Parabolic contact structures with a view towards symplectic geometry

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Conformal and symplectic geometry, Auckland, February 2018

[1]supported by project P27072–N25 of the Austrian Science Fund (FWF)

- Starting from the example of conformal geometry, I will outline several basic features of the theory of parabolic geometries. In particular, I will emphasize the interplay between geometry and representation theory that is typical for the field.
- The basic examples of parabolic geometries that will be discussed are parabolic contact structures. This class already exhibits most of the general features of parabolic geometries and contains well known structures like CR structures.
- In the last part of the talk, I will briefly describe joint work with T. Salač (Prague) on a relation between parabolic contact structures and symplectic geometry.
- This builds on a class of geometric structures refining almost conformally symplectic structures and their realization as symmetry reductions of parabolic contact structures.

Contents

The starting point of conformal geometry is to look for "robust" aspects of Riemannian geometry. Here it will be more natural to view a conformal structure as a geometric structure in its own right. The basic difference between Riemannian and conformal structures can be seen from a property of their automorphisms.

- If $f : M \to M$ is an isometry for a Riemannian metric g on M, then the one-jet of f in a point $x \in M$ uniquely determines the local behavior of f around x.
- This is closely related to the existence of a canonical linear connection on TM (the Levi–Civita connection).
- In contrast, if f is just a conformal isometry, then the local behavior around x is determined by the two-jet in x, but not by the one-jet in x.
- Hence a conformal structure does not determine a canonical connection on TM.

The "naive" approach to conformal geometry of working with a representative metric and then checking how things behave under rescaling quickly gets unmanageably complicated. But there is a canonical connection associated to a conformal structure on an "extended bundle" (two equivalent descriptions).

Cartan connection: One starts from the "homogeneous model" $S^n = G/P$, where $G := SO(n+1,1)$ and $P \subset G$ is the stabilizer of a null–line (a parabolic subgroup). A conformal structure on M^n then canonically determines a principal P–bundle $\mathcal{G} \to M$ and a Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, which nicely trivializes $T\mathcal{G}$.

Representations of $P \cong CO(n) \ltimes \mathbb{R}^{n*}$ determine natural vector bundles, but ω does not induce linear connection on such bundles. There *is* an induced linear connection on bundles if one uses representations of G (restricted to P). This gives rise to *tractor bundles* and *tractor connections*, which equivalently encode the Cartan bundle and connection.

Conformal standard tractors

Given $(M^n, [g])$, this is the associated bundle $\mathcal{T} \to M$ for the representation of P on $\mathbb{R}^{n+2}$. So this comes with a natural bundle metric h of signature $(n+1, 1)$, a distinguished line subbundle $\mathcal{T}^1 \subset \mathcal{T}$ (isotropic for h) and a canonical linear connection $\nabla^{\mathcal{T}}$. Choosing a metric g in the class, $\mathcal{T}$ gets identified with $\mathcal{E}[1] \oplus T^*M[1] \oplus \mathcal{E}[-1]$, and one obtains explicit formulae:

For $\widehat{g} = e^{2\Omega} g$, $\widehat{(\sigma, \mu_a, \rho)} = (\sigma, \mu_a + \Upsilon_a \sigma, \rho - \Upsilon^b \mu_b - \frac{1}{2} \Upsilon^b \Upsilon_b \sigma)$, with $\Upsilon_a = d\Omega$, the metric h is associated to $2\sigma\rho + \mu^a \mu_a$, and $\nabla^{\mathcal{T}}_a (\sigma, \mu_b, \rho) = (\nabla_a \sigma - \mu_a, \nabla_a \mu_b + \mathrm{P}_{ab} \sigma + \mathbf{g}_{ab} \rho, \nabla_a \rho - \mathrm{P}_{ab} \mu^b)$.

Alternatively, one may use the sum together with the conformal transformation law as *defining* a bundle $\mathcal{T} \to M$. Then one can directly verify that the above formulae define canonical objects. (One may then reconstruct $(\mathcal{G}, \omega)$ as an "adapted frame bundle" with connection form.)

