Lie Groups

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Preface

The notion of symmetry is one of the basic concepts in mathematics. A fundamental feature of symmetries is that they form groups. If one deals with symmetries of objects that carry some additional structure, one often gets an induced structure on the group of symmetries. The case we are interested in is that symmetries act on geometric objects, i.e. on smooth manifolds. In many situations of interest, the group of symmetries itself then inherits the structure of a smooth manifold in a way compatible with the group structure. A smooth manifold together with a compatible (i.e. smooth) group structure is called a *Lie group*. The name is derived from the Norwegian mathematician Sophus Lie (1842–1899), who first encountered Lie groups and the associated Lie algebras when studying symmetries of partial differential equations.

Apart from the intrinsic interest, the theory of Lie groups and their representations is used in various parts of mathematics. As groups of symmetries, Lie groups occur for examples in differential geometry, harmonic analysis, dynamical systems, partial differential equations, and in theoretical physics. Via the closely related concept of algebraic groups, the theory of Lie groups also plays an important role in algebraic geometry. Finally, via the concept of homogeneous spaces, Lie groups provide some of the most important examples of smooth manifolds which are used in many areas.

As this basic definition suggests, the theory of Lie groups is located at the border between algebra and differential geometry, and tools from both fields are used to study Lie groups. A fundamental fact in the theory of Lie groups is that almost all of the (rather complicated) structure of a Lie group is already encoded in its *Lie algebra*. This is a much simple object, a finite dimensional vector space (the tangent space at the neutral element) which is naturally endowed with a bilinear operation. Lie algebras can then be studied using purely algebraic tools. This is the subject of the course "Lie algebras and representation theory", see the lecture notes [**Cap:Liealg**].

Here we concentrate on the study of Lie groups rather than Lie algebras. From the place of the course in the curriculum it is natural to assume the basics of analysis on manifolds and differential geometry as prerequisites. All the neccessary material is contained in the lecture notes [**AnaMf**], mostly in the first two chapters there. On the other hand, the course assumes very little background in algebra.

Let me briefly summarize the contents of the individual chapters. Chapter 1 quickly develops the basic concepts and the fundamentals of the Lie group — Lie algebra correspondence using differential geometry methods. Next, we study closed Lie subgroups, proving in particular that any closed subgroup of a Lie group is a Lie subgroup and in particular itself a Lie group. This also provides us with lots of examples, in particular via matrix groups. Then we discuss the more general concept of virtual Lie subgroups, developing the necessary material on initial submanifolds. Finally, we discuss actions and show that the space of left cosets of any Lie group by a closed subgroup canonically is a smooth manifold. We show how to use this to make objects like the space of k–dimensional linear subspaces in $\mathbb{R}^n$ into smooth manifolds.

In chapter one, we mainly discuss passing from the Lie group level to the Lie algebra level. For example, we associate the Lie algebra to a Lie group, show that Lie subgroups correspond to Lie subalgebras and homomorphism of Lie groups induce homomorphisms of Lie algebras. Chapter two focuses on results going the other way round. In particular, we prove that any finite dimensional Lie algebra is obtained from a Lie group, classify connections Lie groups with given Lie algebra and prove existence results on virtual Lie subgroups and Lie group homomorphisms. In the end of the chapter, we discuss the Maurer–Cartan form, and the fundamental theorem of calculus for smooth functions with values in a Lie group. The basic tool for all these results is the Frobenius theorem, for which we give a complete proof in 2.2.

As discussed above, many properties of Lie groups are captured by the Lie algebra. A setting in which remaining at the group level has advantages is the theory of compact Lie groups, which is discussed in chapter 3. The basic advantage here is that integration over a compact group can be used to generate invariant objects. Using this, we prove that compact groups are reductive and deduce the basic facts about representations of compact groups. We show that such representations are always completely reducible, discuss matrix coefficients and the Schur orthogonality relations. Next we move to the theory of maximal tori. The basic conjugacy theorem is proved in 3.8 by computing a mapping degree. Finally, we prove the Peter–Weyl theorem and give a short discussion of infinite dimensional representations.

The material for this course is taken from various sources. Much of the basic material is adapted from Peter Michor's course on differential geometry, see e.g. Chapter II of the book [**Michor**], which is available online. The part on the Maurer–Cartan form and the fundamental theorem of calculus is adapted from R. Sharpe's book [**Sharpe**]. For the other material, I in particular used [**Knapp**], [**Greub–Halperin–Vanstone**], and [**Adams**].

The first version of these notes were prepared when I first taught this course in spring term 2005. I would like to thank the participants of this course and the sequels in fall terms 2006/07, 2009/10, 2012/13, 2013/14, 2015/16, and 2018/19 for corrections and suggestions for improvements, which have been implemented step by step. Slightly bigger changes have been implemented in the preparation of the version for fall term 2020/21. On the one hand, I changed the basic reference for analysis on manifolds and differential geometry to my new lecture notes [**AnaMf**]. On the other hand, I took some motivation for changes from a course on matrix groups I taught in spring term 2018. Lecture notes (in German) for that course are available online, see [**Cap:Mgrp**]. Finally, I have extended the discussion of the use of maximal tori in character theory and added a discussion of the Weyl integration formula. This material is contained in a new section 3.10.

CHAPTER 1

Lie groups and homogeneous spaces

The notion of symmetry is probably one of the most basic notions of mathematics. A fundamental feature of symmetries is that they form groups. If one deals with symmetries of objects having an additional structure, then often one gets an induced structure on the group of symmetries. In the special case of symmetries of geometric objects, one often finds that the group of symmetries itself inherits a differentiable structure such that multiplication and inversion are smooth maps. Such an object is called a Lie group.

The background on differential geometry used in the chapter can be found in the lecture notes [**AnaMf**] or the book [**KMS**], which both are available online.

Lie groups and their Lie algebras

One of the fascinating features of Lie groups is that most of the (rather complicated) structure of a Lie group is encoded into the Lie algebra of the Lie group. This Lie algebra is a much simpler object, a finite dimensional vector space with a certain algebraic structure. Our first task is to develop the basics of the correspondence between Lie groups and Lie algebras.

1.1. Lie groups.

Definition 1.1. (1) A *Lie group* is a smooth manifold G endowed with a group structure with smooth multiplication. This means that we have a smooth multiplication $\mu : G \times G \to G$, an inversion $\nu : G \to G$ and a unit element $e \in G$ such that the usual group axioms are satisfied. If there is no risk of confusion, we will write gh for $\mu(g, h)$, and g^{-1} for $\nu(g)$ for $g, h \in G$.

(2) A *homomorphism* from a Lie group G to a Lie group H is a smooth map $\varphi : G \to H$ which is a group homomorphism.

Example 1.1. (1) $\mathbb{R}$ and $\mathbb{C}$ are evidently Lie groups under addition. More generally, any finite dimensional real or complex vector space is a Lie group under addition.

(2) $\mathbb{R} \setminus \{0\}$, $\mathbb{R}_{>0}$, and $\mathbb{C} \setminus \{0\}$ are all Lie groups under multiplication. Also $U(1) := \{z \in \mathbb{C} : |z| = 1\}$ is a Lie group under multiplication.

(3) If G and H are Lie groups then the product $G \times H$ is a Lie group with the evident product structures. In view of (1) and (2) we conclude that for $n \in \mathbb{N}$ the torus $\mathbb{T}^n := U(1)^n$ is a Lie group. More generally, for $m, n \in \mathbb{N}$ we have a Lie group $\mathbb{R}^m \times \mathbb{T}^n$. It turns out that these exhaust all connected commutative Lie groups.

