RENORMALISATION IN A CLASS OF INTERVAL TRANSLATION MAPS OF d BRANCHES

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ABSTRACT. We generalise results by Bruin & Troubetzkoy [BT] to a class of interval translation maps with arbitrarily many pieces. We show that there is a uncountable set of parameters leading to type ∞ ITMs, but that the Lebesgue measure of these parameters is 0. Furthermore, conditions are given that imply the ITM to have multiple ergodic invariant measures.

1. INTRODUCTION

Interval translation maps (ITMs) were introduced by Boshernitzan & Kornfeld [BK] as a generalisation of interval exchange transformations (IETs). Let the intervals $B_i = [\beta_i, \beta_{i+1})$ for $0 = \beta_0 < \beta_1 < \cdots < \beta_r = 1$ constitute a partition of the unit interval I. An interval translation map $T : I \to I$ is given by

$$T(x) \stackrel{\text{def}}{=} x + \gamma_i \text{ if } x \in B_i,$$

where $\gamma_i \in \mathbb{R}$ are fixed numbers such that T maps I into itself. We also define the image of 1 by $T(1) \stackrel{\text{def}}{=} \lim_{x \to 1^-} T(x)$. Since the images $T(B_i)$ can overlap, it is possible that $\Omega \stackrel{\text{def}}{=} \overline{\cap_n T^n(I)}$ is a Cantor set; in this case T is said to be of type ∞. Boshernitzan & Kornfeld showed, using a renormalisation operator, that a specific ITM has an attracting Cantor set. Bruin & Troubetzkoy [BT] extended this result to a 2-parameter family of ITMs with 3 pieces (or 2 pieces when considered on the circle), and showed that type ∞ map occur for an uncountable set of Lebesgue measure 0 in parameter space. In [SIA], it is shown that type ∞ occurs with Lebesgue measure 0 in the full 3-parameter family of 2-piece ITMs on the circle. In addition, [BT] gives estimates on the Hausdorff dimension of Ω, and it gives conditions under which $T|_\Omega$ is uniquely ergodic, or is not uniquely ergodic.

In this paper, we extend the class of ITMs to a d-parameter family T_α (for a α in a d-dimensional parameter space $\mathcal{U}$), with $d+1$ branches, on which a renormalisation operator G is defined. Similar to [BT], we prove the following theorem:

Theorem 1. *Let $\mathcal{A}_d$ be the set of parameters such that T_α is of type ∞. Then*

1. *$\mathcal{A}_d$ is uncountable, but has d-dimensional Lebesgue measure 0.*
2. *The renormalisation operator $G : \mathcal{A}_d \to \mathcal{A}_d$ acts as a one-sided shift with countably many symbols; the coding map $\alpha \mapsto (k_0, k_1, k_2, \dots)$ is injective, and maps onto $\mathbb{N}^{\mathbb{N} \cup \{0\}} \setminus \mathcal{FT}$ where the exceptional set $\mathcal{FT}$ is given by formula (2) in Section 2.*
3. *The map G eventually maps every $\alpha \in \mathcal{U}$ either into $\mathcal{A}_{d'}$ for some $2 \le d' \le d$ (infinite type) or into a 1-parameter space of circle rotations (finite type).*

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If $\alpha \in \mathcal{A}_d$, i.e., T_α is of infinite type, then the attractor is a minimal Cantor set Ω, and symbolically, T_α acts on it as a substitution shift based on a sequence of substitutions χ_k. The proof of this result is basically unchanged since [BK], see Section 3. Whereas we expect the word complexity of this shift to be sublinear, we have no precise estimates.

It is interesting to know that $T|_\Omega$ need not be uniquely ergodic. The ideas of the proof of this go back to Keane's example [K] of a interval exchange transformation on four pieces that is not uniquely ergodic.

Theorem 2. *If the code of $\alpha \in \mathcal{A}_d$ (for $d \geq 2$) tends to ∞ sufficiently fast, then T_α admits d distinct ergodic probability measures on Ω.*

This fits in nicely with the result of [BH] that an orientation preserving ITM with N branches can preserve at most $2N$ ergodic probability measures, whose total *rank* is $\leq N$. In our case, we are dealing with $d+1$ branches, but on the circle there are only d branches. So Theorem 2 shows that the bound of Buzzi & Hubert is sharp for every N.

Especially in the non-uniquely ergodic case, it would be interesting to find he ergodic invariant measures. Hausdorff measure of the appropriate dimension is always invariant (see [BT]), but it is not always clear that this measure can be normalised to a probability measure, see below. Neither is it clear that Hausdorff measure is unique.

Theorem 3. *If T_α is of infinite type, then the Hausdorff dimension $dim_H(\Omega) < 1$.*

In [BT], it was shown that the Hausdorff dimension of Ω need not be equal to the upper box dimension. In fact $0 = \dim_H(\Omega) = \underline{\dim}_B(\Omega) < \overline{\dim}_B(\Omega)$ is possible. In this case, Hausdorff measure of dimension 0 becomes counting measure which is obviously infinite, and not even σ-finite.

However, if $\alpha \in \mathcal{A}_d$ is periodic under G, then Ω is self-similar, it has Hausdorff dimension strictly between 0 and 1, and Hausdorff measure can be normalised to be the unique ergodic probability measure on Ω.

Let us finish this introduction with some open questions.

Questions:

- What is the Hausdorff dimension of the set $\mathcal{A}_d$ of type ∞ parameters? Since G is ∞-to-1 and not conformal, standard techniques for estimating the Hausdorff dimension repeller are not likely to work.
- Given the fact that exotic behaviour is possible (in the sense of non-unique ergodicity of T_Ω, or Hausdorff dimension different from upper box dimension of Ω), it would be interesting to put a G-invariant measure on $\mathcal{A}_d$ to express how typical, or atypical, this exotic behaviour is. What would be a natural measure on $\mathcal{A}_d$? Is the equilibrium measure for potential $-t \log|\det(DG)|$ for $t = \dim_H(\mathcal{A}_d)$ a reasonable candidate?
- What is the physical measure for non-uniquely ergodic maps, i.e., what is the fate of Lebesgue typical points?

