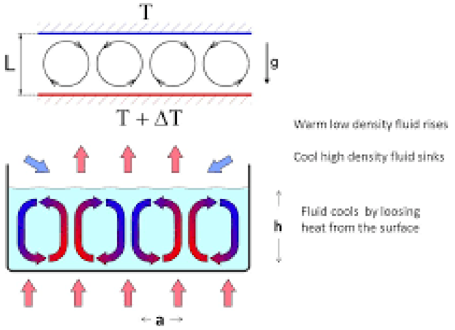
Mixing rates for almost Anosov maps and flows

Henk Bruin

universität wien

October 26, Colloquium, Leiden

Lorenz Flow



The appropriate Navier-Stokes equations reduce (via Galerkin method) to the Lorenz Equations:

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = \rho x - xz - y \\ \dot{z} = xy - \beta z \end{cases}$$

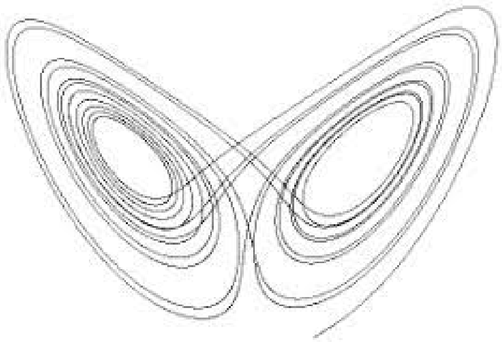
$\sigma, \rho, \beta > 0$ relate to the Prandl and Rayleigh numbers

Lorenz Flow

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = \rho x - xz - y \\ \dot{z} = xy - \beta z \end{cases}$$

$\sigma, \rho, \beta > 0$ relate to the Prandl and Rayleigh numbers

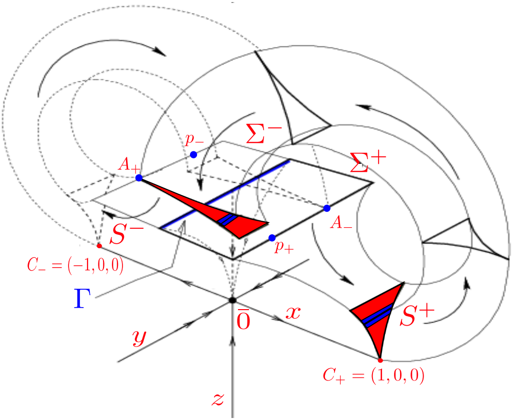
For certain parameters (classical choice $\sigma = 10, \beta = \frac{8}{3}, \rho = 28$) the Lorenz equations have a chaotic attractor.



http://www.malinc.se/m/Lorenz.php

Lorenz Flow

Afraĭmovič, Bykov, & Shilnikov and independently Guckenheimer & Williams made geometrical models to explain the dynamics of the Lorenz system.



Mixing (formal definitions)

A measure μ is invariant for the flow ϕ^t if

$$\mu(\phi^{-t}(A)) = \mu(A) \qquad \text{for all } t > 0 \text{ and measurable } A.$$

Mixing (formal definitions)

A measure μ is invariant for the flow ϕ^t if

$$\mu(\phi^{-t}(A)) = \mu(A) \qquad \text{for all } t > 0 \text{ and measurable } A.$$

We think of physical measures, i.e., those μ for which

$$\mu(A) = \inf_{\varepsilon \to 0} \lim_{T \to \infty} \frac{1}{T} \int_0^T 1_{B(A;\varepsilon)} \circ \phi^t(x)\, dt$$

for x in a set of positive Lebesgue measure.

Mixing (formal definitions)

A measure μ is invariant for the flow ϕ^t if

$$\mu(\phi^{-t}(A)) = \mu(A) \qquad \text{for all } t > 0 \text{ and measurable } A.$$

We think of physical measures, i.e., those μ for which

$$\mu(A) = \inf_{\varepsilon \to 0} \lim_{T \to \infty} \frac{1}{T} \int_0^T 1_{B(A;\varepsilon)} \circ \phi^t(x)\, dt$$

for x in a set of positive Lebesgue measure.

The measure mixing if the correlation coefficients tend to 0:

$$\rho_t(v, w) := \int_X (v \circ \phi^t) \cdot w \, d\mu \to 0$$

for appropriate observables $v, w : X \to \mathbb{R}$.

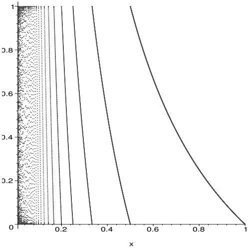
The speed at which ρ_t tends to 0 is called the rate of mixing.