(4) The fundamental example of a Lie group is the group $GL(V)$ of invertible linear maps on a finite dimensional real vector space V. Let us also use this example to see how the interpretation as a group of symmetries leads to the additional structure of a smooth manifold. Any linear map $V \to V$ is uniquely determined by its values on a basis of V. Fixing a basis $\{v_1, \dots, v_n\}$, the map which sends a linear map f to $(f(v_1), \dots, f(v_n))$ induces a bijection from the space $L(V, V)$ of linear maps to V^n. From linear algebra one knows that f is invertible if and only if the elements $f(v_1),\dots,f(v_n)$ form a basis

of V. The set of all bases of V is easily seen to be open in V^n (see also below). Hence we obtain a bijection between $GL(V)$ and an open subset in the vector space V^n, thus making $GL(V)$ into a smooth manifold. Smoothness of the multiplication map in this picture follows from the fact that $(f,v)\mapsto f(v)$ is a smooth map $L(V,V)\times V\to V$.

To make all that more explicit, let us consider the case $V=\mathbb{R}^n$. For the standard basis $\{e_1,\dots,e_n\}$, the element $f(e_i)\in\mathbb{R}^n$ is just the ith column of the matrix of f. Hence the above construction simply maps $f\in L(\mathbb{R}^n,\mathbb{R}^n)$ to its matrix, which is considered as an element of $M_n(\mathbb{R}):=(\mathbb{R}^n)^n=\mathbb{R}^{n^2}$. The determinant defines a smooth function $\det : M_n(\mathbb{R})\to\mathbb{R}$. In particular, $GL(n,\mathbb{R})=\det^{-1}(\mathbb{R}\setminus\{0\})$ is an open subset of $\mathbb{R}^{n^2}$ and thus a smooth manifold. The entries of the product of two matrices are polynomials in the entries of the two matrices, which shows that matrix multiplication defines a smooth map $\mu : GL(n,\mathbb{R})\times GL(n,\mathbb{R})\to GL(n,\mathbb{R})$.

It is an easy exercise to show that the determinant function is regular in each point of $GL(n,\mathbb{R})$. In particular, the subgroup $SL(n,\mathbb{R})$ of all matrices with determinant equal to 1 is a smooth submanifold of $M_n(\mathbb{R})$, so it is itself a Lie group. This is an example of the concept of a *Lie subgroup* that will be discussed in detail later.

As a simple variation, we can consider the group $GL(n,\mathbb{C})$ of invertible complex $n\times n$-matrices, which is an open subset of the vector space $M_n(\mathbb{C})=\mathbb{C}^{n^2}$. Again there is the closed subgroup $SL(n,\mathbb{C})$ consisting of all matrices of (complex) determinant one, which is easily seen to be a submanifold.

1.2. Translations. Let (G,μ,ν,e) be a Lie group. For any element $g\in G$ we can consider the *left translation* $\lambda_g : G\to G$ defined by $\lambda_g(h):=gh=\mu(g,h)$. Smoothness of μ immediately implies that λ_g is smooth, and $\lambda_g\circ\lambda_{g^{-1}}=\lambda_{g^{-1}}\circ\lambda_g=\mathrm{id}_G$. Hence $\lambda_g : G\to G$ is a diffeomorphism with inverse $\lambda_{g^{-1}}$. Evidently, we have $\lambda_g\circ\lambda_h=\lambda_{gh}$. Similarly, we can consider the *right translation* by g, which we write as $\rho^g : G\to G$. Again this is a diffeomorphism with inverse $\rho^{g^{-1}}$, but this time the compatibility with the product reads as $\rho^g\circ\rho^h=\rho^{hg}$. Many basic identities of group theory can be easily rephrased in terms of the translation mappings. For example, the equation $(gh)^{-1}=h^{-1}g^{-1}$ can be interpreted as $\nu\circ\lambda_g=\rho^{g^{-1}}\circ\nu$ or as $\nu\circ\rho^h=\lambda_{h^{-1}}\circ\nu$. The definition of the neutral element can be recast as $\lambda_e=\rho^e=\mathrm{id}_G$.

Lemma 1.2. *Let (G,μ,ν,e) be a Lie group.*
(1) For $g,h\in G$, $\xi\in T_gG$ and $\eta\in T_hG$ we have

$$T_{(g,h)}\mu\cdot(\xi,\eta)=T_h\lambda_g\cdot\eta+T_g\rho^h\cdot\xi.$$

(2) The inversion map $\nu : G\to G$ is smooth and for $g\in G$ we have

$$T_g\nu=-T_e\rho^{g^{-1}}\circ T_g\lambda_{g^{-1}}=-T_e\lambda_{g^{-1}}\circ T_g\rho^{g^{-1}}.$$

In particular, $T_e\nu=-\,\mathrm{id}$.

Proof. (1) Since $T_{(g,h)}\mu$ is linear, we get $T_{(g,h)}\mu\cdot(\xi,\eta)=T_{(g,h)}\mu\cdot(\xi,0)+T_{(g,h)}\mu\cdot(0,\eta)$. Choose a smooth curve $c : (-\epsilon,\epsilon)\to G$ with $c(0)=g$ and $c'(0)=\xi$. Then the curve $t\mapsto(c(t),h)$ represents the tangent vector $(\xi,0)$ and the composition of μ with this curve equals $\rho^h\circ c$. Hence we conclude that $T_{(g,h)}\mu\cdot(\xi,0)=T_g\rho^h\cdot\xi$, and likewise for the other summand.

(2) Consider the function $f : G\times G\to G\times G$ defined by $f(g,h):=(g,gh)$. From part (1) and the fact the $\lambda_e=\rho^e=\mathrm{id}_G$ we conclude that for $\xi,\eta\in T_eG$ we get $T_{(e,e)}f\cdot(\xi,\eta)=(\xi,\xi+\eta)$. Evidently, this is a linear isomorphism $T_eG\times T_eG\to T_eG\times T_eG$, so locally around (e,e), f admits a smooth inverse, $\tilde{f} : G\times G\to G\times G$. By

definition, $\tilde{f}(g,e) = (g,\nu(g))$, which implies that ν is smooth locally around e. Since $\nu \circ \lambda_{g^{-1}} = \rho^g \circ \nu$, we conclude that ν is smooth locally around any $g \in G$.

Differentiating the equation $e = \mu(g,\nu(g))$ and using part (1) we obtain

$$0 = T_{(g,g^{-1})}\mu \cdot (\xi, T_g\nu \cdot \xi) = T_g\rho^{g^{-1}} \cdot \xi + T_{g^{-1}}\lambda_g \cdot T_g\nu \cdot \xi$$

for any $\xi \in T_gG$. Since $\lambda_{g^{-1}}$ is inverse to λ_g this shows that $T_g\nu = -T_e\lambda_{g^{-1}} \circ T_g\rho^{g^{-1}}$. The second formula follows in the same way by differentiating $e = \mu(\nu(g), g)$. □

1.3. Left invariant vector fields. We can use left translations to transport around tangent vectors on G. Put $\mathfrak{g} := T_eG$, the tangent space to G at the neutral element $e \in G$. For $X \in \mathfrak{g}$ and $g \in G$ define $L_X(g) := T_e\lambda_g \cdot X \in T_gG$.

