Li-Yorke chaos
for maps on the interval.

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Li-Yorke, distal and asymptotic pairs

Let $f : X \to X$ be a continuous map on a metric space $(X, d)$.

We call the pair $(x, y)$:

**Distal:** if $\lim \inf_n d(f^n(x), f^n(y)) > 0$.

**Asymptotic:** if $\lim_n d(f^n(x), f^n(y)) = 0$.

**Li-Yorke:** if

$$\lim \inf_n d(f^n(x), f^n(y)) = 0$$

and

$$\lim \sup_n d(f^n(x), f^n(y)) > 0.$$

A set $B$ is **scrambled** if every pair $x \neq y \in B$ is Li-Yorke, and $f$ is **Li-Yorke chaotic** if there is an uncountable scrambled set.

Note: $h_{top}(f) > 0$ implies that $(X, f)$ is Li-Yorke chaotic (Blanchard et al.)
Multimodal maps.

In this talk the interval map $f : I \rightarrow I$ will

- be $C^2$ or $C^3$ multimodal (i.e., finite set of critical points: $f'(c) = 0$);

- have non-flat critical points $c$:

$$f(x) = f(c) + O(|x - c|^{\ell_c})$$

for $x \approx c$, and critical order $\ell_c \in (1, \infty)$.

- Sometimes we will assume that $f$ is topologically mixing, i.e., every iterate $f^n$ has a dense orbit.
$C^1$ maps gives different results

The are $C^1$ interval maps that

- have a scrambled set of positive Lebesgue measure, or

- have a scrambled set of full outer measure (but the scrambled set is non-measurable).

These $C^1$ results are due to Smítal '84.

Jímenez-López '91 proved that no $C^1$ can have a measurable scrambled set of full Lebesgue measure.
What is an Attractor?

For interval maps, the following measure-theoretic definition was introduced by Milnor ’85.

The omega-limit set

$$\omega(x) = \cap_n \bigcup_{m \geq n} f^m(x)$$

is the set of limit points of an orbit.

The basin $$\text{Bas}_A = \{x : \omega(x) \subset A\}$$.

Let $$\lambda$$ be Lebesgue measure. $$A$$ is an attractor (a la Milnor) if

$$\lambda(\text{Bas}_A) > 0$$

and if $$A' \subset A, A' \neq A$$, then $$\lambda(\text{Bas}_{A'}) = 0$$.

Remark: An attractor is closed and forward invariant.
Classification of attractors of multimodal maps.

Theorem 1 If $f : I \to I$ is a non-flat $C^2$ multimodal map, then it has $\leq \#\text{Crit}$ attractors, which are of the following types:

1. $A$ is an attracting periodic orbit.

2. $A$ is a collection of intervals $\{I_j\}_{j=0}^{N-1}$ permuted cyclically. (Topological mixing $\Rightarrow N = 1$.)

3. $A$ is a minimal Cantor set consisting of the infinite intersection of cycles of intervals as above.  
   (solenoidal attractor - infinitely renormalizable, $\text{Bas}_A$ is of $2^{nd}$ Baire category.)

4. $A$ is a minimal Cantor set, but not of the above type.  
   (wild attractor - not Lyapunov stable, $\text{Bas}_A$ is of $1^{st}$ Baire category.)
Some references.

The classification of attractors was proved more or less independently by Blokh & Lyubich '91, Keller '90 and by Martens '90.

Milnor '85 posed the question whether wild attractors really exist. This was proved by Bruin, Keller, Nowicki & van Strien ('96) for the Fibonacci unimodal map with very large critical order $\ell$. Bruin ('98) extended this to more general combinatorial types.

For the quadratic family, Lyubich ('94) proved that there are no wild attractors (neither for $\ell \leq 2 + \varepsilon$, Keller & Nowicki '95).

These result are the starting point of our ‘Li-Yorke chaos classification’.
When attractor $A = I$.