Mixing (heuristics)

Dynamical systems that are **uniformly** hyperbolic, i.e., which have **uniformly** exponential contraction (in stable direction) and exponential expansion (in unstable directions, tend to have exponentially mixing physical measures.

Mixing (heuristics)

Dynamical systems that are **uniformly** hyperbolic, i.e., which have **uniformly** exponential contraction (in stable direction) and exponential expansion (in unstable directions, tend to have exponentially mixing physical measures.



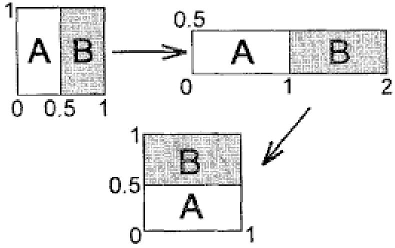


Figure 1 The baker map

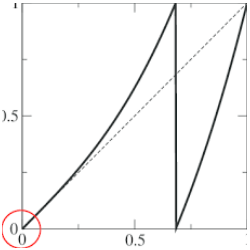
Toy examples: Gauss map: $d\nu = \frac{1}{\log 2} \frac{dx}{1+x}$ and baker map: $\nu = Leb$.

Pomeau-Manneville maps

A classical system with **non-uniform** expansion is the Pomeau-Manneville map with parameter $\alpha > 0$:

$$T_\alpha : [0,1] \to [0,1], \qquad x \mapsto x(1 + x^\alpha) \bmod 1.$$

The neutral fixed point ($T'_\alpha(0) = 1$) makes the expansion non-uniform.

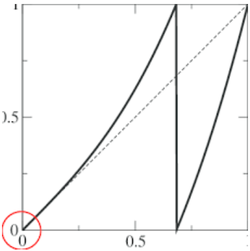


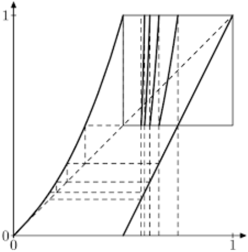
Pomeau-Manneville maps

A classical system with **non-uniform** expansion is the Pomeau-Manneville map with parameter $\alpha > 0$:

$$T_\alpha : [0,1] \to [0,1], \qquad x \mapsto x(1 + x^\alpha) \bmod 1.$$

The neutral fixed point ($T'_\alpha(0) = 1$) makes the expansion non-uniform.

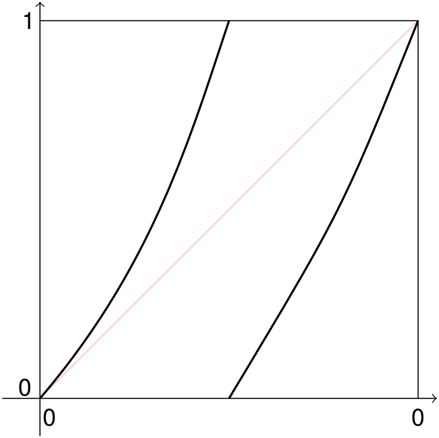




By taking the first return map to the right part, we obtain uniform expansion.