Likewise, we can use left translations to pull back vector fields on G. Recall from [**AnaMf**, Section 2.3] that for a local diffeomorphism $f : M \to N$ and a vector field ξ on N, the pullback $f^*\xi \in \mathfrak{X}(M)$ is defined by $f^*\xi(x) := (T_xf)^{-1} \cdot \xi(f(x))$. In the case of a Lie group G, a vector field $\xi \in \mathfrak{X}(G)$ and an element $g \in G$ we thus have $(\lambda_g)^*\xi \in \mathfrak{X}(G)$.

Definition 1.3. Let G be a Lie group. A vector field $\xi \in \mathfrak{X}(G)$ is called *left invariant* if and only if $(\lambda_g)^*\xi = \xi$ for all $g \in G$. The space of left invariant vector fields is denoted by $\mathfrak{X}_L(G)$.

Proposition 1.3. *Let G be a Lie group and put $\mathfrak{g} = T_eG$. The we have:*

(1) The map $G \times \mathfrak{g} \to TG$ defined by $(g,X) \mapsto L_X(g)$ is a diffeomorphism.

(2) For any $X \in \mathfrak{g}$, the map $L_X : G \to TG$ is a vector field on G. The maps $X \mapsto L_X$ and $\xi \mapsto \xi(e)$ define inverse linear isomorphisms between $\mathfrak{g}$ and $\mathfrak{X}_L(G)$.

Proof. (1) Consider the map $\varphi : G \times \mathfrak{g} \to TG \times TG$ defined by $\varphi(g,X) := (0_g, X)$, where 0_g is the zero vector in T_gG. Evidently φ is smooth, and by part (1) of Lemma 1.2 the smooth map $T\mu \circ \varphi$ is given by $(g,X) \mapsto L_X(g)$. On the other hand, define $\psi : TG \to TG \times TG$ by $\psi(\xi_g) := (0_{g^{-1}}, \xi)$ which is smooth by part (2) of Lemma 1.2. By part (1) of that lemma, we see that $T\mu \circ \psi$ has values in $T_eG = \mathfrak{g}$ and is given by $\xi_g \mapsto T\lambda_{g^{-1}} \cdot \xi$. This shows that $\xi_g \mapsto (g, T\lambda_{g^{-1}} \cdot \xi)$ defines a smooth map $TG \to G \times \mathfrak{g}$, which is evidently inverse to $(g,X) \mapsto L_X(g)$.

(2) By definition $L_X(g) \in T_gG$ and smoothness of L_X follows immediately from (1), so $L_X \in \mathfrak{X}(G)$. By definition,

$$((\lambda_g)^*L_X)(h) = T_{gh}\lambda_{g^{-1}}L_X(gh) = T_{gh}\lambda_{g^{-1}} \cdot T_e\lambda_{gh} \cdot X,$$

and using $T_e\lambda_{gh} = T_h\lambda_g \circ T_e\lambda_h$ we see that this equals $T_e\lambda_h \cdot X = L_X(h)$. Since h is arbitrary, $L_X \in \mathfrak{X}_L(G)$ and we have linear maps between $\mathfrak{g}$ and $\mathfrak{X}_L(G)$ as claimed. Of course, $L_X(e) = X$, so one composition is the identity. On the other hand, if ξ is left invariant and $X = \xi(e)$, then

$$\xi(g) = ((\lambda_{g^{-1}})^*\xi)(g) = T_e\lambda_g \cdot \xi(g^{-1}g) = L_X(g),$$

and thus $\xi = L_X$. □

Remark 1.3. The diffeomorphism $TG \to G \times \mathfrak{g}$ from part (1) of the Proposition is called the *left trivialization* of the tangent bundle TG. For $X \in \mathfrak{g}$, the vector field $L_X \in \mathfrak{X}(G)$ is called the *left invariant vector field generated by* X. Any of the vector fields L_X is nowhere vanishing and choosing a basis of $\mathfrak{g}$, the values of the corresponding left invariant vector fields form a basis for each tangent space.

This for example shows that no sphere of even dimension can be made into a Lie group, since it does not admit any nowhere vanishing vector field. Indeed, the only spheres with trivial tangent bundle are S^1, S^3, and S^7.

1.4. The Lie algebra of a Lie group. Recall from [**AnaMf**, Theorem 2.5] that the pull back operator on vector fields is compatible with the Lie bracket (see also Lemma 1.5 below). In particular, for a Lie group G, left invariant vector fields $\xi, \eta \in \mathfrak{X}_L(G)$ and an element $g \in G$ we obtain

$$\lambda_g^*[\xi, \eta] = [\lambda_g^*\xi, \lambda_g^*\eta] = [\xi, \eta],$$

so $[\xi, \eta]$ is left invariant, too. Applying this to L_X and L_Y for $X, Y \in \mathfrak{g} := T_eG$ we see that $[L_X, L_Y]$ is left invariant. Defining $[X, Y] \in \mathfrak{g}$ as $[L_X, L_Y](e)$, part (2) of Proposition 1.3 show that that $[L_X, L_Y] = L_{[X,Y]}$.

Definition 1.4. Let G be a Lie group. The *Lie algebra* of G is the tangent space $\mathfrak{g} := T_eG$ together with the map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ defined by $[X, Y] := [L_X, L_Y](e)$.

From the corresponding properties of the Lie bracket of vector fields, it follows immediately that the bracket $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is bilinear, skew symmetric (i.e. $[Y, X] = -[X, Y]$) and satisfies the Jacobi identity $[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$. In general, one defines a Lie algebra as a real vector space together with a Lie bracket having these three properties.

Example 1.4. (1) Consider a real vector space V viewed as a Lie group under addition as in example (1) of 1.1. Then the trivialization of the tangent bundle TV by left translations is just the usual trivialization $TV = V \times V$. Hence left invariant vector fields correspond to constant functions $V \to V$. In particular, the Lie bracket of two such vector fields always vanishes identically, so the Lie algebra of this Lie group is simply the vector space V with the zero map as a Lie bracket. We shall see soon that the bracket is always trivial for commutative groups. Lie algebras with the zero bracket are usually called *commutative*.

(2) Let us consider a product $G \times H$ of Lie groups as in example (3) of 1.1. Then $T(G \times H) = TG \times TH$ so in particular $T_e(G \times H) = \mathfrak{g} \oplus \mathfrak{h}$. One immediately verifies that taking left invariant vector fields in G and H, any vector field of the form $(g, h) \mapsto (L_X(g), L_Y(h))$ for $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$ is left invariant. These exhaust all the left invariant vector fields and we easily conclude that the Lie bracket on $\mathfrak{g} \times \mathfrak{h}$ is component-wise, i.e. $[(X, Y), (X', Y')] = ([X, X']_{\mathfrak{g}}, [Y, Y']_{\mathfrak{h}})$. This construction is referred to as the *direct sum* of the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$.

