Normal Weyl structures and solutions of first BGG operators

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ANU, September 2011

- This talk reports on joint work in progress with Rod Gover (Auckland), Matthias Hammerl (Vienna), and Karin Melnick (Maryland).
- The machinery of BGG sequences provides a uniform construction of differential operators which are intrinsic to a parabolic geometry.
- The first operator in each BGG sequence defines a geometric overdetermined system. Prototypical examples of these systems are the infinitesimal automorphism equation for many geometries, the twistor equation on spinors, the equation describing rescalings to Einstein metrics, and the conformal Killing equations on vectors, differential forms and tensors.
- For some of these equations, strong restrictions on the zero sets of solutions are known classically. Questions on zeros of infinitesimal automorphism and the related questions on essential conformal isometries are intensively studied.

- The machinery of BGG sequences also gives rise to a class of special solutions of these systems ("normal solutions"), which are related to parallel sections of so–called tractor bundles.
- Via normal Weyl structures one can analyze parallel sections and hence normal solutions directly. One also obtains a comparison between a general (curved) geometry and the homogeneous model of the geometry, which then relates the objects on the curved geometry to ones on the homogeneous model. This in particular leads to a detailed description of the zero sets of special solutions.
- Large parts of the technical background apply to arbitrary Cartan geometries. In this general setting, our methods allow a non–linear analog, which can be used to study holonomy reductions.
- In the case of infinitesimal automorphisms, one may use ideas from dynamics to go beyond normal solutions.

Contents

Let G be a semisimple Lie group and $P \subset G$ a parabolic subgroup. Then a *parabolic geometry* of type (G, P) on a smooth manifold M with $\dim(M) = \dim(G/P)$ by definition consists of a principal fiber bundle $p : \mathcal{G} \to M$ with structure group P and a Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G.

Assuming the conditions of regularity and normality, such parabolic geometries provide equivalent ways to encode geometric structures. Well known examples of structures that can be described in this way include classical projective, conformal, and almost quaternionic structures, CR structures of hypersurface type, path geometries, quaternionic contact structures, and several types of generic distributions. The details of this equivalence do not play any role for the purpose of this lecture.

normal Weyl structures

The Cartan connection ω in particular gives rise to a trivialization $T\mathcal{G} \cong \mathcal{G} \times \mathfrak{g}$ of the tangent bundle of $\mathcal{G}$, with the vertical subbundle $V\mathcal{G}$ corresponding to $\mathcal{G} \times \mathfrak{p}$. Here $\mathfrak{p} \subset \mathfrak{g}$ denotes the Lie algebra of the subgroup $P \subset G$. Now there always is a subalgebra $\mathfrak{g}_- \subset \mathfrak{g}$ which forms a complementary subspace to $\mathfrak{p}$.

Fixing a point $u_0 \in \mathcal{G}$ and putting $x_0 = p(u_0) \in M$, we get a canonical local section of $p : \mathcal{G} \to M$ mapping x_0 to u_0: For $X \in \mathfrak{g}_-$, we get the constant vector field $\tilde{X} \in \mathfrak{X}(\mathcal{G})$ characterized by $\omega(\tilde{X}) = X$, and we can consider its flow. Restricting to a suitable open neighborhood V of zero in $\mathfrak{g}_-$, $\varphi(X) := \mathrm{Fl}_1^{\tilde{X}}(u_0)$ defines a smooth map $\varphi : V \to \mathcal{G}$ such that $p \circ \varphi$ is a diffeomorphism onto an open neighborhood U of x_0 in M.

Consequences

- Choosing a basis for $\mathfrak{g}_-$, we obtain local coordinates on U ("normal coordinates").
- We can specify a smooth section of $\mathcal{G}|_U$ by mapping $p(\varphi(X))$ to $\varphi(X)$. This gives rise to a trivialization of $\mathcal{G}|_U$.
- Given any representation $\mathbb{W}$ of P, we can form the the associated bundle $\mathcal{G} \times_P \mathbb{W}$, and we get an induced trivialization of this bundle over U ("normal trivializations"). The resulting trivializations are compatible with any natural bundle map, which comes from a P–equivariant map between the inducing representations.
- Choosing a basis in $\mathbb{W}$, we get a canonical local frame of the associated bundle $\mathcal{G} \times_P \mathbb{W}$ over U ("normal frames"). Again, these are compatible with natural bundle maps.

Note that fixing the point $x_0 \in M$, the family of sections obtained in the above way is parametrized by $P \cong p^{-1}(x_0)$, so there is only a finite dimensional family.

For parabolic geometries, there is a special class of natural bundles. These are called *tractor bundles* and are characterized by the fact that they are induced by restrictions to P of representations of G.

