Split signature 4-dim metrics, real totally null planes and (2,3,5) distributions.

ESI 5.09.2012

Joint work with Daniel An, NY

① V_q – 4-dimensional vector space with a real metric g_q of signature $(4-q, q)$

$++++$	$+++-$	$++--$
$dx_1^2+dx_2^2+dx_3^2+dx_4^2$	$dx_1^2+dx_2^2+dx_3^2-dx_4^2$	$dx_1^2+dx_2^2-dx_3^2-dx_4^2$

② Complexification $V_q^{\mathbb{C}}$, $g_q^{\mathbb{C}}$ – by linearity.

totally null planes in $V_q^{\mathbb{C}}$

$$N = \left[\begin{pmatrix}1\\i\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\i\end{pmatrix}\right] \qquad N = \left[\begin{pmatrix}1\\i\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix}\right]$$

$$\bar N = \left[\begin{pmatrix}1\\-i\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\-i\end{pmatrix}\right] \qquad \bar N = \left[\begin{pmatrix}1\\-i\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix}\right]$$

$$N \cap \bar N = \{0\} \qquad N \cap \bar N = \left[\begin{pmatrix}0\\0\\1\\1\end{pmatrix}\right]$$

④ $N = \left[\begin{pmatrix}1\\i\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\\-i\end{pmatrix}\right]$

$N \cap \bar N = \{0\}$

but

$N = \left[\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}\right]$

$\underbrace{N \cap \bar N}_{\dim_{\mathbb{C}} = 2\,!} = N$

③ Fact

$N = [v_1, v_2] \rightsquigarrow$ bivector $v_1 \wedge v_2$

totally null

$*(v_1 \wedge v_2) = \pm \varepsilon_q (v_1 \wedge v_2)$ $\quad \varepsilon_q = 1, i$ depending on q

Two orbits!

Group $SO(4-q, q)$ acts on the space of totally null planes (in case of $\begin{matrix}++++\\+++-\end{matrix}$ 1 orbit of a given selfduality

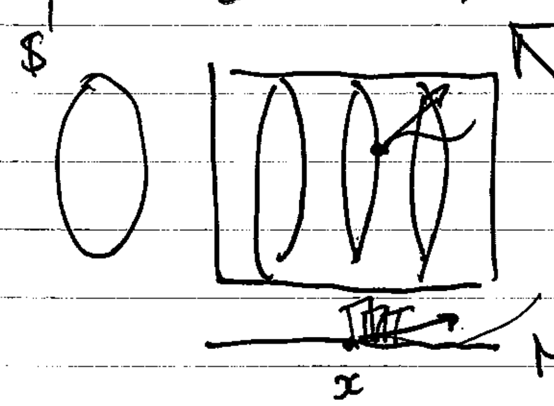
(5) M^4, g metric of signature $(2,2)$

$x \in M^4$, N_x - real selfdual totally null plane at x

act by $SO(2,2)$ take entire orbit $\mathcal{O}(N_x)$

$\mathcal{O}(N_x) \simeq S^1$

$$S^1 \hookrightarrow \mathbb{T} \xrightarrow{\pi} M^4$$



CIRCLE TWISTOR BUNDLE over (M^4, g).

$M^4, g \rightsquigarrow \overset{L.C.}{\nabla}$ - defines horizontal space in $T\mathbb{T} := \mathcal{V} \oplus \mathcal{H}$.

This defines a tautological 2-distribution $\mathcal{D}$ no $\mathbb{T}$.

$\Downarrow$

(6) Canonical distribution

$x \in M^4$ lift every N_x horizontally to its point in the fiber $\pi^{-1}(x)$.

