Cartan geometries
lecture 1

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- This is the first in a series of two lectures on Cartan geometries.
- I will start with the general motivation for the notion of a Cartan geometry, a family of basic examples to get a feeling for the concept, and some general features of Cartan geometries.
- The focus of the lecture will lie on the Cartan description of conformal structures which brings in an important aspect of "higher order".
- I will also discuss the relation to other descriptions of conformal structures (tractors, ambient metric and Poincaré metric) and to conformal compactness.
- Families of more general Cartan geometries and several general tools for Cartan geometries will be discussed in the second lecture.

Contents

- **Basic principle**: Describe manifolds endowed with appropriate geometric structures as "curved analogs" of a "homogeneous model" G/H.
- So one starts with a model structure on G/H, whose automorphisms are exactly the left actions of elements of G.
- The motivation for Cartan's definition then is the following.

- $p : G \to G/H$ is an H-principal bundle that carries the left Maurer-Cartan form $\omega \in \Omega^1(G, \mathfrak{g})$, where $\mathfrak{g} = \mathrm{Lie}(G)$.
- The left actions of elements of G are exactly the diffeomorphisms of G/H that admit an H-equivariant lift $\Phi : G \to G$ such that $\Phi^*\omega = \omega$.
- Observe that $d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)] = 0$ by the Maurer-Cartan equation.

The definition of a Cartan geometry and of its curvature is simply an abstract version of this.

Definition

(1) A *Cartan geometry* of type (G, H) on a smooth manifold M is given by a principal H-bundle $p : \mathcal{G} \to M$ and a *Cartan connection* $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, i.e.

- each $\omega(u) : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism
- $(r^h)^*\omega = \mathrm{Ad}(h)^{-1} \circ \omega$ for all $h \in H$ (equivariancy)
- $\omega(\zeta_X) = X$ for all $X \in \mathfrak{h} \subset \mathfrak{g}$ (fundamental fields)

(2) The *curvature* $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ of the geometry $(\mathcal{G}, \omega)$ is defined by $K(\xi, \eta) = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)]$.

- Such geometries exist only for $\dim(M) = \dim(G/H)$.
- There is an obvious notion of morphisms, and morphisms induce local diffeomorphisms between the base spaces.
- The curvature of a Cartan geometry vanishes identically if and only if it is locally isomorphic to its *homogeneous model* G/H.

A class of examples

Let $H \subset GL(n,\mathbb{R})$ be a closed subgroup and $G := H \ltimes \mathbb{R}^n$, the corresponding group of affine transformations. Then $G/H \cong \mathbb{R}^n$ and $G \to G/H$ is the natural flat H-structure on $\mathbb{R}^n$. In particular for $H = O(n)$ one obtains the flat Riemannian metric on $\mathbb{R}^n$.

Proposition

A Cartan geometry of type $(G = H \ltimes \mathbb{R}^n, H)$ on an n-manifold M is equivalent to an H-structure endowed with a compatible connection; K encodes torsion and curvature of this connection.

- **Sketch of proof**: $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}^n$ (semi-direct sum) and splitting $\omega = \gamma \oplus \theta$ accordingly, γ and θ are H-equivariant.
- θ is strictly horizontal, hence equivalent to a homomorphism to the frame bundle of M; γ is a principal connection.
- Using $d\omega = d\gamma \oplus d\theta$ and the structure of $\mathfrak{g}$, the interpretation of K follows.

This may look disappointing, because the compatible connection looks like a choice. But a slight improvement already leads to interesting examples with type $(H \ltimes \mathbb{R}^n, H)$:

(1) $H = O(n)$: Starting with a Riemannian manifold (M, g) and its ON-frame bundle $(\mathcal{G}, \theta)$, there is a unique Cartan connection ω such that K is torsion-free, i.e. has values in $\mathfrak{o}(n) \subset \mathfrak{g}$.
(2) $H \subset O(n)$: There is a H-invariant linear subspace $N \subset \Lambda^2\mathbb{R}^{n*} \otimes \mathbb{R}^n$ ("normalization condition") such that one similarly gets a canonical Cartan connection with "torsion in N".