(3) Let us consider the fundamental example $G = GL(n, \mathbb{R})$. As a manifold, G is an open subset in the vector space $M_n(\mathbb{R})$, so in particular, $\mathfrak{g} = M_n(\mathbb{R})$ as a vector space. More generally, we can identify vector fields on G with functions $G \to M_n(\mathbb{R})$, but this trivialization is different from the left trivialization. The crucial observation is that for matrices $A, B, C \in M_n(\mathbb{R})$ we have $A(B + tC) = AB + tAC$, so left translation by A is a linear map. In particular, this implies that for $A \in GL(n, \mathbb{R})$ and $C \in M_n(\mathbb{R}) = T_eGL(n, \mathbb{R})$ we obtain $L_C(A) = AC$. Viewed as a function $GL(n, \mathbb{R}) \to M_n(\mathbb{R})$, the left invariant vector field L_C is therefore given by right multiplication by C and thus extends to all of $M_n(\mathbb{R})$. Now viewing vector fields on an open subset of $\mathbb{R}^m$ as functions with values in $\mathbb{R}^m$, the Lie bracket is given by $[\xi, \eta](x) = D\eta(x)(\xi(x)) - D\xi(x)(\eta(x))$. Since right multiplication by a fixed matrix is a linear map, we conclude that $D(L_{C'})(e)(C) = CC'$ for $C, C' \in M_n(\mathbb{R})$. Hence we obtain $[C, C'] = [L_C, L_{C'}](e) = CC' - C'C$, and the Lie bracket on $M_n(\mathbb{R})$ is given by the commutator of matrices.

1.5. The derivative of a homomorphism. Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, and let $\varphi : G \to H$ be a smooth group homomorphism. Then $\varphi(e) = e$, so we have a linear map $\varphi' := T_e\varphi : \mathfrak{g} \to \mathfrak{h}$. Our next task is to prove that this is a homomorphism of Lie algebras, i.e. compatible with the Lie brackets. This needs a bit of preparation.

Let $f : M \to N$ be a smooth map between smooth manifolds. Recall from [**AnaMf**, Section 2.1] that two vector fields $\xi \in \mathfrak{X}(M)$ and $\eta \in \mathfrak{X}(N)$ are called *f-related* if $T_xf \cdot \xi(x) = \eta(f(x))$ holds for all $x \in M$. If this is the case, then we write $\xi \sim_f \eta$. Note that in general it is neither possible to find an f-related ξ for a given η nor to find an f-related η for a given ξ. In the special case of a local diffeomorphism f, for any given $\eta \in \mathfrak{X}(N)$, there is a unique $\xi \in \mathfrak{X}(M)$ such that $\xi \sim_f \eta$, namely the pullback $f^*\eta$.

LEMMA 1.5. *Let $f : M \to N$ be a smooth map, and let $\xi_i \in \mathfrak{X}(M)$ and $\eta_i \in \mathfrak{X}(N)$ be vector fields for $i = 1, 2$. If $\xi_i \sim_f \eta_i$ for $i = 1, 2$ then $[\xi_1, \xi_2] \sim_f [\eta_1, \eta_2]$.*

PROOF. For a smooth map $\alpha : N \to \mathbb{R}$ we have $(Tf \circ \xi) \cdot \alpha = \xi \cdot (\alpha \circ f)$ by definition of the tangent map. Hence $\xi \sim_f \eta$ is equivalent to $\xi \cdot (\alpha \circ f) = (\eta \cdot \alpha) \circ f$ for all $\alpha \in C^\infty(N, \mathbb{R})$. Now assuming that $\xi_i \sim_f \eta_i$ for $i = 1, 2$ we compute

$$\xi_1 \cdot (\xi_2 \cdot (\alpha \circ f)) = \xi_1 \cdot ((\eta_2 \cdot \alpha) \circ f) = (\eta_1 \cdot (\eta_2 \cdot \alpha)) \circ f.$$

Inserting into the definition of the Lie bracket, we immediately conclude that

$$[\xi_1, \xi_2] \cdot (\alpha \circ f) = ([\eta_1, \eta_2] \cdot \alpha) \circ f,$$

and thus $[\xi_1, \xi_2] \sim_f [\eta_1, \eta_2]$. □

Using this, we can now prove

PROPOSITION 1.5. *Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$.*

(1) If $\varphi : G \to H$ is a smooth homomorphism then $\varphi' = T_e\varphi : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of Lie algebras, i.e. $\varphi'([X, Y]) = [\varphi'(X), \varphi'(Y)]$ for all $X, Y \in \mathfrak{g}$.

(2) If G is commutative, then the Lie bracket on $\mathfrak{g}$ is identically zero.

PROOF. (1) The equation $\varphi(gh) = \varphi(g)\varphi(h)$ can be interpreted as $\varphi \circ \lambda_g = \lambda_{\varphi(g)} \circ \varphi$. Differentiating this equation in $e \in G$, we obtain $T_g\varphi \circ T_e\lambda_g = T_e\lambda_{\varphi(g)} \circ \varphi'$. Inserting $X \in T_eG = \mathfrak{g}$, we get $T_g\varphi \cdot L_X(g) = L_{\varphi'(X)}(\varphi(g))$, and hence the vector fields $L_X \in \mathfrak{X}(G)$ and $L_{\varphi'(X)} \in \mathfrak{X}(H)$ are φ-related for each $X \in \mathfrak{g}$. From the lemma, we conclude that for $X, Y \in \mathfrak{g}$ we get $T\varphi \circ [L_X, L_Y] = [L_{\varphi'(X)}, L_{\varphi'(Y)}] \circ \varphi$. Evaluated in $e \in G$ this gives $\varphi'([X, Y]) = [\varphi'(X), \varphi'(Y)]$.

(2) If G is commutative, then $(gh)^{-1} = h^{-1}g^{-1} = g^{-1}h^{-1}$ so the inversion map $\nu : G \to G$ is a group homomorphism. Hence by part (1), $\nu' : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra homomorphism. By part (2) of Lemma 1.2 $\nu' = -\operatorname{id}$ and we obtain

$$-[X, Y] = \nu'([X, Y]) = [\nu'(X), \nu'(Y)] = [-X, -Y] = [X, Y]$$

and thus $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$. □

EXAMPLE 1.5. We have noted in 1.1 that the subset $SL(n, \mathbb{R}) \subset GL(n, \mathbb{R})$ of matrices of determinant 1 is a smooth submanifold, since the determinant function is regular in each matrix A with $\det(A) \neq 0$. As a vector space, we can therefore view the Lie algebra $\mathfrak{sl}(n, \mathbb{R})$ of $SL(n, \mathbb{R})$ as the kernel of $D(\det)(\mathbb{I})$, where $\mathbb{I}$ denotes the identity matrix. It is a nice exercise to show that this coincides with the space of tracefree matrices. Since the inclusion $i : SL(n, \mathbb{R}) \to GL(n, \mathbb{R})$ is a homomorphism with derivative the inclusion $\mathfrak{sl}(n, \mathbb{R}) \to M_n(\mathbb{R})$, we conclude from part (1) of the Proposition and example (3) of 1.4 that the Lie bracket on $\mathfrak{sl}(n, \mathbb{R})$ is also given by the commutator of matrices.

1.6. Right invariant vector fields. It was a matter of choice that we have used left translations to trivialize the tangent bundle of a Lie group G in 1.3. In the same way, one can consider the right trivialization $TG \to G\times\mathfrak{g}$ defined by $\xi_g \mapsto (g, T_g\rho^{g^{-1}}\cdot\xi)$. The inverse of this map is denoted by $(g,X)\mapsto R_X(g)$, and R_X is called the *right invariant vector field generated by* $X\in\mathfrak{g}$. In general, a vector field $\xi\in\mathfrak{X}(G)$ is called *right invariant* if $(\rho^g)^*\xi=\xi$ for all $g\in G$. The space of right invariant vector fields (which is a Lie subalgebra of $\mathfrak{X}(G)$) is denoted by $\mathfrak{X}_R(G)$. As in Proposition 1.3 one shows that $\xi\mapsto\xi(e)$ and $X\mapsto R_X$ are inverse bijections between $\mathfrak{g}$ and $\mathfrak{X}_R(G)$.