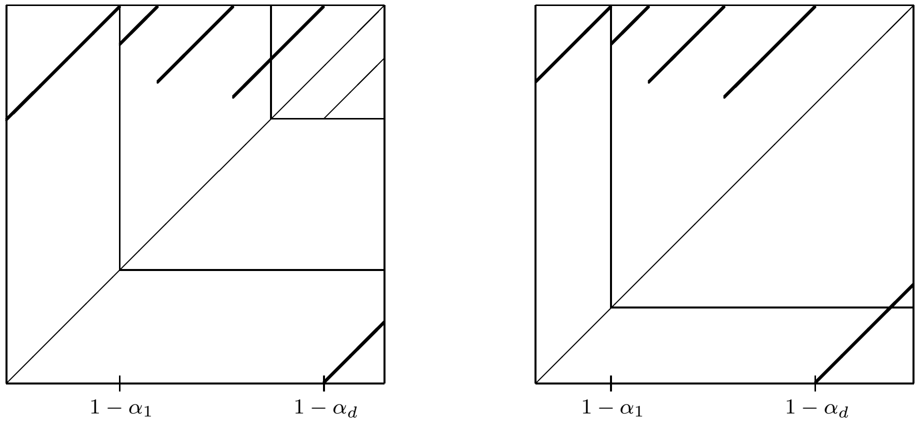

FIGURE 1. Two maps $T_\alpha \in \mathcal{ITM}_4$ and the boxes on which the induced map is defined. In the left picture, the first return reduces the number of branches; rescaled to the smallest box, only two branches remain. In the right picture, the number of branches stays the same.

2. The class of ITMs and its basic properties

Let T_α in the set $\mathcal{ITM}_d$ of interval translation maps defined by

$$T_\alpha(x) = \begin{cases} x+\alpha_1 & \text{for } x \in [0, 1-\alpha_1), \\ x+\alpha_i & \text{for } x \in [1-\alpha_{i-1}, 1-\alpha_i), 1 < i < d, \\ x+\alpha_i - 1 & \text{for } x \in [1-\alpha_d, 1], \end{cases}$$

where the parameter space is

$$\mathcal{U} \stackrel{\text{def}}{=} \{\alpha = (\alpha_1, \dots, \alpha_d) : 1 \geq \alpha_1 \geq \cdots \geq \alpha_d \geq 0\}.$$

We study the map $T = T_\alpha$ using the induced transformation to the interval $[1-\alpha_1, 1]$. One can readily check that this induced transformation, $\tilde{T}$, has a similar shape as T; more precisely, the $i+1$st branch of T becomes the ith branch of $\tilde{T}$ for $i \leq d-2$. The dth branch of T

- either produces a single branch of $\tilde{T}$. In this case, $\tilde{T}$ can be rescaled to a map in $\mathcal{ITM}_{d'}$ for some $d' < d$, see Figure 1, left.
- or splits into two new branches of $\tilde{T}$. In this case, we apply the first (= left-most) branch of T respectively $k-1$ and k times, where $k \stackrel{\text{def}}{=} \lfloor \frac{1}{\alpha_1} \rfloor \in \{1, 2, 3 \dots\}$, see Figure 1, right.

In the latter case, rescaling the domain of $\tilde{T}$ to unit size gives a new map in $\mathcal{ITM}_d$. The corresponding parameter transformation G generalises the *Gauss map* of circle rotations. It is defined as

$$G(\alpha_1, \dots, \alpha_d) \stackrel{\text{def}}{=} \left(\frac{\alpha_2}{\alpha_1}, \frac{\alpha_3}{\alpha_1}, \dots, \frac{\alpha_d}{\alpha_1}, k + \frac{\alpha_d - 1}{\alpha_1}\right), \text{ where } k = \lfloor \frac{1}{\alpha_1} \rfloor. \tag{1}$$

Let

$$\mathcal{A} = \mathcal{A}_d \stackrel{\text{def}}{=} \cap_{n \geq 0} G^{-n}(\text{int } \mathcal{U})$$

be the set of parameters on which G is defined for all iterates. This is the set of parameters corresponding to maps of type ∞ whose induced maps all have d branches.

Let

$$\mathcal{L} \stackrel{\text{def}}{=} \{(\alpha_1, \dots, \alpha_d) : 1 \geq \alpha_1 \geq \cdots \geq \alpha_{d-1} \geq 0 \geq \alpha_d \geq \alpha_{d-1} - 1\}.$$

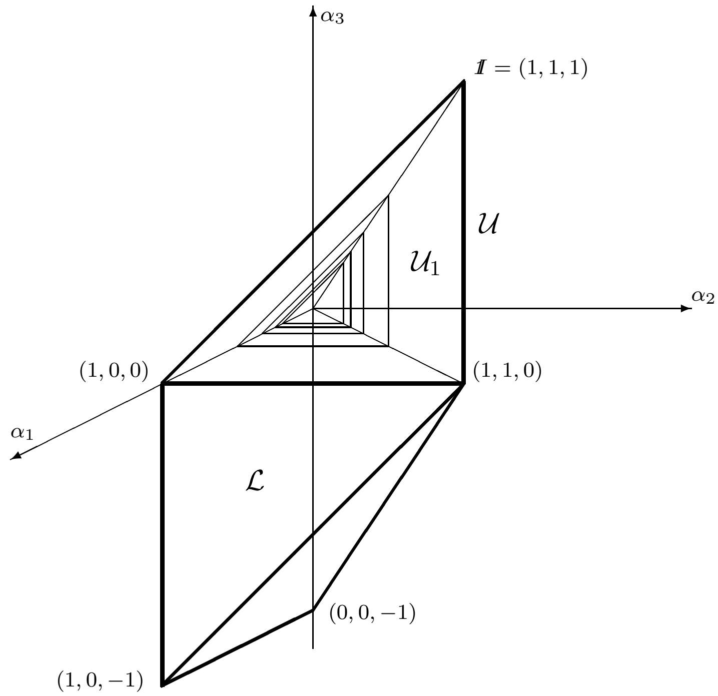

FIGURE 2. The parameter space $\mathcal{U}$ for $d = 3$ and its image $\mathcal{U} \cup \mathcal{L}$. The triangles $\mathcal{V}_r$ are drawn in for $r = 2, 3, 4, 5$. The map $G : \mathcal{U}_1 \to \mathcal{U} \cup \mathcal{L}$ fixes $\mathbb{1}$, maps $(1,1,0)$ onto $(1,0,0)$, $(1,0,0)$ onto the origin and the triangle $\mathcal{V}_1$ onto the triangle spanned by $(1,1,0)$, $(1,0,-1)$ and $(0,0,-1)$.