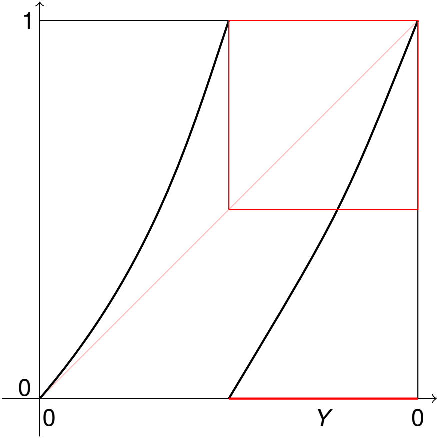
Pomeau-Manneville maps

$$T : x \mapsto x(1 + x^{\alpha}) \bmod 1$$



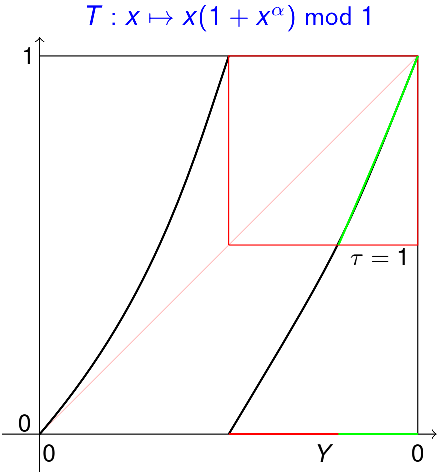
Pomeau-Manneville maps

$T : x \mapsto x(1 + x^{\alpha}) \bmod 1$



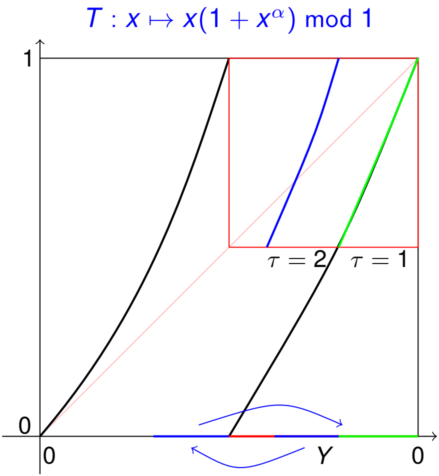
Y is the domain of
the second branch and

Pomeau-Manneville maps



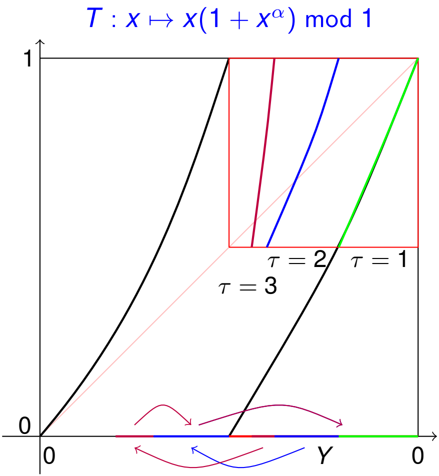
Y is the domain of
the second branch and
$\tau(y) = \min\{n \geq 1 : T^n(y) \in Y\}$.

Pomeau-Manneville maps



Y is the domain of
the second branch and
$\tau(y) = \min\{n \geq 1 : T^n(y) \in Y\}$.
The first return map
$R_Y = T^\tau$
is uniformly expanding,

Pomeau-Manneville maps



Y is the domain of
the second branch and
$\tau(y) = \min\{n \geq 1 : T^n(y) \in Y\}$.
The first return map
$R_Y = T^\tau$
is uniformly expanding,
and preserves a measure
$\nu \approx Leb$.
with big tails
$\nu(\{\tau > n\}) \sim n^{1-1/\alpha}$.

Pomeau-Manneville maps

Theorem [Liverani-Saussol-Vaienti, Young, ...]

The Pomeau-Manneville map has an invariant measure $\mu \approx \mathit{Leb}$.
This measure is

- infinite if $\alpha \geq 1$ (the physical measure is δ_0);
- finite if $\alpha < 1$, and μ is the physical measure.
- the rate of mixing is polynomial if $\alpha < \frac{1}{2}$: $\rho_f(v, w) \leq C_{v,w} n^{1-\frac{1}{\alpha}}$

Pomeau-Manneville maps

Theorem [Liverani-Saussol-Vaienti, Young, ...]

The Pomeau-Manneville map has an invariant measure $\mu \approx \mathit{Leb}$.
This measure is

- infinite if $\alpha \geq 1$ (the physical measure is δ_0);
- finite if $\alpha < 1$, and μ is the physical measure.
- the rate of mixing is polynomial if $\alpha < \frac{1}{2}$: $\rho_f(v, w) \leq C_{v,w} n^{1-\frac{1}{\alpha}}$

The upper bound in the rate of mixing is sharp. Techniques to prove this were developed by Sarig, Gouëzel (and by Melbourne & Terhesiu in infinite measure setting). Requires: "big tails" are regularly varying:

$$\nu(\tau > n) = n^{-\beta} \ell(n)$$

where ℓ is slowly varying (e.g. $\ell(n) \to C \neq 0$ or $\ell(n) = (\log n)^{\gamma}$).

Mixing in the Lorenz system

Theorem [Tucker 2000, rigorous computer-assisted]

The geometric Lorenz model is a valid description of the Lorenz equations. In particular, the Lorenz attractor exists.

Mixing in the Lorenz system

Theorem [Tucker 2000, rigorous computer-assisted]

The geometric Lorenz model is a valid description of the Lorenz equations. In particular, the Lorenz attractor exists.

Theorem [Araújo et al. 2009]

The Lorenz system has a physical measure μ supported on the Lorenz attractor A, and it has a positive Lyapunov exponent χ:

For μ-a.e. $x \in A \; \exists \; v \in T_x\mathbb{R}^3$ s.t. $\lim_t \frac{1}{t} \log \|D\phi^t(v)\| = \chi > 0$.

Mixing in the Lorenz system

Theorem [Tucker 2000, rigorous computer-assisted]

The geometric Lorenz model is a valid description of the Lorenz equations. In particular, the Lorenz attractor exists.