PROPOSITION 1.6. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and inversion $\nu : G\to G$. Then we have*

(1) $R_X=\nu^*(L_{-X})$ for all $X\in\mathfrak{g}$.
(2) For $X,Y\in\mathfrak{g}$, we have $[R_X,R_Y]=R_{-[X,Y]}$.
(3) For $X,Y\in\mathfrak{g}$, we have $[L_X,R_Y]=0$.

PROOF. (1) The equation $(gh)^{-1}=h^{-1}g^{-1}$ can be interpreted as $\nu\circ\rho^h=\lambda_{h^{-1}}\circ\nu$. In particular, if $\xi\in\mathfrak{X}_L(G)$ then

$$(\rho^h)^*\nu^*\xi=(\nu\circ\rho^h)^*\xi=(\lambda_{h^{-1}}\circ\nu)^*\xi=\nu^*\lambda_{h^{-1}}^*\xi=\nu^*\xi,$$

so $\nu^*\xi$ is right invariant. Since $\nu^*\xi(e)=T_e\nu\cdot\xi(e)=-\xi(e)$, the claim follows.

(2) Using part (1) we compute

$$[R_X,R_Y]=[\nu^*L_{-X},\nu^*L_{-Y}]=\nu^*[L_{-X},L_{-Y}]=\nu^*L_{[X,Y]}=R_{-[X,Y]}.$$

(3) Consider the vector field $(0,L_X)$ on $G\times G$ whose value in (g,h) is $(0_g,L_X(h))$. By part (1) of Proposition 1.2, $T_{(g,h)}\mu\cdot(0_g,L_X(h))=T_h\lambda_g\cdot L_X(h)=L_X(gh)$, which shows that $(0,L_X)$ is μ-related to L_X. Likewise, $(R_Y,0)$ is μ-related to R_Y, so by Lemma 1.5 the vector field $0=[(0,L_X),(R_Y,0)]$ is μ-related to $[L_X,R_Y]$. Since μ is surjective, this implies that $[L_X,R_Y]=0$. □

The exponential mapping

1.7. One parameter subgroups. Our next aim is to study the flow lines of left invariant and right invariant vector fields on G. Recall from [**AnaMf**, Sections 2.6 and 2.7] that for a vector field ξ on a smooth manifold M an *integral curve* is a smooth curve $c: I\to M$ defined on an open interval in $\mathbb{R}$ such that $c'(t)=\xi(c(t))$ holds for all $t\in I$. Fixing $x\in M$, there is a maximal interval I_x containing 0 and a unique maximal integral curve $c_x : I_x\to M$ such that $c_x(0)=x$. These maximal integral curves can be put together to obtain the flow mapping: The set $\mathcal{D}(\xi)=\{(x,t): t\in I_x\}$ is an open neighborhood of of $M\times\{0\}$ in $M\times\mathbb{R}$ and one obtains a smooth map $\mathrm{Fl}^\xi:\mathcal{D}(\xi)\to M$ such that $t\mapsto\mathrm{Fl}^\xi_t(x)$ is the maximal integral curve of ξ through x for each $x\in M$. The flow has the fundamental property that $\mathrm{Fl}^\xi_{t+s}(x)=\mathrm{Fl}^\xi_t(\mathrm{Fl}^\xi_s(x))$.

Suppose that $f: M\to N$ is a smooth map and $\xi\in\mathfrak{X}(M)$ and $\eta\in\mathfrak{X}(N)$ are f-related vector fields. If $c: I\to M$ is an integral curve for ξ, i.e. $c'(t)=\xi(c(t))$ for all t, then consider $f\circ c: I\to N$. We have $(f\circ c)'(t)=Tf\cdot c'(t)=\eta(f(c(t)))$, so $f\circ c$ is an integral curve of η. This immediately implies that the flows of ξ and η are f-related, i.e. $f\circ\mathrm{Fl}^\xi_t=\mathrm{Fl}^\eta_t\circ f$.

Recall from [**AnaMf**, Section 2.8] that a vector field ξ is called *complete* if $I_x=\mathbb{R}$ for all $x\in M$, i.e. all flow lines can be extended to all times. Any vector field with compact support (and hence any vector field on a compact manifold) is complete. If there is some $\epsilon>0$ such that $[-\epsilon,\epsilon]\subset I_x$ for all $x\in M$, then ξ is complete. The idea about this is that $t\mapsto\mathrm{Fl}^\xi_{t-\epsilon}(\mathrm{Fl}_\epsilon(x))$ is an integral curve defined on $[0,2\epsilon]$, and similarly one gets an extension to $[-2\epsilon, 0]$, so $[-2\epsilon, 2\epsilon] \subset I_x$ for all x. Inductively, this implies $\mathbb{R} \subset I_x$ for all x.

Now suppose that G is a Lie group and $\xi \in \mathfrak{X}_L(G)$ is left invariant. Then for each $g \in G$, the vector field ξ is λ_g-related to itself. In particular, this implies that $\mathrm{Fl}^\xi_t(g) = g\,\mathrm{Fl}^\xi_t(e)$ for all $g \in G$, so it suffices to know the flow through e. Moreover, for each $g \in G$ we have $I_e \subset I_g$, and hence ξ is complete. Likewise, for a right invariant vector field $\xi \in \mathfrak{X}_R(G)$, we get $\mathrm{Fl}^\xi_t(g) = \mathrm{Fl}^\xi_t(e)g$ and any such vector field is complete. We shall next show that the flows of invariant vector fields are nicely related to the group structure of G.

Definition 1.7. Let G be a Lie group. A *one parameter subgroup* of G is a smooth homomorphism $\alpha : (\mathbb{R}, +) \to G$, i.e. a smooth curve $\alpha : \mathbb{R} \to G$ such that $\alpha(t + s) = \alpha(t)\alpha(s)$ for all $t, s \in \mathbb{R}$. In particular, this implies that $\alpha(0) = e$.

Lemma 1.7. *Let $\alpha : \mathbb{R} \to G$ be a smooth curve with $\alpha(0) = e$, and let $X \in \mathfrak{g}$ be any element. The the following are equivalent:*

(1) α is a one parameter subgroup with $X = \alpha'(0)$.

(2) $\alpha(t) = \mathrm{Fl}^{L_X}_t(e)$ for all $t \in \mathbb{R}$.

(3) $\alpha(t) = \mathrm{Fl}^{R_X}_t(e)$ for all $t \in \mathbb{R}$.

Proof. (1) $\Rightarrow$ (2): We compute:

$$\alpha'(t) = \tfrac{d}{ds}|_{s=0}\alpha(t + s) = \tfrac{d}{ds}|_{s=0}(\alpha(t)\alpha(s)) = T_e\lambda_{\alpha(t)} \cdot \alpha'(0) = L_X(\alpha(t)).$$

Hence α is an integral curve of L_X and since $\alpha(0) = e$ we must have $\alpha(t) = \mathrm{Fl}^{L_X}_t(e)$.

