Global Lie-Tresse theorem

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Group actions and classical Invariants

The classical problem in invariant theory is to describe the action of a Lie group G on a manifold M. The quotient space M/G can have complicated singularities. Invariant functions describe regular obits and in some cases separate them.

Finite generation property for the algebra of all invariants was the topic of Hilbert's XIV problem. For them the Notherian property was proved in the case of reductive groups actions.

For infinite groups of Lie and Cartan the Notherian property generally does not hold for the algebra of differential invariants, which are the invariant of the prolonged action of the pseudogroup to the space of infinite jets.

Instead finiteness is guaranteed by the Lie-Tresse theorem, which uses invariant functions and invariant derivations as generators.

Lie-Tresse theorem

This theorem is a phenomenological statement motivated by Lie and Tresse. It was rigorously proved for un-constrained actions of pseudogroups (on regular strata in the space of jets) by A.Kumpera, and further elaborated by L.Ovsyannikov, P.Olver and others. It was extended for pseudogroup actions on differential equations (which can be, for instance, singular strata of un-constrained actions) by BK and V.Lychagin.

In these approaches the generating property of the algebra $\mathfrak{A}$ of differential invariants holds micro-locally – on open (not always Zariski open or G-invariant) domains in jet-space J^∞.

In our approach we overcome this difficulty by considering algebraic actions (essentially all known examples are such) and restricting to differential invariants that are rational functions by jets of certain order $\leq l$ and polynomial by higher jets.

Rational differential invariants are natural in the classification problems and all known examples are such, even though one can encounter many classical expressions with roots.

Example

The proper motion group $E(2)_+ = SO(2) \ltimes \mathbb{R}^2$ acts on the curves in Euclidean $\mathbb{R}^2(x, y)$. The classical curvature

$$K = \frac{y''(x)}{\sqrt{(1 + y'(x)^2)^3}}$$

is not an invariant of $E(2)_+$ (reflection $(x, y) \mapsto (-x, -y)$ preserves the circle $x^2 + y^2 = R^2$ but transforms $K \mapsto -K$; $K = \pm R^{-1}$), but its square K^2 is a bona fide rational differential invariant. Notice however that the Lie group $E(2)_+$ is connected, and that K is invariant under the action of its Lie algebra.

A Lie pseudogroup $G \subset \mathrm{Diff}_{\mathrm{loc}}(M)$ acting on a manifold M consists of a collection of local diffeomorphisms φ, each bearing own domain of definition $\mathrm{dom}(\varphi)$ and range $\mathrm{im}(\varphi)$, with properties:

- $\mathrm{id}_M \in G$ and $\mathrm{dom}(\mathrm{id}_M) = \mathrm{im}(\mathrm{id}_M) = M$,
- If $\varphi, \psi \in G$, then $\varphi \circ \psi \in G$ whenever $\mathrm{dom}(\varphi) \subset \mathrm{im}(\psi)$,
- If $\varphi \in G$, then $\varphi^{-1} \in G$ and $\mathrm{dom}(\varphi^{-1}) = \mathrm{im}(\varphi)$,
- $\varphi \in G$ iff for every open subset $U \in \mathrm{dom}(\varphi)$ the restriction $\varphi|_U \in G$,
- The pseudogroup is of order k if this is the minimal number such that $\varphi \in G \Leftrightarrow \forall a \in \mathrm{dom}(\varphi) : [\varphi]_a^k \in G^k$.

Here $G^k \subset J^k_{\mathrm{reg}}(M, M)$ is the space of k-jets of the elements from the pseudogroup in the space of k-jets.

We will assume that action of G on M is transitive, i.e. any two points on M can be superposed by a pseudogroup element.

Algebraic pseudogroup

Let $J^k(M,n)$ be the space of k-jets of submanifolds $N \subset M$ of dimension n. If (x,y) are local coordinates in M such that $N : y = y(x)$, then the local coordinates in $J^k(M,n)$ are (x, y_σ) with $|\sigma| \le k$. The space of infinte jets is $J^\infty = \cup_k J^k(M,n)$.

We will consider the lifted action of G on the space of jets

$$G^k \ni \varphi_k : J^k(M,n) \to J^k(M,n).$$

For a point $a \in M$ the stabilizer $G_a^k = \{\varphi \in G^k : \varphi(a) = a\}$ acts on the space $J_a^k(M,n)$ of k-jets at a.