Then G maps $\mathcal{U}$ in a convex ∞-to-1 fashion into $\mathcal{U} \cup \mathcal{L}$. Write

$$\mathcal{U}_r \stackrel{\text{def}}{=} \left\{ \alpha \in \mathcal{U} : \frac{1}{r+1} < \alpha_1 < \frac{1}{r} \right\} \text{ for } r = 1, 2, 3, \ldots.$$

and $\mathcal{V}_r := \{\alpha \in \mathcal{U} : \frac{1}{r} = \alpha_1\}$. Obviously, G has discontinuities at the $d-1$-dimensional "pyramids" $\mathcal{V}_r$ for $r = 2, 3, \ldots$. The transformation acts on the 1-dimensional edges of $\mathcal{U}$ as follows:

$$\begin{array}{lcll}
(t, t, \ldots, t) & \stackrel{t\in[0,1]}{\longmapsto} & (1, 1, \ldots, 1, \lfloor \frac{1}{t} \rfloor + 1 - \frac{1}{t}) & \infty\text{-to-1.} \\
(t, t, \ldots, t, 0) & \stackrel{t\in[0,1]}{\longmapsto} & (1, 1, \ldots, 1, 0, \lfloor \frac{1}{t} \rfloor - \frac{1}{t}) & \infty\text{-to-1.} \\
(t, t, \ldots, t, 0, 0) & \stackrel{t\in[0,1]}{\longmapsto} & (1, 1, \ldots, 1, 0, 0, \lfloor \frac{1}{t} \rfloor - \frac{1}{t}) & \infty\text{-to-1.} \\
\vdots \quad \vdots & \vdots & \quad \vdots & \\
(t, 0, \ldots, 0) & \stackrel{t\in[0,1]}{\longmapsto} & (0, \ldots, 0, \lfloor \frac{1}{t} \rfloor - \frac{1}{t}) & \infty\text{-to-1.}
\end{array}$$

For $\alpha \in \mathcal{V}_r$, we obtain (writing $\frac{1}{r^-}$ for $\lim_{x \nearrow r} \frac{1}{x}$, $r = 1, 2, 3, \dots$),

$$
\begin{array}{lcl}
(\frac{1}{r^-}, \frac{t}{r^-}, 0, \dots, 0, 0) & \stackrel{t \in [0,1]}{\longmapsto} & (t, 0, \dots, 0). \\
(\frac{1}{r^-}, \frac{t}{r^-}, \dots, \frac{t}{r^-}) & \stackrel{t \in [0,1]}{\longmapsto} & (t, t, \dots, t). \\
(\frac{1}{r^-}, \frac{1}{r^-}, \frac{t}{r^-}, \dots, \frac{t}{r^-}) & \stackrel{t \in [0,1]}{\longmapsto} & (1, t, \dots, t). \\
\vdots \qquad \vdots & & \vdots \\
(\frac{1}{r^-}, \frac{1}{r^-}, \dots, \frac{1}{r^-}, \frac{t}{r^-}) & \stackrel{t \in [0,1]}{\longmapsto} & (1, 1, \dots, 1, t, t).
\end{array}
$$

and (writing $\frac{1}{r^+} = \lim_{x \searrow r} \frac{1}{x}$, $r = 2, 3, \dots$),

$$
\begin{array}{lcl}
(\frac{1}{r^+}, \frac{t}{r^+}, 0, \dots, -1) & \stackrel{t \in [0,1]}{\longmapsto} & (t, 0, \dots, 0, -1). \\
(\frac{1}{r^+}, \frac{t}{r^+}, \dots, \frac{t}{r^+}) & \stackrel{t \in [0,1]}{\longmapsto} & (t, t, \dots, t, t-1). \\
(\frac{1}{r^+}, \frac{1}{r^+}, \frac{t}{r^+}, \dots, \frac{t}{r^+}) & \stackrel{t \in [0,1]}{\longmapsto} & (1, t, \dots, t, t-1). \\
\vdots \qquad \vdots & & \vdots \\
(\frac{1}{r^+}, \frac{1}{r^+}, \dots, \frac{1}{r^+}, \frac{t}{r^+}) & \stackrel{t \in [0,1]}{\longmapsto} & (1, 1, \dots, 1, t, t-1).
\end{array}
$$

Lemma 4. *The map $G : \mathcal{A}_d \to \mathcal{A}_d$ acts as an almost full one-sided shift (over the alphabet $\mathbb{N}$), where $(k_i)_{i=0}^\infty$ with*

$$k_i = r \quad \text{if} \quad G^i(\alpha) \in \mathcal{U}_r.$$

is the coding map. With the exception of the following forbidden tails

$$\mathcal{FT} \stackrel{\text{def}}{=} \{(k_0, k_1, k_2, \dots\dots, \underbrace{1, 1, \dots, 1}_{d-1 \ ones} k_t, \underbrace{1, 1, \dots, 1}_{d-1 \ ones}, k_{t+d}, \underbrace{1, 1, \dots, 1}_{d-1 \ ones}, k_{t+2d}, \dots)\} \qquad (2)$$

every $(k_i)_{i=0}^\infty \in \mathbb{N}^{\mathbb{N} \cup \{0\}}$ corresponds to a unique parameter in $\mathcal{A}_d$.