(2) $\Rightarrow$ (1): $\alpha(t) = \mathrm{Fl}^{L_X}_t(e)$ is a smooth curve in G with $\alpha(0) = e$ and $\alpha'(0) = L_X(e) = X$. The basic flow property reads as $\mathrm{Fl}^{L_X}_{t+s}(e) = \mathrm{Fl}^{L_X}_t(\mathrm{Fl}^{L_X}_s(e))$, and by left invariance the last expression equals $\mathrm{Fl}^{L_X}_s(e)\,\mathrm{Fl}^{L_X}_t(e)$. Hence $\alpha(t + s) = \alpha(s)\alpha(t) = \alpha(t)\alpha(s)$.

The equivalence of (1) and (3) can be proved in the same way exchanging the roles of s and t. □

1.8. The exponential mapping. Since we know that the flow of left invariant vector fields is defined for all times, we can use it to define the exponential map.

Definition 1.8. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then we define $\exp : \mathfrak{g} \to G$ by $\exp(X) := \mathrm{Fl}^{L_X}_1(e)$.

Theorem 1.8. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $\exp : \mathfrak{g} \to G$ be the exponential mapping. Then we have:*

(1) The map $\exp$ is smooth, $\exp(0) = e$ and $T_e \exp = \mathrm{id}_{\mathfrak{g}}$, so $\exp$ restricts to a diffeomorphism from an open neighborhood of 0 in $\mathfrak{g}$ to an open neighborhood of e in G.

(2) For each $X \in \mathfrak{g}$ and each $g \in G$ we have $\mathrm{Fl}^{L_X}_t(g) = g \exp(tX)$.

(3) For each $X \in \mathfrak{g}$ and each $g \in G$ we have $\mathrm{Fl}^{R_X}_t(g) = \exp(tX)g$.

Proof. By part (1) of Proposition 1.3, the map $\mathfrak{g} \times G \to TG$ defined by $(X, g) \mapsto L_X(g)$ is smooth. Hence $(X, g) \mapsto (0_X, L_X(g))$ defines a smooth vector field on $\mathfrak{g} \times G$. Its integral curves are evidently given by $t \mapsto (X, \mathrm{Fl}^{L_X}_t(g))$. Smoothness of the flow of this vector field in particular implies that $(X, t) \mapsto \mathrm{Fl}^{L_X}_t(e)$ is a smooth map, so smoothness of exp follows.

If $c : I \to M$ is an integral curve of L_X then clearly for $a \in \mathbb{R}$ the curve $t \mapsto c(at)$ is an integral curve of $aL_X = L_{aX}$. But this implies $\mathrm{Fl}^{L_X}_t(e) = \mathrm{Fl}^{L_{tX}}_1 = \exp(tX)$, so we have proved (2) for $g = e$. Claim (3) for $g = e$ follows in the same way, and the general versions of (2) and (3) follow from 1.7.

Since the integral curves of the zero vector field are constant, we get $\exp(0) = e$. Hence the derivative $T_0 \exp$ can be viewed as a map from $T_0\mathfrak{g} = \mathfrak{g}$ to $T_eG = \mathfrak{g}$. Since $\mathfrak{g}$ is a vector space, we can compute $T_0 \exp \cdot X$ as

$$\tfrac{d}{dt}|_{t=0} \exp(tX) = \tfrac{d}{dt}|_{t=0} \mathrm{Fl}_t^{L_X}(e) = L_X(e) = X.$$

Hence $T_0 \exp = \mathrm{id}_{\mathfrak{g}}$ and exp is a local diffeomorphism. □

Example 1.8. (1) The name exponential map goes back to the example of the (commutative) Lie group $\mathbb{R}_{>0}$ under multiplication. Its Lie algebra is $\mathbb{R}$ with the trivial bracket. Since multiplication is bilinear, the left invariant vector field generated by $s \in \mathbb{R}$ is given by $a \mapsto as$. To get the exponential map, we have to solve the equation $c'(t) = c(t)s$ with initial condition $c(0) = 1$. Of course, the solution is $c(t) = e^{st}$, so $\exp : \mathbb{R} \to \mathbb{R}_{>0}$ is the usual exponential.

(2) This actually generalizes to $G = GL(n,\mathbb{R})$. For $X \in \mathfrak{g} = M_n(\mathbb{R})$ we have to solve the equation $c'(t) = c(t)X$ and its unique solution with intial condition $c(0) = \mathbb{I}$, the unit matrix, is given by the matrix exponential $\exp(tX) = \sum_{k=0}^{\infty} \frac{t^k}{k!} X^k$. (Note that endowing $M_n(\mathbb{R})$ with the operator norm, one has $\|X^k\| \leq \|X\|^k$, so this power series converges absolutely and uniformly on compact subsets.) Note however that $\exp(X + Y) \neq \exp(X)\exp(Y)$ unless the matrices X and Y commute.

Remark 1.8. For a Lie group G with Lie algebra $\mathfrak{g}$, let $V \subset \mathfrak{g}$ be an open neighborhood of zero such that exp restricts to a diffeomorphism from V to $\exp(V) =: U$. Then we can use $(U, (\exp|_V)^{-1})$ as a local chart for G around $e \in G$. The corresponding local coordinates are called *canonical coordinates of the first kind*. For any $g \in G$ we can then use $(\lambda_g(U), (\lambda_g \circ \exp|_V)^{-1})$ as a local chart for G around g.

There are also *canonical coordinates of the second kind*, which are obtained as follows: Choose a basis $X_1, \dots, X_n$ of $\mathfrak{g}$ and consider the map $f : \mathbb{R}^n \to G$ defined by

$$f(t^1, \dots, t^n) := \exp(t^1X_1)\exp(t^2X_2)\cdots\exp(t^nX_n).$$

Evidently, $\frac{\partial f}{\partial t^i}(0) = X_i$, so the tangent map $T_0f : \mathbb{R}^n \to \mathfrak{g}$ of f at 0 is given by $(a^1, \dots, a^n) \mapsto a^1X_1 + \cdots + a^nX_n$. Hence f restricts to a diffeomorphism from a neighborhood of 0 in $\mathbb{R}^n$ to a neighborhood of e in G. The inverse of such a restriction can be used as a local chart which gives rise to canonical coordinates of the second kind.

A surprising application of the exponential map is the following result:

Corollary 1.8. *Let $\varphi : G \to H$ be a continuous homomorphism between two Lie groups. Then φ is smooth.*

Proof. We first show that a continuous one parameter subgroup $\alpha : \mathbb{R} \to G$ is automatically smooth. By the theorem we can find a real number $r > 0$ such that exp restricts to a diffeomorphism from the ball $B_{2r}(0) \subset \mathfrak{g}$ onto an open neighborhood of $e \in G$. Since $\alpha(0) = 0$ and α is continuous, there is an $\epsilon > 0$ such that $\alpha([-\epsilon, \epsilon]) \subset \exp(B_r(0))$, and we define $\beta : [-\epsilon, \epsilon] \to B_r(0)$ as $\beta = (\exp|_{B_r(0)})^{-1} \circ \alpha$. For $|t| < \frac{\epsilon}{2}$ we have $\exp(\beta(2t)) = \alpha(2t) = \alpha(t)^2 = \exp(2\beta(t))$, and hence $\beta(2t) = 2\beta(t)$. Hence $\beta(\frac{s}{2}) = \frac{1}{2}\beta(s)$ for all $s \in [-\epsilon, \epsilon]$. Inductively, this implies that $\beta(\frac{s}{2^k}) = \frac{1}{2^k}\beta(s)$ for all $s \in [-\epsilon, \epsilon]$. Using this, we now compute for $k, n \in \mathbb{N}$

$$\alpha(\tfrac{n\epsilon}{2^k}) = \alpha(\tfrac{\epsilon}{2^k})^n = \exp(\beta(\tfrac{\epsilon}{2^k}))^n = \exp(\tfrac{n}{2^k}\beta(\epsilon)).$$

Together with the fact that $\alpha(-t) = \alpha(t)^{-1}$ and $\exp(-X) = \exp(X)^{-1}$, this implies that $\alpha(t) = \exp(t\frac{1}{\epsilon}\beta(\epsilon))$ holds for all $t \in \{\frac{n\epsilon}{2^k} : k \in \mathbb{N}, n \in \mathbb{Z}\} \subset \mathbb{R}$. Since this set is dense in

$\mathbb{R}$ and both sides of the equation are continuous, it holds for all t, which implies that α is smooth.