$\mathcal{D}$ - 2 dimensional distribution on $S^1 \hookrightarrow (\mathbb{T}, \mathcal{D})$ $\downarrow$ M^4

~~Thm~~

~~1) $\mathcal{D}$ is integrable if and only if the metric g on M^4 is anti-selfdual~~

~~2) If g is not antiselfdual $\underline{C^+ \neq 0}$~~

7) Metric in a null frame

$$g = g_{ij}\theta^i\theta^j = 2\theta^1\theta^2 + 2\theta^3\theta^4$$

$$g_{ij} = \begin{pmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&0&0&1\\ 0&0&1&0 \end{pmatrix} \left(= g^{ij}\right)$$

e_i - dual vector fields:

$$e_i \lrcorner \theta^j = \delta_i^j$$

Circle of selfdual null planes

$$N(\xi) = [\underbrace{e_1 + \xi e_3}_{e_1(\xi)}, \underbrace{e_4 - \xi e_2}_{e_4(\xi)}] \qquad \xi \in \mathbb{R}\cup\{\infty\}$$

Levi-Civita connection 1-forms: $(\Gamma^i{}_j)$

$$\boxed{d\theta^i + \Gamma^i{}_j \wedge \theta^j = 0 \qquad \& \quad \Gamma_{ij} = -\Gamma_{ji}}$$

$$g_{ik}\Gamma^k{}_j = \Gamma_{ij}$$

ξ - coordinate along the fibers.

Distribution $\mathcal{D}$ on $\mathbb{T} \xrightarrow{\pi} M^4$ annihilated by

$$\begin{cases} \omega^3 = d\xi - \Gamma^2{}_4 + (\Gamma^3{}_3 + \Gamma^2{}_2)\xi - \Gamma^1{}_3\xi^2 \\ \omega^4 = \xi\theta^4 + \theta^2 \\ \omega^5 = \theta^3 - \xi\theta^1 \end{cases}$$

$$C = C^+ \oplus C^-$$
$$10 = 5 + 5$$

$$C^+ = (\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4)$$

$$C^+(\xi) = (\Psi_0 + 4\Psi_1\xi + 6\Psi_2\xi^2 - 4\Psi_3\xi^3 + \Psi_4\xi^4) = C(e_1(\xi), e_4(\xi), e_1(\xi), e_4(\xi))$$

(8) Integrability

$$\begin{cases} d\omega^4 \wedge \omega^3 \wedge \omega^4 \wedge \omega^5 \equiv 0 & \checkmark \\ d\omega^5 \wedge \omega^3 \wedge \omega^4 \wedge \omega^5 \equiv 0 & \checkmark \\ d\omega^3 = C^+(\xi)\, \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 \wedge d\xi \end{cases}$$

Thm

1) $\mathcal{D}$ is integrable iff $C^+ \equiv 0 \iff g$ is antiselfdual
2) if C^+ is not zero at a point then in each fiber there are at most 4 points at which $C^+(\xi)$ vanishes.
   - 4 points
   - 3 points
   - 2 points
   - 2 points
   - 1 point
   - zero points $C^+(\xi) = 0$ has no REAL roots.

   ξ_0 real s.t. $C^+(\xi_0) = 0$ principal points in the fiber
3) if C^+ is not zero the distribution $\mathcal{D}$ is $(2,3,5)$ off the principal points in T.

b) Two surfaces

$$(\Sigma_1, g_1) \qquad (\Sigma_2, g_2)$$

two Riemann surfaces

$$M^4 = \Sigma_1 \times \Sigma_2$$

$$2g = g_1 \oplus (-g_2)$$

Circle twistor bundle?

EXAMPLE two surfaces of constant curvatures κ and σ.

$$2g = \frac{dx^2+dy^2}{\left(1+\frac{\kappa}{4}(x^2+y^2)\right)^2} - \frac{du^2+dv^2}{\left(1+\frac{\sigma}{4}(u^2+v^2)\right)^2} =$$

$$= 2\theta^1\theta^2 + 2\theta^3\theta^4$$

$$\begin{cases} \theta^1 = \dfrac{du}{1+\frac{\sigma}{4}(u^2+v^2)} + \dfrac{dx}{1+\frac{\kappa}{4}(x^2+y^2)} \\[2ex] \theta^2 = -\dfrac{du}{1+\frac{\sigma}{4}(u^2+v^2)} + \dfrac{dx}{1+\frac{\kappa}{4}(x^2+y^2)} \\[2ex] \theta^3 = -\dfrac{dv}{1+\frac{\sigma}{4}(u^2+v^2)} + \dfrac{dy}{1+\frac{\kappa}{4}(x^2+y^2)} \end{cases}$$

$$\theta^4 = \frac{dv}{1+\frac{\sigma}{4}(u^2+v^2)} + \frac{dy}{1+\frac{\kappa}{4}(x^2+y^2)}$$