Theorem [Araújo et al. 2009]

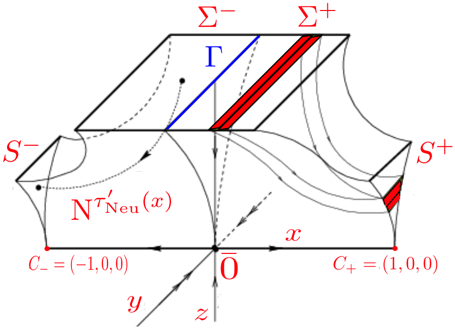
The Lorenz system has a physical measure μ supported on the Lorenz attractor A, and it has a positive Lyapunov exponent χ:

For μ-a.e. $x \in A \ \exists \ v \in T_x\mathbb{R}^3$ s.t. $\lim_t \frac{1}{t} \log \|D\phi^t(v)\| = \chi > 0$.

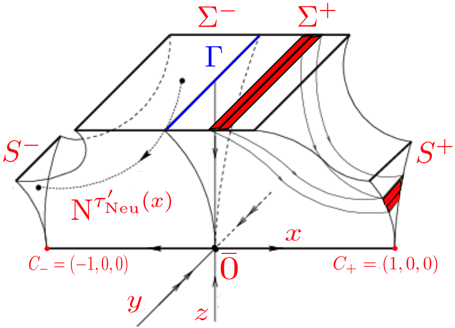
Theorem [Araújo & Melbourne 2016]

This physical measure is mixing at an exponential rate.

Lorenz Flow - Local form at 0



Lorenz Flow - Local form at 0

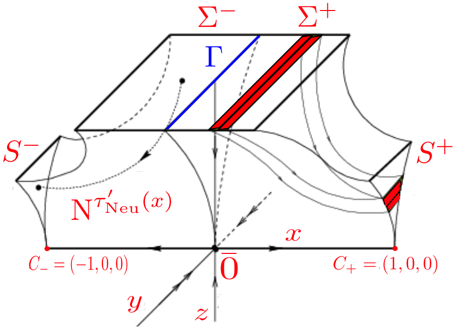


Differential equation near 0 (hyperbolic saddle):

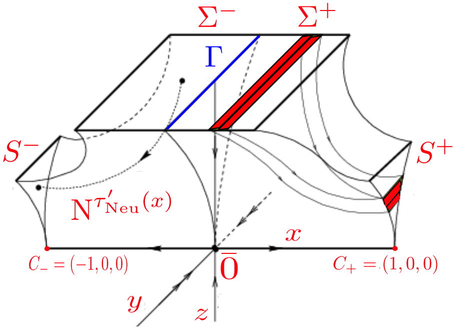
$$\begin{cases} \dot{x} = +\lambda_3 x \\ \dot{y} = -\lambda_1 y \\ \dot{z} = -\lambda_2 z \end{cases} + \mathcal{O}(2) \qquad 0 < \lambda_3 < \lambda_2 < \lambda_1$$

Dulac times have exponential tails.

Lorenz Flow - Surgery at 0



Lorenz Flow - Surgery at 0



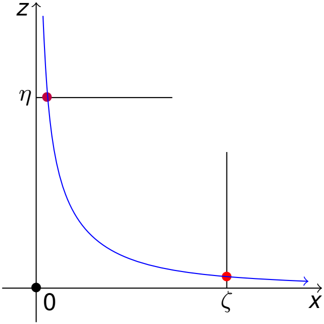
Differential equation near 0 (neutral saddle):

$$\begin{cases} \dot{x} = x(a_0x^2 + a_1xz + a_2z^2) \\ \dot{y} = -\lambda_1 y \\ \dot{z} = -y(b_0x^2 + b_1xz + b_2z^2) \end{cases} + \mathcal{O}(4)$$

$$a_0, a_2, b_0, b_2 \geq 0, \; a_1, b_1 \in \mathbb{R}$$
$$\Delta := a_2b_0 - a_0b_2 \neq 0$$
$$a_1^2 < 4a_0a_2, \; b_1^2 < 4b_0b_2$$
$$\frac{b_1}{a_1} = \frac{a_0b_2 + a_2 + 2b_0b_2}{a_2b_0 + a_2 + 2a_0a_2}$$

Question: Do the Dulac times have polynomial tails?

