From holonomy reductions of Cartan geometries to geometric compactifications

Andreas Čap[1]

University of Vienna
Faculty of Mathematics

Berlin, November 11, 2016

[1] supported by project P27072–N25 of the Austrian Science Fund (FWF)

- This talk is based on joint work with Rod Gover and partly with Matthias Hammerl.
- Given a manifold $\overline{M}$ with boundary ∂M and interior M, the basic topic of this talk are geometric structures on M and ∂M, which are compatible in a certain sense.
- Starting from the example of conformally compact metrics and Poincaré–Einstein metrics, I will discuss how such structures can be obtained from a holonomy reduction of a "background structure" on $\overline{M}$ and from weakenings of this concept.
- In particular, I will discuss the concepts of projective compactness of order one and two and make some remarks on c–projective compactness.

Contents

Background

Throughout this talk we consider manifolds $\overline{M} = M \cup \partial M$ with boundary. A *local defining function* r for ∂M defined on an open subset $U \subset \overline{M}$ is a smooth function $r : U \to \mathbb{R}_{\geq 0}$ such that $r^{-1}(\{0\}) = U \cap \partial M$ and dr is nowhere vanishing on $U \cap \partial M$.

Let g be a Riemannian metric on M. Then

- g is *conformally compact* if locally $r^2 g$ admits a smooth extension to $\overline{M}$ with non–degenerate boundary values for one or equivalently any defining function r for ∂M.
- If in addition g is Einstein with negative Einstein constant, then it is called a *Poincaré–Einstein metric*.

If g is conformally compact, then the boundary values of $r^2 g$ give rise to a well defined conformal class c on ∂M, the *conformal infinity* of g.

If g is conformally compact, then the conformal class $[g]$ on M admits a smooth extension to $\overline{M}$. Try to revert this by taking a conformal class c on $\overline{M}$ as a "background structure" and look for appropriate metrics in c.

Recall that a choice of $g \in c|_M$ is equivalent to the choice of a nowhere vanishing section of the density bundle $\mathcal{E}[w]$ for some $w \neq 0$ over M. Let us take $\sigma \in \Gamma(\mathcal{E}[1]|_M)$ and write $g_\sigma \in c|_M$. Easy exercise: $g_\sigma = \frac{1}{\sigma^2}\mathbf{g}$ is conformally compact iff σ extends by 0 to a defining density for ∂M.

Next, let $\mathcal{T} \to \overline{M}$ be the standard tractor bundle, h the tractor metric and $\nabla^{\mathcal{T}}$ the (metric) tractor connection. There is a projection $\Pi : \mathcal{T} \to \mathcal{E}[1]$ and a natural second order operator $D : \Gamma(\mathcal{E}[1]) \to \Gamma(\mathcal{T})$ (tractor–D) such that $\Pi \circ D = \mathrm{id}$.

holonomy description of P-E metrics

Theorem

The metric g_σ is Einstein on M iff $\nabla_{(a}\nabla_{b)_0}\sigma + \mathsf{P}_{(ab)_0}\sigma = 0$ or equivalently $s = D\sigma \in \Gamma(\mathcal{T}|_M)$ is parallel for $\nabla^{\mathcal{T}}$. The Einstein constant is negative iff $h(s,s)$ is positive on M.
If s is parallel, it extends to a parallel section over $\overline{M}$, so σ admits a smooth extension to all of $\overline{M}$. If g_σ does not extends smoothly to any neighborhood of a boundary point, then the extension of σ is a defining density for ∂M, so g_σ is P-E.

This exhibits P-E–metrics as reductions of the holonomy of $\nabla^{\mathcal{T}}$ ("conformal holonomy") to $O(n,1) \subset O(n+1,1)$. The decomposition $\overline{M} = M \cup \partial M$ is the *curved orbit decomposition* associated to this holonomy reduction. Observe that this suggests intermediate conditions, e.g. that σ is a defining density and $h(s,s)$ is constant (almost–csc).

This fits into the general framework of holonomy reductions of Cartan geometries studied in joint work with R. Gover and M. Hammerl. Any such reduction comes with a decomposition of the underlying manifold into “curved orbits” of different dimensions, each of which inherits a geometric structure. The dimensions of these curved orbits and the induced structures can be read off the homogeneous model of the reduction.

For the reduction $O(n,1) \subset O(n+1,1)$, the homogeneous model is the embedding of the equator into a conformal sphere. So the curved orbits are open with an induced metric or embedded hypersurfaces with an induced conformal structure.

As we shall see below, there are several examples in which the relation of the geometries on the curved orbits to the ambient structure is much less obvious.

