(Non)invertibility of Fibonacci-like unimodal maps restricted to their critical omega-limit sets.

H. Bruin.*

September 16, 2010

Abstract

A Fibonacci(-like) unimodal map is defined by special combinatorial properties which guarantee that the critical omega-limit set $\omega(c)$ is a minimal Cantor set. In this paper we give conditions to ensure that $f|_{\omega(c)}$ is invertible, except at a subset of the backward critical orbit. Furthermore, any finite subtree of the binary tree can appear for some f as the tree connecting all points at which $f|\omega(c)$ is noninvertible.

This technique gives a new way of finding strange adding machines, *i.e.,* nonrenormalizable maps for which $f : \omega(c) \to \omega(c)$ is conjugate to a (triadic) adding machine. This construction of strange adding machines is compatible with $\omega(c)$ being a wild attractor.

1 Introduction

A minimal Cantor systems is continuous map of the Cantor set for which every orbit is dense. Examples in a symbolic contexts (where the Cantor set is represented by a closed shift-invariant subset of the full shift $\{0, 1\}^{\mathbb{N}}$ equipped with product topology and $\mathbb{N} = \{0, 1, 2, 3, \dots\}$) are Toeplitz shifts and Sturmian shift spaces. In fact, there are general methods to express a minimal Cantor system as a substitution shift, as a Bratteli diagram [D], or as a shift emerging from an enumeration scale [GLT].

Minimal Cantor systems can be invertible or not, see *e.g.* [AG, AY] for examples of both cases. Let $X^* := \{x \in X : \#f^{-1}(x) > 1\}$. A dynamical system (X, f) is called *almost invertible* if $X \setminus X^*$ is a dense G_δ set. Kolyada et al. [KST] investigate

*The support of EPSRC grant EP/F037112/1 is gratefully acknowledged

almost invertible systems, showing among other things that if a Cantor system is minimal, then it is almost one-to-one.

If X^* belongs to a single grand orbit, then we can define the *noninvertibility tree* as the smallest directed tree T^* whose vertices contain all points in X^* and whose edges $x \to y$ between vertices x and y are given by the relation $y = f(x)$.

If (X, f) is invertible, then $X^* = T^* = \emptyset$. In the Fibonacci substitution shift (Σ, σ) generated by $\chi : 0 \mapsto 01, 1 \mapsto 1$, which is also the Sturmian sequence associated to the golden mean circle rotation, all points have one preimage except the fixed point of χ which has two preimages. In this case $\Sigma^* = T^*$ is this fixed point. This property is shared by all Sturmian shift spaces.

1.1 Tree structure of minimal omega-limit sets:

The question we want to address in this paper is what noninvertibility trees we may encounter if $X = \omega(c)$ is the critical omega-limit set of a unimodal map. A unimodal map $f : I \to I$ on the unit interval $I = [0, 1]$ is a continuous map with $f(0) = f(1) = 0$ and having a single *critical point* c such that f is increasing on $[0, c)$ and decreasing on $(c, 1]$. The *critical order* is ℓ if there is a diffeomorphism $\varphi : \mathbb{R} \to \mathbb{R}$ fixing 0 such that $f(x) = f(c) + |\varphi(x - c)|^\ell$. The critical point is nonflat if its critical order $\ell < \infty$. An interval W is *wandering* if $f^n|_W$ is monotone for all $n \geq 0$ and $f^n(W)$ does not converge to a periodic orbit. Throughout, we will assume that f has no wandering interval, which is true for all C^2 maps with nonflat critical points, see [MS, Theorem II.6.2.].

For some unimodal maps, $\omega(c)$ is a minimal Cantor set. The best known example is the Feigenbaum map; we discuss this and other infinitely renormalizable maps later on. Another well-known example is the Fibonacci unimodal map, first described in [HK]. This is a unimodal map of a certain combinatorial type (the cutting times are the Fibonacci numbers, see below). For this map, each $x \in \omega(c)$ has one preimage in $\omega(c)$, except for $x = c$ which has two preimages. The similarity with the structure of the Fibonacci substitution shift is not coincidental; both approaches describe isomorphic Cantor systems.

We will focus on maps that are Fibonacci-like in the sense that the cutting times obey recursive rules similar to that the Fibonacci numbers. In such cases, $\omega(c)$ is a minimal Cantor set, [Br1]. This is of more than combinatorial interest, because for Fibonacci-like unimodal maps with sufficiently degenerate critical point, $\omega(c)$ can actually be an attractor (see [Br2]). The noninvertibility tree may give information on the complexity of the subshift produced by $(\omega(c), f)$, cf. [BV], as well as its spectral properties. We will use enumeration scales (see [GLT] and introduced into interval dynamics in [BKS]) to explore the noninvertibility tree of the backward orbit of the critical point $\mathrm{orb}^-(c) := \cup_{i\geq 0} f^{-i}(c) \cap \omega(c)$. There is a priori no reason why X^* should belong to $\mathrm{orb}^-(c)$ but we will give sufficient conditions in Proposition 3.1 under which this is indeed the case. This leads to the main result:

Theorem 1.1. *For every finite subtree T_0 of the binary tree, there is a Fibonacci-like map such that $X^* \subset \mathrm{orb}^-(c)$ and the tree structure of X^* is isomorphic to T_0.*

Questions: In addition to the results in [BV], what are the word complexity and rank of the subshift associated to any of these (stationary or nonstationary) Fibonacci-like maps?

1.2 Adding machines and strange adding machines

We call the unimodal map $f : I \to I$ *renormalizable* if it has a periodic subinterval interval $I_1 \ni c$ of period $p_1 \geq 2$. The p_1-th iterate $f^{p_1} : I_1 \to I_1$ is a new unimodal map which could be renormalizable in its own right. By repeating this infinitely often we arrive at *infinite renormalizable* maps which possess a nested sequence of intervals $I \supset I_1 \supset I_2 \supset I_3 \supset \cdots \ni c$ such that I_k has period $\prod_{i=1}^k p_i$. Under sufficient smoothness conditions, $\cap_k \mathrm{orb}(I_k)$ coincides with the omega-limit set $\omega(c)$ of the critical point, and on this set is a minimal Cantor set of which f is conjugate to a (p_i)-adic adding machine.

Definition 1.1. *Given a sequence $(p_i)_{i\geq 1}$ of integers $p_i \geq 2$, the* (p_i)-adic adding machine *is the dynamical system $\Omega = \{(\omega_j)_{j\geq 1} : 0 \leq \omega_j < p_j\}$ equipped with product topology and the map g of "adding 1 and carry:*

$$g(\omega_1, \omega_2, \dots) = \begin{cases} 0, \dots, 0, \omega_k + 1, \omega_{k+1}, \omega_{k+2}, \dots & \text{if } k = \min\{i : \omega_i < p_i - 1\}; \\ 0, 0, 0, 0, \dots & \text{if } \omega_i = p_i - 1 \text{ for all } i \geq 1. \end{cases}$$

The Feigenbaum map is the simplest example: $p_i = 2$ for all i, and $f : \omega(c) \to \omega(c)$ is conjugate to the *dyadic* adding machine. The combinatorial structure of renormalization has been understood for a long time. More surprising was the recent discovery by Block et al. [BKM] that there are nonrenormalizable maps (*e.g.* within the tent-family) for which $\omega(c)$ is a minimal Cantor set and $f : \omega(c) \to \omega(c)$ are also conjugate to a (p_i)-adic adding machine. These maps are called *strange adding machines*. Their topological structure is fairly well understood as well. A point x is called *approximately periodic* if it is not asymptotically periodic but for every $\varepsilon > 0$, there is a periodic point q and $k_0 \geq 0$ such that $|f^{k_0+j}(x) - f^j(q)| < \varepsilon$ for all $j \geq 0$. The following classification is due to [BJ, Proposition 5.1].

