $e'_i = 1$ if $e_i = 0$ and vice versa. So each block Δ_i repeats the head of the kneading invariant except for the last entry. Let $|\Delta_i|$ denote the length of the block Δ_i, then we can define

$$S_k = 1 + \sum_{j=1}^{k} |\Delta_j|.$$

This is well defined if each block Δ_j has finite length. If however Δ_k is infinitely long, it means that $\sigma^{S_{k-1}}(\nu) = \nu$. Both c and $f^{S_k}(c)$ have the same itinerary, but that means that f has a periodic attractor or a wandering interval. We will see in Lemma 2 that the first is related to period doubling bifurcations.

Suppose that i is the first entry for which $\nu(x)$ and ν differ, then Hofbauer showed that $i \in \{S_k\}$. And conversely, if $i \in \{S_k\}$, there exist points whose itinerary coincide with ν exactly up to the $i-1$th entry. In particular, this means that $|\Delta_k| = S_r$ for some $r < k$, because the itinerary of $f^{S_{k-1}}(c)$ coincides with ν up to $|\Delta_k| - 1$. Therefore it is possible to define a map $Q : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ by

$$S_{Q(k)} = |\Delta_k| = S_k - S_{k-1}.$$

Q satisfies

$$Q(k) < k \text{ or } Q(k) = \infty,$$

and Hofbauer established the following admissibility condition:

$$\{Q(k+j)\}_{j\geq 1} \succeq \{Q(Q^2(k)+j)\}_{j\geq 1}.$$

Here $\succeq$ denotes lexicographical ordering. As we mentioned $Q(k) = \infty$ leads to a periodic attractor of period doubling type. Admissibility means that there exists a unimodal map with this particular kneading invariant. We will not use this, but a more standard admissibility condition, mentioned at Lemma 4.

We introduce some more notation. c_{-n} is called a *closest preimage* of c, if there exists no point $x \in (c_{-n}, \hat{c}_{-n})$ such that $f^m(x) = c$ for some $m \leq n$. Let

$$\Gamma_k = \{x \mid \nu(x) = \nu \text{ up to at least } S_k - 1 \text{ entries.}\}.$$

Clearly $\{\Gamma_k\}_k$ is a nested sequence of open symmetric neighbourhoods of c. Also S_k is the smallest positive iterate such that $f^{S_k}(\Gamma_k) \ni c$. Since this holds for all k, $f^{S_k}(\Gamma_{k+1})$ is an interval adjacent to c. Hence the boundary points of Γ_k are the closest preimages $c_{S_{k-1}}$ and $\hat{c}_{S_{k-1}}$. This implies that c_n can only be a closest preimage if $n \in \{S_k\}$. Let $A_k = int(\Gamma_k \setminus \Gamma_{k+1})$. Due to Hofbauer's observation, the itinerary of each point in A_k coincides with ν up to exactly the $S_k - 1$th position. Moreover $A_k \neq \emptyset$, so there always exist points whose itineraries coincide with ν up to exactly $S_k - 1$. With this information it is not hard to prove the following useful formulas:

$$
\begin{aligned}
f^{S_k}(\Gamma_k) &= [c_{S_k}, c_{S_{Q(k)}}). \\
f^{S_k}(A_k) &= (c, c_{S_{Q(k)}}). \\
f^{S_{k-1}}(\Gamma_k) &= (c, c_{S_{k-1}}]. \\
f^{S_{k-1}}(A_k) &= (c, c_{-S_{Q(k)}}).
\end{aligned} \tag{4}
$$

With respect to the fourth, we mention that $f^{S_{k-1}}(A_k)$ is a largest interval on which $f^{S_{Q(k)+1}}$ is monotone. Indeed, $f^{S_{k-1}}(A_k)$ is a largest interval on which $f^{S_{Q(k)}}$ is monotone and $f^{S_{Q(k)}}(f^{S_{k-1}}(A_k)) \not\ni c$. So $f^{S_{k-1}}(A_k)$ is equal to a component of $\Gamma_{Q(k)+1} \setminus \{c\}$. In the sequel $com(\Gamma_j)$ and $com(A_j)$ will be components of $\Gamma_j \setminus \{c\}$ and A_j. It should be clear from the context which component is meant.

Before giving a combinatorial characterization of long branches, let us first study this notion more precisely. It is sufficient for long-branchedness that branches $f^n_{|T}$ such that $f^n(T) \ni c$, which are called the branches at c-level, are uniformly long. Indeed, let T be a maximal interval of monotonicity of f^n. There exist iterates $r < s < n$ such that $c \in f^r(\partial T), f^s(\partial T)$. Because of long branches at c-level, $|f^s(T)| \geq Z_1$ for some uniform constant $Z_1 > 0$. It easily follows from (3) that we can find $Z > 0$ such that $|f^n(T)| \geq Z > 0$.

Lemma 1. *f has long branches if and only if Q is bounded.*

Proof. First observe that the smaller the distance from a point x to c, the longer its itinerary will coincide with ν. As we have long branches, $f^{S_k}(c)$ cannot be close to c. Assume it lies outside Γ_{B+1} for some fixed number B. Then the itinerary of $f^{S_k}(c)$ coincides with ν up to at most $S_B - 1$, and $|\Delta_{k+1}| = S_{Q(k+1)} \leq S_B$ implying $Q(k+1) \leq B$.