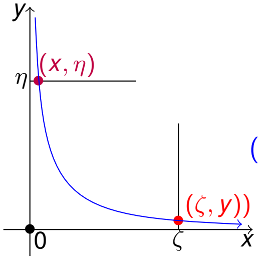
Dulac times



$$\begin{cases} \dot{x} = x(a_0x^2 + a_1xz + a_2z^2) \\ \dot{z} = -y(b_0x^2 + b_1xz + b_2z^2) \end{cases} + \mathcal{O}(4)$$

The Dulac time T is the time needed to flow from an incoming transversal $\{z = \eta\}$ to an outgoing transversal $\{x = \zeta\}$.

Dulac times



$$(*)\begin{cases} \dot{x} = x(a_0x^2 + a_1xz + a_2z^2) \\ \dot{y} = -y(b_0x^2 + b_1xz + b_2z^2) \end{cases} + \mathcal{O}(4)$$

Theorem (regularly varying Dulac times)

Define

$$\Gamma_0 := \frac{2a_0}{a_0 + b_0}, \qquad \Gamma_2 := \frac{2b_2}{a_2 + b_2}.$$

There exists $C_x, C_y > 0$ *such that the Dulac times (as,* $x, y \to 0$*)*

$$T = C_x x^{-\Gamma_2}(1 + \mathcal{O}(x^{\Gamma_2/2})) = C_y y^{-\Gamma_0}(1 + \mathcal{O}(y^{\Gamma_0/2})).$$

Neutral Lorenz flows

Consider the Lorenz flow after surgery; local form near 0:

$$\begin{cases} \dot{x} = x(a_0x^2 + a_1xz + a_2z^2) \\ \dot{y} = -\lambda_1 y \\ \dot{z} = -y(b_0x^2 + b_1xz + b_2z^2) \end{cases} + \mathcal{O}(4)$$

$$a_0, a_2, b_0, b_2 \geq 0,\ a_1, b_1 \in \mathbb{R}$$
$$\Delta := a_2b_0 - a_0b_2 \neq 0$$
$$a_1^2 < 4a_0a_2,\ b_1^2 < 4b_0b_2$$
$$\frac{b_1}{a_1} = \frac{a_0b_2 + a_2 + 2b_0b_2}{a_2b_0 + a_2 + 2a_0a_2}$$

.

Theorem (Bruin & Canales, 2022)

The above neutral Lorenz flow has correlation coefficients:

$$\rho_t(v, w) \leq C \cdot (\|v\|_{C^\eta} + \|v\|_{C^{0,\eta}}) \cdot \|w\|_{C^{m,\eta}} \cdot t^{-\beta}, \qquad \beta = \frac{a_2 + b_2}{2b_2},$$

for $C > 0$ depending only on the flow and η-Hölder functions v and w.

Neutral Lorenz flows

Consider the Lorenz flow after surgery; local form near 0:

$$\begin{cases} \dot{x} = x(a_0x^2 + a_1xz + a_2z^2) \\ \dot{y} = -\lambda_1 y \\ \dot{z} = -y(b_0x^2 + b_1xz + b_2z^2) \end{cases} + \mathcal{O}(4)$$

$$a_0, a_2, b_0, b_2 \geq 0, \; a_1, b_1 \in \mathbb{R}$$
$$\Delta := a_2b_0 - a_0b_2 \neq 0$$
$$a_1^2 < 4a_0a_2, \; b_1^2 < 4b_0b_2$$
$$\frac{b_1}{a_1} = \frac{a_0b_2 + a_2 + 2b_0b_2}{a_2b_0 + a_2 + 2a_0a_2}$$

.

Theorem (Bruin & Canales, 2022)

The above neutral Lorenz flow has correlation coefficients:

$$\rho_t(v, w) \leq C \cdot (\|v\|_{C^\eta} + \|v\|_{C^{0,\eta}}) \cdot \|w\|_{C^{m,\eta}} \cdot t^{-\beta}, \qquad \beta = \frac{a_2 + b_2}{2b_2},$$

for $C > 0$ depending only on the flow and η-Hölder functions v and w.

$$\|v\|_{C^{0,\eta}} = \sup_{x \in M, t>0} \frac{|v(T^t(x)) - v(x)|}{t^\eta}, \qquad \|w\|_{C^{m,\eta}} = \sum_{k=0}^{m} \|\partial_t^k w\|_{C^\eta}.$$

for η-Hölder norm $\|v\|_{C^\eta} = \sup_{x \neq y} \frac{|v(x) - v(y)|}{|x-y|^\eta}$.

Almost Anosov maps

Definition: $T : X \to X$ is called an Anosov map if the whole space X is hyperbolic, i.e., the tangent bundle of X has a continuous splitting into stable and unstable spaces along which the contraction and expansion is uniform.

