MINIMAL CANTOR SYSTEMS AND UNIMODAL MAPS

H. BRUIN

Dedicated to A.N. Sharkovskiĭ

Abstract. We consider the restriction of unimodal maps f to the omega-limit set $\omega(c)$ of the critical point for certain cases where $\omega(c)$ is a minimal Cantor set. We investigate the relation of these minimal systems to enumeration scales (generalized adding machines), to Vershik adic transformations on ordered Bratteli diagrams and to substitution shifts. Sufficient conditions are given for $(\omega(c), f)$ to be uniquely ergodic.

1. Introduction

Let $f : [0,1] \to [0,1]$ be a unimodal map with critical point c. Crucial data in the description of f, as well as of several other objects in this paper, are the *cutting times*. This is an increasing sequence (S_k) of integers, which we define in the next section. But for the occasion, there is an equivalent way of defining them in terms of periods of periodic points: each S_k is the period of a periodic point p with the property that no periodic point in $(p, f(c)]$ has lower period.

If $S_k - S_{k-1} \to \infty$, the critical omega-limit set $\omega(c)$ is a minimal Cantor set. Except for their intrinsic combinatorial and ergodic properties, the system $(\omega(c), f)$ is studied because under certain conditions it can be the attractor of the whole system $([0,1], f)$, see [10, 6].

In this note, we want to explore the relation between the minimal system $(\omega(c), f)$, the enumeration scale (A, T) based on (S_k), Vershik adic transformations (X, τ) and substitution shifts (Σ, σ). The latter two can serve as models for general minimal Cantor systems, but we will only describe them as far as our interests in unimodal maps go. This is the contents of Sections 2-5

One of the questions we would like to answer is when $(\omega(c), f)$ is uniquely ergodic; whereas a complete answer is not known to us, in Section 6 we give sufficient conditions for $(\omega(c), f)$ to be uniquely ergodic, as well as examples where $(\omega(c), f)$ is not uniquely ergodic.

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The system $(\omega(c), f)$ generates a subshift of the kneading theory: to each $x \in \omega(c)$ attach the *itinerary* $(y_i)_{i \geq 0}$, where $y_i = 0$ if $f^i(x) \in [0, c]$ and $y_i = 1$ if $f^i(x) \in [c, 1]$. (The ambiguity in itinerary for $x \in \cup_{n \geq 0} f^{-n}(c)$ is conventional.) Let us call the resulting subshift (Y, σ). It is not hard to show that for any minimal subshift Y_0 of $\{0,1\}^{\mathbb{N}}$, there is a sequence of cutting times (not necessarily satisfying $S_k - S_{k-1} \to \infty$), such that $Y = Y_0$. The *complexity* of (Y, σ), *i.e.* the number $p(n)$ of different n-words, as function of the combinatorics of f, remains a poorly understood area. Except that in our setting $p(n)$ grows subexponentially (see Proposition 1, part 4.) and the proven linear growth of some particular cases, little is known.

2. Unimodal Maps

Let $f : [0,1] \to [0,1]$ be a unimodal map, *i.e.,* f is a continuous map with a single critical point $c \in (0,1)$ such that $f|[0,c]$ is increasing and $f|[c,1]$ is decreasing. We write $c_n := f^n(c)$, the n-th image of the critical point. The omega-limit set of c is given by:

$$\omega(c) = \{x \in [0,1] \mid \exists n_i \to \infty, \ f^{n_i}(c) \to x\}.$$

It is well-known that $\omega(c)$ is compact and strongly invariant: $f(\omega(c)) = \omega(c)$. In general, $\omega(c)$ can be a cycle of intervals, a Cantor set, a countable (or finite) set, or the union of a Cantor set and a countable set, see [5]. We are interested in the case that $\omega(c)$ is a *minimal* Cantor set, *i.e.,* $\omega(c)$ is a Cantor set and $\mathrm{orb}(x)$ is dense in $\omega(c)$ for every $x \in \omega(c)$.

There are various constructions leading to minimal Cantor sets, *e.g.* infinitely renormalizable maps (for example the Feigenbaum-Tresser-Coullet map), maps connected to circle rotations (see [16] and [9, Section 3.6]). Here we use a class of unimodal maps, advocated in *e.g.* [8] (see also [11]), namely maps with kneading map tending to infinity. This class includes the so-called Fibonacci unimodal map, which has been very well studied for its extreme metric and measure theoretical properties, [19, 22, 10].

To describe this class of maps, let us recall a construction due to Hofbauer, [18] and [8]. For each n and each maximal closed interval $J \subset [0,1]$ on which $f^n|J$ is monotone, $f^n : J \to f^n(J)$ is called a *branch* of f^n. Obviously, there are two branches $f^n : J \to f^n(J)$ such that $c \in \partial J$; these are the *central branches* of f^n. The integer n is called a *cutting times* if $c \in f^n(J)$ for (one of) the central branches. (Since the central branches have the same image for $n \geq 2$, we need not specify which central branch is used for those n.) The cutting times will be denoted as $(S_i)_{i \geq 0}$, where

$$1 = S_0 < S_1 < S_2 < \dots$$

Assuming that f has no periodic attractor, f has infinitely many cutting times. It can be shown ([18, 8]) that $S_k - S_{k-1}$ is again a cutting time. We denote it by $S_{Q(k)}$. This gives rise to the so-called kneading map $Q : \mathbb{N} \to \mathbb{N}$. (We take $\mathbb{N} = \{0, 1, 2, \dots\}$; by convention, put $Q(0) = 0$.) Obviously, the sequence of cutting times determines the kneading map and vice versa. Moreover, we have the following admissibility result:

Lemma 1. *There exists a unimodal map with kneading map Q if and only if*

$$\{Q(k+j)\}_{j\geq 1} \succeq \{Q(Q^2(k)+j)\}_{j\geq 1}, \tag{1}$$

where $\succeq$ indicates lexicographical order and $Q^2 = Q \circ Q$.

Proof. See [18, 8]. □

It follows immediately that every non-decreasing kneading map is *admissible i.e.,* corresponds to an actual unimodal map. Some well-known examples of admissible kneading maps are

- $Q(k) = k-1$, which yields the Feigenbaum map, and $S_k = 2^k$.
- $Q(k) = \max\{0, k-2\}$, yields the so-called *Fibonacci unimodal map.* The cutting times are the Fibonacci numbers: $1, 2, 3, 5, 8, 13, \dots$

Take $D_1 := [c_2, c_1]$. For $n \geq 2$, let $D_n = f^n(J)$ be the image of the central branch(es) of f^n. It is a closed interval with endpoints c_n and $c_{\beta(n)}$, where $\beta(n) = n - \max\{S_k \mid S_k < n\}$, see [8]. The following proposition focuses on the class of maps we want to consider in this paper.