Conversely, if $Q(k+1) \leq B$, it follows that $f^{S_k}(c) \notin \Gamma_{B+1}$ for all k. Let U be an arbitrary neighbourhood of c, and i the smallest positive iterate such that $f^i(U) \ni c$. Since $\nu(x)$ coincides with ν up to $S_k - 1$ for some k, we get that $i \in \{S_k\}_k$. It follows that all central branches covering c are long, and therefore f has long branches everywhere. □

In the sequel we will often talk about bounded kneading map instead of long branches, and we assume that $Q(k) \leq B$ for all $k \in \mathbb{N}$. Recall that an interval I is called restrictive if $f^n(I) \subset I$ for some n. A restrictive interval enables us to rescale $f^n_{|I}$ (linearly), and restrict ourselves to this so-called renormalization instead of f. It is not hard to prove long branches are incompatible with arbitrarily small restrictive intervals. So a long-branched map is only finitely renormalizable.

Close returns of the critical point lead to periodic behaviour in the kneading invariant. Due to the next lemma we can indicate the saddle node type returns by comparing the corresponding period and the sequence $\{S_k\}$.

Lemma 2. *Suppose f undergoes a saddle node or a period doubling bifurcation and n is the period of the corresponding one-sided attractor p. If $n = S_k$ for some k, then p is of period doubling type and $Q(k+1) = \infty$. If $n \notin \{S_k\}$, then p is of saddle node type and $S_k < \infty$ for all k.*

Proof. As c is contained in the immediate basin of a periodic attractor, $\sigma^n(\nu) = \nu$. Suppose $S_{k-1} < n < S_k$. First we show that $S_k - n \in \{S_j\}_j$. Indeed, there exist admissible itineraries starting with $e_1 e_2 ... e_{S_k - 1} e'_{S_k}$. Therefore

$$e_1 e_2 ... e_{S_k - n - 1} e_{S_k - n} = e_{n+1} e_{n+2} ... e_{S_{k-1}} e_{S_k}$$

and

$$e_1 e_2 ... e_{S_k - n - 1} e'_{S_k - n} = e_{n+1} e_{n+2} ... e_{S_{k-1}} e'_{S_k}$$

are both the head of admissible itineraries. It follows that $S_k - n = S_j$ for some j.

Let $\vartheta(m)$ be the number of ones in $e_1 e_2 ... e_m$. There is a direct relation between the orientation of f^m in a neighbourhood of c and the parity of $\vartheta(m)$. Let $x \in \Gamma_{k+1} \cap (c, c_1)$. By the chain rule,

$$sign(Df^m(x)) = sign \prod_{i=0}^{m-1} Df(f^i(x)) = - \prod_{i=1}^{m-1} -sign(f^i(x) - c) = -(-1)^{\vartheta(m-1)}.$$

Because $sign(f^m(x) - c) = -(-1)^{e_m}$ we obtain

$$(-1)^{\vartheta(m)} = sign\{Df^m(x)(f^m(x) - c)\}.$$

As $f^{S_k}(\Gamma_k) \ni c$, $f^{S_k}(x) - c$ and $Df^{S_k}(x)$ have different sign. So $\vartheta(S_k)$ is odd. Consequently $\vartheta(n) = \vartheta(S_k) - \vartheta(S_j)$ is even.

From the graph of f^n it is clear that $Df^n(x))$ and $f^n(x) - c$ have the same sign at a saddle node bifurcation, while they have opposite sign at a period doubling bifurcation. So a saddle node bifurcation corresponds to $\vartheta(n)$ even and $n \notin \{S_k\}$. A period doubling bifurcation corresponds to $\vartheta(n)$ odd and $n = S_k$ for some k. In the latter case it is also clear that Γ_{k+1} contains no preimage of c. Hence $\sigma^{S_k}(\nu) = \nu$ and $|\Delta_{k+1}| = \infty$. □

For the proof of Theorem 2 the following lemma is of interest.

Lemma 3. *If Q is bounded, then $\omega(c)$ is nowhere dense.*

Proof. Assume that f is not renormalizable; otherwise we consider the first return map on the smallest restrictive interval. If $\omega(c)$ is finite, or c is contained in a wandering interval, then there is nothing to prove. Therefore we may assume that Q is defined for all $k \in \mathbb{N}$ and $Q(k) \leq B$. Let $r > B$ and consider A_r.

$f^{S_{r-1}}(A_r) = (c, c_{-S_{Q(r)}})$, so there exists $V = (c_{-S_{r-1}}, c_{-2S_{r-1}}) \subset A_r$ such that $f^{S_{r-1}}(V) = com(\Gamma_r)$. We will show that $orb(c) \cap V = \emptyset$.

Indeed, Lemma 2 shows that $c_n \in \Gamma_r$ only if $\vartheta(n)$ is even. Suppose $c_n \in V \subset \Gamma_r$, and $S_{k-1} < n < S_k$. Then $f^n(\Gamma_k) = [c_n, c_{n-S_{k-1}})$ and $\vartheta(n - S_{k-1})$ is odd, so $f^n(\Gamma_k) \ni c_{-S_{r-1}}$. Hence, $f^{S_{r-1}}(c_n, c_{-S_{r-1}}) = (c, c_{n+S_{r-1}}) \subset \Gamma_r$. Consequently $n + S_{r-1} = S_m$ is a cutting time and as $c_{S_m} \in U_r$, $Q(m+1) \geq r > B$. Since $\omega(c)$ is not dense on $[c_2, c_1]$, $\omega(c)$ is nowhere dense. □

4. The induced Markov map.

A map $F : W \to W$ is said to satisfy the Markov properties if

a) F is defined on a countable union of open disjoint intervals I_j. $W \setminus \cup_j I_j$ has zero Lebesgue measure.
b) F^n has uniformly bounded distortion for all $n \geq 1$ and all intervals on which F^n is monotone.
c) If $F(I_i) \cap I_j \neq \emptyset$, then $F(I_i) \supset I_j$.
d) For every i and j, there exists n such that $F^n(I_i) \supset I_j$.
e) There exists $\varepsilon > 0$ such that $|F(I_j)| > \varepsilon$ for each j.