Almost Anosov maps

Definition: $T : X \to X$ is called an Anosov map if the whole space X is hyperbolic, i.e., the tangent bundle of X has a continuous splitting into stable and unstable spaces along which the contraction and expansion is uniform.

Examples are linear toral automorphisms, e.g.

$$T(x) = Mx \bmod 1, \qquad M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

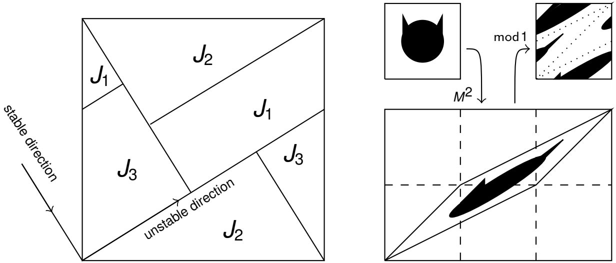


Figure: The Markov partition for $T_M : \mathbb{T}^2 \to \mathbb{T}^2$; the catmap is T_M^2.

Almost Anosov maps

Definition: $T : X \to X$ is almost Anosov if the whole space X is (now non-uniformly hyperbolic, except for finitely many periodic orbits.

Almost Anosov maps

Definition: $T : X \to X$ is almost Anosov if the whole space X is (now non-uniformly hyperbolic, except for finitely many periodic orbits.

The notion of almost Anosov map goes back to Anatoli Katok [1979], who created area preserving examples. Later versions by Hu & Young [1995], Hu [2000] and Young & Zhang [2017]:

Let T be an almost Anosov map with fixed point 0 such that

$$(**) \qquad T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x(1 + a_0 x^2 + a_1 xy + a_2 y^2) \\ y(1 - a_0 x^2 - a_1 xy - b_2 y^2) \end{pmatrix}.$$

in local coordinates near 0.

Almost Anosov maps

Definition: $T : X \to X$ is almost Anosov if the whole space X is (now non-uniformly hyperbolic, except for finitely many periodic orbits.

The notion of almost Anosov map goes back to Anatoli Katok [1979], who created area preserving examples. Later versions by Hu & Young [1995], Hu [2000] and Young & Zhang [2017]:

Let T be an almost Anosov map with fixed point 0 such that

$$(**) \qquad T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x(1 + a_0 x^2 + a_1 xy + a_2 y^2) \\ y(1 - a_0 x^2 - a_1 xy - b_2 y^2) \end{pmatrix}.$$

in local coordinates near 0.

Theorem (Hu 2000)

- *If $a_2 > 2b_2$ and $a_1 = 0 = b_1$ and $a_0 b_2 > a_2 b_0$, then T admits a physical probability measure.*
- *If $a_2 < b_2/2$ and $a_1 \neq 0 \neq b_1$, then T admits an infinite "physical" measure.*

Almost Anosov maps

Hu & Zhang [2017] gave polynomial upper bounds for mixing rates of the physical measure of the almost Anosov map T.

But, T from $(**)$ is the time-1 map of a flow in $(*)$. The tail estimates of $(*)$ are far more precise (exact exponent, slowly varying).

Almost Anosov maps

Hu & Zhang [2017] gave polynomial upper bounds for mixing rates of the physical measure of the almost Anosov map T.

But, T from $(**)$ is the time-1 map of a flow in $(*)$. The tail estimates of $(*)$ are far more precise (exact exponent, slowly varying).

- Exact distinction between finite and infinite “physical” measure:

  T has a physical probability measure iff $\beta = \dfrac{a_2 + b_2}{2b_2} > 1$.

Almost Anosov maps

Hu & Zhang [2017] gave polynomial upper bounds for mixing rates of the physical measure of the almost Anosov map T.

But, T from $(**)$ is the time-1 map of a flow in $(*)$. The tail estimates of $(*)$ are far more precise (exact exponent, slowly varying).

- Exact distinction between finite and infinite "physical" measure:

$$T \text{ has a physical probability measure iff } \beta = \frac{a_2 + b_2}{2b_2} > 1.$$

- Exact mixing rates: upper and lower, finite and infinite measure.

$$\rho_n(v, w) \sim C(v, w)\, n^{1-\beta} \qquad \text{if } \beta > 1,$$

Analogous formula on transfer operator $\mathcal{L}_T$ if measure is infinite.

Almost Anosov maps

Hu & Zhang [2017] gave polynomial upper bounds for mixing rates of the physical measure of the almost Anosov map T.

But, T from $(**)$ is the time-1 map of a flow in $(*)$. The tail estimates of $(*)$ are far more precise (exact exponent, slowly varying).