Proposition 1. *Let f be a unimodal map having no* homtervals *(*i.e., *for each non-degenerate interval J there exists m such that $f^m(J) \ni c$). Assume that the kneading map Q of f satisfies* $\lim_{k\to\infty} Q(k) = \infty$*. Then*

(1) $|D_n| \to 0$ *as $n \to \infty$;*
(2) $\omega(c)$ *is a minimal Cantor set;*
(3) *c is recurrent and* persistently recurrent, i.e., *for every neighborhood $U \ni c$, there are only finitely many integers i such that $f^{i-1} : V_i \to U$ is monotone onto for some neighborhood V_i of c_1;*
(4) *$(\omega(c), f)$ has zero topological entropy.*

Proof. The first three items are demonstrated in [8]. The last item is due to Blokh & Lyubich [4]. In a nutshell, Blokh & Lyubich defined $r_n(x) = |f^n(x) - \partial f^n(J)|$, where J is the branch domain of f^n containing x. If $r_n(x) \to 0$ for all $x \in \omega(c)$, they show, using a result of Ledrappier [20], that the topological entropy of $(\omega(c), f)$ is 0. If $\omega(c)$ is a *Cantor attractor* (see [4]), then indeed $r_n(x) \to 0$ for $x \in \omega(c)$. In [6], it is shown that $Q(k) \to \infty$ implies $r_n(x) \to 0$ uniformly on $\omega(c)$. □

A subclass of the maps with kneading map $Q(k) \to \infty$ are the infinitely renormalizable maps, for which we have the following characterization in terms of kneading maps:

Lemma 2. *A unimodal map f with kneading map Q is renormalizable of period n if and only if $n = S_k$ is a cutting time, and $Q(l) \geq k$ for all $l > k$. As a consequence, f is infinitely renormalizable if there exists a sequence (k_i) such that for all i, $Q(l) \geq k_i$ for all $l > k_i$.*

Proof. See [8]. □

Remark: The sequence $(S_k)_k$ is also extremely useful to describe the combinatorial structure of quadratic polynomials $p(z) = z^2 + c_1$ in the complex plane. In this case $(S_k)_k$ is called the *internal address* [23]; it is a strictly increasing sequence of integers, which coincides with the cutting times if $c_1 \in \mathbb{R}$, but for the general case the map Q cannot be defined, and also *admissibility* observes different rules. There is a complex analog of renormalization: if for some k, S_{k+i} is a multiple of S_k for all $i \geq 1$, then p is complex renormalizable of period S_k. On the other hand, the condition $S_k - S_{k-1} \to \infty$ by itself does not imply minimality of $\omega(0)$ in the complex case. We do not know of general conditions on the internal address that do imply minimality.

To give an idea how fast S_k grows as function of Q, we give some examples in the next lemma.

Lemma 3. *Let $S_0 = 1$ and $S_k = S_{k-1} + S_{Q(k)}$.*

(1) *If $k - Q(k)$ is bounded, then S_k grows exponentially.*
(2) *If $Q(k) = \lfloor k - \alpha k^\beta \rfloor$ for some $\beta \in (0,1)$, then S_k grows in a stretched exponential way, and* $\log(S_k)$ *has leading term* $\frac{\beta}{1-\beta} k^{1-\beta} \log k$.
(3) *If $Q(k) = \lfloor \alpha k \rfloor$, then S_k grows superpolynomially but slower than stretched exponentially: for $r \geq 1$ and $\beta \in (0,1)$ we have* $\lim_k S_k/k^r = \infty$ *and* $\lim_k S_k/e^{k^\beta} = 0$.
(4) *If $Q(k) = \lfloor a k^\beta \rfloor$ for some $a > 0$ and $\beta \in (0,1)$, then S_k grows polynomially, with leading term $k^{1/(1-\beta)} \log^\alpha k$ for $\alpha = -\frac{\log a}{\log \beta}$.*

Proof. 1. For the first statement, assume $Q(k) = \max\{k - B, 0\}$. Then $S_k \approx \lambda^k$, where $\lambda > 1$ is the leading root of the characteristic equation $x^B = x^{B-1} + 1$ of the recurrence relation. For any kneading map Q such that $k - Q(k) \leq B$, the growth is at least as fast.

2. The growth-rate of a smooth function $g(k)$ is best comparable to S_k if g satisfies $g'(t) = g(t - t^\beta)$. Assume that $\log g(t) = rt^\gamma \log^\alpha t + l.o.t.$, where the lower order terms $l.o.t.$ tends to 0 as $t \to \infty$ when divided by $t^\gamma \log^\alpha t$. Then

$$\begin{aligned} g(t - t^\beta) &= \exp[r(\log^\alpha t + \log^\alpha(1 - t^{\beta-1}))(t - t^\beta)^\gamma + l.o.t.] \\ &= \exp[r(t^\gamma \log^\alpha t - \gamma t^{\gamma+\beta-1} \log^\alpha t - t^{\gamma+\alpha(\beta-1)}) + l.o.t.], \end{aligned}$$

while

$$g'(t) = \exp[rt^\gamma \log^\alpha t + (\gamma - 1)\log t + l.o.t.]$$

Therefore, neglecting the $l.o.t.$, we get $\gamma = 1 - \beta$, $\alpha = 1$ and $r = (1-\gamma)/\gamma = \beta/(1-\beta)$.

3. The growth-rate of S_t is comparable to the growth-rate of $f(t)$, where $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is the solution of the differential equation $f'(t) = f(\alpha t)$ with initial condition $f(0) = 1$. Writing f as formal power series $f(t) = \sum_{n \geq 0} a_n t^n$,

we find $a_0 = 1$ and the recurrence relation $a_n = \frac{1}{n}\alpha^{n-1}a_{n-1}$. This gives $a_n = \frac{1}{n!}\alpha^{\frac{n(n-1)}{2}}$.

Clearly $\alpha^{\frac{n(n-1)}{2}} > 0$, and for each $\gamma > 0$, $\alpha^{\frac{n(n-1)}{2}} < \gamma^n$ for n sufficiently large. Therefore, the formal power series converges for all $t \geq 0$, and $f(t)$ grows faster than any polynomial, but slower than any exponential function. For the comparison to $g(t) = e^{pt^{1/q}}$ for fixed $p > 0$ and $0 < q \in \mathbb{N}$, let us write the Taylor series

$$\begin{aligned} g(t) &= \sum_{n\geq 0} \frac{1}{n!} p^n t^{n/q} = \sum_{n\geq 0} t^n \sum_{i=0}^{q-1} \frac{1}{(qn+i)!} p^{qn+i} t^{i/q} \\ &\geq \sum_{n\geq 0} \frac{1}{(qn)!} p^{qn} t^n =: \sum_{n\geq 0} b_n t^n. \end{aligned}$$

Using Stirling's formula, we compute

$$\begin{aligned} \log \frac{a_n}{b_n} &= \log \frac{(qn)!\alpha^{\frac{n(n-1)}{2}}}{n!p^{qn}} \leq \log \frac{(qn)^{qn+1}\alpha^{\frac{n(n-1)}{2}}e^n}{n^n e^{qn} p^{qn}} \\ &= (1+qn)\log qn + \frac{n(n-1)}{2}\log\alpha + n(1-\log n - q - \log p) \\ &\to -\infty, \end{aligned}$$

as $n \to \infty$. Hence $a_n/b_n \to 0$ as $n \to \infty$.

4. Assume that the function $g(t)$ interpolates (S_k) and that $g(t)$ has leading term $t^\gamma \log^\alpha t$, *i.e.*, $\lim_t (g(t) - t^\gamma \log^\alpha t)/(t^\gamma \log^\alpha t) = 0$. Then we get, for some $\xi \in [k-1, k]$,

$$\begin{aligned} g(\lfloor ak^\beta \rfloor) &= S_{Q(k)} = S_k - S_{k-1} = g'(\xi) \\ &= \xi^{\gamma-1}\log^\alpha(\xi) + \xi^{\gamma-1}\alpha\log^{\alpha-1}\xi + l.o.t., \end{aligned}$$

where $l.o.t.$ indicate the lower order terms. Neglecting the difference between k and ξ (it falls under the $l.o.t.$) we need to solve $ak^{\gamma\beta}\beta^\alpha \log^\alpha k = (\log^\alpha k + \alpha \log^{\alpha-1} k)k^{\gamma-1} + l.o.t.$ Therefore $\gamma = 1/(1-\beta)$ and $\alpha = -\log a/\log\beta$ as claimed. □

3. Enumeration scales

Let us start explaining what enumeration scales are. We follow the terminology from [1]; other terms used are (generalized) adding machine or adic transformation, *e.g.* [11].