$\mathcal{D}$ annihilates this.

$$\omega^3 = d\xi + \tfrac{1}{8}(1+\xi^2)\big[(\sigma v + \varkappa y)\,\theta^2 - (\sigma v - \varkappa y)\,\theta^1\big] + \tfrac{1}{8}(1+\xi^2)\big[(\sigma u - \varkappa x)\,\theta^4 - (\sigma u + \varkappa x)\,\theta^3\big]$$

$$\omega^4 = \xi\theta^4 + \theta^2$$

$$\omega^5 = \theta^3 - \xi\theta^1$$

$$\omega^1 = \theta^1$$

$$\omega^2 = \theta^4$$

in $\underline{\underline{\text{u.ub}}}$: $1 \to 4$, $2 \to 5$

$4 \to 2$, $5 \to 1$

$$\boxed{C^+(\xi) = -\tfrac{1}{8}(1+\xi^2)^2(\varkappa - \sigma)}$$

as only $\varkappa \neq \sigma$

$\mathcal{D}$ is $(2,3,5)$ Everywhere!

Conformal $(3,2)$ signature metric

$$\tilde{g} = +(\varkappa+\sigma)(1+\xi^2)(\omega_4^2 + \omega_5^2) + \tfrac{40}{3}\omega_3^2 + \tfrac{5}{2}(\varkappa-\sigma)(1+\xi^2)^2(\omega^5\omega^2 + \omega^4\omega^1)$$

When this is conformally flat??

i.e. when $\mathcal{D}$ has G_2 as a symmetry?

ANSWER

$$\varkappa : \sigma = \begin{cases} 9 \\ \tfrac{1}{9} \end{cases} \quad !!!!$$

Σ_1 with const curvature $\varkappa$

Σ_2 with const curvature σ

Assume that $\varkappa > 0$, $\sigma > 0$ $\Longrightarrow$ two spheres

radii: $r_\varkappa$, r_σ

$r_\varkappa : r_\sigma = \begin{cases} 3 \\ 1/3 \end{cases}$!!!!

ROLLING BALLS!

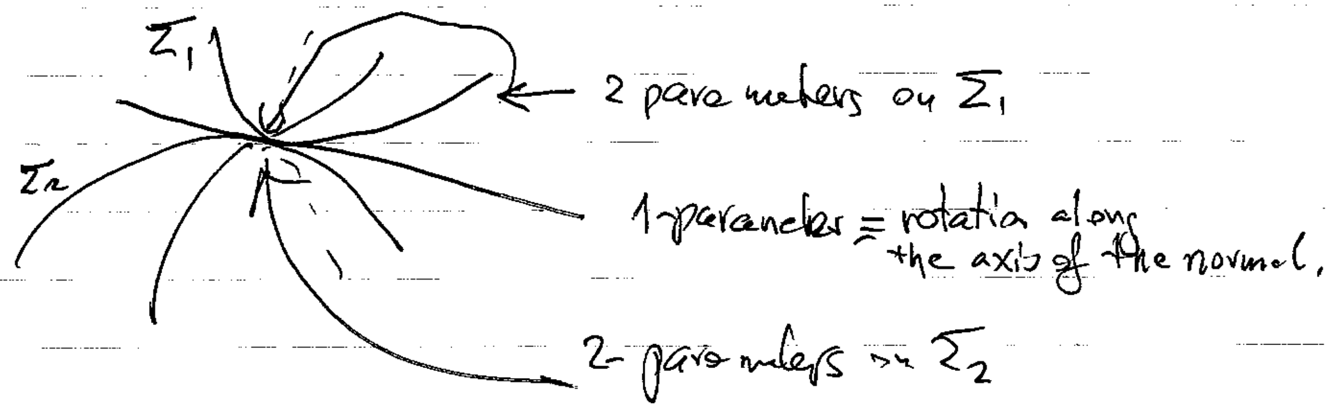
What about two hyperboloids of constant curvatures rolling on each other without slipping?