Even for these simple examples, the Cartan picture provides a beautiful point of view, e.g. for the study of submanifold geometry. Also, specializing the general results on Cartan geometries (see below) to these cases leads to substantial theorems. However, the Cartan point of view is not really needed here, things can also be rephrased in terms of principal connections. This is not true for more general Cartan geometries.

General features of Cartan geometries

- K defines a fundamental and complete invariant
- representations of H induce natural vector bundles
- For the representation on $\mathfrak{g}/\mathfrak{h}$ induced by Ad, one obtains $\mathcal{G} \times_H (\mathfrak{g}/\mathfrak{h}) \cong TM$, so all tensor bundles are associated.
- Starting from distinguished curves in G/H, one obtains general notions of distinguished curves in Cartan geometries.
- Natural notion of infinitesimal automorphisms of a Cartan geometry in $\mathfrak{X}(\mathcal{G})$. Automorphisms of $(\mathcal{G}, \omega)$ form a Lie group of dimension $\leq \dim(G)$ with Lie algebra formed by complete infinitesimal automorphisms.
- Several constructions relating geometries of different type (Correspondence spaces, Fefferman constructions, extension functors).

conformal geometry

Conformal structures can be defined as H-structures for $H := CO(n)$. Equivalently, one prescribes an inner product up to scale on each tangent space T_xM depending smoothly on x. This determines an equivalence class of Riemannian metrics on M where $\tilde{g} \sim g$ iff there is $f \in C^\infty(M, \mathbb{R})$ such that $\tilde{g} = e^{2f} g$.

We restrict to $n \geq 3$ here, for $n = 2$ conformal geometry is an entirely different story because of the relation to holomorphy. There is a 2d-analog of what we discuss ("Möbius structures").

Any conformal structure admits compatible torsion-free connections ("Weyl connections"). Pointwise, these form an n-dimensional affine space. Algebraic results imply that there might be a Cartan geometry description with structure group an n-dimensional extension of $CO(n)$, but getting the "right" groups in this way is hard. So we look for a homogeneous model.

The conformal sphere

Put $G := SO_0(n+1,1)$ for a Lorentzian inner product on $\mathbb{R}^{n+2}$. Then G acts transitively on S^n, viewed as a space of isotropic rays. Hence $S^n = G/P$, where $P \subset G$ is the stabilizer of one such ray. Elementary arguments show that the action of G is conformal for the class of the round metric on S^n.

- Denoting by $o \in S^n$ the point fixed by P, the map $g \mapsto T_o\ell_g$ defines a surjective homomorphism $P \to CO(n) =: G_0$.
- The kernel of this homomorphism is normal subgroup $P_+ \subset P$ isomorphic to $\mathbb{R}^{n*}$ and $P = G_0 \ltimes P_+$.
- For any $g \in P_+$, ℓ_g coincides with id_{S^n} to first order in o, but, for $g \neq e$, not to second order. In particular, there is no G-invariant linear connection on TS^n.

This “higher order issue” is crucial for conformal geometry.

A Cartan geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, P) determines a conformal structure on M: Putting $\mathcal{G}_0 := \mathcal{G}/P_+$ we obtain a principal $CO(n)$-bundle over M and projecting the values of ω from $\mathfrak{g}$ to $\mathfrak{g}/\mathfrak{p} \cong \mathbb{R}^n$ leads to a strictly horizontal, equivariant form $\theta \in \Omega^1(\mathcal{G}_0, \mathbb{R}^n)$.

Theorem (E. Cartan)

Any conformal structure on a smooth n-manifold with $n \geq 3$ can be canonically extended to a Cartan geometry of type (G, P) whose curvature K satisfies a normalization condition.

The classical way to prove this is based on prolongation.