Projective structures

A projective structure on a manifold N is given by an equivalence class $p = [\nabla]$ of torsion–free connections on TN, which have the same geodesics up to parametrization. Via its Levi–Civita connection, any metric on N determines a projective structure, but generic projective classes are not of this form. For $w \in \mathbb{R}$, we denote by $\mathcal{E}(w)$ the bundle of densities of projective weight w.

Equivalently, a projective structure can be described by the (linear) *tractor connection* connection $\nabla^{\mathcal{T}^*}$ on the *cotractor bundle* $\mathcal{T}^* := J^1\mathcal{E}(1)$ or via the dual bundle $\mathcal{T}$. By definition, $\mathcal{T}^*$ comes with a projection Π to $\mathcal{E}(1)$, with kernel isomorphic to $T^*N \otimes \mathcal{E}(1)$.

Any nowhere–vanishing section of some $\mathcal{E}(w)$ with $w \neq 0$ uniquely selects a connection in p for which it is parallel. Connections which preserve some density are called *special*. In particular, this is always true for Levi–Civita connections.

Reductions of projective holonomy

Let us take the sphere viewed as the ray projectivization of $\mathbb{R}^{n+1}$ as the homogeneous model of n–dimensional projective structures. Then there are two simple holonomy reductions relevant for us:

- $SL(n,\mathbb{R}) \subset SL(n+1,\mathbb{R})$ corresponds to the choice of a hyperplane, giving rise to flat metrics on hemispheres and the equator as a totally geodesic embedded hypersurface.
- Curved analogs come from parallel sections $s \in \Gamma(\mathcal{T}^*)$ corresponding to $\sigma \in \Gamma(\mathcal{E}(1))$ satisfying a 2nd order equation. On the open curved orbit $\{\sigma \neq 0\}$, one obtains a Ricci–flat connection in the projective class. The closed curved orbit $\{\sigma = 0\}$ is an embedded hypersurface which inherits a projective structure. In the metric case, this comes with an orthogonal holonomy reduction (see below).

Reductions of projective holonomy (2)

For $SO(p+1, n-p) \subset SL(n+1, \mathbb{R})$, the homogeneous model is given by decomposing into rays which are positive, negative, and null, respectively. Hence we get open curved orbits which inherit hyperbolic metrics of different signature and conformal spheres embedded as hypersurfaces.

Curved analogs are given by non–degenerate parallel sections of $S^2\mathcal{T}^*$ equivalent to $\sigma \in \Gamma(\mathcal{E}(2))$ satisfying a third order equation.

- Open orbits inherit representative connections ∇ for which P_{ab} (Schouten) is non–degenerate with $\nabla \mathrm{P} = 0$. Hence these are Levi–Civita connections of Einstein metrics of two signatures.
- The other curved orbits are separating hypersurfaces which inherit a conformal structure of signature (p, q) via the projective second fundamental form (non–degenerate).

The concept of projective compactness

It turns out that there is a general definition of projective compactness of order α for $\alpha \in (0,2]$ which includes the holonomy reductions described above for $\alpha = 1$ respectively $\alpha = 2$. Recall that connections ∇ and $\hat{\nabla}$ on TN are projectively equivalent iff $\hat{\nabla}_\xi \eta = \nabla_\xi \eta + \Upsilon(\xi)\eta + \Upsilon(\eta)\xi$ for some $\Upsilon \in \Omega^1(N)$ and all $\xi, \eta \in \mathfrak{X}(N)$. In this case, we symbolically write $\hat{\nabla} = \nabla + \Upsilon$.

Definition

For $\overline{M} = M \cup \partial M$ and $\alpha \in (0,2]$, a torsion–free linear connection ∇ on TM is called *projectively compact* of order α if locally around each $x \in \partial M$, there is a defining function ρ such that the projectively modified connection $\hat{\nabla} = \nabla + \frac{d\rho}{\alpha\rho}$ admits a smooth extension to the boundary.

This is easily seen to be independent of ρ and it provides a smooth extension of the projective class $[\nabla]$.

Some basic facts

- The condition $\alpha \leq 2$ ensures that from the point of view of ∇, the boundary ∂M is at infinity.
- Suppose that ∇ is special, and its projective class admits a smooth extension to all of $\overline{M}$. Then projective compactness of order α is equivalent to the fact that any $0 \neq \sigma \in \Gamma(\mathcal{E}(\alpha)|_M)$ which is parallel for ∇ extends to a defining density for ∂M.
- For Ricci–flat connections (non–Ricci–flat Einstein metrics), extendability of the projective class and inextendability of ∇ imply projective compactness of order $\alpha = 1$ ($\alpha = 2$).

Remarks: (1) Extendability of the projective class is equivalent to smooth extendability of the tracefree part of the connection coefficients in local coordinates. (2) The condition on a defining density is equivalent to a specific rate of volume growth.

