C–Projective Compactness

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- This talk reports on joint work with Rod Gover, see arXiv:1603.07039 .
- Broad background: Given a manifold $\overline{M}$ with boundary ∂M and interior M, consider a geometric structure on M, which does not extend to $\overline{M}$. Suppose that some weakening extends to $\overline{M}$ and look for induced structures on ∂M, which are then nicely linked to the interior geometry.
- Examples: conformal compactness, more recently projective compactness (different versions depending on "order").
- My talks discusses an almost complex analog of projective compactness of order two. The weakened structure is an almost c–projective structure, so the initial structure on M is an almost complex structure and a complex affine connection, the main examples come from quasi–Kähler metrics (of any signature).

Contents

1. The concept of c–projective compactness
2. The main results on c–projectively compact metrics
3. Ideas entering the proofs

Almost c–projective structures

Let (N, J) be an almost complex manifold and let ∇ and $\hat{\nabla}$ be two linear connections on TN which preserve J. Then ∇ and $\hat{\nabla}$ are called *c–projectively equivalent* if there is a one–form $\Upsilon \in \Omega^1(N)$ such that

$$\hat{\nabla}_\xi \eta = \nabla_\xi \eta + \Upsilon(\xi)\eta + \Upsilon(\eta)\xi - \Upsilon(J\xi)J\eta - \Upsilon(J\eta)J\xi$$

This implies that $\hat{\nabla}$ and ∇ have the same torsion, and to normalize things, one usually assumes that this torsion is of type $(0,2)$ ("minimal complex connections"). Hence it is essentially the Nijenhuis tensor of J, so in the integrable case, the connections are assumed to be torsion free.

An *almost c-projective structure* on N is given by an almost complex structure J and a c–projective equivalence class of minimal complex connections. The structure is called *c–projective* if J is integrable.

In the situation $\overline{M} = M \cup \partial M$, assume that J is an almost complex structure on M and ∇ is a minimal complex linear connection on TM. Recall that a *local defining function* for ∂M is a smooth function $\rho : U \to [0, \infty)$ defined on an open subset U of $\overline{M}$ such that $U \cap \partial M = \rho^{-1}(\{0\})$ and such that $d\rho(x) \neq 0$ for any $x \in U \cap \partial M$.

The connection ∇ is called *c–projectively compact* iff locally around each $x \in \partial M$, there is a defining function ρ for ∂M, such that the c–projective modification $\hat{\nabla}$ of ∇ corresponding to $\Upsilon = \frac{d\rho}{2\rho}$ admits a smooth extension to the boundary.

It is easy to see that this then works for any local defining function. Next, J admits a smooth extension to the boundary, which defines an almost CR structure on ∂M. The extended connections are c–projectively equivalent, so we get a c–projective structure on $\overline{M}$.

Suppose further that ∇ is special, i.e. it preserves a (real) volume density on M. Then similarly to the projective case, c–projective compactness of ∇ is equivalent to the fact that the almost c–projective structure defined by ∇ extends to $\overline{M}$ plus a specific, uniform growth rate of the preserved volume density towards the boundary.

Suppose now that we have given an almost Hermitian structure (J, g) on M. Then there are complex connections on M which are metric for g, and any such is uniquely determined by its torsion. It turns out that there is a minimal complex connection preserving g iff g is *quasi–Kähler* in the terminology of Gray–Hervella. In this case, c–projective compactness of g is defined via this canonical connection.

I will mainly discuss c–projectively compact metrics in this talk.

A fundamental example

For $n \geq 2$ consider $\mathbb{C}P^{n+1}$ as a homogeneous space of the group $\tilde{G} := SL(n+2, \mathbb{C})$, which is the group of automorphisms of the obvious flat c–projective structure.

Choosing a Lorentzian Hermitian form $\langle\ ,\ \rangle$ on $\mathbb{C}^{n+2}$, we obtain $G := SU(n+1, 1) \subset \tilde{G}$ (a reduction of c–projective holonomy). Under the action of G, the space $\mathbb{C}P^{n+1}$ splits as $M_- \cup M_0 \cup M_+$ according to the signature of the restriction of $\langle\ ,\ \rangle$ to a complex line. Here $M_\pm$ are open in M while $M_0 \cong S^{2n+1}$ is an embedded hypersurface, on which G acts as the group of CR diffeomorphisms.

By construction $\overline{M} = M_- \cup M_0$ is a manifold with boundary M_0. This is biholomorphic to the closed unit disk in $\mathbb{C}^{n+1}$ whose group of biholomorphisms thereby gets identified with G.

A fundamental example (continued)

There is a metric aspect to this picture: The interior M_- carries a complete Kähler Einstein metric, for which G is the isometry group, while M_0 can be interpreted as a CR boundary at infinity.