- For a conformal structure $(\mathcal{G}_0, \theta)$, the Weyl connections can be viewed as principal connections on $\mathcal{G}_0$. In each point $u_0 \in \mathcal{G}_0$ their values form a n-dimensional affine space. (Observe that torsion-freeness is a point-wise condition on principal connection forms.)

- Attaching this to u_0, one obtains a fiber bundle $\mathcal{G} \to M$. The action of G_0 on $\mathcal{G}_0$ can be extended to an action of P on $\mathcal{G}$ making it into a principal P-bundle.
- Since a point in $\mathcal{G}$ is the value of a principal connection form in some point of $\mathcal{G}_0$, one obtains a tautological one-form $\omega_0 \in \Omega^1(\mathcal{G}, \mathfrak{g}_0)$. For each $u \in \mathcal{G}$ over u_0, $\theta(u_0) \oplus \omega_0(u)$ defines a linear isomorphism $T_{u_0}\mathcal{G}_0 \to \mathfrak{g}/\mathfrak{p}_+$.
- Still in a point $u \in \mathcal{G}$, the possible lifts of this to a linear isomorphism $T_u\mathcal{G} \to \mathfrak{g}$ which are compatible with fundamental fields form an affine space. Any such lift determines a point wise "curvature" which has values in $\mathfrak{g}_0$.
- An algebraic computation then shows that there is a unique such lift for which this has vanishing Ricci-type contraction.
- These lifts are easily seen to fit together smoothly, and thus define a canonical Cartan connection.

tractor bundles

Starting from the Cartan geometry associated to a conformal structure, representations of P give rise to natural vector bundles. Since $P/P_+ \cong CO(n)$ this in particular applies to representations of $CO(n)$, so in particular tensor bundles are obtained in this way. Representations on which P_+ acts non-trivially lead to a different kind of geometric objects that "see" higher order data.

Taking a representation $\mathbb{V}$ of G and restricting to P gives rise to so-called *tractor bundles*, e.g. $\mathcal{T}M$ via $\mathbb{V} = \mathbb{R}^{n+2}$ ("standard tractors") and $\mathcal{A}M$ via $\mathbb{V} = \mathfrak{g}$ ("adjoint tractors"). On any tractor bundle, ω induces a linear connection, the *tractor connection*.

This is similar to the construction of an induced connection. Conversely, one can recover $(\mathcal{G}.\omega)$ from $(\mathcal{T}M, \nabla^{\mathcal{T}})$ or $(\mathcal{A}M, \nabla^{\mathcal{A}})$ as an "adapted frame bundle" with Cartan connection.

explicit description

Choosing a metric g in the conformal class, the Levi-Civita connection ∇ defines a section of the bundle $\mathcal{G} \to \mathcal{G}_0$. Via this, any associated bundle $\mathcal{G} \times_P \mathbb{V}$ can be viewed as associated to $\mathcal{G}_0$ and hence described explicitly.

E.g. the choice of g identifies $\mathcal{T}M$ with $\mathcal{E}[1] \oplus T^*M[1] \oplus \mathcal{E}[-1]$ and there is an explicit formula for the change of this isomorphism under a conformal rescaling. The natural bundle metric on $\mathcal{T}M$ and the tractor connection can be described explicitly via g and ∇.

Now one may turn things around, forget the Cartan geometry in the background, and directly define a tractor bundle, tractor metric and tractor connection associated to a conformal structure via these explicit formulae. (This goes back to T. Thomas in the 1930's.) From above, we know that this provides a description equivalent to the Cartan geometry.