It is a well known result, which dates back to Renyi [R] and was extended by others, that a Markov map has an acip. We'll give a short proof of this, following [MS1], that shows that the density of this measure is bounded and bounded away from 0. Property d) guarantees that this measure is ergodic. Notice also that due to b), d) and e), F^N is uniformly expanding for N sufficiently large. In particular, F admits no periodic attractor.

Using the notation of the previous section we will induce from f a piecewise continuous map F and show that F satisfies properties a) to e). Let as before the kneading map Q be bounded by B. Let W be an interval containing c, with closest preimages of c as boundary points. These preimages will be specified in a minute. We can partition W into the sets A_j. Since $W \setminus \cup_j A_j$ is countable, property a) is satisfied. Usually $A_j \cap W$ has two components. On each component we define

$$F_{|A_j} = f^{S_{j-1}}_{|A_j}.$$

From (4) we have $f^{S_{j-1}}(A_j) = (c, c_{-S_{Q(j)}})$. So property c) is fulfilled. As Q is bounded property e) follows too.

Again by the boundedness of the kneading map $F(A_j) \supset com(\Gamma_{B+1})$ for all j. Let $W = \cup_{j>0} F^j(A_B)$, then it is easy to see that F is surjective on W, and that property d) is fulfilled. We will now prove property b).

Proof of b). First we prove b) for $n = 1$. Put $\delta = \min(\frac{|\Gamma_{B+1} \cap (c,c_1)|}{|(c_2,c)|}, \frac{|\Gamma_{B+1} \cap (c_2,c)|}{|(c,c_1)|})$. Let $I = com(A_j)$ for some j, and $T \supset I$ the largest interval on which $f^{S_{j-1}}$ is monotone. It is easy to see from the previous section that $T = com(\Gamma_{j-1})$ and $f^{S_{j-1}}(T) = (c_{S_{j-1}}, c_{S_{Q(j-1)}}) \ni c$. Due to boundedness of the kneading map $f^{S_{j-1}}(T)$ contains a one-sided δ-scaled neighbourhood of $f^{S_{j-1}}(I)$ on the side of c. Now let $T' \supset I$ be the maximal interval such that $f^{S_j}_{|T'}$ is monotone, we see that $T' = com(\Gamma_j)$. $f^{S_j}(T') = (c_{S_j}, c_{S_{Q(j)}})$ and $f^{S_j}(I) = (c, c_{S_{Q(j)}})$. $f^{S_j}(T' \setminus I) =$

$(c_{S_j}, c) \supset com(\Gamma_{B+1})$, so $f^{S_j}(T')$ contains a one-sided δ-scaled neighbourhood of $f^{S_j}(I)$. Since $S_j - S_{j-1} = S_{Q(j)} \leq S_B$, $f^{S_{j-1}}(T)$ contains a one-sided δ'-scaled neighbourhood of $f^{S_{j-1}}(I)$ on the side of $c_{S_{Q(j-1)}}$ for $\delta' \geq \frac{\delta}{L^{S_B}}$. Together this is a δ'-scaled neighbourhood of $f^{S_{j-1}}(I)$, so the Koebe Principle yields the bounded distortion of $f^{S_{j-1}}_{|I} = F_{|I}$.

Now we need to prove that F^n has bounded distortion as well. Let $I \subset I' \subset I''$ be intervals of continuity of F^n, F^{n-1} and F^{n-2}. $F^n_{|I} = f^k_{|I}$ and $F^{n-1}_{|I'} = f^{k'}_{|I'}$. Let $J \supset I$ and $J' \supset I'$ be the maximal intervals of monotonicity of f^k and $f^{k'}$. By construction $F^{n-1}(I') = com(\Gamma_j)$ and $F^{n-1}(I) = com(A_i)$, where $j \leq i$. If $j < i$, then $F^{n-1}(J) = com(\Gamma_{i-1}) \subset com(\Gamma_j)$. The previous argument shows that $f^k(J)$ contains a δ'-scaled neighbourhood of $f^k(I)$.

So we're left with the case that $i = j$. $F^{n-2}(I') = com(A_{i'})$ and $F^{n-2}(I'') = com(\Gamma_{j'})$. Again $j' \leq i'$. Because $F^{n-2}_{|I''}$ is continuous, $F^{n-2}(J') \supset com(\Gamma_{i'})$ and $f^{k'}(J') \supset (c, c_{S_{i'-1}})$. This means that $F^{n-1}(J')$ contains a δ'-scaled neighbourhood of $F^{n-1}(I)$. It follows that $F^n_{|I} = F \circ F^{n-1}_{|I}$ has bounded distortion. □

Since F is a Markov map, any weak accumulation point

$$m = \lim_{j\to\infty} \frac{1}{n_j} \sum_{i=0}^{n_j-1} F^i_* \lambda$$

is an acip. Here λ is the normalized Lebesgue measure on W and $F^i_*(A) = \lambda(F^{-i}(A))$ for every measurable set A. Invariance and finiteness of m are clear, and the properties b) and e) provide absolute continuity. Indeed, let A be a measurable set and I a maximal interval of monotonicity of F^n, then

$$\frac{|F^{-n}(A) \cap I|}{|I|} \leq K \frac{|A \cap F^n(I)|}{|F^n(I)|} \leq \frac{K}{\varepsilon}|A|.$$

Summing over all intervals of the partition generated by F^n, we obtain $|F^{-n}(A)| \leq \frac{K}{\varepsilon}|A|$. This is independent of n, so also $m(A) \leq \frac{K}{\varepsilon}|A|$ and m has bounded density. In our case the density of m is also bounded away from 0 in a neighbourhood of c. Let A a measurable set such that $A = \hat{A}$ and argue as above:

$$\begin{aligned}\frac{|F^{-n}(A) \cap I|}{|I|} &\geq \frac{1}{K} \frac{|A \cap F^n(I)|}{|F^n(I)|} \\ &\geq \frac{1}{K} min\{|A \cap \Gamma| \mid \Gamma \text{ a component of } \Gamma_{B+1} \setminus \{c\}\},\end{aligned}$$

because $F^n(I)$ covers one component of $\Gamma_{B+1} \setminus \{c\}$. It follows that $m(A) \geq \frac{1}{K'}|A|$ for some constant $K' > 0$.