Given an increasing sequence of integers $(S_k)_{k\geq 0}$, we can construct a *greedy* representation $a = a_0a_1a_2\dots$ of any integer $n_0 \geq 0$ as follows. Start with the sequence a of all zeroes. Take the maximal $S_k \leq n_0$, replace a_k with $a_k = \lfloor n_0/S_k \rfloor$ and continue with the remainder $n_1 := n_0 - a_kS_k$. That is, find k' maximal such that $S_{k'} \leq n_1$ and let $a_{k'} = \lfloor n_1/S_k \rfloor$, etc. After a finite number of steps, $n_i = 0$ and $n_0 = \sum_k a_kS_k$. We write $\langle n \rangle = a$.

Let $A^* = \{\langle n \rangle \mid n \geq 0\}$ be the set of greedy representations of nonnegative integers. Obviously, each $a \in A^*$ is an infinite string of integers, with only finitely many nonzero entries. Let A be the closure of A^* in the product topology on $\mathbb{N}^{\mathbb{N}}$.

We think of the case where the S_k are the cutting times of a unimodal map. This implies that $S_k \leq 2S_{k-1}$ for each k, so for each $a \in A$ and $i \geq 0$, $a_i \in \{0, 1\}$. In the language of [1], this gives rise to a *low* enumeration scale (*échelle basse*). As was shown in [11],

$$A = \{a \in \{0,1\}^{\mathbb{N}} \mid a_k = 1 \Rightarrow a_j = 0 \text{ for } Q(k+1) \leq j < k\}. \tag{2}$$

General number systems of this construction are of course possible, see [17, 1, 2] for more information. From now on, let (S_k) be a sequence of cutting times.

Lemma 4. *The map $T : A^* \to A^*$ defined by $T(\langle n \rangle) = \langle n+1 \rangle$ is uniformly continuous if and only if $Q(k) \to \infty$.*

Proof. See [11] or [17]. □

This allows us to extend T to A by continuity. It acts by "add and carry" on the sequences $a \in A$. One can show that T is invertible, except (possibly) at $\langle 0 \rangle = 0, 0, 0, \dots$ For example, if (S_k) is the sequence of Fibonacci numbers, then $A = \{a | a_k = 1 \Rightarrow a_{k+1} = 0\}$, and there are two maximal sequences $a = 1, 0, 1, 0, 1, 0, 1, \dots$ and $a' = 0, 1, 0, 1, 0, 1, 0, 1, \dots$ such that $T(a) = T(a') = 0, 0, 0, \dots$ In general, $\#T^{-1}(\langle 0 \rangle)$ equals the number of infinite paths in the *arbre de retenues* of [1]. In [7, Lemma 3] a condition is given on (S_k) that renders the map T invertible, and in [5] it is shown that this condition is necessary and sufficient.

If (A, T) is a number system based on the sequence of cutting times (S_k) and the corresponding kneading map $Q(k) \to \infty$, then (A, T) factorizes over $(\omega(c), f)$ as follows:

Lemma 5. *Let $h : A^* \to \operatorname{orb}(c)$ be defined as $h(\langle n \rangle) = c_n$. Then h is uniformly continuous. The extension $h : A \to \omega(c)$ is continuous onto and satisfies $h \circ T = f \circ h$.*

Proof. See [11, Theorem 1]. There it is also shown that if $a \in A \setminus A^*$, then

$$h(a) = \cap_i D_{g(i)}, \text{ for } g(i) = \sum_{j=0}^{i} a_j S_j. \tag{3}$$

It should be remarked that the intervals D_n have a nested structure in the sense that $D_n \subset D_{\beta(n)}$ for each $3 \leq n \notin (S_k)_k$, see [11, Lemma 5]. Therefore the intersection in (3) is an intersection of nested intervals. In fact, a stronger property holds is easy to prove: If $Q(k)$ is non-decreasing and $k < l$, then $D_{1+S_k} \subset D_{1+S_l}$. For later use, we state the corollary:

$$\text{If } n > m \notin (S_k) \text{ are such that } \beta(m) = \beta(n), \text{ then } D_m \subset D_n. \tag{4}$$

To see this, let $K = \beta(m) = \beta(n)$ and observe that D_{m-K} and D_{n-K} are the last levels before D_m resp. D_n containing the critical point. By the above statement, $D_{n-K+1} \subset D_{m-K+1}$, and since $f^{K-1}|D_{m-K+1}$ is monotone, (4) follows. □

In general the map $h : A \to \omega(c)$ is not one-to-one. As an example, if $Q(k) = \max\{0, k-3\}$, then 0 has two preimages under f in $\omega(c)$, but $\langle 0 \rangle$ has three preimages under T. Hence, at least two preimages of $\langle 0 \rangle$ must be mapped to the same preimage of 0 in $\omega(c)$. In [7, Theorem 2] conditions are given for h to be one-to-one, provided T is one-to-one. The techniques of [7] however extend to show when h is injective on $A \setminus \cup_{n>0} T^{-n}(\langle 0 \rangle)$. This can be used to construct examples of kneading maps such that h and $f|\omega(c)$ are one-to-one except for a finite or countable set.

In [11] this is a key construction to find topological groups as factor of $(\omega(c), f)$. Among other things, it is shown that for the kneading maps $Q(k) = \max\{0, k-d\}$ for $d \in \{2, 3, 4\}$, $(\omega(c), f)$ is semiconjugate to a rotation on a $d-1$-dimensional torus, while if $d \geq 5$, $(\omega(c), f)$ is weakly mixing.

4. Bratteli diagrams and Vershik adic transformations

An *ordered Bratteli diagram* is an infinite graph consisting of

- a sequence of nonempty collections of vertices V_i, $i \geq 1$, such that V_1 consists of a single vertex v_0;
- a sequence of collections of edges E_i, $i \geq 1$, such that each edge $e \in E_i$ connects a vertex $\mathbf{s}(e) \in V_i$ to a vertex $\mathbf{t}(e) \in V_{i+1}$. For every $v \in V_i$, there exists at least one *outcoming edge* $e \in E_i$ with $v = \mathbf{s}(e)$, and for every $v \in V_{i+1}$ there exists at least one *incoming edge* $e \in E_i$ with $v = \mathbf{t}(e)$;
- for each $v \in \cup_{i \geq 2} V_i$, a total order $<$ between the incoming edges to v.

To each E_i we associate an incidence matrix M^i of size $\#V_i \times \#V_{i+1}$, where $M^i_{k,l}$ the number of edges from the k-th vertex in V_i to the l-th vertex in V_{i+1}. Instead of taking the collections of edges E_i separately, we can also consider all the paths from V_i to V_{j+1} for some $j \geq i$. We denote this collection of paths by $E_{i,j}$. The incidence matrix associated to $E_{i,j}$ can be shown to be the matrix product $M^{i,j} = M^i \cdots M^j$. This process is called *telescoping.* It will turn out useful to telescope Bratteli diagrams in such a way that all incidence matrices become strictly positive. This is possible if and only if, for every i, there exists $j \geq i$ such that for every $v \in V_i$ and $w \in V_{j+1}$, there is a path from v to w. In the Bratteli diagrams that we will study, this is always satisfied.

Let X be the collection of all infinite paths starting from v_0, endowed with product topology. If $x = x_1 \dots x_N$ and $y = y_1 \dots y_N$, with $x_i, y_i \in E_i$ are finite paths, then we can compare x and y if they have the same endpoint in V_{N+1}. Let $m < N$ be the largest index such that $x_m \neq y_m$. This means that $\mathbf{t}(x_m) = \mathbf{t}(y_m)$. In this case we say that $x < y$ if $x_m < y_m$. This gives a partial order on the set of N-paths.

For every $v \in V_{i+1}$ there is a unique minimal path from v up to v_0, and at least one $e \in E_{i+1}$ with $\mathbf{s}(e) = v$. From this it follows that there are infinite paths $x \in X$ such that the initial N-path $x^{\min}_{[1,N]}$ is minimal among all N-paths with the same terminal vertex. Let us assume, and this is true in all the examples we encounter in this paper, that this minimal infinite path is unique, and denote it by $x^{\min}$.