The theory of parabolic geometries basically deals with geometric structures that have a homogeneous model G/P with G any semisimple Lie group and $P \subset G$ any parabolic subgroup. These underlying structures can be described explicitly from Lie algebraic data. In particular, one obtains

- families of distinguished connections (Weyl structures)
- tractor bundles endowed with canonical connections, and a general description in terms of underlying data
- various general constructions for differential operators intrinsic to the structure (e.g. BGG sequences – representation theory)
- natural notions of distinguished curves generalizing conformal circles and the chains used in CR geometry
- several natural constructions relating geometries of different types (twistor spaces, Fefferman constructions, etc.)

Let $\mathfrak{g}$ be a simple Lie algebra, which contains a highest root vector, let $\mathfrak{g}_2$ be the highest and $\mathfrak{g}_{-2}$ the lowest root space. For a standard generator $E \subset [\mathfrak{g}_2, \mathfrak{g}_{-2}]$ we get a decomposition into eigenspaces for ad_E as $\mathfrak{g} = \oplus_{i=-2}^{2} \mathfrak{g}_i$, so $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$. Structure theory easily implies the following:

- The bracket $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is non–degenerate, so $\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ is a Heisenberg algebra.
- $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}$ acting by derivations on $\mathfrak{g}_-$, and hence gets identified with a subalgebra of $\mathfrak{der}_{gr}(\mathfrak{g}_-) \cong \mathfrak{csp}(\mathfrak{g}_{-1})$.

The model example is $\mathfrak{g} = \mathfrak{su}(p+1, q+1)$ for which $\mathfrak{g}_{-1}$ inherits a complex structure and $\mathfrak{g}_0$ consists of all complex linear maps in $\mathfrak{csp}(\mathfrak{g}_{-1})$. The case that $\mathfrak{g}$ itself is a symplectic algebra is exceptional since then $\mathfrak{g}_0 = \mathfrak{csp}(\mathfrak{g}_{-1})$. This corresponds to contact projective structures, which are interesting but will not be discussed here.

Given a group G with Lie algebra $\mathfrak{g}$, define $P \subset G$ to be the stabilizer of the line $\mathfrak{g}_2$ under the adjoint action. This has Lie algebra $\mathfrak{p} := \oplus_{i=0}^2 \mathfrak{g}_i$ and is a parabolic subgroup of G. Now $\mathfrak{g}_{-1}$ defines a P–invariant subspace in $\mathfrak{g}/\mathfrak{p}$, hence inducing a G–invariant hyperplane distribution on G/P, which is contact. Moreover, G/P gets identified with a closed G–orbit in $\mathcal{P}(\mathfrak{g})$ and thus is compact.

The subalgebra $\mathfrak{p}_+ := \mathfrak{g}_1 \oplus \mathfrak{g}_2$ corresponds to a closed subgroup $P_+ \subset P$, so $G_0 := P/P_+$ has Lie algebra $\mathfrak{g}_0$.

Given a contact manifold (M, H), put $Q := TM/H$. Then $H \oplus Q$ has a natural frame bundle with structure group $CSp(2n, \mathbb{R})$. Thus the concept of a reduction of structure group of this bundle to G_0 makes sense, and this defines a *parabolic contact structure* of type (G, P). It turns out that the corresponding principal G_0–bundle can be canonically extended to a P–bundle and endowed with a Cartan connection.

For the model example $\mathfrak{g} = \mathfrak{su}(p+1, q+1)$ the reduction to G_0 is equivalent to a complex structure J on H, for which $\mathcal{L} : \Lambda^2 H \to Q$ is Hermitian. Then in each $x \in M$, this is the imaginary part of a Hermitian form, and we require this to have signature (p, q). So this is partially integrable almost CR structure.

The other parabolic contact structures admit similar descriptions. One always has some additional structure on H, say a quaternionic structure or a decomposition as a tensor product of auxiliary bundles, which is compatible with $\mathcal{L}$ in an appropriate sense.

From the equivalent description via a Cartan geometry, one directly gets, for example:

- families of distinguished connections (c.f. Webster–Tanaka)
- fundamental invariants (harmonic curvatures)
- canonical curves generalizing Chern–Moser chains
- Fefferman's construction, including relations between tractor bundles, canonical curves, etc.

BGG sequences and complexes

Starting with a type (G, P) of structures, any representation V of G gives rise to a tractor bundle $VM \to M$ over each manifold M with such a structure. This carries a natural connection ∇^V that can be coupled to the exterior derivative to obtain the *twisted de-Rham sequence* $(\Omega^*(M, VM), d^V)$ corresponding to V.