Definition

The G action on M is called algebraic if G_a^k is an algebraic group acting algebraically on the algebraic manifold J_a^k.

More generally, consider the pseudogroup action on a differential equation $\mathcal{E}$ considered as a submanifold in jets, so that $\mathcal{E}^k \subset J^k(M, n)$ is G-invariant. We will assume that $\mathcal{E}_a^k$ as a manifold is algebraic and irreducible.

Notice that this property concerns the behavior only with respect to the derivatives, so for instance sin-Gordon $u_{xy} = \sin u$ or Liouville $u_{xy} = e^u$ are algebraic differential equations from this perspective.

Alternatively instead of the pseudogroup G we can consider its Lie algebra sheaf $\mathfrak{g}$. The flow of vector fields from $\mathfrak{g}^k$ preserves $\mathcal{E}^k$.

Now we increase the order k for both pseudogroup and the equation, using the prolongations, and calculate the invariants of the prolonged action.

Differential Invariants and Invariant Differentiations

A function $I \in C^\infty_{\mathsf{loc}}(\mathcal{E}^k)$ constant on the orbits of the above action is called a differential invariant: $\varphi_k^*(I) = I \ \forall \varphi_k \in G^k$. For the Lie sheaf $\mathfrak{g}$ the defining equation is $L_X(I) = 0$. Uniting the invariants by k we get the subspace of G-invariant functions in $C^\infty_{\mathsf{loc}}(\mathcal{E}^\infty)$.

We will not consider the general smooth/analytic functions, but restrict to the algebra of invariant functions $\mathfrak{A}^l = \cup_k \mathfrak{A}^l_k$ that are smooth by the base variables, rational by the fibers of $\pi_l : J^l \to M$ and polynomial by higher jets of order $k > l$.

Vector fields $v \in C^\infty(\mathcal{E}^\infty) \otimes_{C^\infty(M)} D(M)$ invariant under the action of pseudogroup G and having rational coefficients act as follows:

$$v : \mathfrak{A}^l_{k-1} \to \mathfrak{A}^l_k.$$

More generally we consider rational invariant derivations of the above form.

A closed subset $S \subset \mathcal{E}^k$ is called Zariski closed if its intersection S_a with every fiber $\mathcal{E}_a^k$, $a \in M$, is Zariski closed. Our first main result says that regularity is guaranteed by a finite number of conditions.

Theorem

Consider an algebraic action of a pseudogroup G on a formally integrable irreducible differential equation $\mathcal{E}$ over M such that G acts transitively on M. There exists a number l and a Zariski closed invariant proper subset $S_l \subset \mathcal{E}^l$ such that the action is regular in $\pi_{\infty,l}^{-1}(\mathcal{E}^l \setminus S_l) \subset \mathcal{E}^\infty$, i.e. for any $k \geq l$ the orbits of G^k on $\mathcal{E}^k \setminus \pi_{k,l}^{-1}(S_l)$ are closed, have the same dimension and algebraically fiber the space. In other words, there exists a rational geometric quotient

$$\left(\mathcal{E}^k \setminus \pi_{k,l}^{-1}(S_l) \right) / G^k \simeq Y_k.$$

Our second main result gives finiteness for differential invariants.
We keep the assumptions of the previous theorem.

Theorem

There exists a number l and a Zariski closed invariant proper subset $S_l \subset \mathcal{E}^l$ such that the algebra $\mathfrak{A}^l$ of differential invariants on $\mathcal{E}^\infty$ separates the regular orbits from $\mathcal{E}^\infty \setminus \pi_{\infty,l}^{-1}(S_l)$ and is finitely generated in the following sense.
There exists a finite number of functions $I_1, \ldots, I_t \in \mathfrak{A}^l$ and a finite number of rational invariant derivations $\nabla_1, \ldots, \nabla_s : \mathfrak{A}^l \to \mathfrak{A}^l$ such that any function from $\mathfrak{A}^l$ is a polynomial of differential invariants $\nabla_J I_i$, where $\nabla_J = \nabla_{j_1} \cdots \nabla_{j_r}$ for some multi-indices $J = (j_1, \ldots, j_r)$, with coefficients being rational functions of I_i.