What about any 2 surfaces for which

$$\mathrm{Weyl}(\tilde{g}) \equiv 0 \, ?$$

Reinterpretation of $\mathbb{T}$.

$$S^1 \hookrightarrow \mathbb{T} \xrightarrow{\pi} \Sigma_1 \times \Sigma_2$$



Find surfaces for which

$\mathrm{Weyl}(\tilde{g}) = 0$ where $\tilde{g}$ is (3,2) conf structure

[If we write this for general g_1 and g_2 the problem is hopeless. meaning beyond our calculational skills.]

but

$$\begin{cases} \Sigma_1: & g_1 = dx^2 + dy^2 \\ \Sigma_2: & g_2 = S(u)^2 (du^2 + dv^2) \end{cases}$$ flat surface

$$g = g_1 - g_2 \rightsquigarrow T, \mathcal{D} \rightsquigarrow \tilde{g}$$

$$\mathrm{Weyl}(\tilde{g}) \equiv 0 \iff$$

$$\boxed{S''' S' S^2 - 3 S''^2 S^2 + S'' S'^2 S + S'^4 = 0} \quad (*)$$

Σ_2 is a surface of revolution.

Indeed

$$u = u(S)$$

$$g_2 = S^2 u'^2 dS^2 + S^2 dv^2$$

$$\begin{cases} x = S \cos v \\ y = S \sin v \\ z = f(S) \end{cases}$$

$$dx^2 + dy^2 + dz^2 = dS^2 + S^2 dv^2 + f'^2 dS^2 =$$
$$= (1 + f'^2) dS^2 + S^2 dv^2$$

$$S^2 u'^2 = 1 + f'^2$$

$$\boxed{f'^2 = S^2 u'^2 - 1}$$ given $u = u(S)$ f gives the surface.

$$s' = \frac{1}{u'}$$

$$\frac{d}{du} = \frac{ds}{du}\frac{d}{ds} = \frac{1}{u'}\frac{d}{ds}$$

$$s'' = \frac{1}{u'}\frac{d}{ds}\left(\frac{1}{u'}\right) = \frac{-u''}{u'^3}$$

$$s''' = \frac{1}{u'}\frac{d}{ds}\left[\frac{1}{u'}\frac{d}{ds}\left(\frac{1}{u'}\right)\right] = -\frac{1}{u'}\;\frac{u'''u'^3 - u''3u'^2u''}{u'^6} =$$

$$= \frac{3u''^2 - u'''u'}{u'^5}$$

$$\cancel{\frac{3u''^2 - u'''u'}{u'^6}} s^2 - \cancel{3\frac{u''^2}{u'^6}} s^2 - \frac{u''}{u'^5} s + \frac{1}{u'^4} = 0$$

$$-u''' s^2 - u'' s + u' = 0$$

$$\boxed{u''' s^2 + u'' s + u' = 0}$$

$$u' = 2as + \frac{b}{s}$$

$$\boxed{u = as^2 + b\log s + c}$$

S satisfies (*) $\iff$ u satisfies

$$u''' S^2 + u'' S - u' = 0$$

$$u' = U$$

$$U'' S^2 + U' S - U = 0 \quad \text{Euler's equation}$$

$$\lambda(\lambda-1) + \lambda - 1 = 0$$

$$\lambda^2 = 1 \implies \lambda = \pm 1$$

$$U = 2aS + b/S$$

$$\boxed{u = aS^2 + b \log S + c}$$

Surface:

$$f'^2 = (2aS^2 + b)^2 - 1$$

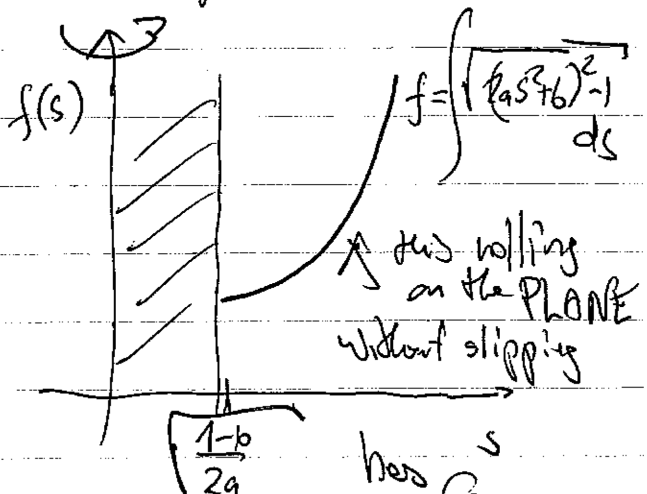
$$f' = \sqrt{(2aS^2+b)^2 - 1}$$

$a \neq 0$ otherwise Σ_2 is cylinder $C^+ \equiv 0$

$$f = \int \sqrt{(2aS^2+b)^2 - 1}\, dS \quad \text{elliptic integral}$$

e.g. $a > 0$ $\quad 0 < b \leq 1$

$$S \geq \sqrt{\frac{1-b}{2a}}$$



has G_2^s as a symmetry of $\mathcal{D}$ on relative space

⑨ Which $(2,3,5)$ distributions come from this construction?

a) **Plebański $(2,2)$ metrics**

split signature 4-metrics generated by 1-function of 4-variables $\Theta = \Theta(x,y,z,w)$

$$g = dw\,dx + dz\,dy - \Theta_{xx}dz^2 - \Theta_{yy}dw^2 + 2\Theta_{xy}dw\,dz$$

$$\begin{cases} \theta^1 = dx - \Theta_{yy}dw + \Theta_{xy}dz \\ \theta^2 = dw \\ \theta^3 = dy - \Theta_{xx}dz + \Theta_{xy}dw \\ \theta^4 = dz \end{cases}$$

$$\begin{cases} \omega^3 = d\xi - \left[(\partial_x + \xi\partial_y)^3\Theta\right]dz \\ \omega^4 = dw + \xi dz \\ \omega^5 = dy - \xi dx - \left[(\partial_x + \xi\partial_y)^2\Theta\right]dz \end{cases} \qquad \Phi(\xi) = (\partial_x + \xi\partial_y)^4\Theta$$

New coordinates (Goursat – Ian Anderson) in $\mathbb{T}$.

$$\underline{x_1} = z, \quad \underline{x_2} = w, \quad \underline{x_3} = -\xi, \quad \underline{x_4} = y - \xi x, \quad \underline{x_5} = x$$

$$\begin{cases} \cancel{\omega^3 \sim dx_2 - x_3 dx_1} \\ \cancel{\omega^4 \sim dx_3 + \Theta_{555}dx_1} \\ \omega^5 \sim \cancel{dx_4 - x_5 dx_3} - \Theta_{555}\,dx_1 \end{cases}$$

~~Note~~

$\cancel{\Theta_3 = -x_5 \Theta_4}$

$$\begin{cases} \omega^3 = dx_2 - x_3\,dx_1 \\ \omega^4 = dx_3 + \Theta_{555}\,dx_1 \\ \omega^5 = dx_4 - (\Theta_{55} - x_5\Theta_{555})\,dx_1 \end{cases}$$

$C^+ = \Theta_{5555} \neq 0$

$\Theta_5 \equiv 0 \Downarrow$ ~~…~~

$0 = \Theta_3 + x_5\Theta_4$

(Plebański equation:

$$\Theta_{14} + \Theta_{25} + x_3\Theta_{24} + (\Theta_{55} + 2x_3\Theta_{45} + x_3^2\Theta_{44})\Theta_{44} - (\Theta_{45} + x_3\Theta_{44})^2 = 0$$)

$\boxed{\Theta_3 = x_5\Theta_5}$

Example

(1) $\Theta_{555} = -x_5$ simplest to guarantee $C^+ \neq 0$

$$\begin{cases} \omega^3 = dx_2 - x_3\,dx_1 \\ \omega^4 = dx_3 - x_5\,dx_1 \\ \omega^5 = dx_4 - \left(\frac{1}{2}x_5^2 + h\right)dx_1 \end{cases}$$