- Exact distinction between finite and infinite "physical" measure:

$$T \text{ has a physical probability measure iff } \beta = \frac{a_2 + b_2}{2b_2} > 1.$$

- Exact mixing rates: upper and lower, finite and infinite measure.

$$\rho_n(v, w) \sim C(v, w)\, n^{1-\beta} \qquad \text{if } \beta > 1,$$

  Analogous formula on transfer operator $\mathcal{L}_T$ if measure is infinite.

- Various other statistical limit theorems.

Almost Anosov maps

Set T as the time-1 map of

$$(*) \qquad \begin{cases} \dot{x} = x(a_0x^2 + a_1xy + a_2y^2) \\ \dot{y} = -y(b_0x^2 + b_1xy + b_2y^2) \end{cases} + \mathcal{O}(4)$$

with restrictions on parameters as before.

Almost Anosov maps

Set T as the time-1 map of

$$(*) \qquad \begin{cases} \dot{x} = x(a_0x^2 + a_1xy + a_2y^2) \\ \dot{y} = -y(b_0x^2 + b_1xy + b_2y^2) \end{cases} + \mathcal{O}(4)$$

with restrictions on parameters as before.

Theorem (Bruin & Terhesiu 2017, mixing for ∞ measure)

Let T be an almost Anosov map of the torus as in $(**)$ with $\beta \in (\frac{1}{2}, 1)$, i.e., there is an infinite "physical" measure μ.

Then for all observables $v, w \in C^1$ supported outside a neighbourhood of the fixed point, we have

$$\lim_{n\to\infty} n^{1-\beta} \int v \cdot w \circ T^n \, d\mu = \frac{C_0}{\Gamma(\beta)(1-\Gamma(\beta))} \int v \, d\mu \int w \, d\mu$$

for Γ-function Γ and C_0 comes from the big tails $\mu(\tau > n) \sim C_0 n^{-\beta}$.

Almost Anosov maps

In case that T is the time-1 map of a cubic $(*)$, i.e., without higher order terms, then the tail estimates can be improved to estimates of the small tails:

$$\mu(\tau = n) = C_0 n^{-(\beta+1)}(1 + o(1)).$$

Almost Anosov maps

In case that T is the time-1 map of a cubic $(*)$, i.e., without higher order terms, then the tail estimates can be improved to estimates of the small tails:

$$\mu(\tau = n) = C_0 n^{-(\beta+1)}(1 + o(1)).$$

Theorem (Bruin & Terhesiu 2017, mixing for ∞ measure)

Let T be an almost Anosov map above, with $\beta \in (\frac{1}{2}, 1)$ and

$$q = \max\{j \in \mathbb{N} : (j+1)\beta > j\}.$$

Then for all observables as previously there are (generically nonzero) real constants $d_1, \dots, d_q \in \mathbb{R}$ such that

$$\int v \cdot w \circ T^n \, d\mu \quad \sim \quad \left(d_0 n^{\beta-1} + d_1 n^{2(\beta-1)} + \dots + d_q n^{(q+1)(\beta-1)}\right) \times \int v \, d\mu \int w \, d\mu.$$

Almost Anosov flows (volume-preserving)

The almost Anosov flow can be made volume preserving by setting the divergence of the vector field equal to zero. For the local parameters this means:

$$3a_0 = b_0 \qquad a_1 = b_1 \qquad a_2 = 3b_2.$$

Then the condition $\frac{b_1}{a_1} = \frac{a_0 b_2 + a_2 + 2b_0 b_2}{a_2 b_0 + a_2 + 2a_0 a_2}$ is automatically satisfied.

Use the change of coordinates

$$\bar{x} = \sqrt{a_0}x,\ \bar{y} = \sqrt{b_2}y,\ \bar{\gamma} = a_1/\sqrt{a_0 b_2} \in (-4, 4)$$

to transform the differential equation into

$$\begin{pmatrix} \dot{\bar{x}} \\ \dot{\bar{y}} \\ \dot{\bar{z}} \end{pmatrix} = \begin{pmatrix} \bar{x}(\bar{x}^2 + \bar{\gamma}\bar{x}\bar{y} + 3\bar{y}^2) \\ -\bar{y}(3\bar{x}^2 + \bar{\gamma}\bar{x}\bar{y} + \bar{y}^2)) \\ 1 + \bar{w}(\bar{x}, \bar{y}) \end{pmatrix} + \mathcal{O}(4),$$

for some real-valued function $\bar{w}$.