Proof. The codes of $\mathcal{FT}$ correspond to finite type parameters. The reason for this exclusion is that the edges of $\mathcal{U}$ get permuted in a cyclic way:

$$
\begin{array}{rcl}
(1, 1, \dots, 1, t) & \stackrel{G}{\longmapsto} & (1, 1, 1, \dots, 1, t, t) \\
 & \stackrel{G}{\longmapsto} & (1, 1, \dots, 1, t, t, t) \\
 & \vdots & \vdots \\
 & \stackrel{G}{\longmapsto} & (t, t, t, \dots, t) \\
 & \stackrel{G}{\longmapsto} & (1, 1, \dots, 1, 1 + \lfloor \frac{1}{t} \rfloor - \frac{1}{t}),
\end{array}
$$

where the first $d-1$ steps are injective and last step is ∞-to-1. Since G is a proper ∞-to-1 surjection otherwise, any other code $(k_i)_{i=0}^\infty$ is attained by all parameters in the the non-empty set set $\cap_i (G^{-i}(\mathcal{U}) \cap \mathcal{U}_{k_i})$. Let us show that this set consists of a single point, by showing that at every $\alpha \neq \mathbb{1}$ some iterate of G is expanding.

Let $G_j^{-1} : \mathcal{U} \cup \mathcal{L} \to \mathcal{U}_j$ be the jth inverse branch of G. We can compute

$$G_j^{-1}(\alpha) = \frac{1}{j + \alpha_{d-1} - \alpha_d} \big(1, \alpha_1, \alpha_2, \dots, \alpha_{d-1}\big).$$

Therefore

$$H_j(\alpha) \stackrel{\text{def}}{=} G_j^{-1} \circ G(\alpha) = \frac{1}{1+(j-k)\alpha_1} \, (\alpha_1, \alpha_2, \dots, \alpha_d),$$

for $k = \lfloor \frac{1}{\alpha_1} \rfloor$. Write $\alpha_1 = \frac{1}{k+\varepsilon}$ for $\varepsilon \in [0,1)$. Then if $j < k$,

$$\frac{1}{1+(j-k)\alpha_1} = \frac{k+\varepsilon}{j+\varepsilon} > \frac{k+1}{j+1},$$

so H_j expands all distances with a factor at least $\frac{k+1}{j+1} > 1$. Furthermore,

$$G_1^{-1} \circ G_j^{-1}(\alpha) = \frac{1}{j+\alpha_{d-2}-\alpha_{d-1}} \, (1, 1, \alpha_1, \dots, \alpha_{d-2}),$$

which is more contracting as j increases. This means that if the code of α starts with $(k_0, k_1, \dots, k_{d-1})$, then we claim that there is an $\tilde{\alpha}$ with code starting $(1, 1, \dots 1)$ such that $G^d(\alpha) = G^d(\tilde{\alpha})$, and the derivative $DG^d(\alpha)$ is more expanding than $DG^d(\tilde{\alpha})$. To see this, consider the following diagram:

$$\begin{array}{ccccccccc}
\alpha \in \mathcal{U}_{k_0} & \xrightarrow{1} & \mathcal{U}_{k_1} & \xrightarrow{1} & \dots & \xrightarrow{1} & \mathcal{U}_{k_{d-1}} & \xrightarrow{1} & \mathcal{U} \\
 & \nearrow_2 & & \nearrow_2 & & \nearrow_2 & & \nearrow_2 & \\
\tilde{\alpha} \in \mathcal{U}_1 & \xrightarrow{3} & \mathcal{U}_1 & \xrightarrow{3} & \dots & \xrightarrow{3} & \mathcal{U}_1 & \xrightarrow{3} & \mathcal{U}
\end{array}$$

The path taking arrows $1, 1, , \dots, 1$ is at least as expanding as the path $2, 1, \dots, 1$, because of the expansion of H_{k_0}. Next the path $2, 1, 1, \dots, 1$ is as least as expanding as the path $3, 2, 1, \dots, 1$, because of the contraction of $G_1^{-1} \circ G_{k_1}^{-1}$, which is the inverse of G^2 along the path $2, 1$ in the diagram. Continuing by induction, we see that the path taking arrows $1, 1, , \dots, 1$ is as least as expanding as the path $3, 3, \dots, 3, 2$. This proves the claim. So let us now estimate compute the derivative $DG^d(\alpha)$. Assuming again that α has code $(k_i)_{i=0}^{\infty}$, a straightforward computation shows that

$$\begin{aligned}
G^d(\alpha) = \Big(& \frac{k_0\alpha_1 + \alpha_d - 1}{\alpha_d}, \quad \frac{k_1\alpha_2 + (k_0-1)\alpha_1 + \alpha_d - 1}{\alpha_d}, \\
& \frac{k_2\alpha_3 + (k_1-1)\alpha_2 + (k_0-1)\alpha_1 + \alpha_d - 1}{\alpha_d}, \dots\dots \\
& \dots, \frac{k_{d-2}\alpha_{d-1} + (k_{d-3}-1)\alpha_{d-2} + \dots + (k_0-1)\alpha_1 + \alpha_d - 1}{\alpha_d}, \\
& \frac{(k_{d-1}+1)\alpha_d + (k_{d-2}-1)\alpha_{d-1} + \dots + (k_0-1)\alpha_1 - 1}{\alpha_d} \Big).
\end{aligned}$$

The derivative $DG^d(\alpha)$ is

$$\frac{1}{\alpha_d} \begin{pmatrix}
k_0 & 0 & 0 & \dots & & \frac{1-k_0\alpha_1}{\alpha_d} \\
k_0 - 1 & k_1 & 0 & & & \frac{1-k_1\alpha_2-(k_0-1)\alpha_1}{\alpha_d} \\
k_0 - 1 & k_1 - 1 & k_2 & & & \vdots \\
\vdots & & & \ddots & & \\
\vdots & & & & k_{d-2} & \frac{1-k_{d-2}\alpha_{d-1}-(k_{d-3}-1)\alpha_{d-2}-\dots-(k_0-1)\alpha_1}{\alpha_d} \\
k_0 - 1 & k_1 - 1 & k_2 - 1 & \dots & k_{d-2} - 1 & \frac{1-(k_{d-2}-1)\alpha_{d-1}-\dots-(k_0-1)\alpha_1}{\alpha_d}
\end{pmatrix}.$$

By the previous claim, the least expansion is achieved if $k_0 = \dots = k_{d-2} = 1$, but then this matrix is upper triangular, and all eigenvalues are ≥ 1 with equality if and only if $\alpha = \mathbb{1}$. Hence on this subset of $\mathcal{U}_1$, G^d is uniformly expanding outside every neighbourhood of $\mathbb{1}$.

