Quaternionic Complexes

Andreas Čap

University of Vienna

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- The machinery of BGG sequences offers a uniform construction for most of these operators and a calculus relating them to differential forms with values in auxiliary bundles.
- If the almost quaternionic structure is quaternionic (i.e. satsifies an integrability condition), then we obtain a large number of natural complexes, many of which are elliptic.

Structure

1. Basic notions and motivation
2. Invariant differential operators
3. Quaternionic complexes

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There are two possible versions of a quaternionic analogue of this concept, since (in contrast to $\mathbb{C}$) the skew field $\mathbb{H}$ of quaternions has many automorphisms.

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These structures can be equivalently defined as first order G–structures corresponding to the subgroups $GL(n, \mathbb{H})$ respectively $S(GL(1, \mathbb{H})GL(n, \mathbb{H}))$ of $GL(4n, \mathbb{R})$. Integrability then is the standard concept for G–structures.

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motivation

(1) One of the two maximal irreducible special Riemannian holonomies is $Sp(1)Sp(n) \subset SO(4n)$, corresponding to quaternion–Kähler (qK) manifolds. These have an underlying quaternionic structure (but not an underlying complex structure in general).

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(2) For $n = 1$ almost quaternionic structure are equivalent to conformal structures in dimension 4, and integrability corresponds to self duality. In many respects, almost quaternionic structures are the "right" higher dimensional analog of four dimensional conformal structures (e.g. for Penrose transforms and twistor theory).

Structure

An almost quaternionic manifold (M, Q) carries a G–structure with structure group $G_0 := S(GL(1, \mathbb{H})GL(n, \mathbb{H})) \subset GL(4n, \mathbb{R})$. Hence any representation of G_0 gives rise to a natural vector bundle on M, and tensor bundles may admit a finer decomposition according to the restriction of the corresponding representation of $GL(4n, \mathbb{R})$ to the subgroup G_0.

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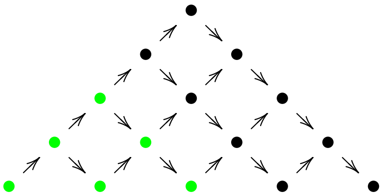
This gives rise to a decomposition of the de–Rham complex of the form (written out for $n = 1$ and $n = 2$):



The components $d^{1,0} : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)$ and $d^{0,1}$ of the exterior derivative are differential operators intrinsic to an almost quaternionic structure. To understand more general examples of such operators, we have to first study the special case $\mathbb{H}P^n$.

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the homogeneous model

Consider $G := SL(n+1, \mathbb{H})$, and let $P \subset G$ be the stabilizer of a quaternionic line in $\mathbb{H}^{n+1}$, so $\mathbb{H}P^n \cong G/P$. Denoting by $o = eP \in HP^n$ the base point, we get $T_o\mathbb{H}P^n \cong \mathfrak{g}/\mathfrak{p} \cong \mathbb{H}^n$. Mapping $g \in P$ to $T_o\ell_g$ induces a surjection $P \to G_0 = S(GL(1,\mathbb{H})GL(n,\mathbb{H}))$. In particular, any representation of G_0 canonically extends to P, thus giving rise to a homogeneous vector bundle on $\mathbb{H}P^n$.

For a homogeneous vector bundle $E \to \mathbb{H}P^n$, the space $\Gamma(E)$ of smooth sections carries a canonical representation of G ("parabolic induction"). These are principal series representations of G. A differential operator intrinsic to the almost quaternionic structure has to define a G–equivariant map $\Gamma(E) \to \Gamma(F)$. Using a dualization argument, G–equivariant differential operators are equivalent to homomorphisms of generalized Verma modules. Using central character and Harish–Chandra's theorem, this leads to strong restrictions

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- any irreducible representation of G_0 occurs in exactly one position in one pattern only

Existence of homomorphisms between generalized Verma modules was proved in the 1970's by Bernstein–Gelfand–Gelfand and Lepowsky, so on the level of $\mathbb{H}P^n$ one has a fairly complete understanding (in an unusual equivalent picture) of invariant differential operators.

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These proofs are purely combinatorial and even the results are difficult to translate into the language of differential operators. In joint work with J. Slovák and V. Souček (improved later by D. Calderbank and T. Diemer), we gave an independent construction phrased directly in terms of differential operators. This construction generalizes without changes to arbitrary almost quaternionic structures using the fact that they can be described as Cartan geometries, i.e. as "curved analogs" of the homogeneous space $\mathbb{H}P^n = G/P$.

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tractor bundles

Hence any representation of P gives rise to a natural vector bundle on almost quaternionic manifolds. In particular we can use restrictions of representations of G. The corresponding bundles are called *tractor bundles*. These bundles define unusual geometric objects but have the advantage that they carry canonical linear connections induced by the Cartan connection ω.

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- by Kostant's version of the BBW theorem the pointwise homologies are exactly the bundles in the pattern of differential operators corresponding to V

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the BGG construction

Using algebraic tools, one constructs higher order natural operators

$$\Gamma(H_k(T^*M, \mathcal{V}M)) \to \Omega^k(M, \mathcal{V}M),$$

which split the tensorial projections. Using these, the covariant exterior derivatives can be compressed to higher order differential operators

$$D^V : \Gamma(H_k(T^*M, \mathcal{V}M)) \to \Gamma(H_{k+1}(T^*M, \mathcal{V}M)),$$

and decomposing the homology bundles into irreducibles, one arrives at the patterns of operators.

Structure

For structures locally isomorphic to $\mathbb{H}P^n$, the tractor connections are flat, so the twisted de-Rham sequence is a resolution of the constant sheaf V. It is then easy to show that also the BGG sequence $(\Gamma(H_*(T^*M, \mathcal{V}M)), D^V)$ is a complex which computes the same cohomology. This can be used to show that in the locally flat case, we recover the classical BGG resolution.

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For general structures, the composition of two covariant exterior derivatives is given by the action of the curvature of the tractor connection. Hence also $D^V \circ D^V \neq 0$ in general, but individual components of the operators still may have trivial composition. For torsion free (i.e. quaternionic) geometries we were able to deduce an explicit condition on the representations inducing the two bundles which ensures vanishing of the composition of two subsequent components of D^V.

Subcomplexes

The weights of the representations inducing the bundles in a BGG pattern are described in terms of the action of the Weyl group of $\mathfrak{g}$. The vanishing criterion can be applied systematically to show that, for quaternionic structures, each BGG sequence contains a number of subcomplexes. In the triangular shape, the composition of any two upwards directed arrows or any two downwards directed arrows is zero. Explicitly, in the case $n = 2$, we have the following subcomplexes in each BGG sequence:

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Ellipticity

A highest weight for $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{H})$ is given by $2n$ non–negative integers (writing it as a linear combination of fundamental weights). If only the first and last of these coefficients are nonzero, then we were able to prove that the subcomplexes along the edges of the triangle are elliptic.
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A particularly important example is the following

Theorem. (C, 2005) Let V be the adjoint representation $\mathfrak{g}$. Then for a quaternionic manifold (M, Q), the elliptic subcomplex along the left edge of the triangle can be interpreted as a deformation complex in the category of quaternionic structures.