Restricting the representation V to $\mathfrak{g}_-$, there are the Lie algebra cohomology spaces $H^k(\mathfrak{g}_-, V)$. These are naturally representations of G_0 that are explicitly described (in representation theory terms) by Kostant's theorem.

The construction of BGG sequences realizes the associated bundle $\mathcal{H}_k^V$ corresponding to $H^k(\mathfrak{g}_-, V)$ as a subquotient of $\Lambda^k T^*M \otimes VM$ and "compresses" the d^V to a sequence of higher order invariant operators between these bundles. On locally flat geometries both sequences are complexes that compute the same cohomology.

General parabolic geometries correspond in a similar fashion to arbitrary gradings $\mathfrak{g} = \oplus_{i=-k}^{k}\mathfrak{g}_i$ on semisimple Lie algebras. They can be equivalently described as Cartan geometries of type (G, P) with $\mathfrak{p} = \oplus_{i=0}^{k}\mathfrak{g}_i$ or as underlying structures that consist of

- A distribution of rank $\dim(\mathfrak{g}_{-1})$ giving rise to a filtration $\{T^iM\}_{i=-1}^{-k}$ with associated graded isomorphic to $\mathfrak{g}_-$.
- A reduction of structure group of the associated graded $\mathrm{gr}(TM)$ to the structure group $G_0 := P/P_+$.

For $k = 1$, one obtains standard G_0–structures, for example conformal structures. It may also happen that G_0 is the full automorphism group of the graded Lie algebra $\mathfrak{g}_-$. In that case, the whole geometry is equivalent to the distribution $T^{-1}M \subset TM$. This includes generic rank k distributions in dimension n for $(k, n) = (2, 5)$, $(3, 6)$, $(4, 7)$, $(4, 8)$, and $(k, \frac{k(k+1)}{2})$.

Let us return to a contact grading $\mathfrak{g} = \oplus_{i=-2}^{2} \mathfrak{g}_i$ of a simple Lie algebra $\mathfrak{g}$ (not of type C_n). This led to a parabolic contact structure in dimension $2n+1$, where $2n = \dim(\mathfrak{g}_{-1})$. Since $\mathfrak{g}_0 \subset \mathfrak{csp}(\mathfrak{g}_{-1})$, this also defines an ordinary G_0–structure in dimension $2n$, which has an underlying almost conformally symplectic structure (a non–degenerate line subbundle in $\Lambda^2 T^*M$).

Such a PACS structure on M turns out to determine a canonical linear connection on TM. Its torsion naturally splits into two parts, one obstructing the ACS structure from being CS. The other part resembles the harmonic torsion from the parabolic contact case.

For the $\mathfrak{su}$–algebras, one obtains a weakening of almost Kähler structures, with the canonical connection preserving all ingredients. The explicit description of the other PACS structures is closely parallel the description of parabolic contact structures.

Given a parabolic contact structure endowed with a transversal infinitesimal automorphism of this structure, one can form local leaf spaces for the corresponding foliation. These inherit a natural PCS structure. Conversely, it turns out that any PCS structure can locally be realized (uniquely up to isomorphism) in this way. This admits an analog in type C_n via conformally Fedosov structures (which again determine a canonical connection).

For a PCS structure, the canonical connection locally is symplectic. This relates to special symplectic connections in the sense of Cahen and Schwachhöfer, a class, which in particular contains all affine connections with exotic holonomy that preserve a symplectic form. It turns out that these special symplectic connections are exactly the distinguished connections of PCS structures that can locally be realized as reductions of locally flat parabolic contact structures by a transversal infinitesimal automorphism.

For a symmetry reduction $\tilde{M} \to M$ as above, one obtains a direct relation between natural vector bundles, and sections of such a bundle over M naturally form a subspace of sections of the corresponding bundle over $\tilde{M}$. This easily implies that natural differential operators for parabolic contact structures can be "descended" to natural operators on PCS structures.

Since there is a canonical connection for PCS structures, constructing natural operators is not difficult. However, starting from a complex upstairs, one obtains a complex downstairs. Applying this to BGG sequences, one obtain families of differential complexes on manifolds endowed with special symplectic connections, in particular on $\mathbb{C}P^n$.

In some cases, one may descend relative BGG sequences that are complexes or subcomplexes in curved BGG sequences, to obtain natural differential complexes associated in more general situations, for example on Kähler manifolds.