Now consider the general case $\varphi : G \to H$. Let us use canonical coordinates of the second kind on G, i.e. take a basis $X_1, \dots, X_n$ for $\mathfrak{g}$ and use $f(t_1, \dots, t_n) := \exp(t_1X_1) \cdots \exp(t_nX_n)$ as the inverse of a chart. Then

$$(\varphi \circ f)(t_1, \dots, t_n) = \varphi(\exp(t_1X_1)) \cdots \varphi(\exp(t_nX_n)).$$

Each factor in this product is a continuous one parameter subgroup in H and thus smooth by the first part of the proof. Hence $\varphi \circ f$ is smooth, so φ is smooth locally around $e \in G$. Since φ is a homomorphism, we have $\varphi \circ \lambda_{g^{-1}} = \lambda_{\varphi(g)^{-1}} \circ \varphi$ for all $g \in G$. The right side is smooth locally around e, so φ is smooth locally around g. □

1.9. We can now use the exponential mapping to prove that smooth homomorphisms on connected Lie groups are uniquely determined by their derivative. For a Lie group G let $G_0 \subset G$ be the connected component of G containing the neutral element e. This is usually referred to as the *connected component of the identity* of G. Clearly, G_0 is a manifold and it is a subgroup of G: for $g, h \in G_0$ there are smooth curves connecting g and h to the unit element e. But then the pointwise product of these two curves is a smooth curve connecting gh to e, so $gh \in G_0$, and similarly the pointwise inverse of the curve connects g^{-1} to e. In particular, G_0 is itself a Lie group. Finally, for $g \in G_0$ and $h \in G$ and a curve c connecting g to e, the curve $t \mapsto hc(t)h^{-1}$ shows that $hgh^{-1} \in G_0$, so G_0 is even a normal subgroup in G. Hence the quotient G/G_0 is a (discrete) group, called the *component group* of G.

Theorem 1.9. *Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ and exponential mappings $\exp^G$ and $\exp^H$. Then we have:*

(1) For a smooth homomorphism $\varphi : G \to H$ with derivative $\varphi' : \mathfrak{g} \to \mathfrak{h}$ we have $\varphi \circ \exp^G = \exp^H \circ \varphi'$.

(2) The subgroup of G generated by the image $\exp(\mathfrak{g})$ of the exponential map coincides with the connected component G_0 of the identity.

(3) If $\varphi, \psi : G \to H$ are smooth homomorphisms such that $\varphi' = \psi'$ then the restrictions of φ and ψ to G_0 coincide.

Proof. (1) In the proof of Proposition 1.5 we have seen that for $X \in \mathfrak{g}$ the vector fields $L_X \in \mathfrak{X}(G)$ and $L_{\varphi'(X)} \in \mathfrak{X}(H)$ are φ-related. Hence their flows are φ-related, which in particular implies that

$$\varphi(\exp(X)) = \varphi(\mathrm{Fl}_1^{L_X}(e)) = \mathrm{Fl}_1^{L_{\varphi'(X)}}(\varphi(e)) = \exp(\varphi'(X)).$$

(2) Let $\tilde{G}$ be the subgroup of G generated by $\exp(\mathfrak{g})$. Since $t \mapsto \exp(tX)$ is a smooth curve connecting e to $\exp(X)$ we see that the image of exp and hence $\tilde{G}$ is contained in G_0.

Conversely, since exp is a local diffeomorphism, there is an open neighborhood U of e in G which is contained in $\exp(\mathfrak{g})$ and hence in $\tilde{G}$. But then for $g \in \tilde{G}$, the set $\{gh : h \in U\}$ is an open neighborhood of g contained in $\tilde{G}$, so $\tilde{G} \subset G$ is open. On the other hand, $\{h^{-1} : h \in U\}$ is an open neighborhood of e in G. For $g \notin \tilde{G}$ the set $\{gh^{-1} : h \in U\}$ must evidently have trivial intersection with $\tilde{G}$. Hence $\tilde{G}$ is closed, and $\tilde{G} = G_0$ follows.

(3) By part (1) φ and ψ coincide on the image of $\exp^G$, and since they both are homomorphisms they therefore coincide on the subgroup generated by this image. Now the result follows from (2). □

1.10. The adjoint representation. A *representation* of a Lie group G on a finite dimensional vector space V is a smooth homomorphism $\varphi : G \to GL(V)$. A representation of a Lie algebra $\mathfrak{g}$ on V is a Lie algebra homomorphism $\mathfrak{g} \to L(V, V)$. In particular, for any representation $\varphi : G \to GL(V)$ of a Lie group, one obtains a representation $\varphi' : \mathfrak{g} \to L(V, V)$ of its Lie algebra.

Now any Lie group G has a canonical representation on its Lie algebra, called the *adjoint representation*. For $g \in G$, consider the conjugation by g, i.e. the map $\mathrm{conj}_g : G \to G$ defined by $\mathrm{conj}_g(h) := ghg^{-1}$. Evidently, conj_g is a group homomorphism, so we can form the derivative, which is a Lie algebra homomorphism $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$. Note that $\mathrm{conj}_g = \lambda_g \circ \rho^{g^{-1}} = \rho^{g^{-1}} \circ \lambda_g$, and therefore $\mathrm{Ad}(g) = T_{g^{-1}}\lambda_g \circ T_e\rho^{g^{-1}} = T_g\rho^{g^{-1}} \circ T_e\lambda_g$.

We clearly have $\mathrm{conj}_{gh} = \mathrm{conj}_g \circ \mathrm{conj}_h$ and differentiating this, we obtain $\mathrm{Ad}(gh) = \mathrm{Ad}(g) \circ \mathrm{Ad}(h)$. Likewise, $\mathrm{conj}_{g^{-1}} = (\mathrm{conj}_g)^{-1}$, and hence $\mathrm{Ad}(g^{-1}) = \mathrm{Ad}(g)^{-1}$, whence $\mathrm{Ad} : G \to GL(\mathfrak{g})$ is a homomorphism. To see that the map Ad is smooth, it suffices to show that $(g, X) \mapsto \mathrm{Ad}(g)(X)$ is smooth. Consider the smooth map $\varphi : G \times \mathfrak{g} \to TG \times TG \times TG$ defined by $\varphi(g, X) := (0_g, X, 0_{g^{-1}})$. By Lemma 1.2 we obtain

$$T\mu \circ (\mathrm{id}_{TG} \times T\mu)(\varphi(g, X)) = T_{g^{-1}}\lambda_g \circ T_e\rho^{g^{-1}} \cdot X = \mathrm{Ad}(g)(X),$$

and smoothness follows. The corresponding representation $\mathrm{ad} : \mathfrak{g} \to L(\mathfrak{g}, \mathfrak{g})$ is called the adjoint representation of the Lie algebra $\mathfrak{g}$.