Now we derive a measure for f itself. Let m_j be the measure restricted to an element A_j of the partition of W. So for any m-measurable set B, $m_j(B) = m(A_j \cap B)$ and $m(B) = \sum_j m_j(B)$. It can be shown [MS1] that

$$\mu = \sum_j \sum_{i=0}^{S_{j-1}-1} f^i_* m_j$$

is an absolutely continuous invariant measure for f. For the whole interval $\mathcal{N}$ we have

$$\mu(N) = \sum_j \sum_{i=0}^{S_{j-1}-1} f_*^i m_j(\mathcal{N}) = \sum_j S_{j-1} m(A_j) \leq C \sum_j S_{j-1}|A_j|.$$

where the last inequality follows because m has a bounded density. Similarly, $\mu(\mathcal{N}) \geq C' \sum_j S_{j-1}|A_j|$. So μ is an acip if and only if $\sum_j S_{j-1}|A_j| < \infty$. Using again the boundedness of the kneading map we can simplify this condition to

$$\sum_j j|A_j| < \infty. \tag{2}$$

In general (2) doesn't hold as we show in Theorem 2. In Section 8 we shall derive topological conditions that guarantee (2). We remark that (2) follows easily from the metrical condition

$$\sum_j S_{j-1}|Df^{S_{j-1}-1}(c_1)|^{-\frac{1}{\ell}} < \infty.$$

This condition is slightly stronger than (1), but it clearly includes all long-branched maps satisfying the Collet-Eckmann condition. The proof relies on a one-sided version of the Koebe Principle [MS1]. We omit the details.

5. The combinatorics of the example.

We will now construct inductively the kneading invariant of our example. Using Lemmas 1 and 2 we can verify that this kneading invariant corresponds to a long-branched map and that it is the limit map of a saddle node cascade. The approximating saddle node map are alternated by maps for which the critical point eventually hits the orientation reversing fixed point p. Two sequences of integers, which can still be chosen freely, will indicate how close the limit map approaches the approximating saddle node and Misiurewicz maps.

We start with the kneading invariant

$$\nu_1 = 101101101101101....$$

The period of ν_1 is 3, and because $S_0, S_1, S_2... = 1, 2, 4...$, it is clear from Lemma 2 that ν_1 corresponds to a saddle node bifurcation. In fact this is easy to check because ν_1 is assumed by the function $f(x) = (1+\sqrt{8})x(1-x)$. Notice that the corresponding kneading map is bounded by 1. This will be the case during the whole construction, yielding a limit map with bounded kneading map.

Let Δ_1 be the block 101, so $|\Delta_1|$ is the period of ν_1. We define inductively

$$\nu_i = E_i E_i E_i ...$$

and

$$E_{i+1} = \underbrace{E_i E_i ... E_i E_i}_{a_i \text{ times}} \underbrace{111...111}_{2b_i \text{ times}}.$$

The map f with kneading invariant

$$\nu = \lim_i \nu_i$$

is the goal of our construction. Lemma 4 below and its corollary show that the kneading invariants ν_i and therefore the limit kneading invariant are admissible for all choices of a_i and b_i. But first we study $\{\nu_i\}$ more precisely.

It is clear that $f^{a_i|E_i|}(c)$ is very close to p, the orientation reversing fixed point, if b_i is very large. In fact, the sequence $E_iE_i...E_iE_i11111...$ is also an admissible kneading invariant, and it corresponds to a map where c is eventually mapped on p. So we can consider ν_i as kneading invariants of perturbations of maps with eventually fixed critical point. On the other hand, f is very close to a map at a saddle node bifurcation if a_i is very large. So f may also be considered as a perturbation of a saddle node map.

Define k_i by $S_{k_i-1} < |E_i| \leq S_{k_i}$ and l_i by $S_{l_i-1} < a_i|E_i| \leq S_{l_i}$. Clearly $k_i < l_i < k_{i+1}$. We will use these sequences in our final estimation. They more or less indicate the periods of the saddle node maps and the time it takes the critical point to reach p in the Misiurewicz maps. Let us show that ν_i indeed corresponds to a saddle node map for each i, and that the corresponding kneading map is universally bounded. We claim that

$$\begin{aligned} |E_i| &= S_{k_i} - 1 \\ a_i|E_i| &= S_{l_i} - 1 \end{aligned} \tag{5}$$

and argue by induction. For $i = 1$ it is clear from the form of ν_1. So assume that it is true for i. $e_{a_i|E_i|}$ is followed by $2b_i$ ones, but as $E_{i+1} = 10...$ it is in fact followed by $2b_i + 1$ ones. That means $e_{S_{l_i}}$ is followed by $2b_i$ ones, b_i basic blocks 11. Hence

$$S_{k_{i+1}} = S_{l_i+b_i} = |E_{i+1}| + 1.$$

Consequently all blocks E_{i+1} are split into basic blocks in the same way. Indeed, it is not hard to see that for ν_{i+1}, $S_{k_{i+1}+j} = S_j$ for all $j > 0$. Therefore

$$S_{l_{i+1}} = S_{a_i k_i} = a_{i+1}|E_{i+1}| + 1,$$

which proves the claim.