**Theorem 2** If $A = I$, and $\lambda(B) > 0$. Then

- there are $x \neq y \in B$ such that $f^n(x) = f^n(y)$ for some $n \geq 0$. Hence, there is no scrambled set of positive Lebesgue measure.

- there are $x \neq y \in B$ such that $(x, y)$ is not asymptotic. Hence, there is no asymptotic set of positive Lebesgue measure.

- there are $n > m \geq 0$ such that $\lambda(f^n(B) \cap f^m(B)) > 0$: there are no strongly wandering sets (Blokh & Misiurewicz).

- For all $x \in I$ there is a set $C_x$ of full measure such that for all $y \in C_x$

  \[
  \liminf_{n \to \infty} |f^n(x) - f^n(y)| = 0, \quad \text{and} \quad \limsup_{n \to \infty} |f^n(x) - f^n(y)| \geq \text{diam}(I)/2.
  \]

  so $f$ is Li-Yorke sensitive.
Dense orbits for 
\( f \times f : I^2 \rightarrow I^2 \).

**Proposition 3** Let \( f \) be a non-flat \( C^3 \) unimodal map, then the following are equivalent:

1. \( f \) has an invariant prob. measure \( \ll \lambda \);

2. \( \lim \inf_n \lambda(f^n(A)) > 0 \) for every \( A \subset I \), \( \lambda(A) > 0 \).

3. \( \lim_n \lambda(f^n(A)) = 1 \) for every \( A \subset I \), \( \lambda(A) > 0 \).

In this case two-dimensional \( \lambda_2 \)-a.e. \( (x, y) \) has a dense orbit. (cf. \( \mu \) weak mixing)

**Conjecture 4** There exists a top. mixing \( f \in C^\infty(I) \) such that \( \lambda_2 \)-a.e. \( (x, y) \) is Li-Yorke, but does not have a dense orbit. In fact, \( \lambda_2 \) is dissipative, whereas \( \lambda \) is conservative.
When attractor $\mathcal{A}$ is a solenoidal Cantor set.

A point $x$ is **approximately periodic** if $f^n \not\to$ periodic orbit, but for every $\epsilon > 0$, there is a periodic point $p$ such that

$$|f^j(x) - f^j(p)| < \epsilon$$

for all $j$ sufficiently large.

**Lemma 5 (Barrio-Blaya, Jiménez-López)**

$(\omega(x), f)$ is conjugate to some $(p_i)$-adic adding machine if and only if $x$ is approximately periodic.

**Theorem 6** If $\mathcal{A}$ is solenoidal Cantor attractor, then

$$\lambda_2(\text{Distal pairs}) = \lambda_2(\text{Bas}_\mathcal{A} \times \text{Bas}_\mathcal{A})$$

and there are no Li-Yorke pairs in $\text{Bas}_\mathcal{A}$.
When \( \mathcal{A} \) is wild.

**Theorem 7** Let \( \mathcal{A} \) be a wild attractor (with positive drift and basin of full measure), then we have these possibilities:

(a) \( \lambda_2(\text{Distal pairs}) = 1 \) and every point in the basin is approximately periodic. 

*(Strange adding machine!)*

(b) \( \lambda_2(\text{Distal pairs}) = 1 \), but no point in the basin is approximately periodic.

(c) \( \lambda_2(\text{LY pairs}) = 1 \).

(d) \( \lambda_2(\text{Distal pairs}) > 0 \) and 
\[
\lambda_2(\text{LY pairs}) > 0.
\]

Each of these cases occurs. In cases (b)-(d) there is \( \varepsilon > 0 \) such that \( \text{Bas}_\mathcal{A} \) contains uncountable \( \varepsilon \)-scrambled sets, and \( f \) is Li-Yorke sensitive on \( \text{Bas}_\mathcal{A} \).
Cutting Times for Unimodals

The $n$-th iterate of $f$ has central branch

$$f^n : J \to f^n(J) =: D_n \subset f^n(J),$$

where $J$ is a maximal interval adjacent to $c$ on which $f^n$ is monotone.