The Vershik adic transformation $\tau : X \to X$ is defined as follows [24, 25]: For $x \in X$, let i be minimal such that $x_i \in E_i$ is not the maximal incoming edge. Then put $\tau(x)_j = x_j$ for $j > i$, let $\tau(x)_i$ be the successor of x_i among the incoming edges of this level, and let $\tau(x)_1 \dots \tau(x)_{i-1}$ be the minimal path connecting v_0 with $\mathbf{s}(\tau(x)_i)$. If no such i exists, then x is a maximal path in X, and we set $\tau(x) = x^{\min}$. Assuming that $\#V_i < \infty$ for all i, it can be shown that $\tau : X \to X$ is continuous onto, and invertible, except that $x^{\min}$ can have several preimages, [14].

Proposition 2. *The enumeration scale* (A, T) *based on the sequence of cutting times* $(S_k)_{k \geq 0}$*, with kneading map* Q*, is isomorphic to the following ordered Bratteli diagram:*

- $V_1 = \{v_0\} = \{1\}$*, and* $V_i = \{k \geq i \mid Q(k-1) + 1 < i\}$ *for* $i \geq 2$*;*
- $E_i = \{i \to i+1\} \sqcup \{k \to k \mid i < k \in V_i\} \sqcup \{i \to k \mid k \in V_{i+1}, Q(k-1) + 1 = i\}$ *for* $i \geq 1$*;*
- $(i \to i+1) < (i+1 \to i+1)$*. (Note that* $i+1 \in V_{i+1}$ *is the only vertex with more than one incoming edge.)*

In addition, give the edge $(i+1 \to i+1) \in E_i$ *the label* 1*. If the edge* $(i+1 \to i+1)$ *does not exist in* E_i*, then* E_i *contains two edges* $i \to i+1$*; one gets the label* 0 *the other one* 1*. All other edges in* E_i *get the label* 0*. Write* $l(x) = a_0 a_1 a_2 \cdots \in \{0,1\}^{\mathbb{N}}$ *with* a_i *the label of* x_{i+1}*, then every path* $x \in X$ *has a unique infinite label. There is a unique minimal path* $x^{\min} = 1 \to 2 \to 3 \to \dots$ *with label* $0000\dots$*, and* $l \circ \tau(x) = T \circ l(x)$*, see Figure 1.*

Let us call the minimal path $x^{\min} = 1 \to 2 \to 3 \to \dots$ the *spine* of the Bratteli diagram. It passes through $i \in V_i$ for each i. The vertex i is called the *first vertex* of V_i. The Bratteli diagram thus consists of a spine and, for each k, paths from first vertex $Q(k-1)+1$ to first vertex k. The last edge of these paths is labeled 1, all other edges have label 0.

Proof. Each vertex $k \in V_{i+1}$ has one incoming edge $k \to k$ or $i \to k$ if $Q(k-1) + 1 = i$. Only $i + 1 \in V_{i+1}$ has two incoming edges $i \to i+1$ and $i+1 \to i+1$, the latter with label 1. (If $Q(i+1) = i$, then instead of $i+1 \to i+1$, the second incoming edge is $i \to i+1$, so there are two edges

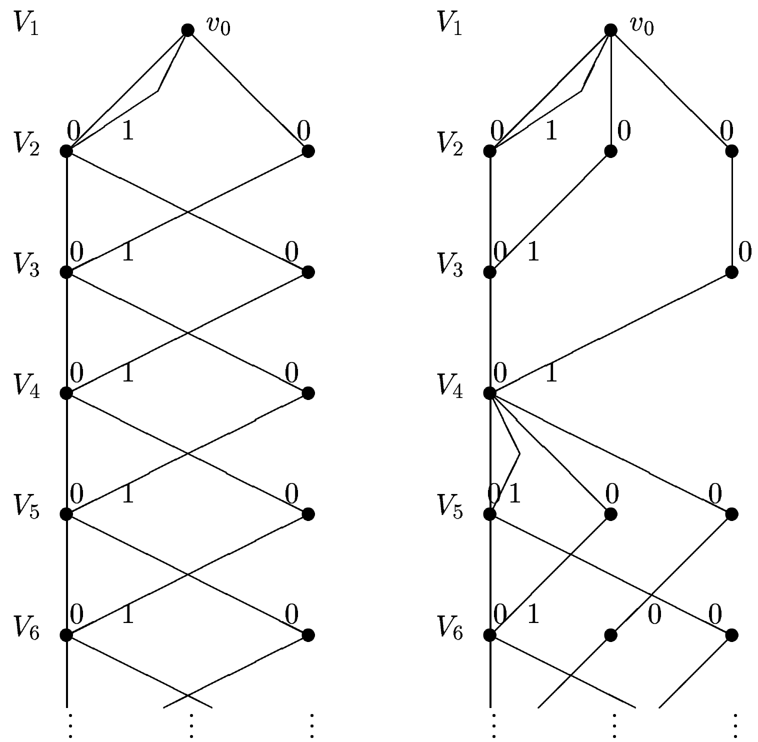

FIGURE 1. The Fibonacci Bratteli diagram (left) and a renormalizable diagram (right)

$i \to i+1$ one with label 1 and one with label 0, see Figure 1.) All other edges have label 0.

Every vertex $k \in V_{i+1}$ has at least one outgoing edge, either $k \to k$ if $k > i+1$, or $k \to k+1$ if $k = i+1$. In the latter case, additional edges $k \to l$ are possible if $Q(l-1)+1 = k = i+1$. Therefore each enumeration scale yields a bona fide Bratteli diagram. (Note also that $Q(k) \to \infty$ implies $\#V_{i+1} < \infty$ for all i.)

Clearly every path has a unique label. On the other hand, each infinite path passes through the first vertex $k \in V_k$ for infinitely many k. If x and y are different path, there must be at least one index i such that x_i has label 1 (and $\mathbf{t}(x_i) = i+1 \in V_{i+1}$) while y_i is labeled 0, or vice versa. Hence different paths have different labels.

Now if x_i is labeled 1, then for $Q(i)+1 \leq j \leq i$ we have $\mathbf{t}(x_j) = i+1$ and hence x_j is labeled 0 for $Q(i)+1 \leq j < i$. Now let $a_p = l(x_{p+1})$ and rewrite $i = p+1$, $j = q+1$, then we get $a_p = 1$ implies $a_q = 0$ for $Q(p+1) \leq q < p$. This is precisely condition (2) in the characterization of A. Hence $l : X \to A$ is one-to-one onto.

Finally, we need to show that $l \circ \tau(x) = T \circ l(x)$. We use the following claim: Given $i \geq 1$, let $x \in X$ be the path such that $x_1 \dots x_i$ is the maximal path connecting v_0 and first vertex $i+1 \in V_{i+1}$. Let $a = a_0 a_1 a_2 \dots a_{i-1}$ be its label. Then

$$S_i = 1 + \sum_{j=0}^{i-1} a_j S_j. \tag{5}$$

We show this by induction. Indeed, for $i = 1$, we have $x = 1 \to 2$ with label 1, and $S_1 = 2 = 1 + \sum_{j=0}^{0} a_j S_j = 1 + S_0$.

Suppose now that (5) is true for all $j < i$, and let $x_1 \dots x_i$ be the maximal path connecting v_0 and $i+1 \in V_{i+1}$. Then x_i has label 1, and following the path upwards, it passes a vertex $j \in V_j$ for the first time for $j = Q(i) + 1$. The path $x_1 \dots x_{j-1}$ is maximal connecting v_0 to $j \in V_j$, so by induction

$$\begin{aligned} 1 + \sum_{k=0}^{i-1} a_k S_k &= S_{i-1} + 1 + \sum_{k=0}^{Q(i)-1} a_k S_k \\ &= S_{i-1} + S_{Q(i)} = S_i. \end{aligned}$$

This proves the induction step.