As E_{i+1} consists of the same basic blocks as E_i (though there are more copies of each basic block and a lot of basic blocks 11 are added), the kneading map of ν_{i+1} has the same bound as the kneading map of ν_i. In particular the kneading map of $\lim_i \nu_i$ is bounded by 1. Furthermore, since $|E_i| \notin \{S_j\}$ for all i, ν_i is indeed of saddle node type.

In the next lemma, let $\preceq$ denote the usual order relation between itineraries. Recall that a kneading invariant is called admissible if there exists a unimodal map which assumes this kneading invariant. The standard admissibility condition for kneading invariants reads $\sigma^n(\nu) \preceq \nu$ for all n. An itinerary $\nu(x)$ is admissible if $\sigma^n(\nu(x)) \preceq \nu$ for all $n \geq 0$ [D]. A point q is called *nice*, if its forward orbit is disjoint from $(q, \hat{q})$.

Lemma 4. *Let f be in a full family of unimodal maps, ν its kneading invariant. Then there exists a nice point q with itinerary ν' if and only if $\nu' \preceq \nu$ is an admissible kneading invariant.*

Proof. If q is a nice point, then $f^n(q) \leq f(q)$ for all $n > 0$. So $\sigma^{n-1}(\nu') \preceq \nu'$ for all $n > 0$. This is precisely the admissibility condition for kneading invariants.

Conversely, if $\nu' \preceq \nu$ is an admissible kneading invariant, $\sigma^n(\nu') \preceq \nu' \preceq \nu$. So ν' is an admissible itinerary with respect to ν. This yields the existence of q, and it is easy to see that q is nice as well. □

Corollary. *If ν_i is admissible, ν_{i+1} is also admissible.*

Proof. Let f_i be the map corresponding to ν_i. According to the lemma we need to find a nice (periodic) point q of f_i which has itinerary $\nu(q) = \nu_{i+1}$. Recall $S_{l_i} = a_i|E_i| + 1$. Since $Q(1 + l_i) = 0$, ν_i coincides with $\nu(q)$ up to $S_{1+l_i} - 1 = S_{l_i}$. So the periodic point q, if it exists, is contained in A_{1+l_i}. $f_i^{S_{l_i}}(A_{1+l_i}) = (c, c_{-S_{Q(1+l_i)}}) = (c, c_{-1})$. Hence $f_i^{S_{l_i}+1}(A_{1+l_i}) = (c, c_1)$ covers a neighbourhood of the fixed point p.

Let $V = \{x \mid f_i^j(x) > c \text{ for } 0 \leq j \leq 2b_i\}$. So V is a neighbourhood of p and for all $x \in V$ the itinerary starts with b_i basic blocks of length 2. It is clear that at least one end-point of V is mapped by $f_i^{2b_i}$ on c. Also V contains p, so $f_i^{2b_i}(V) \supset (c, p) \supset com(A_{1+l_i})$.

Summarizing, $f_i^{S_{l_i}+1}(A_{1+l_i})$ covers $f_i(V)$ and $f_i^{2b_i}(V)$ covers a component of A_{1+l_i}. So there exists a point $q \in A_{1+l_i}$ such that $f_i^{S_{l_i}+2b_i}(q) = f_i^{|E_{i+1}|}(q) = q$ and $\nu(q) = \nu_{i+1}$. Moreover, $f^j(A_{1+l_i}) \cap \Gamma_{1+l_i} = \emptyset$ for $0 < j < S_{l_i}$, $f^{S_{l_i}}(q)$ is close to $\hat{p}$ or p. $f^j(V) \cap \Gamma_{1+l_i} = \emptyset$ for $0 \leq j < 2b_i$. So $orb(q) \cap (q, \hat{q}) = \emptyset$; q is indeed nice. □

6. The proof of Theorem 2.

Johnson [J] already constructed unimodal maps without acip, and Hofbauer and Keller [HK1,HK2] showed that the Bowen-Ruelle-Sinai measure can be quite unexpected instead. The main idea was to use almost restrictive intervals. This implies, just as in our example, that c returns very close to itself, violating summability condition (1). However, one can show that almost restrictive intervals correspond to the period doubling type of return. So in a combinatorial sense this example is different. Furthermore, and unlike Johnson's example, its properties are persistent under topological conjugacies as long as we stay in $\mathcal{G}$.

Before proving Theorem 2 we need a statement on conservativity of f, the proof of which can be found in [BL2]. If $f \in \mathcal{G}$ admits no periodic attractor or restrictive intervals, then f is conservative. This is equivalent to the non-existence of Cantor attractors. It has been proved under much weaker long-branchedness conditions, cf. [GJ], that such a Cantor attractor doesn't exist. Recently Lyubich [Ly2] claimed that a Cantor attractor cannot exist at all in the quadratic setting.

We will give a short proof of the claim using Markov property b). Assume that f is not infinitely renormalizable, because this is not compatible with long-branchedness. Consider f on the orbit of its smallest restrictive interval. Suppose by contradiction that for a set X, $f(X) \subset X$ and $0 < |X| < |\mathcal{N}|$. Let $x \notin \cup_n f^{-n}(c)$

be a density point of X and I_n the interval of continuity of F^n containing x. By taking n sufficiently large, we can get $\frac{|X \cap I_n|}{|I_n|}$ as close to 1 as we want. So due to the bounded distortion of F^n, $\frac{|F^n(X \cap I_n)|}{|F^n(I_n)|} = \frac{|X \cap F^n(I_n)|}{|F^n(I_n)|}$ can be as close to 1 as we want. But since $F^n(I_n) = com(\Gamma_k)$ for some $k \leq B$ and f is not renormalizable, this implies that X has full measure.