Properties of tractor bundles

- The Cartan connection induces a linear connection ("(normal) tractor connection") on any tractor bundle.
- If $\mathbb{V}$ is an irreducible representation of G, then is is canonically filtered by P–invariant subspaces, and in particular it has a canonical P–irreducible quotient $\mathbb{H}_0$.
- Correspondingly, the tractor bundle $\mathcal{V} := \mathcal{G} \times_P \mathbb{V}$ has a natural quotient bundle $\mathcal{H}_0 := \mathcal{G} \times_P \mathbb{H}_0$. We denote by $\Pi : \mathcal{V} \to \mathcal{H}_0$ the corresponding natural bundle map.

The machinery of BGG–sequences uses the tractor connection and the associated covariant exterior derivative to construct higher order natural differential operators defined on sections of $\mathcal{H}_0$ and other natural subquotients of the bundles of $\mathcal{V}$–valued differential forms. We only need very limited information:

- The first BGG operator D is defined on sections of $\mathcal{H}_0$. The target bundle, the order, and the principal part of the operator can be easily computed from representation theory data of $\mathbb{V}$.
- If $s \in \Gamma(\mathcal{V})$ is a section which is parallel for the tractor connection, then $\Pi(s)$ lies in the kernel of D.
- The map Π is injective on parallel sections, and identifies them with a subspace of solutions of D ("normal solutions").
- On G/P (and geometries locally isomorphic to G/P) the tractor connection is flat and any solution is normal.

The main technical ingredient needed for the further study of normal solutions is easy to prove:

Proposition

Let $s \in \Gamma(\mathcal{V})$ be a section of a tractor bundle, which is parallel for the tractor connection. Then the function $f : U \to \mathbb{V}$ corresponding to s via a normal trivialization is given by $f(p(\varphi(X))) = \exp(-X) \cdot f(x_0)$.

This simple observation has rather strong consequences, however. First, we can get quite detailed information about the structure of potential solutions and for simple tractor bundles, we can even easily give a complete list of all potential solutions:

Polynomiality

Theorem

The coordinate functions of any solution of a first BGG operator with respect to a normal frame are polynomials in the normal coordinates. The degree of these polynomials is bounded by the length of the natural P–invariant filtration of the representation $\mathbb{V}$.

Idea of proof: For a parallel section $s \in \Gamma(\mathcal{V})$ polynomiality of the coordinates in any normal frame follows from the fact that any $X \in \mathfrak{g}_-$ acts by a nilpotent endomorphism on $\mathbb{V}$ with nilpotence index bounded by the filtration length. Thus the result follows for normal frames of $\mathcal{H}_0$ which are obtained by projections from $\mathcal{V}$, and then the general result follows easily.

Lists of potential solutions

To get the flavor, here is the list of potential normal solutions of the equation $\nabla_{(a}\sigma_{bc)} = 0$ on sections of $S^2T^*M(4)$ on a manifold endowed with a projective structure. Here $\{\varphi_{ij}\}$ is a normal frame of the bundle, the x_i are normal coordinates, and on $\mathbb{R}P^n$ any element in the list is a solution. (The tractor bundle in question consists of tractors with curvature tensor type symmetries.)

φ_{ij}	$i \le j$
$x_k\varphi_{ij} - x_j\varphi_{ik}$	$i \le j < k$
$x_k\varphi_{ij} - x_i\varphi_{jk}$	$i < j \le k$
$x_jx_k\varphi_{ii} - x_ix_k\varphi_{ij} - x_ix_j\varphi_{ik} + {x_i}^2\varphi_{jk}$	$i < j \le k$
$x_kx_\ell\varphi_{ij} - x_jx_k\varphi_{i\ell} - x_ix_\ell\varphi_{jk} + x_ix_j\varphi_{k\ell}$	$i < j < k \le \ell$
$x_jx_\ell\varphi_{ik} - x_jx_k\varphi_{i\ell} - x_ix_\ell\varphi_{jk} + x_\ell x_k\varphi_{j\ell}$	$i < j \le k < \ell$

We can interpret the above results in a slightly different way. Namely, we can combine the inverse of a normal chart for the homogeneous model $G \to G/P$ with a normal chart of a curved geometry $\mathcal{G} \to M$, to obtain a diffeomorphism $\psi : U \to \tilde{U}$ where $U \subset M$ is the domain of a normal chart and $\tilde{U} \subset G/P$ is an appropriate open neighborhood of $o = eP$ in G/P.

Theorem

For any normal solution α of a first BGG operator on M, the diffeomorphism ψ intertwines the coordinate functions of α with respect to a normal frame with the coordinate functions of a normal solution $\tilde{\alpha}$ of the same first BGG operator on G/P.

In particular, ψ intertwines the zero sets of α and $\tilde{\alpha}$ so the zero set of α cannot be more complicated than the one of $\tilde{\alpha}$.

Important points for the applicability

- On G/P tractor bundles are always trivial with flat tractor connections, so parallel sections and hence normal solutions of first BGG operators on G/P are very well understood.
- While the comparison map cannot be an isomorphism of parabolic geometries (since one geometry is flat and the other isn't) it is more than just a diffeomorphism. In particular, one can prove compatibility with certain distinguished curves, which e.g. leads to proofs for zero sets being totally geodesic.
- Normal solutions which are sections of density bundles select (outside of their zero sets) one of the distinguished connections for the geometry. One can carry over properties of this connection related to completeness from G/P to M.