From the complex analytic point of view, this is the Bergmann metric, which can be constructed via a potential obtained from a certain defining function for the boundary.
Alternatively, M_- can be identified with the complex hyperbolic space $\mathcal{H}_{\mathbb{C}}^{n+1}$ so there is the complex version of the hyperbolic metric, which also induces the geometry on M_0.

As the picture suggests, one proves that the Levi–Civita connection of the hyperbolic metric is c–projectively compact and the CR structure on S^{2n+1} is exactly the induced boundary geometry.

In the discussion of equivalent conditions, we will always at least assume that the almost complex structure J is defined on all of $\overline{M}$. Then ∂M is a real hypersurface in an almost complex manifold, thus inheriting an almost CR structure $H \subset T\partial M \subset T\overline{M}|_{\partial M}$. Let $\mathcal{L} : \Lambda^2 H \to T\partial M/H$ be the induced Levi bracket. Then we will always assume that

- the almost CR structure is non–degenerate, i.e. $\mathcal{L}_x$ is non–degenerate for each $x \in \partial M$.
- If J is non–integrable, then along ∂M, $\mathcal{N}$ has values in H, which implies partial integrability.

Under these assumptions, a local defining function ρ gives rise to a one form $\theta := -d\rho \circ J$ which restricts to a contact form on ∂M. This has the property that $d\theta$ is Hermitian on $T\overline{M}|_{\partial M}$.

One main result is that, under the above assumptions on the boundary, c–projective compactness of a quasi–Kähler metric g is equivalent to existence of a certain asymptotic form in terms of any local defining function ρ. The asymptotic form in question is

$$g = \frac{C}{\rho^2}\left(d\rho \odot d\rho + \theta \odot \theta\right) + \frac{1}{\rho} h$$

for a non–zero real constant C and a Hermitian form h which is smooth up to the boundary and s.t. for ζ with $d\rho(\zeta) = \theta(\zeta) = 0$ and any ξ, $h(\xi, J\zeta)$ approaches $Cd\theta(\xi, \zeta)$ at the boundary.

For a non–degenerate domain $\Omega \subset \mathbb{C}^{n+1}$ with smooth boundary $\partial\Omega$ choose a defining function ρ for the boundary. Then using $-\log(\rho)$ as a potential, one obtains a complete pseudo–Kähler metric g_ρ on Ω. This is easily seen to admit an asymptotic form as above (with $C = -1$ and $h = d\theta(\ ,J\ )$), and hence is c–projectively compact.

The role of scalar curvature

In the usual setup $(\overline{M} = M \cup \partial M, J)$ assume that we have given a quasi–Kähler metric g (of any signature) on M such that the c–projective structure defined by the canonical connection ∇ of g admits a smooth extension to $\overline{M}$. Then one shows that the scalar curvature S of ∇ admits a smooth extension to all of $\overline{M}$ and that

∇ is c–projectively compact locally around $x \in \partial M$ if and only the boundary value of S is non–zero at x and ∇ does not admit a smooth extension to any neighborhood of x. If this is the case, then the boundary value of S is constant on a neighborhood of x.

One can describe the curvature asymptotics of ∇ in more detail. In particular, while both g and the Ricci tensor diverge as $1/\rho^2$, the tracefree part of Ricci admits a smooth extension to the boundary, so g is asymptotically Einstein.

- Sufficiency of the asymptotic form for c–projective compactness is proved by more or less direct computation.
- The other results are not proved separately, but depend on each other in a subtle way. For example, properties of S are important in the proof of necessity of the asymptotic form.
- The basic tools come from the description of c–projective structures as parabolic geometries and the resulting tractor calculus. The main objects are the real tractor bundle $\mathcal{H} = \mathrm{Herm}(\mathcal{T})$ and its dual $\mathcal{H}^* = \mathrm{Herm}(\mathcal{T}^*)$.
- If the c–projective structure extends $\overline{M}$, these bundles and the related tools (splitting operators, tractor connections, prolongation connections, etc.) are defined on all of $\overline{M}$. Assuming c–projective compactness, one even knows explicit connections and thus splittings of the tractor bundles into slots that are smooth up to the boundary.

- Via the metricity equation, a quasi–Kähler metric g gives rise to an explicitly known section $\Phi \in \Gamma(\mathcal{H}^*)$ such that $\nabla^p\Phi = 0$.
- As a Hermitian form on $\mathcal{T}^*$, Φ has a well defined determinant, which turns out to be S (up to a non–zero constant).
- If $\Phi \in \Gamma(\mathcal{H}^*)$ is non–degenerate, one can form an inverse, call this $\Psi \in \Gamma(\mathcal{H})$. If $\nabla^p\Phi = 0$, then one derives restrictions on $\nabla^{\mathcal{H}}\Psi$.
- The irreducible quotient of $\mathcal{H}$ is a density bundle, and c–projective compactness is related to existence of defining densities of that weight.
- Assuming c–projective compactness of g, one can analyze Ψ in terms of splittings into slots which are smooth up to the boundary to get the asymptotic form.