This renders the coding map $\mathcal{A} \mapsto \mathbb{N}^{\mathbb{N}\cup\{0\}} \setminus \mathcal{FT}$ injective. $\square$

Proof of Theorem 1. By Lemma 4, G acts on $\mathcal{A}_d$ as a one-sided shift, proving part 2. In particular, $\mathcal{A}_d$ is uncountable. Let us show that $\mathcal{A}_d$ has zero Lebesgue measure.

The derivative of G is

$$DG = \frac{1}{\alpha_1} \begin{pmatrix} -\frac{\alpha_2}{\alpha_1} & 1 & & & \\ -\frac{\alpha_3}{\alpha_1} & 0 & 1 & & \\ \vdots & & & \ddots & \\ -\frac{\alpha_d}{\alpha_1} & & & 0 & 1 \\ \frac{1-\alpha_d}{\alpha_1} & & & 0 & 1 \end{pmatrix}$$

and the characteristic polynomial is

$$\begin{aligned} p_d(\lambda) &\stackrel{\text{def}}{=} \det(\lambda I - DG) \\ &= \lambda^d - \frac{\alpha_1 - \alpha_2}{\alpha_1^2}\lambda^{d-1} - \frac{\alpha_2 - \alpha_3}{\alpha_1^2}\lambda^{d-2} - \cdots - \frac{\alpha_{d-1} - \alpha_d}{\alpha_1^d}\lambda - \frac{1}{\alpha_1^{d+1}}. \end{aligned}$$

It follows that $\det(DG) = (-\alpha_1)^{-(d+1)}$, so $|\det DG(\alpha)| \geq 1$, with equality attained only in the top-most corner $\mathbb{1} = (1, 1, \dots 1)$ of the parameter space. We will study the distortion properties of $\det DG^n(\alpha)$ to estimate the Lebesgue measure of $\mathcal{A}_d$. Let

$$J(\alpha) := \det|DG \circ G_k^{-1}(\alpha)| = \Big(\frac{1}{k + \alpha_{d-1} - \alpha_d}\Big)^{d+1},$$

so if $\beta, \beta' \in \mathcal{U}_k$, then

$$\frac{J(\beta)}{J(\beta')} = \Big(\frac{k + \beta'_{d-1} - \beta'_d}{k + \beta_{d-1} - \beta_d}\Big)^{d+1} = \Big(1 + \frac{(\beta'_{d-1} - \beta_{d-1}) - (\beta'_d - \beta_d)}{k + \beta_{d-1} - \beta_d}\Big)^{d+1}.$$

Claim: There is a constant K such that $K \geq \sum_{j=1}^n |G^j(\beta) - G^j(\beta')|$ for every n, whenever β and β' have the same code up to $n-1$.

Since G^d is expanding away from a neighbourhood of $\mathbb{1}$, β and β' must be exponentially (in n) close to each other when these codes do not contain long strings of 1s. In this case, $\sum_{j=1}^n |G^j(\beta) - G^j(\beta')|$ can be majorised by a geometric series which is bounded independently of n. If there are long strings of ones, that is, there are iterates i and large r such that $G^i(\beta), G^{i+1}(\beta), \dots, G^{i+r}(\beta) \in \mathcal{U}_1$ and $G^i(\beta'), G^{i+1}(\beta'), \dots, G^{i+r}(\beta') \in \mathcal{U}_1$, then $\sum_{j=i}^{i+r} |G^j(\beta) - G^j(\beta')| \leq |C \cdot |G^{i+r}(\beta) - G^{i+r}(\beta')|$. After iterate $i+r$, there will be a period of uniform expansion before a new close visit to $\mathbb{1}$ can occur. So summing $|G^{i+r}(\beta) - G^{i+r}(\beta')|$ over all close visit times i still gives a uniformly bound. This proves the claim.

Let $\alpha \in \mathcal{A}_d$ be arbitrary, and let $C_n(\alpha) := \{\beta \in \mathcal{U} : k_0(\beta) \dots k_{n-1}(\beta) = k_0(\alpha) \dots k_{n-1}(\alpha)\}$ be the n-cylinder set at α. Since G acts as an almost full one-sided shift, $G^n(C_n(\alpha))$

contains the interior of $\mathcal{U} \cup \mathcal{L}$. For $\beta, \beta' \in C_n(\alpha)$ we have

$$\begin{aligned}
\frac{|\det(DG^n(\beta))|}{|\det(DG^n(\beta'))|} &= \prod_{j=1}^{n} \frac{J(G^j(\beta)}{J(G^j(\beta')} \\
&= \prod_{j=1}^{n} \left(1 + \frac{(G^j(\beta')_{d-1} - G^j(\beta)_{d-1}) - (G^j(\beta')_d - G^j(\beta)_d)}{k_j + G^j(\beta)_{d-1} - G^j(\beta)_d}\right)^{d+1} \\
&\le \left(\exp \sum_{j=1}^{n} |G^j(\beta')_{d-1} - G^j(\beta)_{d-1}| + |G^j(\beta')_d - G^j(\beta)_d|\right)^{d+1} \\
&\le \exp(2K(d+1)).
\end{aligned}$$

Therefore

$$\mathrm{Leb}(\ G^{-n}(\mathcal{L}) \cap C_n(\alpha)\ ) \ge e^{-2K(d+1)} \ \frac{\mathrm{Leb}(\mathcal{U})}{\mathrm{Leb}(\mathcal{U} \cup \mathcal{L})} > 0.$$

This shows that α has arbitrarily small neighbourhoods, a definite proportion of which is eventually mapped outside $\mathcal{U}$. Thus α cannot be a Lebesgue density point of $\mathcal{A}_d$, and since $\alpha \in \mathcal{A}_d$ was arbitrary, $\mathrm{Leb}(\mathcal{A}_d) = 0$. This proves part 1.