Almost Anosov flows (volume-preserving)

Theorem (Bruin, 2020)

Let $\bar{\gamma} \in (-4, 4)$ *and* $v = v_0 + o(\rho)$, *where* $\int_0^\tau v_0 \circ \phi^t \, dt$ *is homogeneous of order* $\rho > -2$ *in local coordinates.*

1. *If* $\rho \in (0, \infty)$, *then* v *satisfies the Central Limit Theorem, i.e.,*

$$\frac{\int_0^t v \circ \phi^s \, ds - t \int v \, dVol}{\sigma \sqrt{t}} \Rightarrow_{dist} \mathcal{N}(0, 1) \quad \text{as } t \to \infty,$$

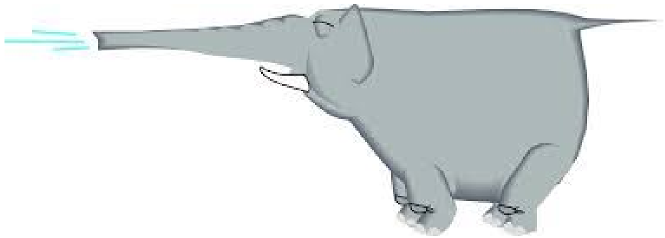
   where the variance $\sigma^2 > 0$ *unless* $\int_0^\tau v \circ \phi^t \, dt$ *is a coboundary.*

2. *If* $\rho = 0$, *then* v *satisfies the Central Limit Theorem with non-standard scaling* $\sqrt{t \log t}$, *i.e.,*

$$\frac{\int_0^t v \circ \phi^s \, ds - t \int v \, dVol}{\sigma \sqrt{t \log t}} \Rightarrow_{dist} \mathcal{N}(0, 1) \quad \text{as } t \to \infty,$$

   and the variance $\sigma^2 > 0$ *unless* $\int_0^\tau v \circ \phi^t \, dt$ *is a coboundary.*

3. *If* $\rho \in (-2, 0)$ *then* v *satisfies a Stable Law of order* $\frac{4}{2-\rho} \in (1, 2)$.



En toen kwam er een olifant met een heel erg lange snuit....

- Afraĭmovič, V., Bykov, V. and Shilnikov, L., *The origin and structure of the Lorenz attractor.*, Dokl. Akad. Nauk SSSR, Vol. 234, 2, 336–339, 1977.
- Araújo, V. and Melbourne, I., *Exponential decay of correlations for non-uniformly hyperbolic flows with a $C^{1+\alpha}$ stable foliation, including the classical Lorenz attractor*, Annales Henri Poincaré, Springer, Vol. 17, 11, 2975–3004, 2016.
- Araújo, V., Pacifico, M., Pujals, E., Viana, M., *Singular-hyperbolic attractors are chaotic.* Trans. Amer. Math. Soc. **361** (2009), 2431–2485.
- Bruin, H., *On volume preserving almost Anosov flows*, Monatsh. Math. DOI:10.1007/s00605-022-01807-w appeared electronically in December 2022
- Bruin, H., Canales Farias H. H., *Mixing rates of the geometrical neutral Lorenz model*, Preprint 2023 arXiv:2305.07502.
- Bruin, H. and Terhesiu, D., *Regular variation and rates of mixing for infinite measure preserving almost Anosov diffeomorphisms*, Ergodic Theory and Dynamical Systems, Cambridge University Press, Vol. 48, 3, 663–698 2020.

- Gouëzel, S., *Sharp polynomial estimates for the decay of correlations*. Israel J. Math. **139** (2004) 29–65.
- Guckenheimer, J. and Williams, R., *Structural stability of Lorenz attractors*, Inst. Hautes Etudes Sci. Publ. Math. **50** (1979), 59–72.
- Hu,H, *Conditions for the existence of SBR measures of "almost Anosov" diffeomorphisms,* Trans. Amer. Math. Soc. **352** (2000) 2331–2367.
- Hu, H., Young, L-S., Nonexistence of SBR measures for some diffeomorphisms that are "almost Anosov", *Ergodic Theory and Dynamical Systems,* **15**, (1995) 67–76.
- Hu, H., Zhang, X., Polynomial decay of correlations for almost Anosov diffeomorphisms, Ergod. Th. Dynam. Sys. published online 2017 doi:10.1017/etds.2017.45
- Katok, A., *Bernoulli diffeomorphisms on surfaces,* Ann. of Math. **110** (1979), 529–547.
- Melbourne, I., Terhesiu, D., Operator renewal theory for continuous time dynamical systems with finite and infinite measure. *Monatsh. Math.* **182** (2017) 377–432.

Sarig, O., *Subexponential decay of correlations*. Invent. Math. **150** (2002) 629–653.