Proof of Theorem 2. The proof of the second statement follows from Lemma 3 and Martens work [Ma2], if f has no periodic attractor. In fact we can prove that the measure μ in Section 4 is σ-finite. Let U be a component of the complement of $\omega(c)$ and V its middle third interval. We may assume that f is not a Misiurewicz map, so $orb(c) \subset \omega(c)$. It follows that if $f^i(A_k) \cap V \neq \emptyset$ for some $0 \leq i < S_{k-1}$, $|f^i(\Gamma_k)| \geq \frac{1}{3}|U|$. Due to (3), there exists $N = N(\frac{1}{3}|U|)$ such that $f^{i+j}(\Gamma_k) \ni c$ for some $j < N$. Hence

$$\begin{aligned}
\mu(V) &= \sum_j \sum_{i=0}^{S_{j-1}-1} m(f^{-i}(V) \cap A_j) \\
&\leq C \sum_j \sum_{i=0}^{S_{j-1}-1} |f^{-i}(V) \cap A_j| \\
&\leq C \sum_j N|A_j| \\
&\leq CN|\mathcal{N}| < \infty.
\end{aligned}$$

Now let $V_0 = V$, $V_1 = f^{-1}(V_0) \setminus V_0$ and in general $V_j = f^{-j}(V_0) \setminus \cup_{i=0}^{j-1} V_i$. Clearly $\mu(V_j) \leq \mu(f^{-j}(V)) = \mu(V) < \infty$. Furthermore $|\mathcal{N} \setminus \cup_j V_j|$ is forward invariant, so its Lebesgue measure must be zero. Hence $\{V_0, V_1, ...\}$ is countable partition in finite μ-measured sets; μ is σ-finite.

Next we show that the measure μ is unique. Suppose μ' is a σ-finite f-invariant measure which is absolutely continuous with respect to the Lebesgue measure. We first claim that μ' is also absolutely continuous with respect to μ. Suppose by contradiction that for a set W, $\mu'(W) > 0$ and $\mu(W) = 0$. As μ' is absolutely continuous with respect to the Lebesgue measure, $|W| > 0$. We define the sets W_j just as we did for V above. Since $\mu(W_j) \leq \mu(W) = 0$, it follows that $\mu(\mathcal{N}) = 0$, a contradiction. Now it follows from the Radon-Nikodým Theorem and ergodicity that the density of μ' with respect to μ is constant μ-almost everywhere. Hence μ and μ' coincide, up to a multiplicative constant [M].

If f has a periodic attractor, Theorem 2 holds too, though no uniqueness can be established. We simply construct a fundamental domain V in the immediate basin of the attractor, and put a constant density on V. Since every point visits $f^k(V)$ at most once for each $k \in \mathbb{Z}$, and almost every point visits $f^k(V)$ for some k, one can extend this density to $\mathcal{N}$.

Now to prove the first statement, we construct a saddle node cascade, along the lines of the previous section, such that the limit kneading invariant satisfies the following property:

$$\text{For all } \lambda < 1, \limsup_i a_i |E_i| \lambda^{|E_i|} = \infty.$$

Take M arbitrary. We will show that $\sum_j S_{j-1}|A_j| > M$, so that (2) is not satisfied. We know a priori that there exists $\lambda_0 < 1$ such that, unless f has a periodic attractor, $d(c_n, c) > \lambda_0^n$ for all $n \geq 1$. Indeed, if this were not the case, then we can find n such that $|c - c_n|^{\ell-1} O_2 \ell 2^\ell L^{n-1} \leq 1$. Here $L = \max_{x \in I} |Df(x)|$. Let x be such that $c_n \in (c, x)$ and $|c - x| = 2|c - c_n|$, then

$$\begin{aligned} |f^n(c) - f^n(x)| &\leq |\int_c^x Df^n(z) dz| \\ &\leq |c - x| O_2 \ell |c - x|^{\ell-1} L^{n-1} \\ &\leq |c - c_n| O_2 \ell 2^\ell |c - c_n|^{\ell-1} L^{n-1} \\ &\leq |c - c_n|. \end{aligned}$$

Hence $f^n((c,x)) \subset (c,x)$, yielding a periodic interval. Long-branchedness prohibits infinite renormalization, so for n large enough, (c,x) contains a periodic attractor.

Choose i such that $(a_i - 1)|E_i|\lambda_0^{|E_i|} > M$. Since $\sigma^{|E_i|}(\nu) = \nu$ up to at least $(a_i - 1)|E_i|$ entries, $c_{|E_i|} \in \Gamma_k$ for some k such that $S_{k-1} \geq (a_i - 1)|E_i|$. So $|\Gamma_k| \geq d(c, c_{|E_i|}) \geq \lambda_0^{|E_i|}$ and

$$\sum_j S_{j-1}|A_j| \geq \sum_{j \geq k} S_{j-1}|A_j| \geq S_{k-1}|\Gamma_k| \geq (a_i - 1)|E_i|\lambda_0^{|E_i|} > M.$$

Uniqueness of μ shows that there is no finite measure. □

7. The derivative $|Df^n(c_1)|$.

In this section we will merely give a proof that $\liminf_n |Df^n(c_1)|$ is finite in our example, provided we choose the integers a_i, and consequently l_i, properly. We claim that if the map f is close enough to a map with a saddle node bifurcation of period n, $|Df^n(c_1)|$ is bounded. This is of course clear from continuity; at the saddle node bifurcation $|Df^n(c_1)| < 1$. The next lemma states that $|Df^n(c_1)|$ is still small, if the iterates $f^{jn}(c_1)$ are close to c_1 for $j \leq m$, where $m > n$. For example $m = 2n$ will do. In view of the proof of Theorem 1 in the next section, this is small enough to derive (2).