Finally, to prove part 3, if $G^n(\alpha) \in \mathcal{L}$ for some minimal n, then the n-th induced map has only $d' < d$ branches (any $1 \le d' < d$ is possible), and can be rescaled to a map in $\mathcal{ITM}_{d'}$. A similar analysis of $\mathcal{ITM}_{d'}$ shows that there is a countable alphabet one-sided shift of type ∞ maps with d' branches, whereas Lebesgue-a.e. $T \in \mathcal{ITM}_{d'}$ is eventually maps into $\mathcal{ITM}_{d''}$ for $d'' < d'$ under renormalisation, etc. □

3. The Hausdorff dimension of Ω

In this section we prove our results on the Hausdorff dimension.

Proof of Theorem 3. We have studied (Ω, T_α) using first return maps to a nested sequence of intervals; let Δ_k be the k-th interval of this nest, so $\Delta_0 = [0,1]$, $\Delta_1 = [1-\alpha_1, 1]$ and $\Delta_2 = [1-\alpha_1\alpha_2, 1]$, etc. In general, the length of Δ_n is $\pi_n := |\Delta_n| = \prod_{j=0}^{n-1} G^j(\alpha)_1$. In order to compute the upper box dimension, we will construct a cover Ω_n with intervals of length $\pi_{k,j} \le \pi_n$ and count the number we need. Let $\pi_{k,j}$, $j = 0, \dots d$, be the length of the domain B_i of the $j+1$-st branch of the first return map to Δ_n. Hence $\sum_{j=0}^{n-1} G^j(\alpha)_1 = \pi_n$ and more precisely:

$$\pi_{n,j} = \pi_n \cdot \begin{cases} (1 - G^n(\alpha)_1) & \text{for } j = 0; \\ (G^n(\alpha)_j - G^k(\alpha)_{j+1}) & \text{for } 1 \le j < d; \\ G^n(\alpha)_d & \text{for } j = d. \end{cases}$$

Let $l_{n,j}$ be the number of intervals of length $\pi_{n,j}$ used in the cover. Then $l_{0,j} = 1$ for $j = 0, \dots, d$ and each interval of length

$$\pi_{n,j} \text{ is covered by } \begin{cases} k_n \text{ intervals of length } \pi_{n+1,d-1} & \\ \qquad \text{and } k_n - 1 \text{ of length } \pi_{n+1,d} & \text{if } j = 0; \\ \text{one interval of length } \pi_{n+1,j-1} & \text{if } 1 \le j < d; \\ \text{one interval of length } \pi_{n+1,d-1} & \\ \qquad \text{and one of length } \pi_{n+1,d} & \text{if } j = d, \end{cases}$$

the numbers $l_{n,j}$ satisfy the recursive linear relation:

$$\begin{pmatrix} l_{n+1,0} \\ l_{n+1,1} \\ \vdots \\ \vdots \\ \vdots \\ l_{n+1,d} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \dots & & & \\ 0 & 0 & 1 & \dots & & & \\ \vdots & & & \ddots & & & \\ 0 & & & & & 1 & 0 \\ k_n & 0 & \dots & & & 0 & 1 \\ k_n - 1 & 0 & \dots & & & 0 & 1 \end{pmatrix} \begin{pmatrix} l_{n,0} \\ l_{n,1} \\ \vdots \\ \vdots \\ \vdots \\ l_{n,d} \end{pmatrix}. \tag{3}$$

Let M_k be the above $d+1 \times d+1$ matrix for $k = k_n$. Its characteristic polynomial is

$$m_k(\lambda) \stackrel{\text{def}}{=} \det(\lambda I - M_k) = \lambda^{d+1} - \lambda^d - k\lambda + 1.$$

Let $\overline{r}_k \geq 1$ be the leading eigenvalue; $\overline{r}_1 = 1$ and $\overline{r}_k > 1$ if $k > 1$. Write $\rho_d = \log \overline{r}_2 / \log 2$. Then $\rho = \rho_2 \approx 0.84955\dots$, and ρ_d is decreasing in d, so $\rho_d \leq 0.84955\cdots < 1$ for all d. It can be shown that $\overline{r}_k \leq k^\rho$ for all $k \in \mathbb{N}$ and $d \geq 2$. If $\alpha_1 \in \mathcal{U}_k$, we have $\frac{1}{\alpha_1} \geq k$, and hence $\frac{1}{\pi_n} = \prod_{j=0}^{n-1} 1/G^j(\alpha)_1 \leq \prod_{j=0}^{n-1} k_j$. Therefore we can estimate the upper box dimension of Ω as

$$\begin{aligned} \overline{\dim}_B(\Omega) &\leq \limsup_n \frac{\log \sum_{j=0}^d l_{n,j}}{-\log \pi_n} \\ &\leq \limsup_n \frac{\log (d+1) \sum_{j=0}^{n-1} \overline{r}_{k_j}}{\sum_{j=0}^{n-1} \log k_j} \\ &\leq \limsup_n \frac{\rho \sum_{j=0}^{n-1} \log k_j}{\sum_{j=0}^{n-1} \log k_j} = \rho < 1. \end{aligned}$$

This proves the theorem. □

Remark: Note that for each $k \in \mathbb{N}$, G has a unique fixed point in $\mathcal{U}_k$. The coordinates of this fixed point $\alpha := \alpha(k)$ satisfy

$$\alpha_1^{d+1} = \alpha_1^d + k\alpha_1 - 1 \quad \text{and} \quad \alpha_i = \alpha_1^i. \tag{4}$$

For these parameters, we have complete self-similarity of the attractor Ω, and the Hausdorff dimension of Ω is $-\frac{\log \overline{r}_k}{\log \alpha_1}$.