Proposition 1.1. *If $(\omega(x), f)$ is conjugate to some (p_i)-adic adding machine if and only if x is approximately periodic.*

Note that adding machines are not subshifts (cf. [BK, Proposition 3.1]), and if we want to encode $(\omega(c), f)$ using the standard "kneading" symbolic dynamics, we run into problems in coding c. The intervals $[0, c)$ and $(c, 1]$ get symbols 0 and 1 respectively, but there is no unambiguous assignment of a symbol 0 or 1 to c that makes the shift continuous and/or compact. In [Br3] this problem is avoided by the construction of unimodal map for which the orbit of c approaches c only from one side, and $\omega(c)$ is a minimal Cantor set. This paper gives several examples where $(\omega(c), f)$ is an invertible minimal Cantor system, and more examples and background information are given in [BB, Br3, BOT].

The technique of enumeration scales leads to new examples of strange adding machines, which can be realized as attractors without the need of renormalization. The set $\omega(c)$ is called a *wild attractor* if f is not infinitely renormalizable but $\omega(x) = \omega(c)$ for Lebesgue a.e. x. In Section 5, we present a combinatorial type, realized by a map f with the following properties:

1. f is nonrenormalizable and has minimal Cantor omega-limit set $\omega(c)$ which is realized as a *wild* attractor, provided the critical order is sufficiently large.
2. The system (X, f) is conjugate to a triadic adding machine. (This produces a strange adding machines in a different way from the construction in [BKM].)

Acknowledgement: This research was supported by EPSRC grant EP/F037112/1. I want to thank Leslie Jones, Brian Raines and Víctor Jiménez López for their remarks on this and previous versions. Also the hospitality of the University of Murcia is gratefully acknowledged.

2 Preliminaries

Let $f : I \to I$, $I = [0, 1]$ be a unimodal map such that $f(0) = f(1) = 0$ and c is the unique critical point; hence f is increasing on $[0, c]$ and decreasing on $[c, 1]$. Write $c_n = f^n(c)$. We assume that $c_2 < c < c_1$ and $c_3 \geq c_2$. The omega limit set of the critical point is defined as $\omega(c) = \cap_i \overline{\{c_j : j > i\}}$; if c is recurrent, this set coincides with the closure $\overline{\text{orb}(c)}$. The omega limit set is closed and invariant, so $(\omega(c), f)$ is a surjective dynamical system in its own right.

The system (I, f) can be described symbolically by a subshift of $\{0, 1\}^{\mathbb{N}}$ where each $x \in I$ is assigned an *itinerary* $i(x) = i_0(x)i_1(x)i_2(x)\dots$ where

$$i_k(x) = \begin{cases} 0 & \text{if } f^k(x) \in [0, c), \\ * & \text{if } f^k(x) = c, \\ 1 & \text{if } f^k(x) \in (c, 1]. \end{cases}$$

Write $\kappa = i(c_1)$ for the *kneading invariant* of f. Two itineraries i and $\tilde{i}$ can be compared in *parity lexicographical order* $\prec_p$: If $k = \min\{j \geq 0 : i_j \neq \tilde{i}_j\}$ then

$$i \prec_p \tilde{i} \text{ if } \begin{cases} i_k < \tilde{i}_k & \text{and } \#\{j < k : e_j = 1\} \text{ is even,} \\ i_k > \tilde{i}_k & \text{and } \#\{j < k : e_j = 1\} \text{ is odd.} \end{cases}$$

Here $0 < * < 1$. Let σ be the left-shift. The map $x \mapsto i(x)$ is order preserving, whence $\sigma(\kappa) \preceq_p i(x) \preceq_p \kappa$ for every $x \in [c_2, c_1]$. Applied to the kneading invariant itself, we obtain an *admissibility condition* for kneading invariants:

$$\sigma(\kappa) \preceq_p \sigma^n(\kappa) \preceq_p \kappa \text{ for all } n \geq 0. \tag{1}$$

The kneading invariant of any unimodal map satisfies (1) and conversely, for any κ satisfying (1), there is a unimodal map having kneading invariant κ.

For our purposes, the kneading invariant is the only information that we require from the unimodal map. However, we will use an equivalent combinatorial description given by *cutting times* and the *kneading map.* These ideas were introduced by Hofbauer, see *e.g.* [H]. A survey can be found in [Br1].

If J is a maximal (closed) interval on which f^n is monotone, then $f^n : J \to f^n(J)$ is called a *branch.* If $c \in \partial J$, $f^n : J \to f^n(J)$ is a *central branch.* Obviously f^n has two central branches, and they have the same image. Denote this image by D_n.

If $D_n \ni c$, then n is called a *cutting time.* Denote the cutting times by $\{S_i\}_{i \geq 0}$, $S_0 < S_1 < S_2 < \dots$ For interesting unimodal maps (such as tent maps with slope > 1) $S_0 = 1$ and $S_1 = 2$. The sequence of cutting times completely determines the tent map and vice versa. It can be shown that $S_k \leq 2S_{k-1}$ for all k. Furthermore, the difference between two consecutive cutting times is again a cutting time. Therefore we can write

$$S_k = S_{k-1} + S_{Q(k)}, \tag{2}$$

for some integer function Q, called the *kneading map.* Each unimodal map therefore is characterized by its kneading map. Conversely, each map $Q : \mathbb{N} \setminus \{0\} \to \mathbb{N}$ satisfying $Q(k) < k$ and the *admissibility condition* (equivalent to (1))

$$\{Q(k+j)\}_{j \geq 1} \succeq \{Q(Q^2(k)+j)\}_{j \geq 1} \tag{3}$$

(where $\succeq$ denotes the lexicographical ordering) is the kneading map of some unimodal map. Well-known examples are the *Feigenbaum map*, where $Q(k) = k - 1$ and the *Fibonacci map*, where $Q(k) = \max\{k-2, 0\}$. We call f *Fibonacci-like* if there is N such that $k - Q(k) \leq N$ for all $k \in \mathbb{N}$.

Using cutting times and kneading map, the following properties of the intervals D_n are easy to derive:

$$D_{n+1} = \begin{cases} f(D_n) & \text{if } c \notin D_n, \\ [c_{n+1}, c_1] & \text{if } c \in D_n. \end{cases}$$

Equivalently:

$$D_n = [c_n, c_{\beta(n)}], \text{ where } \beta(n) = n - \max\{S_k; S_k < n\}, \tag{4}$$

and in particular $D_{S_k} = [c_{S_k}, c_{S_{Q(k)}}]$.