Remark on holonomy reductions

Given a subgroup $H \subset G$ and a Cartan geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, P), one may form the associated bundle $\mathcal{G} \times_P (G/H)$. Since the P–action on G/H extends to G, this carries a canonical (non–linear) connection induced by ω. Holonomy reductions of the geometry to the subgroup H are then equivalent to parallel sections of this bundle.

Via the comparison map, a holonomy reduction of a curved geometry is then related to a holonomy reductions of G/P. The holonomy reduction of G/P can be nicely understood in terms of the action of H on G/P. In particular, the decomposition of G/P into H–orbits carries over to the so–called *curved orbit decomposition* of the curved geometry. Also the natural flat Cartan geometries on the orbits carry over to curved geometries on the curved orbits.

The infinitesimal automorphism equation is related to the *adjoint tractor bundle* $\mathcal{A}M = \mathcal{G} \times_P \mathfrak{g}$. Recall that

- Sections of $\mathcal{A}M$ can be naturally identified with right invariant vector fields on $\mathcal{G}$, so these are infinitesimal principal bundle automorphisms.
- $TM \cong \mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})$, so there is a natural projection $\Pi : \mathcal{A}M \to TM$ corresponding to projecting vector fields.
- The curvature of the Cartan connection ω can be naturally viewed as $\kappa \in \Omega^2(M, \mathcal{A}M)$.
- $s \in \Gamma(\mathcal{A}M)$ is an infinitesimal automorphism of the Cartan geometry if and only if $\nabla s + i_{\Pi(s)}\kappa = 0$. The BGG machinery relates this to an equation on $\Pi(s) \in \mathfrak{X}(M)$ (or some quotient). In particular, normal solutions are those which insert trivially into κ.

In joint work in progress with Karin Melnick we use ideas related to comparison combined with ideas from dynamics to study certain infinitesimal automorphisms which are not assumed to be normal.

The basic question

Suppose a geometry admits an infinitesimal automorphism which has a "higher order zero", or equivalently that the corresponding one–parameter group of local automorphisms has a higher order fixed point. Does this imply vanishing of the curvature on an open subset with the fixed point in its closure?

If an infinitesimal automorphism is given by $s \in \Gamma(\mathcal{A}M)$ having a higher order zero at $x_0 \in M$ means that the corresponding function $\mathcal{G} \to \mathfrak{g}$ has values in $\mathfrak{p}_+$ on the fiber over x_0. This then gives rise to a well defined element $\psi \in T^*_{x_0}M$ called the *isotropy* of s at x_0.

Definition

Let φ_t be a one–parameter group of automorphisms of $(\mathcal{G}, \omega)$. For a point $u \in \mathcal{G}$, a path $t \mapsto g(t) \in P$ is a *holonomy path* at u with attractor $u_o \in \mathcal{G}$ if there exists a path $u(t)$ in $\mathcal{G}$ such that $u(t)$ converges to u and $\varphi_t(u(t)) \cdot g(t)^{-1}$ converges to u_0 for $t \to \infty$.

Knowing such a holonomy path leads to restrictions on the values of the curvature at u_0 and at u. For the curvature function $\kappa : \mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$, we then get $\kappa(\varphi_t(u(t)) \cdot g(t)^{-1}) \to \kappa(u_0)$ for $t \to \infty$. But of course, $\kappa \circ \varphi_t = \kappa$ for all t, which together with P–equivariancy implies that

$$\kappa(\varphi_t(u(t)) \cdot g(t)^{-1}) = g(t) \cdot \kappa(u(t))$$

for all t and $\kappa(u(t)) \to \kappa(u)$ for $t \to \infty$.

One can obtain holonomy paths using only algebra starting from $Z := s(u_0) \in \mathfrak{p}_+$. This can be also understood as comparison to the model infinitesimal automorphism of G/P corresponding to Z or the one–parameter group given by multiplication by $\exp(tZ)$. In particular:

- The subspace $\{X \in \mathfrak{g}_- : [X, Z] = 0\}$ locally gives rise to a submanifold of fixed points along which the isotropy remains unchanged.
- If $X \in \mathfrak{g}_-$ is such that $(X, A := [Z, X], Z)$ is a $\mathfrak{sl}_2$–triple, then one obtains holonomy paths of the form $\exp(\alpha(t)A)$ with $\alpha(t) \to \infty$ along the curve in M corresponding to the line $\mathbb{R}X$ in normal coordinates. The resulting restrictions on curvature can be described in terms of the grading on $\Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ determined by A.

Using these methods, one obtains a positive answer to the main question in many cases (conformal structures of all signatures, CR structures except when the isotropy lies in the contact subbundle and is null, Grassmannian and quaternionic structures, etc.). I will talk more about this in Wollongong.