To finish the proof, let x be any path, and let $j \geq 1$ be the minimal index such that x_j is not the maximal incoming edge. Then $x_1 \dots x_{j-1}$ is the maximal path connecting v_0 to $\mathbf{s}(x_j)$. Let $a = l(x)$ and $b = l \circ \tau(x)$. By (5), $\sum_{k=0}^{j-1} a_k S_k = \sum_{k=0}^{j-2} a_k S_k = S_{j-1} - 1$. The edge $\tau(x)_j$ is the second (is maximal) incoming edge to $\mathbf{t}(x_j)$, whereas $\tau(x)_1 \dots \tau(x)_{j-1}$ is the minimal path connecting v_0 to $\mathbf{s}(\tau(x)_j)$. Therefore $\sum_{k=0}^{j-1} b_k S_k = b_{j-1} S_{j-1} = S_{j-1}$. Thus $T(a) = b$.

Finally, if no minimal index j exists, x is a maximal sequence in X. Thus $\tau(x) = x^{\min}$. At the same time, $\sum_{k=0}^{j-1} a_k S_k = S_{j-1} - 1$ for every j such that $l(x_j) = 1$. It follows that $T(a) = \langle 0 \rangle$ which is indeed equal to $l \circ \tau(x)$. $\square$

Remarks: 1. Condition (5) implies that the number of paths connecting v_0 with the first index $i+1 \in V_{i+1}$ is S_i.
2. If f is renormalizable, say with period $n = S_k$, then V_{k+1} consists of a single vertex, see Figure 1. The Bratteli diagram of the renormalization can be found by cutting off the upper part $(V_i, E_i)_{i \leq k}$ from the original Bratteli diagram. If f is infinitely renormalizable, there will be an infinite sequence of levels V_i consisting of single vertices. Telescoping between these vertices gives rise to a Bratteli diagram for which $\#V_i = 1$ and $\#E_i \geq 2$ for each $i \geq 1$. This is the standard form of a Bratteli diagram of an odometer, and indeed $(\omega(c), f)$ is isomorphic to an odometer in this case. These systems are well-known to be uniquely ergodic.
3. Any enumeration scale can be translated into a Bratteli diagram, see *e.g.* [1, 17]. The interest of this theorem is rather the precise description of the Bratteli diagram, which in fact can easily be generalized.

5. Substitution shifts

Substitutions are a common way to build minimal subshift; in fact, if we are allowed to build the substitution shift on a countable collection of substitutions, then every minimal Cantor system can be expressed in this way. For each $i \geq 2$, let V_i be a finite alphabet, and let V_i^* denote the collection of finite words in this alphabet. For $i \geq 1$, let $\chi_i : V_{i+1} \to V_i^*$ be a substitution; hence, to each $v \in V_{i+1}$, χ_i assigns a string of letters from V_i. The substitution acts on strings by concatenation:

$$\chi_i(v_1 v_2 \dots v_N) = \chi_i(v_1)\chi_i(v_2)\dots\chi_i(v_N).$$

To each substitution χ_i we associate the incidence matrix M^i of size $\#V_i \times \#V_{i+1}$, where the entry $M^i_{k,l}$ denotes the number of appearances of the k-th letter from V_i in the χ_i-image of the l-th letter of V_{i+1}. By iterating the substitutions, we can construct an infinite string

$$s = \lim_{i\to\infty} \chi_2 \circ \chi_3 \circ \dots \chi_i(v),$$

where v is taken from V_i. Using the irreducibility conditions:

$$\forall i \; \forall w \in V_i \; \exists j > i \; \forall v \in V_{j+1} \, \chi_i \circ \dots \circ \chi_j(l) \text{ starts with } w,$$

which holds for all our results, the limit can be shown to exist, independently of the choice of $v \in V_i$. Moreover, s is a uniformly recurrent string in $V_2^{\mathbb{N}}$. It generates a minimal subshift (Σ, σ), where σ is the left-shift and $\Sigma = \overline{\sigma^n(s)}$, the closure taken with respect to product topology.

The best-known examples are of course the *stationary* substitutions, *i.e.,* $V_i \equiv V$ and $\chi_i \equiv \chi$. For example, the Fibonacci substitution acts on the alphabet $\{0, 1\}$ by

$$\chi : \begin{cases} 0 \to 01 \\ 1 \to 0 \end{cases}$$

and $s = 0100101001001\dots$. This string is equal to the string of first labels of $\{\tau^n(x^{min})\}_{n\geq 0}$ from the Fibonacci Bratteli diagram from Figure 1. As a result, the Fibonacci substitution is isomorphic to the Fibonacci Bratteli diagram, which in its turn is isomorphic to the Fibonacci enumeration scale.

The following result is well-known, see *e.g.* [14].

Lemma 6. *Every substitution shift such that each letter $v \in V_i$, $i \geq 1$ appears in some word $\chi_i(w)$, $w \in V_{i+1}$, is isomorphic to a Vershik transformation on an ordered Bratteli diagram and vice versa.*

If the substitutions (χ_i) are such that for every $i \geq 2$, there is $j_0 \geq 1$ such that

$$\chi_i \circ \dots \circ \chi_j(v) \text{ starts with the same symbol for all } j \geq j_0, v \in V_{j+1}, \quad (6)$$

then the corresponding Bratteli diagram has a unique minimal element.

Proof. We only give a sketch of the construction. The vertices of Bratteli diagram coincide with the alphabets V_i (for this reason we chose the same notation), except that the Bratteli diagram has a first level $V_1 = \{v_0\}$. Let $E_1 = \{v_0 \to v \mid v \in V_2\}$. (In fact, in the construction of Proposition 1, there is an extra edge $(1 \to 2) \in E_1$. This gives rise to an isomorphic Bratteli diagram.) For each $v \in V_{i+1}$, $i \geq 2$, there is an incoming edge $w \to v$ for each appearance of $w \in V_i$ in $\chi_i(v)$, and the ordering of the incoming edges in v is the same as the order of the letters in $\chi_i(v)$. It follows that the incidence matrices of the substitution χ_i coincide with incidence matrices associated to the edges E_i.

Clearly, the Bratteli diagrams and substitutions $(\chi_i)_{i \geq 2}$ are in one-to-one correspondence, provided every $w \in V_i$ appears in at least one $\chi_i(v)$, $v \in V_{i+1}$.

Let $i \in V_i$ be the symbol indicated by (6), then it easily follows that v_{i-1} is the first symbol of $\chi_i(v_i)$, and that $x^{\min} := 1 \to 2 \to 3 \to \dots$ is the unique minimal element.

The sequence $s = \lim_j \chi_2 \circ \cdots \circ \chi_i(v)$ can be read off as $s_i = \mathbf{t}(\tau^i(x^{\min})_2)$. In other words, s_i records the vertex in V_2 that the i-th τ-image of $x^{\min}$ goes through. The way to see this is the following: Since the incoming edges to $w \in V_3$ are ordered as in $\chi_2(w)$, a path starting with $\chi_2(w)_1 \to w$ is followed by a path starting with $\chi_2(w)_2 \to w$, etc. Because this is true for every vertex in every level V_i, the required sequence s will emerge.

In general, if $t \in \Sigma$ corresponds to a path $x \in X$, then t_i is the vertex in V_2 that the i-th τ-image of x goes through. □

6. Unique ergodicity

The aim of this section is to give sufficient conditions for the critical omega limit sets to be uniquely ergodic. In the previous sections we have seen that the enumeration scale, the Bratteli diagram and the substitution shift based on the same kneading map Q are all one-to-one equivalent to each other, while $(\omega(c), f)$ is a factor of each of these systems. Therefore, (A, T) is uniquely ergodic if and only (X, τ) is uniquely ergodic if and only if (Σ, σ) is uniquely ergodic, and each of these statements implies that $(\omega(c), f)$ is uniquely ergodic. In this section, we speak of *the system based on* Q and choose the one that serves our argument best. Only when we show that the system is not uniquely ergodic, we need an additional argument that $(\omega(c), f)$ is not uniquely ergodic either. This happens in Theorem 4.