Let $z_k < c < \hat{z}_k$ be the boundary points of the domains the two central branches of $f^{S_{k+1}}$. Then z_k and $\hat{z}_k$ lie in the interiors of the domains of the central branches of f^{S_k} and $f^{S_k}(z_k) = f^{S_k}(\hat{z}_k) = c$. Furthermore, f^j is monotone on (z_k, c) and $(c, \hat{z}_k)$ for all $0 \le j \le S_k$. These points are called *closest precritical points*, and the relation (2) implies

$$f^{S_{k-1}}(c) \in (z_{Q(k)-1}, z_{Q(k)}] \cup [\hat{z}_{Q(k)}, \hat{z}_{Q(k)-1}). \tag{5}$$

We will use these relations repeatedly without specific reference.

Lemma 2.1. *The domains of the central branches of f^{S_k} are $J_{S_k} = [z_{k-1}, 0]$ and $\hat{J}_{S_k} = [0, \hat{z}_{k-1}]$. If $Q(k) \to \infty$, then*

- $|D_n| \to 0$ *as $n \to \infty$;*
- $\omega(c)$ *is a minimal Cantor set.*

Proof. These statements appear somewhat implicitly in e.g. [Br1], so let us give a full proof. Since z_{k-1} and z_k are consecutive closest precritical points, there is no point $z \in (z_k, 0)$ such that $f^n(z) = 0$ for $n < S_k$. Therefore $[z_{k-1}, 0]$ is a maximal interval of monotonicity and $f^{S_k}([z_{k-1}, 0]) \ni 0$.

First we claim that $c_{1+S_k} \to c_1$. Indeed, if this were not the case, then (recalling from (4) that $D_{1+S_k} = [c_{1+S_k}, c_1]$) there would be a subsequence $(k_i)_i$ such that $B := \cap_i D_{1+S_{k_i}}$ is a nondegenerate interval. But $f^m(D_{1+S_k}) \not\ni c$ for $m = 1, \dots, S_{k+1} - S_k - 1$ and because $S_{k+1} - S_k - 1 = S_{Q(k+1)} - 1 \to \infty$, we find that B is actually a wandering interval. Thus nonexistence of wandering intervals proves the claim. Consequently, $|D_{S_k}| = |c_{S_k} - c_{S_{Q(k)}}| \to 0$ as $k \to \infty$.

Since f has no wandering intervals, the following "noncontraction principle" holds, see [MS, Chapter IV]:

> For every $\varepsilon > 0$ there is a $\delta > 0$ such that if J is an interval of length $|J| > \varepsilon$, then $|f^n(J)| > \varepsilon$ for all $n \ge 0$.

Now let $\varepsilon > 0$ be arbitrary and choose δ accordingly. Take K such that $|D_{S_k}| < \delta$ for all $k > K$. If $n > S_K$, then there is j such that $f^j(D_n) \subset D_{S_k}$ for some $k \ge K$, so $|D_n| < \varepsilon$.

By definition, $\mathrm{orb}(c)$ is dense in $\omega(c)$, and since $|c - c_{S_k}| < |D_{S_k}| \to 0$, c is recurrent as well. If x is such that $\mathrm{orb}(x)$ accumulates on c, then $\omega(c) \subset \omega(x)$. Therefore $(\omega(c), f)$ is minimal if and only if $c \in \omega(x)$ for every $x \in \omega(c)$.

Suppose by contradiction that $(\omega(c), f)$ is not minimal, and that $x \in \omega(c)$ and $\varepsilon > 0$ are such that $\mathrm{orb}(x) \cap B(c; \varepsilon) = \emptyset$. We know that $|D_n| \to 0$ as $n \to \infty$, so there is K such that $D_{S_k} \subset B(c; \varepsilon)$ for all $k \geq K$.

For each $m < S_K$, let $\gamma(m) \geq S_K$ be such that $\beta(\gamma(m)) = m$ and $D_{\gamma(m)}$ is the largest interval among all D_n with $n \geq S_K$ and $\beta(n) = m$. We claim that $\omega(c) \subset \cup_{m<S_K} D_{\gamma(m)}$. Indeed, if $n \geq S_K$, then there is a nested sequence of intervals $D_n \subset D_{\beta(n)} \subset \cdots \subset D_{\beta^r(n)}$, where $r \geq 1$ is minimal such that $m := \beta^r(n) < S_K$. But then $D_n \subset D_{\beta^{r-1}(n)} \subset D_{\gamma(m)}$. Therefore

$$\omega(c) \ \subset \ \overline{\cup_{n \geq S_K} D_n} \ \subset \ \overline{\cup_{m<S_K} D_{\gamma(m)}} \ = \ \cup_{m<S_K} D_{\gamma(m)},$$

where the last equality follows because $\cup_{m<S_K} D_{\gamma(m)}$ is the finite union of closed sets. It follows that $x \in D_{\gamma(m)}$ for some $m < S_K$, and because $\gamma(m) \geq S_K$, there is $j \geq 0$ such that $f^j(x) \in f^j(D_{\gamma(m)}) \subset B(c; \varepsilon)$. This contradiction proves the final statement. □

The relation between cutting times and kneading invariant is easy to make if we define

$$\tau : \mathbb{N} \to \mathbb{N}, \ \ \tau(n) = \min\{m > 0 \ ; \ \kappa_m \neq \kappa_{m+n}\}. \tag{6}$$

We retrieve the cutting times as follows:

$$S_0 = 1 \text{ and } S_{k+1} = S_k + \tau(S_k) = S_k + S_{Q(k+1)} \text{ for } k \geq 0.$$

In other words, $\kappa_1 \dots \kappa_{S_k} = \kappa_1 \dots \kappa_{S_{k-1}} \kappa_1 \dots \kappa'_{S_{Q(k)}}$, writing $\kappa'_i = 0$ if $\kappa_i = 1$ and vice versa. For the proofs of these statements, we refer to [Br1].

An *enumeration scale* is an adding machine-like number system based on a strictly increasing sequence of nonnegative integers $\{S_k\}_{k \geq 0}$. Any nonnegative integer n can be written in a canonical way as a sum of cutting times: $n = \sum_j e_j S_j$, where

$$e_j = \begin{cases} 1 & \text{if } j = \max\{k; S_k \leq n - \sum_{k>j} e_k S_k\}, \\ 0 & \text{otherwise.} \end{cases}$$

In particular $e_j = 0$ if $S_j > n$. In this way we can code the nonnegative integers $\mathbb{N}$ as zero-one sequences with a finite number of ones: $n \mapsto \langle n \rangle \in \{0,1\}^{\mathbb{N}}$. Let $E_0 = \langle \mathbb{N} \rangle$ be the set of such sequences, and let E be the closure of E_0 in the product topology.

In our case, we let $\{S_k\}_k$ be the cutting time of a unimodal map, and assume that $Q(k) \to \infty$. This results in

$$E = \{e \in \{0,1\}^{\mathbb{N}} \ ; \ e_i = 1 \Rightarrow e_j = 0 \text{ for } Q(i+1) \leq j < i\}.$$

The condition in this set follows because if $e_i = e_{Q(i+1)} = 1$, then this should be rewritten to $e_i = e_{Q(i+1)} = 0$ and $e_{i+1} = 1$. It follows immediately that for each $e \in E$ and $j \geq 0$,

$$e_0 S_0 + e_1 S_1 + \cdots + e_j S_j < S_{j+1}. \tag{7}$$

There exists the standard addition of 1 by means of 'add and carry'. Denote this action by g. Obviously $g(\langle n \rangle) = \langle n+1 \rangle$. It is known (see *e.g.* [BKS, GLT]) that $g : E \to E$ is continuous if and only if $Q(k) \to \infty$, and that g is invertible on $E \setminus \{\langle 0 \rangle\}$. The next lemma describes the inverses of $\langle 0 \rangle$ precisely.