An important issue (not appearing micro-locally) is that some of the derivations ∇_j may not be represented by horizontal vector fields.

Provided $\mathcal{E}^k$ is a Stein space the obtained finite generation property holds for the bigger algebra $\mathfrak{M}(\mathcal{E}^\infty)^G$ of meromorphic G-invariant functions. For smooth functions the global Lie-Tresse theorem fails, while the micro-local one follows via the implicit functions theorem. The polynomial version of the theorem also fails.

The proof uses the following ingredients:

- vanishing of the Spencer cohomology for the Lie equation,
- Rosenlicht theorem for rational invariants,
- affine property of the prolongation of the action,
- micro-local Lie-Tresse theorem.

Arnold's Conjecture

We can also calculate and bound the asymptotic for the number r_k of independent differential invariants of pure order k in a neighborhood of a regular point. By virtue of our results r_k is a polynomial (the Hilbert function) for large $k \gg 1$.

This implies rationality of the Poincaré function, answering the question of V.Arnold (1994):

Theorem (BK & V.Lychagin)

For transitive actions the Poincaré series $\sum_{k=0}^{\infty} r_k \cdot z^k$ equals
$P(z) = \dfrac{R(z)}{(1-z)^d}$ *for some polynomial $R(z)$ and integer $d \in \mathbb{Z}_+$.*
Moreover $P(z)$ is rational in z for all points outside the stratum
$\mathcal{S}_\infty \subset \mathcal{E}_\infty$ *of codimension ∞.*

Further results

Some other results proved in the same manner are: finiteness theorem for invariant derivations, differential forms and other natural geometric objects (tensors, differential operators, connections etc).

We also obtain finiteness for differential syzygies, and interpret this as G-equivariant Cartan-Kuranishi theorem.

We can informally summarize the main results as follows:

Remark

Hilbert and Rosenlicht theorems allow to treat the quotient of an algebraic variety by an algebraic group action as an algebraic variety. Our global version of the Lie-Tresse theorem allows to treat the quotient of a differential equation by an algebraic pseudogroup action as a differential equation.

There are examples of pseudogroup actions, when some of invariant derivatives act trivially, or their number is less than n. Let us show that invariant derivations $\neq$ invariant vector fields.

Example. Consider the pseudogroup G on $\mathbb{R}^3 = \mathbb{R}^2(x, y) \times \mathbb{R}^1(u)$ with the transitive Lie algebra sheaf

$$\mathfrak{g} = \langle f(x, y)\, \partial_x,\ \partial_y, \partial_u \rangle.$$

This algebra lifted to $J^\infty(\mathbb{R}^2)$ acts transitively in the complement to the equation $\mathcal{E} = \{u_x = 0\}$, so there are no differential invariants. The invariant horizontal fields are

$$\frac{1}{u_x}\mathcal{D}_x,\ \mathcal{D}_y - \frac{u_y}{u_x}\mathcal{D}_x.$$

Restriction to $\mathcal{E}$ yields non-trivial algebra of differential invariants $\mathfrak{A} = \langle u_y, u_{yy}, u_{yyy}, \dots \rangle$, generated by u_y and the invariant derivation $\mathcal{D}_y \mod \langle \mathcal{D}_x \rangle$.

However there are no invariant horizontal vector fields at all!

Non-algebraic situation

Let $\mathfrak{g} = \langle \partial_{x^1}, \ldots, \partial_{x^4}, (x^2 \partial_{x^1} - x^1 \partial_{x^2}) - \lambda(x^4 \partial_{x^3} - x^3 \partial_{x^4}) \rangle$ act upon $M = \mathbb{R}^4(x^1, x^2, x^3, x^4)$, $\lambda \in \mathbb{R} \setminus \mathbb{Q}$. Extend the action trivially to $J^0(M) = \mathbb{R}^5(x^1, x^2, x^3, x^4, u)$ and prolong it to $J^\infty(M)$.