Asymptotic forms

One can directly prove existence of many local examples of projectively compact metrics:

Theorem

For $1 \leq n \in \mathbb{N}$, suppose that g is a pseudo–Riemannian metric on M such that there are local defining functions ρ for ∂M for which g admits an asymptotic form $\boxed{g = C\frac{d\rho^2}{\rho^{2n}} + \frac{h}{\rho^n}}$, where h is smooth up to the boundary and non–degenerate on $T\partial M$, and C is a smooth function asymptotic to a non–zero constant on ∂M. Then g is projectively compact of order $\alpha = 2/n$.

For $\alpha = 2$, this form is available (with the same C and conformally related h) for any defining function. For other values of α, it singles out specific defining functions.

Suppose that a projective class contains the Levi–Civita connection of g_{ab}. Then one obtains an obvious section σ^{ab} of $S^2TM(-2)$, which solves the *metricity equation* and via a splitting operator a section H of $S^2\mathcal{T}$. As a bundle metric on $\mathcal{T}^*$, H has a well defined determinant, which coincides with $Scal(g)$. This implies

If the projective class of ∇^g admits a smooth extension to $\overline{M}$, then the same holds for σ^{ab}, H, and $Scal(g)$.

One can also analyze the curvature asymptotics of projectively compact connections directly:

- For $\alpha = 1$, ρR extends to the boundary with boundary value the curvature tensor determined by the projective second fundamental form of the boundary.
- For $\alpha \neq 1$, $\rho^2 R$ extends to the boundary with boundary value the rank–one curvature tensor determined by $d\rho$. The next order, relates P_{ab} to the projective second fundamental form.

Using the extension of scalar curvature, one obtains a surprising characterization of projective compactness of order 2 for metrics and an improved description of curvature asymptotics:

Theorem

Let g be a metric on M such that $[\nabla^g]$ extends to $\overline{M}$ and let $S : \overline{M} \to \mathbb{R}$ be the extension of $Scal(g)$. Then for $x \in \partial M$, TFAE

1. $S(x) \neq 0$
2. g is projectively compact of order $\alpha = 2$ locally around x.
3. Locally around x, $S|_{\partial M}$ is constant and g admits an asymptotic form $g = C\frac{d\rho^2}{\rho^2} + \frac{h}{\rho}$ with constant C.

If these are satisfied, then $Ric_0(g)$ admits a smooth extension to the boundary, so g is asymptotically Einstein.

The metric h represents the projective second fundamental form and one obtains a detailed description of the boundary geometry.

(Almost) c–projective structures are an (almost) complex analog of classical projective structures. Both the holonomy reductions of projective structures described above have an analog in the almost c–projective setting. These two analogs look quite different, however, and we will focus to the analog of the order–two case.

The model of c–projective geometry is $\mathbb{C}P^n$ with the c–projective class defined by the Fubini–Study metric, which is homogeneous under $SL(n+1, \mathbb{C})$.

- The holonomy reduction is to $SU(p+1, n-p)$ corresponding to complex hyperbolic spaces (of different signatures) and the CR–sphere embedded into $\mathbb{C}P^n$.
- Curved analogs come with Hermitian metrics, which are Kähler–Einstein in the integrable case, and embedded hypersurfaces with the induced CR–structure.

For $\overline{M} = M \cup \partial M$ and an almost complex structure J on M, c–projective compactness can then be defined for a linear connection ∇ on TM such that $\nabla J = 0$. To get a relation to tractors, one has to assume that ∇ is *minimal*, i.e. its torsion is of type $(0, 2)$. The definition is via specific c–projective modifications admitting a smooth extension to the boundary.

- If ∇ is c–projectively compact, then J admits a smooth extension to $\overline{M}$, and thus ∂M inherits a CR–structure.
- The concept applies to *quasi–Kähler metrics* (Kähler in the integrable case) via the canonical connection.
- Formulated in tractor terms, things are closely parallel to the projective case.
- In particular, in the metric case, one gets a c–projective interpretation of scalar curvature (of the canonical connection) which leads to extendability results.

Results

- The complete Kähler metrics on smoothly bounded, non–degenerate domains constructed from defining functions all are c–projectively compact.
- For Levi–non–degenerate boundary with a condition slightly stronger than partial integrability, c–projective compactness of a quasi–Kähler metric is equivalent to an asymptotic form. This is form more restricted than in the projective case, since the boundary value of h has to be related to the Levi–form.
- There is also an equivalent characterization in terms of the boundary value of scalar curvature (of the canonical connection).

Thank you

und

Alles Gute zum Geburtstag, Andreas!