Lemma 2.2. *For a sequence $e \in E$, let $\{q_j\}_{j \geq 0}$ be the index set (in increasing order) such that $e_{q_j} = 1$. We have $g(e) = \langle 0 \rangle$ if and only if $e \notin E_0$, $Q(q_0 + 1) = 0$ and $Q(q_j + 1) = q_{j-1} + 1$ for $j \geq 1$.*

Proof. This follows immediately from the add and carry construction, because the condition on $\{q_j\}$ is the only way the addition of 1 carries 'to infinity'. □

Corollary 2.1. *The cardinality $\#g^{-1}(\langle 0 \rangle)$ is equal to the number of distinct infinite Q^{-1}-orbits $\{k_j\}_{j \geq 0}$ with $Q(k_j) = k_{j-1}$ and $k_0 > Q(k_0) = 0$.*

Proof. Given a sequence $\{k_j\}_{j \geq 0}$ satisfying the properties of this corollary, take $q_i = k_i - 1$. Then $Q(q_0 + 1) = 0$ and $Q(q_j + 1) = q_{j-1} + 1$ for $j \geq 1$. Lemma 2.2 implies that if $e_i = 1$ if and only if $i = q_j$ for some j, then $g(e) = \langle 0 \rangle$.

Conversely, any e with $g(e) = \langle 0 \rangle$ has this form, and hence corresponds to a backward orbit $\{k_j\}_{j \geq 0}$ of Q. □

Example: If $Q(k) = \max\{0, k - d\}$ for some fixed $d \geq 1$, then the number system (E, g) where $\langle 0 \rangle$ has exactly d preimages; namely the sequences where all 1s are exactly d entries apart map to $\langle 0 \rangle$.

Lemma 2.3. *If $Q(k) \to \infty$, then (E, g) factorizes over $(\omega(c), f)$,* i.e., *there is a continuous map $\pi : E \to \omega(c)$ such that $\pi \circ g = f \circ \pi$.*

Proof. Recall that for $n \in (S_{k-1}, S_k]$, we defined $\beta(n) = n - S_{k-1}$. It is easy to check that $\langle \beta(n) \rangle$ is $\langle n \rangle$ with the last nonzero entry changed to 0. The map β also has a geometric interpretation in the Hofbauer tower: It was shown in [BKS, Lemma 5] that for all $n \geq 2$,

$$D_n \subset D_{\beta(n)}. \tag{8}$$

In fact, D_n and $D_{\beta(n)}$ have the boundary point $c_{\beta(n)}$ in common. Recall that for $e \in E$, $\{q_j\}_j$ is the index sequence of the nonzero entries of e. Define

$$b(i) := \sum_{j \leq q_i} e_j S_j.$$

We have $b(i) \geq S_{q_i}$ by definition of q_i and $b(i) < S_{q_i+1}$ by (7). It follows that $\beta(b(i)) = b(i-1)$. By a *nest* of levels will be meant a sequence of levels $D_{b(i)}$. By (8) and the fact that $\beta(b(i)) = b(i-1)$, these levels lie indeed nested, and because $Q(k) \to \infty$ implies that $|D_n| \to 0$ (see [Br1]), each nest defines a unique point $x = \cap_i D_{b(i)} \in \omega(c)$. Therefore the following projection (see [BKS]) makes sense:

$$\pi(\langle n \rangle) = c_n$$

and

$$\pi(e \notin E_0) = \cap_i D_{b(i)}. \tag{9}$$

Obviously $f \circ \pi = \pi \circ g$ and it can be shown (see [BKS, Theorem 1]) that $\pi : E \to E$ is continuous and onto. $\square$

3 Sufficient Conditions for $X^* \subset \mathrm{orb}^-(c)$

Note that a nest contains exactly one cutting level $D_{b(0)}$. If $\{D_{b(i)}\}_i$ is some nest converging to x, then $\{f(D_{b(i)})\}_i$ is a nested sequence of levels converging to $f(x)$. To obtain a nest of $f(x)$, we may have to add or delete some levels, but $\{f(D_{b(i)})\}$ asymptotically coincides with a nest converging to $f(x)$.

Proposition 3.1. *If Q is a kneading map and there is K such that*

$$Q(k+1) > Q(Q^2(k)+1) + 1 \tag{10}$$

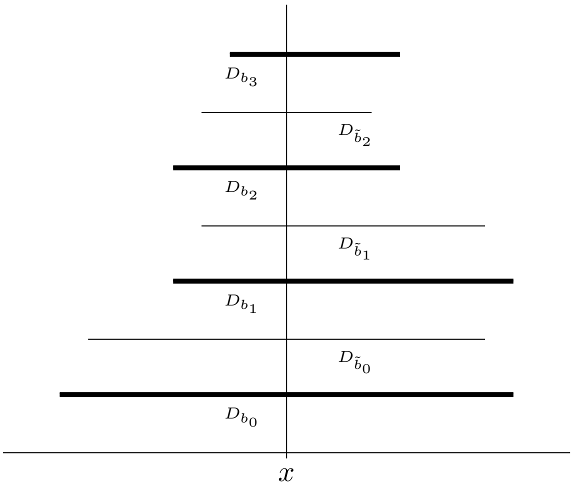
and

$$\#Q^{-1}(k) = 1 \tag{11}$$

for all $k \geq K$, then every point in $\omega(c) \setminus \mathrm{orb}^-(c)$ has only one preimage in $\omega(c)$.

Proof. Since $g : E \to E$ is invertible, except (possibly) at $\langle 0 \rangle$, if suffices to show that $\pi : E \to \omega(c)$ is one-to-one. First note (see also [BKS, Theorem 1]) that $\pi^{-1}(c) = \langle 0 \rangle$. Indeed, if $e \neq \langle 0 \rangle$ and $\pi(e) = c$, then, taking $k < l$ the first nonzero entries of e, $c \in D_{S_k + S_l}$. Then $S_k + S_l$ is a cutting time S_m and we have $m = l+1$ and $k = Q(l+1)$. This would trigger a carry to $e_k = e_l = 0$ and $e_m = 1$, contradicting that e_k and e_l are the first nonzero entries of e. Because g is invertible, also $\#\pi^{-1}(x) = 1$ for each $x \in f^{-n}(c)$ and $n \geq 0$. Assume from now on that $x \in \omega(c) \setminus \cup_{n \geq 0} f^{-n}(c)$. We need one more lemma:

Lemma 3.1. *Let f be a unimodal map whose kneading map Q satisfies* (10) *and $Q(k) \to \infty$. Then there exists K' such that for any $n \notin \{S_i\}_i$ such that $\beta(n)$ is a cutting time (*i.e., $n = S_r + S_t$ *for some $r < t$ with $r < Q(t+1)$), and every $k \geq K'$, D_n does not contain both c_{S_k} and a point from $\{z_{Q(k+1)-1}, \hat{z}_{Q(k+1)-1}\}$.*