now really explicit in abstract index notation

Given a choice of metric g, let ∇_a be the Levi-Civita connection, and P_{ab} be the Schouten tensor of g. Write sections of $\mathcal{T}M$ as column vectors with the $\mathcal{E}[1]$-component on top and let h be the tractor metric. For a rescaling $\hat{g} = e^{2f}g$ put $\Upsilon = df$ and let $\mathbf{g}_{ab}$ be the (canonical) *conformal metric* with inverse $\mathbf{g}^{ab}$. Then

$$\begin{pmatrix} \hat{\sigma} \\ \hat{\mu}_a \\ \hat{\rho} \end{pmatrix} = \begin{pmatrix} \sigma \\ \mu_a + \Upsilon_a \sigma \\ \rho - \mathbf{g}^{ab}(\Upsilon_a \mu_b + \frac{1}{2}\Upsilon_a \Upsilon_b \sigma) \end{pmatrix}, \quad \nabla^{\mathcal{T}}_a \begin{pmatrix} \sigma \\ \mu_b \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + \mathbf{g}_{ab}\rho + \mathrm{P}_{ab}\sigma \\ \nabla_a \rho - \mathbf{g}^{ij}\mathrm{P}_{ai}\mu_j \end{pmatrix}.$$

$$h\left(\begin{pmatrix} \sigma \\ \mu_a \\ \rho \end{pmatrix}, \begin{pmatrix} \tilde{\sigma} \\ \tilde{\mu}_a \\ \tilde{\rho} \end{pmatrix} \right) = \sigma\tilde{\rho} + \mathbf{g}^{ab}\mu_a \tilde{\mu}_b + \rho\tilde{\sigma}.$$

This now allows one to directly construct invariant differential operators, e.g. $D^A : \Gamma(\mathcal{E}[w]) \to \Gamma(\mathcal{T}M \otimes \mathcal{E}[w])$ defined by

$$D^A \tau := \begin{pmatrix} w(n+2w-2)\tau \\ (n+2w-2)\nabla_a \tau \\ -\mathbf{g}^{ij}(\nabla_i \nabla_j + \mathrm{P}_{ij})\tau \end{pmatrix}.$$

The ambient metric

The homogeneous model arises as light-like rays in an ambient flat Lorentzian vector space. The ambient metric provides, formally up to some order, an analog for general conformal structures.

- A conformal structure on M defines a line subbundle $\mathcal{C} \subset S^2T^*M$ endowed with a tautological degenerate metric.
- Try to extend this to a Ricci-flat Lorentzian metric $g_\#$ on $M_\# := \mathcal{C} \times (-\epsilon, \epsilon)$.
- This can be done formally along $\mathcal{C}$ to infinite order for odd n and up to order $n/2$ for even n. The result is unique up to that order and up to a diffeomorphism fixing $\mathcal{C}$.

[Č,Gover,03]: The standard tractor bundle $\mathcal{T}M$ can be obtained by a quotient construction from $TM_\#$ and $g_\#$ descends to a bundle metric on $\mathcal{T}M$. If $g_\#$ is Ricci-flat along $\mathcal{C}$ then this coincides with the tractor metric and $\nabla^{g_\#}$ descends to the tractor connection.

Poincaré metric and conformal compactness

For the homogeneous model, the space of time-like rays can be identified with hyperbolic space $\mathcal{H}^{n+1}$, which leads to the picture of the model structure on S^n as the conformal infinity of $\mathcal{H}^{n+1}$. The (formal) *Poincaré metric* extends this to general structures (closely related to the ambient metric). General conformally compact metrics admit a nice tractorial description:

Consider a manifold $\overline{M} = M \cup \partial M$ with boundary and a conformally compact metric g on M. Then the conformal structure $[g]$ extends to all of $\overline{M}$ and gives the conformal infinity on ∂M.

- g determines a tractor I, which extends smoothly to $\overline{M}$. If g is Einstein, then I is parallel.
- Along ∂M, I is related to the normal tractor, which allows one to relate $\mathcal{T}\overline{M}|_{\partial M}$ and $\mathcal{T}\partial M$.

References

For the classical construction of the conformal Cartan Connection:

- S. Kobayashi: "Transformation Groups in Differential Geometry", Springer 1972

For generalities on Cartan geometries:

- R.W. Sharpe: "Differential geometry. Cartan's generalization of Klein's Erlangen program.", Springer 1997
- Chapter 1 of A. Čap, J. Slovák: "Parabolic Geometries I. Background and general theory", Amer. Math. Soc. 2009, this also contains a construction of the conformal Cartan connection.