$\Theta_{55} = -\frac{1}{2}x_5^2 + h(x_1, x_2, x_3, x_4)$

$\Theta_5 = -\frac{1}{6}x_5^3 + h x_5 + f_1$

$\Theta_3 = x_3\left[-\frac{1}{6}x_5^3 + h x_5 + f\right]$

$h_3 x_5 + f_3 = -\frac{1}{2}x_3 x_5^2 + h x_3$

e.g. $h = h(x_1, x_3)$

all such distributions have conf. (3,2) sign. metrics with Fefferman-Graham metrics expressible in terms of elementary functions. (Banff talk)

2) $\Theta = \Theta(x_5) \Longrightarrow$

$\omega^1 = dx_1$

$\omega^2 = dx_5$

$\omega^3 = -dx_3 - \Theta_{555}\,dx_1$

$\omega^4 = dx_2 - x_3\,dx_1$

$\omega^5 = dx_4 - x_5\,dx_3 - \Theta_{55}\,dx_1$

$$\tilde{g} = 20\,\omega_3^2\,\Theta^{(4)2} + 30\,\omega_2\omega_4\,\Theta^{(4)3} + 30\,\omega_1\omega_5\,\Theta^{(4)3} + 10\,\omega_3\omega_5\,\Theta^{(4)}\Theta^{(5)} + 5\,\omega_5^2\,\Theta^{(5)2} - 3\,\omega_5^2\,\Theta^{(4)}\Theta^{(6)}$$

skip

2) $H = H(x_5)$

$$\begin{cases} \omega^1 = dx_1 \\ \omega^2 = dx_5 \\ \omega^3 = dx_2 - x_3 dx_1 \\ \omega^4 = dx_3 + H_{555} dx_1 \\ \omega^5 = dx_4 - x_5 dx_3 - H_{55} dx_1 \end{cases}$$

conf metric:

$$\tilde{g} = 20\, H^{(4)2} \omega_4^2 + 30\, H^{(4)3} \omega_2 \omega_3 + 30\, H^{(4)3} \omega_1 \omega_5 - 10\, H^{(4)} \omega_3 \omega_5 + \left(5\, H^{(5)2} - 3\, H^{(4)} H^{(6)}\right) \omega_5^2$$

When conformally flat?

$$\boxed{175\, H^{(5)4} - 280\, H^{(4)} H^{(5)2} H^{(6)} + 49\, H^{(4)2} H^{(6)2} + 70\, H^{(4)2} H^{(5)} H^{(7)} - 10\, H^{(4)3} H^{(8)} = 0}$$

easiest sol:

- $H_{5555} = \text{const.}$
- $H^{(4)} = x_5^n \Rightarrow n = -4, -\frac{5}{2}, -\frac{3}{2}.$

$$\cancel{H_{5555} = n(n-1)(n-2)(n-3)\, x_5^{n-4}}$$

$$\cancel{H_{5555} = x_5^n \qquad n = -8, -\frac{13}{2}, \frac{-11}{2}}$$

(e.g.)

$$H^{(4)} = f(x_5)^{-3/2} \Rightarrow \boxed{20 f^{IV} f + 10 f''' f' - 3 f''^2 = 0}$$

$$H^{(4)} = a x^n$$

$$H^{(5)} = n a x^{n-1}$$

$$H^{(6)} = n(n-1) a x^{n-2}$$

$$H^{(7)} = n(n-1)(n-2) a x^{n-3}$$

$$H^{8} = n(n-1)(n-2)(n-3) a x^{n-4}$$

$$175 n^4 a^4 x^{4n-4} - 280 a^4 n^3 (n-1) x^{4n-4} + 49 a^4 n^2 (n-1)^2 x^{4n-4} + 70 a^4 n^2 (n-1)(n-2) x^{4n-4} - 10 a^4 n(n-1)(n-2)(n-3) x^{4n-4} = 0$$

$$\boxed{175 n^4 - 280 n^3 (n-1) + 49 n^2 (n-1)^2 + 70 n^2 (n-1)(n-2) - 10 n (n-1)(n-2)(n-3) = 0}$$

$$4n^4 + 32 n^3 + 79 n^2 + 60 n = 0$$

$$n = 0,\ n = -4,\ n = -\frac{3}{2},\ n = -\frac{5}{2}$$