If $X^* \not\subset \mathrm{orb}^-(c)$, then there are two points $e, \tilde{e} \in E$ such that

$$\pi(e) = \pi(\tilde{e}) = x$$

and

$$g^n(e), g^n(\tilde{e}) \neq \langle 0 \rangle \quad \forall n \geq 0.$$

Geometrically, this shows as two nests of intervals

$$D_{b_0} \supset D_{b_1} \supset D_{b_2} \supset \cdots \text{ (bold)}$$

and

$$D_{\tilde{b}_0} \supset D_{\tilde{b}_1} \supset D_{\tilde{b}_2} \supset \cdots$$

such that $x = \bigcap_i D_{b_i} = \bigcap_i D_{\tilde{b}_i}$

Figure 1: Interconnected nests to be prevented in Proposition 3.1

Proof of Lemma 3.1. Assume the contrary. Write $n = S_r + S_t$ with $r < Q(t+1)$ and let k but such that $z_{Q(k+1)-1}$ or $\hat{z}_{Q(k+1)-1} \in D_n \subset D_{S_r}$. Formula (5) implies that $Q(r+1) < Q(k+1)$, see Figure 2.

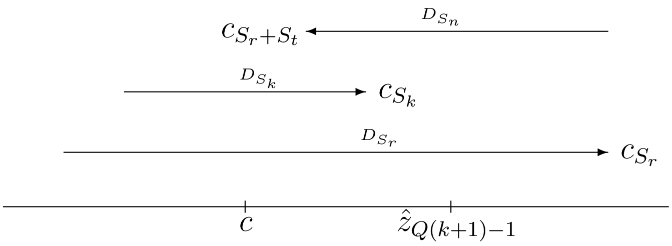


Figure 2: The levels D_{S_k} and $D_{S_r+S_t}$

It follows that also $z_{Q(r+1)}$ or $\hat{z}_{Q(r+1)} \in D_{S_r+S_t}$ and therefore $S_r+S_t+S_{Q(r+1)} = S_{t+1}$. This gives

$$r + 1 = Q(t+1), \tag{12}$$

and $S_r + S_t = S_{t+1} - S_{Q^2(t+1)}$. If also $z_{Q(r+1)+1}$ or $\hat{z}_{Q(r+1)+1} \in D_{S_r+S_t}$, then

$$S_{t+2} = S_{t+1} + (S_{Q(r+1)+1} - S_{Q(r+1)} = S_{t+1} + S_{Q(Q^2(t+1)+1)},$$

which yields $Q(Q(r+1)+1) = Q(t+2)$. Using (12) for $t+1$, this gives $Q(t+1+1) = Q(Q^2(t+1)+1)$. This contradicts (10), if $t \geq K-1$ is sufficiently large. For $t < K-1$, because there are only finitely many pairs r, t with $r < Q(t+1)$ and $Q(k+1) \to \infty$ (so $c_{S_k} \to c$), there is K' such that $D_{S_r+S_t} \not\ni c_{S_k}$ for $k \geq K'$. Hence

$$D_{S_r+S_t} \text{ contains at most one closest precritical point.} \tag{13}$$

Therefore, as $c_{S_k} \in D_{S_r+S_t}$, $Q(r+1) = Q(k+1) - 1$. Take the $S_{Q(r+1)}$-th iterate of $D_{S_r+S_t}$ and $[c, c_{S_k}]$ to obtain $D_{S_k+S_{Q(r+1)}} \cap D_{S_{t+1}} \neq \emptyset$, see Figure 3.

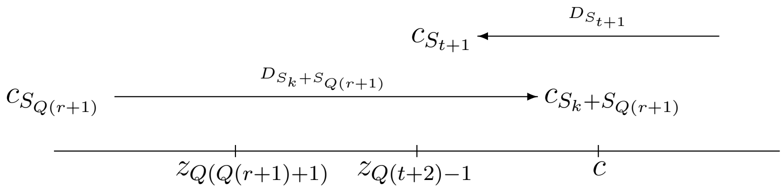


Figure 3: The levels $D_{S_{Q(r+1)}+S_k}$ and $D_{S_{t+1}}$

By (12) we have $Q(Q(r+1)+1) = Q(Q^2(t+1)+1)$, and using (10) on $t+1$, we obtain

$$Q(Q^2(t+1)+1) < Q(t+1+1) - 1 = Q(t+2) - 1.$$

Hence there are at least two closest precritical points contained in $D_{S_k+S_{Q(r+1)}}$. This contradicts the arguments leading to (13). $\square$

We continue the proof of Proposition 3.1. Observe that if $\pi(e) = \pi(\tilde{e}) = x$ for some $e \neq \tilde{e}$, then the corresponding nests $\{D_{b(i)}\}$ and $\{D_{\tilde{b}(i)}\}$ are different, but both nests converge to x, see Figure 1. Because $x \notin \cup_{n \geq 0} f^{-n}(c)$, $\#\pi^{-1}(f^n(x)) > 1$ for all $n \geq 0$. We will derive a contradiction.

> **Claim 1:** By taking an iterate, we can assume that $q_0 \neq \tilde{q}_0$, where as before q_0 and $\tilde{q}_0$ are the indices of the first nonzero entries of e and $\tilde{e}$.

Let i be the smallest integer such that $b(i) \neq \tilde{b}(i)$, say $b(i) < \tilde{b}(i)$. Then also $q_i < \tilde{q}_i$. Let $l = S_{\tilde{q}_i+1} - \tilde{b}(i)$. By (7), l is nonnegative. By the choice of l, $(g^l(\tilde{e}))_j = 0$ for all $j \leq \tilde{q}_i$, but because $b(i) < \tilde{b}(i)$ and $l + b(i) < S_{\tilde{q}_i+1}$, there is some $j \leq \tilde{q}_i$ such that $(g^l(e))_j = 1$.

Replace x by $f^l(x)$, and the corresponding sequences e and $\tilde{e}$ by $g^l(e)$ and $g^l(\tilde{e})$. Then for this new point, $q_0 < \tilde{q}_0$ and $b(0) < \tilde{b}(0)$. This proves Claim 1. Note that by the same argument we can take $q_0 \neq \tilde{q}_0$ arbitrarily large.

> **Claim 2:** By taking an iterate, we can assume that $Q(q_0+1) \neq Q(\tilde{q}_0+1)$

This follows from assumption (11), *i.e.,* Q is eventually injective, plus the fact that q_0 can be taken arbitrarily large.

Then we are in the situation of Lemma 3.1, which tells us that $D_{b(1)} \cap D_{\tilde{b}(0)} = \emptyset$. Therefore the two nests cannot converge to the same point. This contradiction concludes the proof. $\square$

Condition (11) is true for the maps in Section 4, but is too strong for many other examples. The following lemma gives a weaker condition, which is sufficient for the examples in Section 5.

Lemma 3.2. *If the kneading map Q satisfies* (10) *and (instead of* (11)*) satisfies $Q(n) \to \infty$ and for all $n, \tilde{n}$ sufficiently large $Q(n+1) = Q(\tilde{n}+1)$ implies the following conditions:*

(a) $Q^3(n) \neq Q(\tilde{n})$,

(b) if $Q(n) \neq Q(\tilde{n})$, then $Q^2(n) \neq Q^2(\tilde{n})$.