The action on J^1 exhibits irrational winding on T^2, so micro-local and global invariants differ. The latter have the following basis (in complex coordinates $z = x_1 + ix_2$, $w = x_3 + ix_4$):

$$u, \ I_1' = |u_z|^2, \ I_1'' = |u_w|^2 \text{ and } I_{pqrs} = \frac{1}{u_z^p u_{\bar{z}}^q u_w^r u_{\bar{w}}^s} \frac{\partial^{p+q+r+s} u}{\partial z^p \partial \bar{z}^q \partial w^r \partial \bar{w}^s}.$$

The algebra $\mathfrak{A}$ is finitely generated by the differential invariants u, I_1', I_1'', I_{1100}, I_{1010}, I_{0011} and the invariant derivatives

$$\nabla_z = \frac{1}{u_z}\mathcal{D}_z, \ \nabla_{\bar{z}} = \frac{1}{u_{\bar{z}}}\mathcal{D}_{\bar{z}}, \ \nabla_w = \frac{1}{u_w}\mathcal{D}_w, \ \nabla_{\bar{w}} = \frac{1}{u_{\bar{w}}}\mathcal{D}_{\bar{w}}.$$

Thus the Lie-Tresse finiteness holds, but $\mathfrak{A}$ does not separate the G-orbits.

Singular systems

Consider the equivalence problem for Hamiltonian systems near a non-degenerate linearly stable equilibrium point 0. Here the pseudogroup G consists of germs of symplectic diffeomorphisms of $(\mathbb{R}^{2n}, \omega)$ preserving 0. Its subgroup acts on the space $\mathcal{E}$ of the germs of functions H vanishing at 0 to order 2 such that the operator $\omega^{-1} d_0^2 H$ has purely imaginary spectrum and no resonances.

Differential invariants in this problem occur only in even orders, and the Poincaré function is equal to

$$P(z) = \frac{1}{(1-z^2)^n}.$$

Consequently the algebra of invariants $\mathfrak{A}$ is not finitely generated (even in a generalized sense), though it separates the orbits.

Consider the group $\mathrm{SDiff}(2)$ of volume-preserving transformations of $M = \mathbb{R}^2$. Its Lie algebra $\mathfrak{g}$ consists of divergence-free vector fields, and it is isomorphic to the Poisson algebra $\mathcal{P}$ of functions on $(\mathbb{R}^2, \varpi)$, $\varpi = dt \wedge dz$.

This algebra naturally lifts its action to $TM = \mathbb{R}^4(t, z, x, y)$, where the symplectic form identified from T^*M by raising the indices is $\omega = dx \wedge dz + dt \wedge dy$. This lift is related to the total derivative $\nabla : C^\infty(M) \to C^\infty(TM)$, and it is the unique 1st order operator preserving the Poisson brackets, but it is defined only up to adjoint action of $\mathcal{P}$:

$$A \mapsto \nabla A + \{q, A\}, \qquad \nabla = x\partial_t + y\partial_z.$$

This leads to extension of the Poisson algebra to the graded Lie algebra $\mathcal{P}_0 \oplus \mathcal{P}_1$ ($\mathcal{P}_i \simeq \mathcal{P}$): $(A_0, A_1) \mapsto \nabla A_0 + A_1$.

Extending the action to $J^0(TM) = TM \times \mathbb{R}^1(u)$ is equivalent to calculating the cohomology group $H^1(\mathfrak{g}, C^\infty(TM)) = \mathbb{R}^2$ (non-trivial element ∇^3 representing this cohomology is $\mathrm{SDiff}(2)$-analog of the Gelfand-Fuks cocycle).

Trying to extend this to the algebra $\mathcal{H}_1 = \mathcal{P}_0 \oplus \mathcal{P}_1$ violates the Jacobi identity because of the cut tails, and we shall allow more gradings. The process naturally stops with $\mathcal{H}_3 = \mathcal{P}_0 \oplus \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \mathcal{P}_3$, representation of this algebra in $\mathfrak{D}(J^0(TM))$ being (unique modulo some singular cases)

$$
\begin{aligned}
V_0(A^0) &= A_z^0 \partial_t - A_t^0 \partial_z + (A_{tz}^0 x + A_{zz}^0 y)\partial_x - (A_{tt}^0 x + A_{tz}^0 y)\partial_y \\
&\qquad - \frac{1}{6}\nabla^3(A^0)\partial_u,
\end{aligned}
$$

$$
V_1(A^1) = A_z^1 \partial_x - A_t^1 \partial_y - \frac{1}{2}\nabla^2(A^1)\partial_u,
$$