In [2, Theorem 7] the following result (tailored to cutting times) is proven:

Theorem 1. *Write* $b_k = S_k \sum_{m \geq k} 1/S_m$. *If* $Q(k) \to \infty$ *and* $\sup_k b_k < \infty$, *then* (A, T) *is uniquely ergodic.*

An immediate corollary is

Corollary 1. *Let* f *be a unimodal map with kneading map* Q *satisfying* $\sup_k k - Q(k) = B < \infty$. *Then* $(\omega(c), f)$ *is uniquely ergodic.*

Proof. In this case it is easy to check that the S_k's grow exponentially fast, so Theorem 1 applies. □

Remark: In the setting of Corollary 1, we can telescope the Bratteli diagram in order to make the resulting incidence matrices strictly positive. Since $k - Q(k) \leq B$, it follows that for every vertex $l \in V_k$ there is a path to every vertex $l' \in V_{k-2B+1}$. Thus by telescoping (V_i, E_i) between level 1 and $2B$, between level $2B$ and $4B-1$, between $4B-1$ and $6B-2$, etc., we find a new Bratteli diagram $(\tilde{V}_j, \tilde{E}_j)$, and the corresponding incidence matrices $\tilde{M}^j$ have entries $1 \leq \tilde{M}^j_{a,b} \leq 2^{2B}$. Hence the $\tilde{M}^j$ (and substitutions $\tilde{\chi}_j$), are chosen from a finite collection of primitive matrices (and substitutions).
It follows that (χ_j) is *S-adic* in the sense of Ferenczi [15]. Durand [13] (in a corrected version of the proof) gives additional conditions under which substitutions of this class are *linearly recurrent.* This notion (coined in [12]) is a stronger version of uniform recurrence; it states that there exists K such that every finite words $w \in t \in \Sigma$ reappears in t within $K|w|$ entries. Note that K is independent of w, t and the position in which w appears in t. A corollary of linear recurrence is that (Σ, σ) has linear complexity [13]. Furthermore, any factor of a linear recurrent subshift is linearly recurrent. This applies in particular to the subshift $Y \subset \{0,1\}^{\mathbb{N}}$ that emerges from $(\omega(c), f)$ using the partition $I_0 = [0, c]$ and $I_1 = [c, 1]$. The subshift (Y, σ), therefore, has linear complexity as well. As a concrete example, each kneading map $Q(k) = \max\{k - B, 0\}$ gives rise to a subshift (Y, σ) of linear complexity.

Theorem 1 does not apply if S_k grows polynomially: if $S_k \approx k^\alpha$ for some $\alpha > 1$, then $b_k \approx \frac{k}{\alpha} \to \infty$. Also stretched exponential growth, *i.e.,* $S_k \approx e^{\alpha k^\beta}$ for some $\beta \in (0,1)$, is insufficient to apply Theorem 1. Indeed

$$\begin{aligned} b_k &\approx e^{\alpha k^\beta} \sum_{m \geq k} e^{-\alpha k^\beta} \approx e^{\alpha k^\beta} \int_k^\infty e^{-\alpha t^\beta} dt \\ &= e^{\alpha k^\beta} \left[-t e^{-\alpha t^\beta} |_k^\infty - \int_k^\infty \alpha\beta t^\beta e^{-\alpha t^\beta} dt \right] \\ &= k - e^{\alpha k^\beta} \int_{k^\beta}^\infty \alpha u^{1/\beta} e^{-\alpha u} du \geq k - e^{\alpha k^\beta} \int_{k^\beta}^\infty \alpha u e^{-\alpha u} du \\ &\geq k - k^\beta \to \infty. \end{aligned}$$

Let (Σ, σ) be the substitution shift generated by the primitive sequence of substitutions (χ_i), and let M_i be the corresponding incidence matrices. Each M^n represents a linear transformation from $\mathbb{R}^{\#V_{n+1}}$ to $\mathbb{R}^{\#V_n}$, mapping the cone $C_{n+1} := \mathbb{R}^{\#V_{n+1}}_{\geq 0}$ into the cone $C_n = \mathbb{R}^{\#V_n}_{\geq 0}$.

Theorem 2. *The substitution shift is uniquely ergodic if and only if, for each $k \geq 1$, the composition $M^k \cdots M^n$ contracts the cone C_{n+1} to a half-line in C_k as $n \to \infty$.*

Proof. First assume that $M^k \cdots M^n$ contracts the cone C_{n+1} to a half-line, say l_k, as $n \to \infty$. Let (Σ_k, σ) be the subshift generated by the sequence $s_k = \lim_{i \to \infty} \chi_k \circ \dots \chi_i(v)$ for $v \in V_{i+1}$. (This subshift is independent of the choices of v.) Let $f_k \in C_k$ be the vector whose components add up to 1. Then f_k indicates the frequency of the letters $v \in V_k$ of sequences in Σ_k.

Let $\varepsilon > 0$ be arbitrary and C be any finite word in $\Sigma = \Sigma_2$. Each $t \in \Sigma$ is a concatenation of words of the form $W(i) = \chi_2 \circ \cdots \circ \chi_N(w)$, for $w \in V_{N+1}$ and some fixed N. More precisely,

$$t = VW(w_1)W(w_2)W(w_3)\dots,$$

where V is a suffix of some word $W(w_0)$. Let us say that an occurrence of C in t *N-overlaps* if $t_{[k,k+M]} = C$ and $k < |VW(i_1)\dots W(i_r)| < k+M$ for some r. By taking N sufficiently large, we can assume that

$$\limsup_n \frac{1}{n}\#\{i \mid C = t_{[i,i+M]} \text{ and } N\text{-overlaps}\} \leq \varepsilon,$$

uniformly over all $t \in \Sigma$. Therefore

$$\lim_n \frac{1}{n}\#\{0 \leq i < n \mid C = t_{[i,i+M]} \text{ and does not } N\text{-overlap}\}$$

differs from $v(C,t) := \lim_n \frac{1}{n}\#\{0 \leq i < n \mid C = t_{[i,i+M]}\}$ by no more than ε.

Write the frequency vector $f_N = (f(v) \mid v \in V_N)$. For each ε', there exists $M > N$ such that for each word $v \in V_{N+1}$ and $w \in V_{M+1}$ and the word $W'(w) = \chi_{N+1} \circ \cdots \circ \chi_M(w)$

$$\left| \frac{\#\{0 < j \leq |W'(w)| \mid W'(w)_j = v\}}{|W'(w)|} - f(v) \right| < \varepsilon'.$$

For each $v \in V_{N+1}$, let C_v be the number of occurrences of C in the word $W(v)$. Then

$$v(C,t) = \frac{\sum_{v \in V_{N+1}} C_v \cdot (f(v) + \mathcal{O}(\varepsilon'))}{\sum_{v \in V_{N+1}} |W(v)|(f(v) + \mathcal{O}(\varepsilon'))} + \mathcal{O}(\varepsilon).$$

Since ε and ε' are arbitrary, we see that $v(C,t)$ is independent of the string t. Thus unique ergodicity follows.

Conversely, if $M^2 \cdots M^n$ does not contract the cone C_{n+1} to a half-line as $n \to \infty$, then let G_n be the intersection of $M^2 \cdots M^n(C_{n+1})$ with the unit simplex in C_2. We have $G_n \supset G_{n+1}$, but $\lim_n \operatorname{diam}(G_n) =: \delta > 0$.

The extremal points of G_n are the normalized images under $M^2 \cdots M^n$ of the standard unit vectors $e_w := (0, \dots, 0, 1, 0, \dots, 0)$, with the 1 at entry $w \in V_{n+1}$. These extremal points indicate the frequency of the symbols $v \in V_2$ in the words $\chi_2 \circ \cdots \circ \chi_n(w)$. In the above circumstances, there is at least one symbol $v \in V_2$ whose frequency in words $\chi_2 \circ \cdots \circ \chi_n(w)$, for any $n \geq 1$, depends on the choice of $w \in V_{n+1}$.