Then every point in $\omega(c) \setminus \mathrm{orb}^-(c)$ has only one preimage in $\omega(c)$.

Proof. By Lemma 3.1, there is no interval $D_{S_k+S_l}$ for $k < Q(l+1)$ containing both a closest precritical point $z_{Q(k+1)}$ and a point $c_{S_{\tilde{k}}}$ with $Q(\tilde{k}+1) = Q(k+1)+1$. In view of Figure 1 we need to prevent three cases from occurring:

Case A: $z_{Q(k+1)}$ or $\hat{z}_{Q(k+1)} \in D_{S_k}, D_{S_k+S_l}, D_{S_k+S_l+S_m}$ and $D_{S_{\tilde{k}}}$, but $z_{Q(k+1)} \notin D_{S_{\tilde{k}}+S_{\tilde{l}}}$ and $D_{S_k+S_l+S_m} \cap D_{S_{\tilde{k}}+S_{\tilde{l}}} \neq \emptyset$.

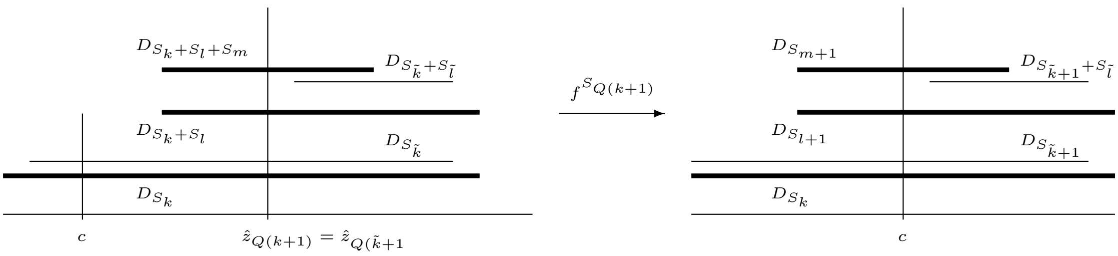


Figure 4: Case A

In this case $Q(k+1) = Q(k'+1)$ and applying $f^{S_{Q(k+1)}}$ to the left part of Figure 5 we find $S_{l+1} = S_l + S_k + S_{Q(k+1)}$, so $Q(l+1) = k+1$. Similarly, $S_{m+1} = S_k + S_l + S_{Q(k+1)}$, so $Q(m+1) = l+1$.

This shows that $Q^3(m+1) = Q(\tilde{k}+1)$. At the same time $Q(m+2) = Q(\tilde{k}+2)$ by Lemma 3.1. But these two condition together violate (a) for $n = m+1$, so this cannot happen.

Case B: $z_{Q(k+1)}$ or $\hat{z}_{Q(k+1)} \in D_{S_k}, D_{S_k+S_l}, D_{S_{\tilde{k}}}$ and $D_{S_{\tilde{k}}+S_{\tilde{l}}}$.

In this case $Q(k+1) = Q(k'+1)$ and applying $f^{S_{Q(k+1)}}$ to the left part of Figure 5 we find $S_{l+1} = S_l + S_k + S_{Q(k+1)}$, so $Q(l+1) = k+1$. Similarly, $S_{\tilde{l}+1} = S_{\tilde{k}} + S_{\tilde{l}} + S_{Q(\tilde{k}+1)}$, so $Q(\tilde{l}+1) = \tilde{k}+1$. By Lemma 3.1, $Q(l+2) = Q(\tilde{l}+2)$. Also $Q(l+1) \neq Q(\tilde{l}+1)$, but since $Q^2(l+1) = Q^2(\tilde{l}+1)$, condition (b) is violated with $n = l+1$, $\tilde{n} = \tilde{l}+1$.

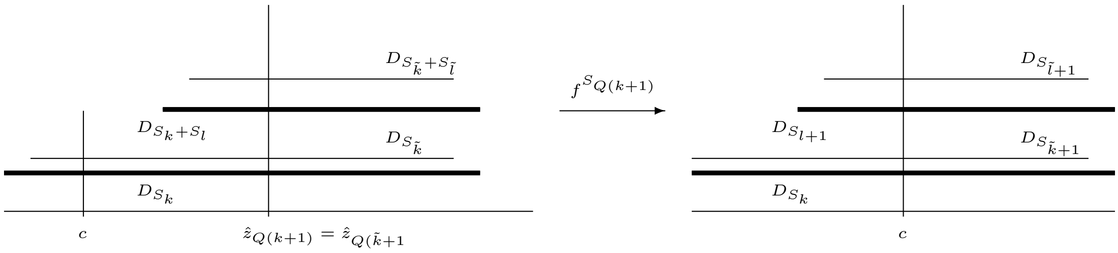


Figure 5: Case B

Case C: $z_{Q(k+1)}$ or $\hat{z}_{Q(k+1)} \in D_{S_k}, D_{S_{\tilde{k}}}$, but $z_{Q(k+1)}, \hat{z}_{Q(k+1)} \notin D_{S_k+S_l}$ and $D_{S_{\tilde{k}}+S_{\tilde{l}}}$.

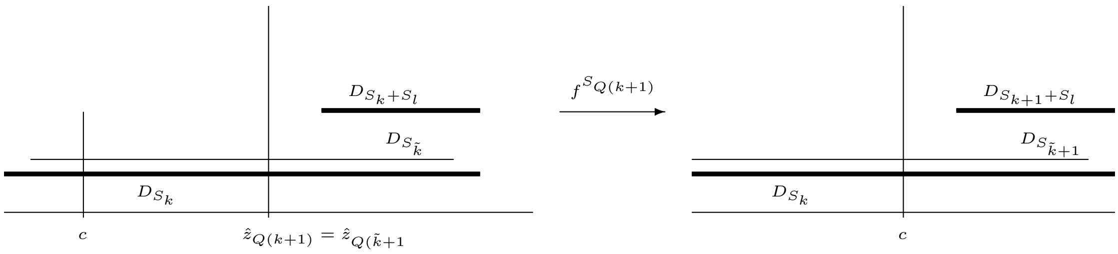


Figure 6: Case C

Applying $f^{S_{Q(k+1)}}$ to the left part of Figure 6 cannot produce Cases A or B, because they have been excluded already, so Case C reappears. But that means that $Q(k+2) = Q(\tilde{k}+2)$. Proceeding by induction, we find $Q(k+j) = Q(\tilde{k}+j)$ for all $j \geq 1$, so Q is periodic. But this contradicts that $Q(k \to \infty$. This completes the proof. □

4 The preimage tree of the critical point

Consider the full binary tree rooted at v, where each edge is labeled by a symbol 0 or 1. Let $\mathcal{B}$ be any finite subtree also rooted at v. Each vertex b in the binary tree can be coded by a finite string $e(b)$ of zeroes and ones according to the labels of the path connecting b to the root of the tree, see Figure 7. Let $\mathcal{B}^* = \{b \in \mathcal{B} :$ b has two outgoing edges in $\mathcal{B}\}$.