The number $n$ is a cutting time if $c \in f^n(J)$.

Cutting times are denoted as

$$1 = S_0 < S_1 < S_2 < \ldots$$
Enumeration Scales

Given the integer sequence \( \{ S_k \}_{k \geq 0} \) with \( S_0 = 1 \) and \( S_k \leq 2S_{k-1} \) let

\[ \langle n \rangle \in \{0, 1\}^{\mathbb{N} \cup \{0\}} \text{ such that } \sum_{k \geq 0} \langle n \rangle_k S_k = n \]

be the **greedy** representation of \( \mathbb{N} \).

Let \( g \) be ‘add one and carry’:

\[ g : \langle \mathbb{N} \cup \{0\} \rangle \rightarrow \langle \mathbb{N} \rangle, \quad g(\langle n \rangle) = \langle n + 1 \rangle. \]

The extension to the closure

\[ g : E := \overline{\langle \mathbb{N} \cup \{0\} \rangle} \rightarrow E \]

is well-defined and continuous, provided there is \( \alpha_k \rightarrow \infty \) such that

\[ S_k = \sum_{j \geq \alpha_k} S_j. \]
Factors of Enumeration Scales

Let \((E, g)\) be the enum. scale based on \(\{S_k\}_{k \geq 0}\).

Let
\[
h(x) = x - \text{round}(x) \in \left[-\frac{1}{2}, \frac{1}{2}\right)
\]
be the signed distance to the nearest integer.

Assume that there is \(\rho \in \mathbb{R} \setminus \mathbb{Q}\) such that
\[
\sum_{k} k |h(\rho S_k)| < \infty.
\] (1)

Then
\[
\pi_{\rho} : E \to S^1, \quad \pi(e) = \sum_{k} e_k h(\rho S_k) \quad \text{(mod } 1)\]
is well-defined, onto and satisfies
\[
\pi_{\rho} \circ g = R_{\rho} \circ \pi_{\rho}.
\]
Idea of Proof part (b).

Assume that (1) holds.

Assume that $f$ has a wild attractor with positive drift.

Define

$$b_n(x) = \max_j \{ j : f^j(Z_j(x)) = D_n \}$$

where $Z_j(x)$ is order $j$ cylinder containing $x$, and

$$\pi_n(x) = -\sum_k h(\rho (b_n(x) - n) S_k) \pmod{1}.$$ 

Then $\{\pi_n(x)\}_n$ is Cauchy sequence in $S^1$, defined $\lambda$-a.e. $x$ in the basin of $\omega(c)$.

For the limit $\pi = \lim_n \pi_n$:

$$\pi \circ f = R_\rho \circ \pi_\rho, \quad \lambda - \text{a.e.}$$
Existence wild attractors.

Wild attractors exist for unimodal (i.e. one critical point $c$) maps with specific Fibonacci-like combinatorics and sufficiently large critical order $\ell_c$.

Idea of proof:

- Construct pairs of symmetric intervals $U_k$ converging to $c$.

- Use induced map

$$F_{\mid U_k} = f_{S_k\mid U_k}$$

and consider its dynamics as random walk:

$$\chi_n(x) = k \text{ if } F^n(x) \in U_k.$$
Existence of wild attractors: idea of proof continued:

- If combinatorics are good and $\ell_c$ is large, then this random walk has **positive drift**:

$$\mathbb{E}(\chi_{n+1} - k | \chi_n = k) \geq \eta > 0$$

w.r.t. $\lambda$, uniformly in $n$ and $k$.

- Then $\chi_n \to \infty$ $\lambda$-a.s., so $F^n(x) \to c$ $\lambda$-a.e. and $f^j(x) \to \omega(c)$ $\lambda$-a.e.