Lemma 5. *Let $f \in \mathcal{G}$ have kneading invariant ν. Then there exists $M = M(f) > 0$ with the following property: If n is such that $\sigma^n(\nu) = \nu$ for at least mn entries and $m > n$ then $|Df^n(c_1)| < M$.*

Proof. Let $V \ni c_1$ be the maximal interval on which f^n is monotone and let $U \subset [c, c_1]$ be the maximal interval adjacent to c_1 such that $f^n(U) \subset V$. If $\sigma^n(\nu) = \nu$ up to at least mn entries, it follows that $f^{mn}(c_1) \in U$. So we need to find un upper bound for $|Df^n(c_1)|$ if $f^{mn}(c_1) \in U$.

If $|Df^n(c_1)| < 1$ then there is nothing to prove, so assume $1 \leq |Df^n(c_1)| = |Df(f^{n-1}(c_1))||Df^{n-1}(c_1)| \leq O|f^n(c_1)-c_1|^{\frac{\ell-1}{\ell}}|Df^{n-1}|(c_1)$, where the last inequality follows from non-flatness. Taking $L' = L^{\frac{\ell}{\ell-1}}$ and $O' = O^{\frac{\ell}{\ell-1}}$, (notice that we use explicitly that $\ell > 1$) this yields

$$|f^n(c_1) - c_1| \geq \frac{1}{|O\,Df^{n-1}(c_1)|^{\frac{\ell}{\ell-1}}} \geq \frac{1}{O'}(\frac{1}{L'})^n.$$

As c_1 is the global maximum of f, $f^n_{|U}$ is increasing. Due to negative Schwarzian derivative, $|Df^n_{|U}|$ is decreasing. Let $M = O'L'$ and assume $|Df^n(c_1)| > M$. Let $g(x) = M(x - c_1) + c_1$, then the graph of $f^n_{|U}$ lies below the graph of g, which in its turn lies below the diagonal. Consequently, $f^n(x) \leq g(x)$ and by induction $f^{(m-1)n}(x) \leq g^{m-1}(x)$ if $f^{kn}(x) \in U$ for $0 \leq k < m - 1$. A straightforward calculation gives $|g^{m-1}(x) - c_1| = M^{m-1}|x - c_1|$, so

$$f^{mn}(c_1) \leq c_1 - M^{m-1}|c_1 - f^n(c_1)|,$$

i.e. $M^{m-1}|c_1 - f^n(c_1)| \leq |c_1 - f^{mn}(c_1)|$. Because $f^{mn}(c_1) \in U$, we get

$$|U| \geq |c_1 - f^{mn}(c_1)| \geq M^{m-1}|c_1 - f^n(c_1)| \geq M^m \frac{1}{O'}(\frac{1}{L^{\frac{\ell}{\ell-1}}})^n.$$

$|U| < 1$, so by definition of M, $m \leq n$. If both $m > n$ and $f^{mn}(c_1) \in U$, $|Df^n(c_1)|$ can only be less than M. □

8. The proof of Theorem 1.

The next stage is to investigate methods to get a bound for (2). Because $A_j \subset \Gamma_j$, it is enough to prove that $|\Gamma_j|$ decreases exponentially. This is why the following lemma is of interest.

Lemma 6. *Let $f \in \mathcal{G}$, B the maximum of the kneading map of f and $R > B$ some integer. There exists $\lambda < 1$ with the following property. If $c_n \notin \Gamma_R$ for $j_0 \leq n \leq j_1$, where j_0 is taken minimal for this property, then $|f(\Gamma_k)| \leq \lambda^{k-m}|f(\Gamma_m)|$ for all $2j_0 + S_B \leq S_m \leq S_k \leq j_1$.*

Proof. Choose ε and η such that $|x - c| \geq \varepsilon$ if $x \notin \Gamma_R$ and $|x - c| \geq \eta$ if $x \notin \Gamma_{B+1}$. From (3) we obtain that there exists $\delta = \delta(\varepsilon)$ such that $|f^n(I)| \geq \delta$ whenever $|I| > \varepsilon$ and $f^n_{|I}$ is monotone. We may assume that $\delta \leq \varepsilon \leq \eta$.

Take m minimal such $S_m \geq 2j_0 + S_B$ and k, $S_m \leq S_k \leq j_1$, arbitrary. Let $U_k \supset f(\Gamma_k)$ be the largest interval on which $f^{S_{k-1}-1}$ is monotone. We will show that $f^{S_{k-1}-1}_{|f(\Gamma_k)}$ has bounded distortion, by showing that $f^{S_{k-1}-1}(U_k)$ contains a δ-scaled neighbourhood of $f^{S_{k-1}-1}(f(\Gamma_k)) = (c, c_{S_{k-1}}]$. Indeed, by long-branchedness, it contains a one-sided η-scaled neighbourhood on the side of c. Let $J = U_k \setminus \mathcal{N}$ and let b be the iterate such that $c \in \partial f^{b-1}(J)$. If $b \geq j_0$, then $f^b(c) \notin \Gamma_R$, so $|f^{b-1}(J)| \geq \varepsilon$ and $|f^{S_{k-1}}(J)| \geq \delta$. This yields the required δ-scaled neighbourhood and the distortion of $f^{S_{k-1}-1}_{|f(\Gamma_k)}$ is bounded by $K = K(\delta)$. So it suffices to show that $b < j_0$ is impossible. By minimality of j_0, ν and $\sigma^b(\nu)$ differ at some entry

$n \leq j_0 + S_B$. So $f^n((c, c_b)) \ni c$, but S_k is the smallest positive iterate such that $f^{S_k}(\Gamma_k) \ni c$. It follows that $S_k \leq b - 1 + n < 2j_0 + S_B$ contradicting our choice of k.