Theorem 4.1. *Given $\mathcal{B}$ as above, there exists a unimodal map (with kneading invariant ν) such that*

1. *$\omega(c)$ is a minimal Cantor set.*
2. *$f|\omega(c)$ is one-to-one, except at points in the backward critical orbit of c.*
3. *The points that have two preimages in $\omega(c)$ are arranged precisely according to $\mathcal{B}$. That is, a vertex $b \in \mathcal{B}^*$ if and only if there is a point $x \in \omega(c)$ with two preimages in $\omega(c)$, and the itinerary of x is $e(x) = e(b) * \nu$.*

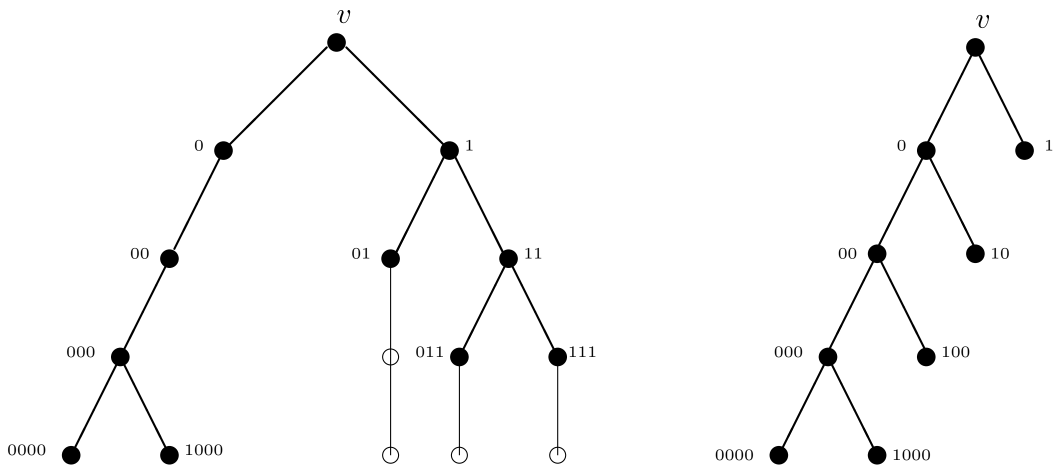


Figure 7: Left: A sample backward tree $\mathcal{B}$ (thick edges and vertices) with codes $e(b)$. The thin edges and vertices extend $\mathcal{B}$ to $\mathcal{B}^\sharp$ as explained in Step 1 of the construction below.
Right: The preimage tree of the Fibonacci-like map with kneading map $Q(k) = \{k-d, 0\}$ for $d = 5$.

Remarks: 1) We will construct the kneading map Q of f and it will turn out that $Q(k) \to \infty$ and $k - Q(k)$ is bounded. Thus $\omega(c)$ is a minimal Cantor set. By taking the critical order of f sufficiently large, we can assure that $\omega(c)$ is a metric attractor of f, that is: $\omega(x) = \omega(c)$ for Lebesgue almost every x, see [Br2].

2) A special case is the Feigenbaum map with $Q(k) = k - 1$; for this map, $f|\omega(c)$ is one-to-one. Another special case is the Fibonacci map with $Q(k) = \max\{k-2, 0\}$. In this case, c is the only point in $\omega(c)$ with two preimages. More generally, if $Q(k) = \max\{k-d, 0\}$, then the preimage tree has exactly $d-1$ branch points, with itineraries $0^i * \nu$ for $0 \leq i < d-2$, see Figure 7. This follows by inspection of the kneading invariant ν, and the fact that $\#g^{-1}(\langle 0 \rangle) = d$, so there can be at most $d-1$ branch points in the preimage tree.

Proof. **Step 1: the construction of $\boldsymbol{\nu}$:** Extend $\mathcal{B}$ to a finite tree $\mathcal{B}^\sharp$ with the same set of vertices with two outgoing edges, but such that maximal path downwards from the root v all have the same length, see Figure 7. Let $B_1, \dots, B_N$ be the codes corresponding to those vertices without outgoing edge, so $|B_k| = L$ for all k. Start ν as

$$\nu = 10^{N+3} B_1 e_1 11 B_2 e_2 11 \dots B_N e_N 11,$$

where $e_k \in \{0, 1\}$ is such that the position of e_k (*i.e.*, $p_k := N+4+k(L+3)$) is a cutting time. As the block 0^{N+3} does not reappear after position 2, there are admissible kneading invariants starting as ν. Next let $C_k = 10^{N+3} B_1 e_1 11 B_2 e_2 11 \dots B_k e_k$ be the

p_k-initial block of ν, and let C'_k be the same block with the last symbol switched. Continue ν as

$$\nu = 10^{N+3} B_1 e_1 11 B_2 e_2 11 \dots B_N e_N 11 C'_1 C'_2 \dots C'_N.$$

In this way, the end-position of each block C'_k in ν is a cutting time. Let K be such that S_K is the length of ν created so far; this determines the kneading map $Q(k)$ for all $k \le K$. Continue Q (and hence ν) by

$$Q(k) = k - N \text{ for } k > K.$$

Remark: If B has no branch points, then $N = 1$, so the dynamics of f is comparable to the Feigenbaum map. In this case, the fact that $f : \omega(c) \to \omega(c)$ is invertible is not surprising.

Step 2: $\boldsymbol{\nu}$ is admissible: By construction, $10^{N+3} B_1 e_1 11 B_2 e_2 11 \dots B_N e_N 11$ doesn't contain the block 0^{N+3} except the block starting at position 2. This means that

$$\sigma\nu \prec \sigma^k 10^{N+3} B_1 e_1 11 B_2 e_2 11 \dots B_N e_N 11 \prec \nu$$

for each $k \ge 2$. By condition (1), this means that $10^{N+3} B_1 e_1 11 B_2 e_2 11 \dots B_N e_N 11$ is admissible, and hence the kneading map implied by this block must satisfy (3). Since the length of the block $|10^{N+3} B_1 e_1 11 B_2 e_2 11 \dots B_N e_N 11|$ is a cutting time S_K by construction, and ν continues with $C'_1 C'_2 \dots$ so that $Q(k) = k - N$ for $k > K$, condition (3) also holds for the entire kneading sequence ν. So ν is admissible.

Step 3: $\boldsymbol{\# f^{-1}(x) \cap \mathrm{orb}^-(c) = 2}$ iff $\boldsymbol{e(x) = e(b) * \nu}$ for some $\boldsymbol{b \in \mathcal{B}^*}$: By construction, $B_i e'_i$ is a suffix of C'_i for each i. Since Therefore $Q(k) = k - N$ for $k > K$, $B_i e_i$ is a suffix of $\nu_1 \dots \nu_{S_{K+i}}$ and either $B_i e_i$ or $B_i e'_i$ is a suffix of $\nu_1 \dots \nu_{S_{K+jN+i}}$ for every $j \in \mathbb{N}$. Therefore there exists a sequence of iterates $n_j = S_{K+j}N - L$ such that $f^{n_j}(c) \to x \in \mathrm{orb}^-(c)$ and $e(x) = B_i * \nu$. It follows that for each $b \in \mathcal{B}^\sharp$, there is $x \in \omega(c)$ such that $e(x) = e(b) * \nu$, and this applies in particular to $b \in \mathcal{B}^*$.

Now for the only if direction, recall that $\#\mathcal{B}^* = N$, and $Q(k) = k - N$ eventually. By Lemma 2.2 this means that $\langle 0 \rangle$ has only N preimages, so $\# f^{-j}(c) \le \# g^{-1}\langle 0 \rangle \le N$ for all $j \ge 0$, and hence there can be at most N points in orb^{-1} with two preimages in $\omega(c)$.