Thus, if $s = s_0 s_1 \cdots = \lim_{i \to \infty} \chi_2 \circ \cdots \circ \chi_i(v)$, then the limit $\frac{1}{m}\#\{k \leq i \leq k+m \mid s_i = v\}$ does not exists uniformly in k. Hence the subshift (Σ, σ) is not uniquely ergodic. $\square$

For the incidence matrix M^i define

$$\rho_i = \max_{l,l' \in V_{i+1}} \sqrt{\frac{\max\{M^i_{k,l}/M^i_{k,l'} \mid k \in V_i\}}{\min\{M^i_{k,l}/M^i_{k,l'} \mid k \in V_i\}}}, \tag{7}$$

where $\rho_i = \infty$ is allowed when $M^i_{k,l}$ or $M^i_{k,l'} = 0$.

Proposition 3. *Let $(\chi_i)_{i\geq 2}$ be a primitive sequence of substitutions with incidence matrices M^i, generating the substitution shift (Σ, σ). If $\sum 1/\rho_i = \infty$, then (Σ, σ) is uniquely ergodic.*

Proof. According to Theorem 2 we need to show that $M^k \cdots M^n(C_{n+1})$ tends to a half-line as $n \to \infty$.

For the proof we will use the Hilbert metric on a cone C. Given $v, w \in \mathcal{C}$, define

$$\Theta(v, w) = \log\left(\frac{\inf\{\mu \mid \mu v - w \in \mathcal{C}\}}{\sup\{\lambda \mid w - \lambda v \in \mathcal{C}\}}\right).$$

To be precise, Θ is a metric on the projective space over C: $\Theta(v, w) = \Theta(\lambda v, \mu w)$ for all $\lambda, \mu > 0$, and hence $\Theta(v, w) = 0$ if and only if v is a multiple of w. Let $T : \mathcal{C} \to \tilde{\mathcal{C}}$ be a linear map. It is shown in *e.g.* [3] that $\Theta(T(v), T(w)) \leq \tanh(D/4)\Theta(v, w)$ for $D = \sup_{v',w' \in T(\mathcal{C})} \Theta(v', w')$. In particular, T is a contraction if T maps $\partial\mathcal{C} \setminus \{0\}$ into the interior of $\mathcal{C}$.

To find the diameter of $M^n(C_{n+1})$ in the Hilbert metric on C_n, we compute, for unit vectors e_l and $e_{l'}$, $l, l' \in V_{n+1}$, the distance $\Theta(M^n(e_l), M^n(e_{l'}))$ and then maximize over all $l, l' \in V_{n+1}$. By (7), this gives $D_n = \operatorname{diam}(M^n(C_{n+1})) = 2\log\rho_n$. By the previous remarks, M^n contracts the Hilbert metric by a factor

$$\begin{aligned} r_n \quad &:= \quad \tanh(D_n/4) \leq \frac{\sqrt{\rho_n} - \sqrt{1/\rho_n}}{\sqrt{\rho_n} + \sqrt{1/\rho_n}} \\ &= \quad \frac{\rho_n - 1/\rho_n}{\rho_n + 2 + 1/\rho_n} = 1 - 2\frac{1 + 1/\rho_n}{\rho_n + 2 + 1/\rho_n} \approx 1 - 2/\rho_n. \end{aligned}$$

Therefore, we need to verify that $0 = \prod_{n=k}^{\infty} r_n = \prod_{n=k}^{\infty}(1 - 2/\rho_n)$, which is equivalent to $\sum_n 1/\rho_n = \infty$. □

If all the matrices M^i have at least one zero entry, then the above proposition gives no direct information. However, by telescoping the sequence of substitutions (or the Bratteli diagram), we can make the incidence matrices strictly positive. This is the strategy of the following result.

Theorem 3. *Let $Q(k)$ be a non-decreasing kneading map satisfying $k - Q(k) \leq C\sqrt{k}$ for some fixed constant C. Then the corresponding systems are uniquely ergodic.*

As before, the unique ergodicity (non-unique ergodicity) for the Vershik transformation, the enumeration scale and $(\omega(f), c)$ follows immediately. Since $Q(k) = \lfloor k - \sqrt{k} \rfloor$ leads to stretched exponential growth of the S_k, see Lemma 3, Theorem 3 covers cases that were not covered by Theorem 1.

The proof below shows that the fact that Q is non-decreasing can be relaxed. For example, if $\sup_k |Q(k) - \tilde{Q}(k)| < \infty$ and $\tilde{Q}$ satisfies Theorem 3, then the the substitutions based on Q and $\tilde{Q}$ behave similarly as far as unique ergodicity is concerned.

Proof. We start again by building the enumeration scale and Bratteli diagram (V_i, E_i). By telescoping, we want to make all incidence matrices M^i primitive. To this end, define recursively:

$$\begin{cases} k_1 = 2, \text{ and for } n \geq 2 \\ \tilde{k}_n = \max\{k > k_{n-1} \mid Q(k-1)+1 < k_{n-1}\}, \\ k_n = 1 + \max\{k > \tilde{k}_n \mid Q(k-1)+1 < \tilde{k}_n\}. \end{cases}$$

In this way, we get that for every $l \in V_{k_n}$ there is at least one path to each $l' \in V_{k_{n-1}}$. Note that by the assumption that $Q(k)$ is non-decreasing, $Q^2(k_n) \approx k_{n-1} < Q(k_n)$. Note also that the maximal number of paths from some $l \in V_{k_n}$ to an $l' \in V_{k_{n-1}}$ is $1 + k_n - \tilde{k}_n \leq k_n - Q(k_n)$, and this maximum is assumed when $l = k_n$ and $l' = k_{n-1}$ are first vertices.

Telescope the Bratteli diagram between levels k_n and k_{n-1} for each n, to arrive at a Bratteli diagram $(\tilde{V}_j, \tilde{E}_j)$, where $1 \leq \tilde{M}^n_{a,b} \leq \tilde{M}^n_{1,1} =: m_n = 1 + k_n - \tilde{k}_n \leq k_n - Q(k_n)$. It follows easily that $m_n \geq \rho_n$ with ρ_n as in (7).

Now if $Q(n) \geq n - C\sqrt{n}$, then $Q^2(n) \geq n - 2C\sqrt{n}$, so that $k_{n-1} \approx Q^2(k_n) \geq k_n - 2\sqrt{k_n}$. From $k_n - k_{n-1} \leq 2\sqrt{k_n}$ it follows that $k_n \leq C'n^2$ for some $C' > 0$. Hence $m_n = 1 + k_n - \tilde{k}_n \approx k_n - Q(k_n) \leq C\sqrt{k_n} \leq C\sqrt{C'}n$, and $\sum 1/\rho_n \geq \sum 1/m_n = \infty$, as required. $\square$

The following theorem gives an example of a kneading map Q such that the corresponding system (Bratteli diagram, etc.) is not uniquely ergodic.

Theorem 4. *Let (a_i) be a sequence of positive integers such that $\sum_i \sqrt{\frac{1}{a_i}}$ is sufficiently small and $\frac{a_{i+2}a_i}{a_{i+1}^2} \leq 1$. Define recursively $k_1 = 3$, $k_2 = 4$ and $k_i = k_{i-1} + a_i$. Let the kneading map Q be defined by*

$$Q(k) = \begin{cases} 0 & \text{if } k \in \{1,2\}; \\ 1 & \text{if } k = 3; \\ 2 & \text{if } k = 4; \\ k_{i-2} - 1 & \text{if } k_{i-1} \leq k < k_i, \ i \geq 3. \end{cases}$$

Then the corresponding systems are not uniquely ergodic.