$f^{S_{k-1}}(\Gamma_k) = (c, c_{S_{k-1}})$ and $f^{S_{k-1}}(\Gamma_{k+1}) = (c_{-S_{Q(k)}}, c_{S_{k-1}})$. This means that $|f^{S_{k-1}}(\Gamma_k \setminus \Gamma_{k+1})| \geq \eta \geq \eta |f^{S_{k-1}}(\Gamma_k)|$. Since distortion is bounded we obtain $|f(\Gamma_k \setminus \Gamma_{k+1})| \geq \frac{\eta}{K} |f(\Gamma_k)|$, or in other words $|f(\Gamma_{k+1})| \leq (1 - \frac{\eta}{K}) |f(\Gamma_k)|$. Now it is also clear that $|f(\Gamma_k)| \leq (1 - \frac{\eta}{K})^{k-m} |f(\Gamma_m)|$. Non-flatness ensures the exponential decrease of $|\Gamma_j|$. $\square$

Finally, we shall give an upper bound for expression (2). We will fix $\{k_i\}$ and $\{l_i\}$ by an inductive procedure that only depends on S_B, where $B \geq Q(k)$ for all k. Then we fix $f \in \mathcal{G}$ such that it has the limit kneading invariant ν. The constants κ, C_1 and C_2 of the final estimation depend on f, but if $f \in \mathcal{G}$, $\kappa < 1$ and $C_1, C_2 < \infty$. The choice of k_i and l_i determines a certain partition of $\mathbb{N}$. This partition contains pieces where $|\Gamma_j|$ decreases exponentially, and in pieces where $|\Gamma_j|$ decreases, but the rate of decrease is unknown.

From the definition of ν it is clear that $\sigma^j(\nu)$ starts with 11 whenever $S_{l_i} \leq j < S_{k_{i+1}}$ for some i. So $c_j \notin \Gamma_3$ for $S_{l_i} \leq j < S_{k_{i+1}}$. This is the assumption of the previous lemma. Suppose we have found l_i. Take m_i such that $S_{m_i} > 2S_{l_i} + S_B$ and k_{i+1} such that $k_{i+1} - m_i > m_i$. Hence, from Lemma 6 we get $|f(\Gamma_k)| \leq \lambda^{k-m_i} |f(\Gamma_{m_i-1})|$ whenever $m_i - 1 \leq j < k_{i+1}$. It follows that $|\Gamma_{k_i}|$ decreases exponentially in k_i. To be precise,

$$\begin{aligned} |\Gamma_{k_j}| &\leq C {\lambda'}^{\sum_{i=1}^{j-1} k_{i+1} - m_i} |\Gamma_{k_1}| \\ &\leq C {\lambda'}^{\frac{1}{2} \sum_{i=1}^{j-1} k_{i+1} - k_i} |\Gamma_{k_1}| \\ &\leq C \kappa^{k_j - k_1} |\Gamma_{k_1}|, \end{aligned}$$

where $\kappa = \sqrt{\lambda'} = \lambda^{\frac{1}{2\ell}}$ and C emerges from non-flatness. Since $|\Gamma_{k_1}|$ and k_1 are fixed numbers, we may write $|\Gamma_{k_i}| \leq C_1 \kappa^{k_i}$.

Statement (5) gives $|E_i| = S_{k_i} - 1$, and by Lemma 5 it is enough to repeat the block E_i at least $|E_i| + 1$ times to make sure that $Df^{|E_i|}(c_1)$ is uniformly bounded. Suppose we have found k_i. We took $S_{m_i} > 2S_{l_i} + S_B$, but for $l_i > B$ we may also assume $S_{m_i} \leq 3S_{l_i}$. Due to the boundedness of the kneading map, $j \leq S_j \leq S_B j$. So $m_i \leq 3S_B l_i$ and we may take l_i such that $S_B k_i^2 < l_i < m_i < P(k_i)$ for an appropriate quadratic polynomial P. In fact, any subexponential function will do. Let C_2 be un upper bound of $\sum_j j\lambda_j$. Then

$$\begin{aligned}\sum_j j|A_j| \le \sum_j j|\Gamma_j| &\le \sum_i \sum_{j=k_i}^{m_i-2} j|\Gamma_j| + \sum_i \sum_{j=m_i-1}^{k_{i+1}-1} j|\Gamma_j| \\ &\le \sum_i \sum_{j=k_i}^{m_i-2} j|\Gamma_{k_i}| + \sum_i \sum_{j=m_i-1}^{k_{i+1}-1} j\lambda^{j-(m_i-1)}|\Gamma_{m_i-1}| \\ &\le \sum_i m_i^2|\Gamma_{k_i}| + \sum_i (\frac{m_i-1}{1-\lambda} + C_2)|\Gamma_{k_i}| \\ &\le \sum_i P(k_i)^2 C_1\kappa^{k_i} + \sum_i (\frac{P(k_i)}{1-\lambda} + C_2)C_1\kappa^{k_i} \\ &< \infty.\end{aligned}$$

This concludes the proof of the finiteness of $\sum_j j|A_j|$.

In order to check that Theorem 1 holds for all maps having ν as kneading invariant, we check the dependence of the constants in the proof. Clearly the integers B and R of Lemma 6 depend only on ν. The constants M of Lemma 5 and κ depend on the non-flatness constants O_1, O_2 and ℓ. M depends on $L = max_x|Df(x)|$ and κ depends on $\delta = \delta(\varepsilon)$ of (3). ε in its turn appears in the proof of Lemma 6, and depends only on L, O_1, O_2, ℓ and R.

For any map in $\mathcal{G}$, we have for the corresponding constants $0 < O_1, \delta$ and $O_2, L, \ell < \infty$, implying $M < \infty$ and $\kappa < 1$. So Theorem 1 holds. It is possible to find a sequence $\{f_n\}$ of conjugate maps such that the corresponding constants $\kappa \to 1$ or $M \to \infty$. But for each map in this sequence Theorem 1 holds and the limit map, if it exists, lies outside $\mathcal{G}$.

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