Step 4: no point in $\boldsymbol{\omega(c) \setminus \mathrm{orb}^-(c)}$ has two preimages in $\boldsymbol{\omega(c)}$: Since $Q(k) = k - N$ for k sufficiently large, this follows immediately from Proposition 3.1. $\square$

5 Making $f_{|\omega(c)}$ invertible.

If we apply the previous sections to construct an example for which $f_{|\omega(c)}$ is one-to-one, we arrive at the Feigenbaum-like map (*i.e.*, $Q(k) = k-1$ for k sufficiently large).

The point of the "strange adding machines" paper [BKM] is creating unimodal maps for which $f_{|\omega(c)}$ is conjugate to an adding machine, even though f is not renormalizable. This shows that adding machines can be found in the standard tent family, which does not contain infinitely renormalizable maps. In this section, we show that how a minor adaptation of the cutting times can create maps for which $f_{|\omega(c)}$ is indeed one-to-one, without resorting to renormalizable maps. The method allows $k - Q(k)$ to be bounded, and is in fact compatible with $\omega(c)$ being a wild attractor, see [Br2], which in turn produces highly nontrivial maps with nontrivial Lebesgue maximal automorphic factors, cf. [BH].

Lemma 5.1. *Let $Q(k) \to \infty$ be the kneading map of a unimodal map, and let (E, g) be the corresponding enumeration scale. Suppose that there exists an increasing sequence $(k_i)_{i \in \mathbb{N}}$ such that for every i the following holds:*

$$Q(k) \leq k_i < k \textit{ implies } Q(Q(l) - 1) - 1 \neq k \textit{ for all } l \in \mathbb{N}. \tag{14}$$

Then $\#g^{-1}(\langle 0 \rangle) = 1$.

Example: Take

$$Q(k) = \begin{cases} 0 & \text{if } k \in \{1, 2, 4\}; \\ 1 & \text{if } k = 3; \\ 3l - 4 & \text{if } k = 3l - 1 \text{ or } 3l + 1 \text{ and } l \geq 2; \\ 3l - 2 & \text{if } k = 3l \text{ and } l \geq 2. \end{cases}$$

This results in the following table:

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	...
$Q(k)$		0	0	1	0	2	4	2	5	7	5	8	10	8	11	13	...
S_k	$\underline{1}$	2	$\underline{3}$	5	6	$\underline{9}$	15	18	$\underline{27}$	45	54	$\underline{81}$	135	162	$\underline{243}$	405	...

where we have underlined the powers of 3 to clarify the pattern. There is a factor map $\pi_{tri} : E \to \{0, 1, 2\}^{\mathbb{N}}$ onto the triadic adding machine defined by

$$\pi_{tri}(e)_k = \begin{cases} e_0 + 2(e_1 + e_3) & \text{if } k = 0; \\ e_{3k-1} + e_{3k} + 2(e_{3k+1} + e_{3k+3}) & \text{if } k \geq 1. \end{cases}$$

In the table below, we decompose the cutting times in sums of powers of 3, which shows that $\pi_{tri}(e)_k$ contains e_j as many times as 3^k appears in the triadic decomposition of S_j. Note also that the "add and carry" rule guarantees that $\pi_{tri}(e)_k \le 2$.

k	$Q(k)-k$	S_k	1	3	9	27	81	243	729
0		1	1	0	0	0	0	0	0
1	-1	2	2	0	0	0	0	0	0
2	-2	3	0	1	0	0	0	0	0
3	-2	5	2	1	0	0	0	0	0
4	-4	6	0	2	0	0	0	0	0
5	-3	9	0	0	1	0	0	0	0
6	-2	15	0	2	1	0	0	0	0
7	-5	18	0	0	2	0	0	0	0
8	-3	27	0	0	0	1	0	0	0
9	-2	45	0	0	2	1	0	0	0
10	-5	54	0	0	0	2	0	0	0
11	-3	81	0	0	0	0	1	0	0
12	-2	135	0	0	0	2	1	0	0
13	-5	162	0	0	0	0	2	0	0
14	-3	243	0	0	0	0	0	1	0
15	-2	405	0	0	0	0	2	1	0

The map π_{tri} is clearly continuous and since it maps $\langle \mathbb{N} \rangle$ onto the triadic representation of the integers, π_{tri} extends to a surjective map. Let us show by contradiction that π_{tri} is injective. Suppose that $\pi_{tri}(e) = \pi_{tri}(\tilde{e})$, and let j be minimal such that $e_j \ne \tilde{e}_j$. Since the contributions of e and $\tilde{e}$ to each power of 3 is the same, we have two cases (after changing the role of e and $\tilde{e}$ if necessary):

- The contribution of S_j to 3^k is 1. Then $j = 3k-1$, $e_j = 1$, $\tilde{e}_j = 0$, but $\tilde{e}_{j+1} = 1$. Then $\tilde{e}$ contributes 2 to 3^{k-1}, and $e_{j-1} = 1$ to match that contribution. But $e_{j-1} = 1$ and $e_j = 1$ would give a carry.
- The contribution of S_j to 3^k is 2. Then $j = 3k+1$, $e_j = 1$, $\tilde{e}_j = 0$, but $\tilde{e}_{j+2} = 1$. Then $\tilde{e}$ contributes 1 to 3^{k+1}, and $e_{j+1} = 1$ to match that contribution. But $e_j = 1$ and $e_{j+1} = 1$ would give a carry.

It follows that π_{tri} is indeed a conjugacy between (E, g) and the triadic adding machine. One can check that both Lemma 5.1 (namely for $k_i = 3i+1$) and Lemma 3.2 apply, so we can conclude that $\pi_{tri} \circ \pi^{-1}$ conjugates $f_{|\omega(c)}$ is to the triadic adding machine, while at the same time f is nonrenormalizable, $k - Q(k)$ is bounded and $Q(k+1) > Q^2(k)+1$ for k sufficiently large. It follows from [Br2, Theorem 6.1] that if f is a unimodal map with this kneading map and sufficiently large critical order, then $\omega(c)$ is a wild attractor.

Proof of Lemma 5.1. Any maximal sequence e satisfies

$$\text{If } e_k = 1, \text{ then } e_{Q(k)-1} = 1 \text{ and } e_j = 0 \text{ for } Q(k+1) \le j < k. \tag{15}$$

This is just a rephrasing of Lemma 2.2. Suppose that e is maximal with $e_{k_i} = 0$ for some i. Let $k = \min\{j > k_i : e_j = 1\}$. Then $e_{Q(k+1)-1} = 1$ by maximality of e. By minimality of k, $Q(k+1) - 1 < k_i$, so $Q(k+1) \leq k_i$. By (14) this means that there is no l such that $Q(Q(l) - 1) - 1 = k$. Combine this with (15), this shows that e cannot be maximal.

Hence for any maximal sequence e and any i, $e_{k_i} = 1$. Using (15) once more, this determines e uniquely. □

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Department of Mathematics
University of Surrey
Guildford, Surrey, GU2 7XH, UK
h.bruin@surrey.ac.uk
http://personal.maths.surrey.ac.uk/st/H.Bruin/