Note that in this example $Q(k)$ is non-decreasing. Moreover, if we have $a_i = \lfloor Ci^{2\gamma} \rfloor$ for some (any) $\gamma > 1$ and $C > 0$ large, then $k_i \approx i^{2\gamma+1}$ and $K_1 k^{(2\gamma)/(2\gamma+1)} \leq k - Q(k) \leq K_2 k^{(2\gamma)/(2\gamma+1)}$ for some constants $0 < K_1 < K_2$. According to Lemma 3, this corresponds to $\log S_k = 2\gamma k^{1/(2\gamma+1)} \log k + l.o.t.$

Proof. Build the Bratteli diagram to this kneading map, see Figure 2. Let $S_k(v)$ denote the number of paths from the first vertex of V_{k+1} to v_0 passing through to vertex $v \in V_2$. For example, we have $S_2(2) = 2$, $S_2(3) = 1$

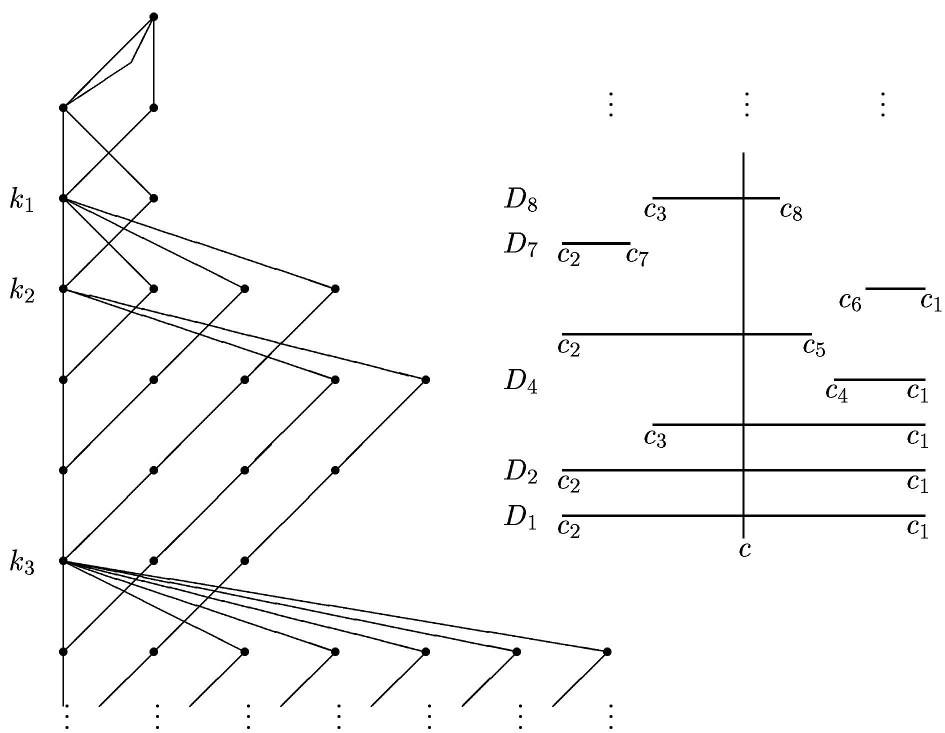

FIGURE 2. Bratteli diagram and Hofbauer tower for a non-uniquely ergodic system.

and $S_3(2) = 4$, $S_3(3) = 1$. In general, $S_k = \sum_{v \in V_2} S_k(v)$ and we have the recursion $S_k(v) = S_{k-1}(v) + S_{Q(k)}(v)$. Applied to the subsequence (k_i), we find $S_{k_i}(v) = S_{k_{i-1}}(v) + a_i S_{k_{i-2}}(v)$. Write $p_i = S_{k_i-1}(2)$ and $q_i = S_{k_i-1}(3)$. Then we have

$$\begin{cases} p_1 = 2, \ p_2 = 4, \ p_i = a_i p_{i-2} + p_{i-1}, \\ q_1 = 1, \ q_2 = 1, \ q_i = a_i q_{i-2} + q_{i-1}. \end{cases} \tag{8}$$

We will show that q_i/p_i does not converge as $i \to \infty$. First we claim that $p_i/p_{i-1} \sim \sqrt{a_i}$. Indeed, $\frac{p_i}{p_{i-1}} = 1 + a_i \frac{p_{i-2}}{p_{i-1}}$, so by introducing new coordinates $u_i = \frac{1}{\sqrt{a_i}} \frac{p_i}{p_{i-1}}$, we have to iterate

$$u_i = g(u_{i-1}) := \frac{1}{\sqrt{a_i}} + \sqrt{\frac{a_i}{a_{i-1}}} \frac{1}{u_{i-1}}.$$

Obviously we have $u_{i+1} = g(u_i) \geq \sqrt{\frac{a_{i+1}}{a_i}} \frac{1}{u_i}$ and

$$u_{i+2} = g(u_{i+1}) \leq \frac{1}{\sqrt{a_{i+2}}} + \sqrt{\frac{a_{i+2} a_i}{a_{i+1} a_{i+1}}} u_i \leq u_i + \frac{1}{\sqrt{a_{i+2}}}$$

Because $\sum 1/\sqrt{a_i} < \infty$, we get that the orbit (u_i) is bounded, and consequently also bounded away from 0. This proves the claim.

Now by (8),

$$\begin{aligned} \left|\frac{q_i}{p_i} - \frac{q_{i-2}}{q_{i-2}}\right| &= \frac{|q_{i-1}p_{i-2} - p_{i-1}q_{i-2}|}{p_i p_{i-2}} \\ &= \frac{p_{i-1}}{p_i}\left|\frac{q_{i-1}}{p_{i-1}} - \frac{q_{i-2}}{p_{i-2}}\right| \leq \frac{p_{i-1}}{p_i}. \end{aligned}$$

Therefore the sequences $(\frac{q_{2i}}{p_{2i}})$ and $(\frac{q_{2i-1}}{p_{2i-1}})$ are both Cauchy. Because $\frac{q_1}{p_1} = \frac{1}{2}$, $\frac{q_2}{p_2} = \frac{1}{4}$, and $\sum \frac{1}{\sqrt{a_i}}$ is small, the limits of these Cauchy sequences are different.

It follows that the relative frequencies of symbols 2 and 3 in $\chi_2 \circ \cdots \circ \chi_i(i)$ do not converge as $i \to \infty$. Therefore the system (Vershik transformation, etc.) cannot be uniquely ergodic. The system $(\omega(c), f)$ is a factor of the other systems. In the language of (A, T),

$$\frac{q_i}{p_i} = \frac{\#\{0 \leq j \leq S_{k_i} \mid \langle j \rangle = 0, 1, \dots\}}{\#\{0 \leq j \leq S_{k_i} \mid \langle j \rangle = 0, 0, \dots \text{ or } \langle j \rangle = 1, 0, \dots\}}$$

But from (3) we can derive that

$$a \in A \text{ starts with } \begin{cases} 1, 0 & \text{if } h(a) \in D_4; \\ 0, 1 & \text{if } h(a) \in D_7; \\ 0, 0 & \text{if } h(a) \in [c_3, c_5]. \end{cases}$$

Indeed, if a starts with $1, 0$, then $h(a) = c_1$ or $h(a) \subset D_r$ for some r with $\beta(r) = 1$. By (4), any such interval D_r is contained in D_4. For a starting with $0, 1$ or $0, 0$, the argument is similar. One can check that these intervals are disjoint, see the Hofbauer tower in Figure 2. It follows that the visit frequencies of points $x \in \omega(c)$ to these disjoint intervals $D_4 \cup [c_3, c_8]$ and D_7 do not converge, prohibiting unique ergodicity. □

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Department of Mathematics, University of Groningen, P.O. Box 800, 9700 AV Groningen, the Netherlands

E-mail address: `bruin@math.rug.nl`