Let r_k be the root of (4) between $\frac{1}{k}$ and $\frac{1}{k+1}$. If the code $(k_j)_{j\geq 0}$ consists of blocks of sufficiently fast increasing length of, say, $k_j = 2$ and $k_j = 3$ alternately, then the upper box dimension will be $\log \overline{r}_2 / \log r_2$ and the lower box dimension is at most $\log \overline{r}_3 / \log r_3$. This shows that $\overline{\dim}_B(\Omega) > \underline{\dim}_B(\Omega)$ is possible.

4. Symbolic dynamics and non-unique ergodicity of $T|_\Omega$.

The use of 'old' branches to produce the 'new' branches can be expressed symbolically by the substitution

$$\chi_k : \begin{cases} 0 & \to & 1 \\ 1 & \to & 2 \\ & \vdots & \\ d-2 & \to & d \\ d-1 & \to & d1^k \\ d & \to & d1^{k-1} \end{cases} \tag{5}$$

This substitution has associated $d+1 \times d+1$-matrix M_k, i.e., the same matrix as used in (3).

Proposition 5. *If $\alpha \in \mathcal{A}$ has code $(k_0, k_1, k_2, \dots)$, where $k_i = r$ if $G^i(\alpha) \in \mathcal{U}_r$, then T_α has an attracting Cantor set Ω and $T_\alpha|_\Omega$ is isomorphic to the substitution shift space (Σ, σ) generated by*

$$s \stackrel{\text{def}}{=} \lim_{i\to\infty} \chi_{k_0} \circ \chi_{k_1} \circ \cdots \circ \chi_{k_i}(d).$$

Proof. The argument is the same as in [BK] and [BT]. □

Let $C = \{x = (x_0, \dots, x_d) : x_i \geq 0\}$ be the nonnegative cone in $\mathbb{R}^{d+1}$ and $S = \{x \in C : \sum_{i=0}^d x_i = 1\}$ the unit simplex.

$$C_\infty \stackrel{\text{def}}{=} \cap_i M_{k_0}^t \cdot M_{k_1}^t \cdots M_{k_i}^t(C), \tag{6}$$

where $(k_i)_{i\geq 0}$ is the code of $\alpha \in \mathcal{A}$, the matrices M_{k_i} are those of equation (3) and M^t indicates the transpose of the matrix M.

Lemma 6. *The system (Σ, σ) is uniquely ergodic if and only if $C_\infty = \ell$ is a half-line. In this case, the point $v = \ell \cap S$ is the vector of frequencies of the symbols $0, \dots, d$ appearing in s, or equivalently, v_i is the invariant mass of the domain of the $i+1$st branch of T.*

Regardless of whether $C_\infty = \ell$ or not, the intersection of C_∞ and the unit simplex S is a convex polytope, and its corners correspond to the ergodic measures of (Σ, σ) and hence of (Ω, T).

Proof. Let $B_{n,j}$, $n \geq 0, j \in \{0, \dots, d\}$, be the domain of the $j+1$-st branch of the n-th renormalisation of T. If μ is a T-invariant measure, then

$$\mu(B_{n,j}) = \begin{cases} k_n\ \mu(B_{n+1,d-1}) + k_{n-1}\ \mu(B_{n+1,d}) & \text{if } j = 0; \\ \mu(B_{n+1,j-1}) & \text{if } 1 \leq j < d; \\ \mu(B_{n+1,d-1}) + \mu(B_{n+1,d}) & \text{if } j = d, \end{cases}$$

so

$$\Big(\mu(B_{n,j})\Big)_{j=0}^d = \frac{1}{N_n} M_{k_n}^t \Big(\mu(B_{n+1,j})\Big)_{j=0}^d,$$

for a normalising constant N_n. Since $M_{k_n}^t : C \to C$ is linear for each n, C_∞ is a convex set and $C_\infty \cap S$ is the intersection of convex polytopes and hence a convex polytope itself, with at most $d+1$ extrema. (In fact, there will be at most d extrema, as the proof of Theorem 2 suggests.) If v and v' are distinct extremal points in $C_\infty \cap S$, then there are symbols $a, a' \in \{0, \dots, d\}$ and arbitrarily large n such that the appearance frequencies of the symbols in $\chi_{k_1} \circ \cdots \circ \chi_{k_n}(a)$ and $\chi_{k_1} \circ \cdots \circ \chi_{k_n}(a')$ are arbitrarily close to v and v', and hence uniformly bounded away from each other. This implies that the itinerary of $1 \in [0,1]$ (or any other $x \in \Omega$) has arbitrarily long subwords of which the appearance frequencies of the symbols is arbitrarily close to v and similarly for v'. This contradict unique ergodicity, cf. Proposition 4.2.8 of [P].

Conversely, if C_∞ is a single line, then also $C_{n,\infty} \stackrel{\text{def}}{=} \cap_i M_{k_n}^t \cdot M_{k_{n+1}}^t \cdots M_{k_{n+i}}^t(C)$ is a single line. If μ and μ' are different T-invariant measures, then there is $n \geq 0$ and j such that $\mu(B_{n,j}) \neq \mu'(B_{n,j})$. But $(\mu(B_{n,j}))_j, (\mu'(B_{n,j}))_j \in C_{n,\infty} \cap S$, contradicting that $C_{n,\infty} \cap S$ is a single point.