$$
V_2(A^2) = \nabla(A^2)\partial_u,
$$

$$
V_3(A^3) = A^3 \partial_u.
$$

The lowest order (fundamental) differential invariant of the action of the Lie pseudogroup G, corresponding to $\mathcal{H}_3$, in $J^\infty(TM)$ is

$$I = u_{ty} - u_{xz} + u_{xx}u_{yy} - u_{xy}^2.$$

The whole algebra $\mathfrak{A}$ of differential invariants is generated by I,

$$J = E_2E_4 - E_1E_3 - u_{xx}E_3^2 + 2\,u_{xy}E_2E_3 - u_{yy}E_2^2,$$

where $E_1 = \mathcal{D}_t(I)$, $E_2 = \mathcal{D}_x(I)$, $E_3 = \mathcal{D}_y(I)$, $E_4 = \mathcal{D}_z(I)$, and the invariant derivations

$$\mathbb{D}_1 = E_3\mathcal{D}_x - E_2\mathcal{D}_y, \quad \mathbb{D}_3 = E_6\mathcal{D}_x - E_5\mathcal{D}_y,$$
$$\mathbb{D}_2 = E_3\mathcal{D}_t + E_4\mathcal{D}_x - E_1\mathcal{D}_y - E_2\mathcal{D}_z,$$
$$\mathbb{D}_4 = E_6\mathcal{D}_t - E_7\mathcal{D}_x + E_8\mathcal{D}_y - E_5\mathcal{D}_z.$$

with

$$E_5 = u_{xx}E_3 - u_{xy}E_2 + E_1, \quad E_6 = u_{xy}E_3 - u_{yy}E_2 + E_4,$$
$$E_7 = (E_2I_2(u_{yy}E_2 - E_4) - E_2E_3E_4E_5 + E_3^2E_6^2)\,E_2^{-2}E_3^{-1},$$
$$E_8 = (I_2(u_{yy}E_2^2 - E_1E_3 - E_2E_4) + E_3^2(E_6^2 - E_1E_5))E_2^{-1}E_3^{-2}$$

The equation $I = 0$ is known as the second Plebański equation, and it is the basic equation of self-dual gravity. It is a rare integrable equation possessing a huge local symmetry algebra.

All integrable Monge-Ampère equations of Hirota type in 4D were classified by Doubrov and Ferapontov. Nonlinear such equations are: first and second Plebański equations, modified heavenly, Husain equation and the general heavenly.

We checked that all of them possess 4 copies of $\mathrm{SDiff}(2)$ as the local symmetry group, arising naturally and with different grading. We conjecture this uniquely characterizes integrability.

Consider a 2nd order PDE of Hirota type in $\mathbb{R}^3(t, x, y)$ on one dependent variable

$$F(u_{tt}, u_{tx}, u_{ty}, u_{xx}, u_{xy}, u_{yy}) = 0.$$

Its linearization $\ell_F(v) = \frac{d}{d\epsilon}\big|_{\epsilon=0} F(u + \epsilon v)$ defines a canonical conformal bi-vector g^{ij} and hence the conformal structure g_{ij}, depending on the 2-jet of the solution u of $F = 0$.

Can one read the integrability of a given PDE
off the geometry of its formal linearization?

Given a conformal structure $g = g_{ij}(u)dx^i dx^j$ let us introduce a covector $\omega = \omega_s dx^s$ by the formula

$$\omega_s = 2g_{sj}\mathcal{D}_{x^k}(g^{jk}) + \mathcal{D}_{x^s}(\ln \det g_{ij}).$$

Theorem

Equation $F = 0$ is linearizable by a transformation from the equivalence group, here $\mathrm{Sp}(6)$, if and only if the conformal structure g is flat on any solution.

Theorem

Equation $F = 0$ is integrable by the method of hydrodynamic reductions if and only if, on any solution, the conformal structure g and the covector ω satisfy the Einstein-Weyl equation:

$$Ric = \Lambda g.$$

Here $\mathbb{D}$ is the symmetric connection, uniquely determined by the equation $\mathbb{D}g = \omega \otimes g$, and $Ric = R_{(ij)}dx^i dx^j$ is the symmetrized Ricci tensor of $\mathbb{D}$ (Λ is some scalar function).

These results generalize to other classes of dispersionless integrable systems in 3D.

Happy birthday MikE !