Finally, the ergodic measures must clearly correspond to the extremal points of $C_\infty \cap S$. $\square$

Define $F_k : S \to S$ by

$$F_k(x) \stackrel{\text{def}}{=} \frac{(kx_{d-1} + (k-1)x_d,\ x_0,\ x_1,\ \ldots,\ x_{d-2},\ x_{d-1} + x_d)}{k(x_{d-1} + x_d) + x_0 + \cdots + x_{d-1}},$$

that is the intersection of the simplex S and the line connecting 0 to $M_k^t x$. Introduce new coordinates:

$$\left\{ \begin{array}{rcl} \zeta_1 & = & x_0 + \cdots + x_{d-1}, \\ \zeta_2 & = & x_{d-1} + x_d, \\ \zeta_3 & = & x_{d-2} + x_{d-1} + x_d, \\ \vdots & & \vdots \\ \zeta_d & = & x_1 + \cdots + x_d. \end{array} \right. \quad \text{whence} \quad \left\{ \begin{array}{rcl} x_0 & = & 1 - \zeta_d, \\ x_1 & = & \zeta_d - \zeta_{d-1}, \\ x_2 & = & \zeta_{d-1} - \zeta_{d-2}, \\ \vdots & & \vdots \\ x_{d-2} & = & \zeta_3 - \zeta_2 \\ x_{d-1} & = & \zeta_1 + \zeta_2 - 1 \\ x_d & = & 1 - \zeta_1. \end{array} \right.$$

In these coordinates, F obtains the form:

$$\tilde{F}_k(\zeta) = \frac{1}{k\zeta_2 + \zeta_1} \left((k-1)\zeta_2 + \zeta_1, \zeta_3, \zeta_4, \ldots, \zeta_d, 1\right),$$

acting on

$$Z \stackrel{\text{def}}{=} \{\zeta = (\zeta_1, \ldots, \zeta_d) \ : \ 0 \le \zeta_2 \le \zeta_3 \le \cdots \le \zeta_d \le 1,\ 1 - \zeta_2 \le \zeta_1 \le 1\}.$$

Let us now prove Theorem 2, i.e., that T_α with code $(k_0, k_1, \ldots)$ preserves d distinct ergodic measures on its attractor provided $k_i \to \infty$ sufficiently fast.

Proof of Theorem 2. It suffices to examine the infinite intersection

$$Z_\infty \stackrel{\text{def}}{=} \bigcap_i \tilde{F}_{k_1} \circ \tilde{F}_{k_1} \circ \cdots \circ \tilde{F}_{k_i}(Z)$$

of closed d-dimensional polytopes. If this intersection has non-empty $d-1$-dimensional interior, then the d extrema of the intersection represent the d ergodic T-invariant measures on Ω. Recall $\mathbb{1} = (1, \ldots, 1)$ is the top corner of the polytope Z, and let Z_+ be the $d-1$-dimensional face of Z opposite to (and hence not containing) $(0, 1, 1, \ldots, 1)$. Straightforward calculation reveals that

$$\begin{aligned} \tilde{F}_{k_0}(\mathbb{1}) &= \frac{1}{k_0 + 1}\big(k_0, 1, \ldots, 1\big), \\ \tilde{F}_{k_0} \circ \tilde{F}_{k_1}(\mathbb{1}) &= \frac{1}{k_0 + k_1}\big(k_0 + k_1 - 1, 1, \ldots, 1, k_1 + 1\big), \\ \tilde{F}_{k_0} \circ \tilde{F}_{k_1} \circ \tilde{F}_{k_2}(\mathbb{1}) &= \frac{1}{k_0 + k_1 + k_2 - 1}\big(k_0 + k_1 + k_2 - 2, 1, \ldots, 1, k_1 + 1, k_1 + k_2\big), \end{aligned}$$

and for general $0 \le r < d$,

$$\begin{aligned} \tilde{F}_{k_0} \circ \cdots \circ \tilde{F}_{k_r}(\mathbb{1}) = \frac{1}{k_0 + \cdots + k_r + 1 - r}\big(&k_0 + \cdots + k_r - r, 1, \ldots \\ &\ldots,\ 1,\ \underset{\substack{\uparrow \\ \text{position } d-r+1}}{k_1 + 1},\ k_1 + k_2,\ \ldots,\ \underset{\substack{\uparrow \\ \text{position } d}}{k_1 + \cdots + k_r + 2 - r}\big), \end{aligned}$$

Therefore, if $k_{d-1} \gg k_{d-2} \gg \cdots \gg k_0$, we find that $\mathbb{1}$ is almost periodic under consecutive applications of $\tilde{F}_{k_i}$, closely visiting the other vertices of Z_+ along the way. A similar argument applies to the other vertices of Z_+. Since $\tilde{F}_k$ is continuous in ζ, we can make

$$\mathrm{dist}_{\mathrm{H}} \left(\tilde{F}_{k_0} \circ \tilde{F}_{k_1} \circ \cdots \circ \tilde{F}_{k_{d-1}}(Z_+) \ , \ Z_+ \right)$$

arbitrarily close to 0, where $\mathrm{dist}_{\mathrm{H}}$ indicates Hausdorff distance, by choosing $k_{d-1} \gg k_{d-2} \gg \cdots \gg k_0$. Choosing the next d elements of the code so that $k_{2d-1} \gg k_{2d-2} \gg \cdots \gg k_d$ and $k_d \gg k_{d-1}$ sufficiently large again, we can make

$$\mathrm{dist}_{\mathrm{H}} \left(\tilde{F}_{k_0} \circ \tilde{F}_{k_1} \circ \cdots \circ \tilde{F}_{k_{d-1}}(Z_+) \ , \ \tilde{F}_{k_0} \circ \tilde{F}_{k_1} \circ \cdots \circ \tilde{F}_{k_{2d-1}}(Z_+) \right)$$

arbitrarily small again. Repeating this way, we can assure that $\mathrm{dist}_{\mathrm{Hausd.}}(Z_\infty, Z_+)$ is small and hence Z_∞ has nonempty $d-1$-dimensional interior. □

Remark: If $d = 2$, then the condition $k_{i+1} > \lambda k_i$ for some fixed $\lambda > 1$ and all i sufficiently large suffices to conclude non-unique ergodicity for $\alpha \in \mathcal{A}_2$, see [BT].

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