Example 1.10. Consider $G = GL(n, \mathbb{R})$. Then the conjugation map $\mathrm{conj}_A(B) := ABA^{-1}$ is linear in B, so its derivative is given by $\mathrm{Ad}(A)(X) = AXA^{-1}$ for each $X \in M_n(\mathbb{R})$. As we shall see in the proposition below, the derivative ad of this is given by $\mathrm{ad}(X)(Y) = [X, Y] = XY - YX$.

Proposition 1.10. *Let G be a Lie group with Lie algebra $\mathfrak{g}$.*
(1) For $X \in \mathfrak{g}$ and $g \in G$ we have $L_X(g) = R_{\mathrm{Ad}(g)(X)}(g)$.
(2) For $X, Y \in \mathfrak{g}$ we have $\mathrm{ad}(X)(Y) = [X, Y]$.
(3) For $X \in \mathfrak{g}$ and $g \in G$ we have $\exp(t\,\mathrm{Ad}(g)(X)) = g\exp(tX)g^{-1}$.
(4) For $X, Y \in \mathfrak{g}$ we have

$$\mathrm{Ad}(\exp(X))(Y) = e^{\mathrm{ad}(X)}(Y) = Y + [X, Y] + \tfrac{1}{2}[X, [X, Y]] + \tfrac{1}{3!}[X, [X, [X, Y]]] + \dots$$

Proof. (1) We have $\lambda_g = \rho^g \circ \mathrm{conj}_g$ and hence

$$L_X(g) = T_e\lambda_g \cdot X = T_e\rho^g \cdot T_e\,\mathrm{conj}_g \cdot X = R_{\mathrm{Ad}(g)(X)}(g).$$

(2) Choosing a basis $\{v_1, \dots, v_n\}$ of $\mathfrak{g}$, we can consider the matrix representation $(a_{ij}(g))$ of $\mathrm{Ad}(g)$ and each $a_{ij} : G \to \mathbb{R}$ is smooth. For $X \in \mathfrak{g}$, the matrix representation of $\mathrm{ad}(X)$ is given by $(X \cdot a_{ij})$. We can compute $X \cdot a_{ij}$ as $(L_X \cdot a_{ij})(e)$. For $Y = \sum Y_i v_i \in \mathfrak{g}$ we get $L_Y(g) = R_{\mathrm{Ad}(g)Y}(g) = \sum_{i,j} a_{ij}(g)Y_jR_{v_i}$. Forming the bracket with L_X, we obtain

$$[L_X, L_Y] = \sum_{i,j} Y_j[L_X, a_{ij}R_{v_i}] = \sum_{i,j} Y_j(L_X \cdot a_{ij})R_{v_i},$$

where we have used that $[L_X, R_{v_i}] = 0$. Evaluating in e, we get

$$[X, Y] = \sum_{i,j}(X \cdot a_{ij})Y_jv_i = \mathrm{ad}(X)(Y).$$

(3) This follows directly from applying part (1) of Theorem 1.9 to the homomorphism $\mathrm{conj}_g : G \to G$ and its derivative $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$.

(4) This is part (1) of Theorem 1.9 applied to the homomorphism $\mathrm{Ad} : G \to GL(\mathfrak{g})$ and its derivative $\mathrm{ad} : \mathfrak{g} \to L(\mathfrak{g}, \mathfrak{g})$ together with the description of the exponential map for matrix algebras in 1.8. $\square$

(Virtual) Lie subgroups

1.11. Recall from [**AnaMf**, Section 1.16] that a subset N of a smooth manifold M of dimension n is called a *submanifold* of dimension k if for each $x \in N$ there is a chart (U, u) for M with $x \in U$ such that $u(U \cap N) = u(U) \cap \mathbb{R}^k$, where we view $\mathbb{R}^k$ as a subspace of $\mathbb{R}^n$ in the obvious way. If $N \subset M$ is a submanifold, then we can use the restrictions of charts of the above form as an atlas on N, thus making N into a smooth manifold.

Let us consider the case of a Lie group G and a subgroup H which at the same time is a submanifold of G. Then $H \subset G$ is called a *Lie subgroup* of G. Since the multiplication on H is just the restriction of the multiplication of G it is smooth, so H itself then is a Lie group. Hence the following result in particular provides us with a huge number of examples of Lie groups.

THEOREM 1.11. *Let G be a Lie group and let $H \subset G$ be a subgroup which is closed in the topological space G. Then H is a Lie subgroup of G and in particular it is itself a Lie group.*

PROOF. Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{h} \subset \mathfrak{g}$ be the set of all $c'(0)$, where $c : \mathbb{R} \to G$ is a smooth curve with $c(0) = e$ which has values in H. If c_1 and c_2 are two such curves and a is a real number, then consider $c(t) := c_1(t)c_2(at)$. Clearly this is smooth and since $H \subset G$ is a subgroup it has values in H, so $c'(0) \in \mathfrak{h}$. Now $c'(0) = T_{(e,e)}\mu \cdot (c_1'(0), ac_2'(0))$, which equals $c_1'(0) + ac_2'(0)$ by Lemma 1.2. Thus $\mathfrak{h}$ is a linear subspace of $\mathfrak{g}$.

Claim 1: Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{g}$ which converges to $X \in \mathfrak{g}$ and let $(t_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}_{>0}$ converging to 0. If $\exp(t_nX_n) \in H$ for all $n \in \mathbb{N}$, then $\exp(tX) \in H$ for all $t \in \mathbb{R}$.

Fix $t \in \mathbb{R}$, and for $n \in \mathbb{N}$ let m_n be the largest integer $\leq \frac{t}{t_n}$. Then $m_nt_n \leq t$ and $t - m_nt_n < t_n$, and hence the sequence m_nt_n converges to t. By continuity of exp we get

$$\exp(tX) = \lim_n \exp(m_nt_nX_n) = \lim_n (\exp(t_nX_n)^{m_n}),$$

which lies in the closed subgroup H.

Claim 2: $\mathfrak{h} = \{X \in \mathfrak{g} : \forall t \in \mathbb{R} : \exp(tX) \in H\}$.

By definition, the right hand set is contained in $\mathfrak{h}$. Conversely, let $c : \mathbb{R} \to G$ be a smooth curve with values in H and put $X = c'(0)$. On a neighborhood of 0, we obtain a well defined smooth curve v with values in $\mathfrak{g}$ such that $c(t) = \exp(v(t))$ for sufficiently small t. Then $X = c'(0) = \frac{d}{dt}|_{t=0} \exp(v(t)) = v'(0) = \lim_n nv(\frac{1}{n})$. Putting $t_n = \frac{1}{n}$ and $X_n = nv(\frac{1}{n})$ for sufficiently large n, we obtain $\exp(t_nX_n) = c(\frac{1}{n}) \in H$. Since $X_n \to X$ in $\mathfrak{g}$, we obtain $\exp(tX) \in H$ for all $t \in \mathbb{R}$ from claim 1, and claim 2 follows.

Claim 3: Let $\mathfrak{k} \subset \mathfrak{g}$ be a linear subspace which is complementary to $\mathfrak{h}$. Then there is an open neighborhood W of zero in $\mathfrak{k}$ such that $\exp(W) \cap H = \{e\}$.

If not, then we can find a sequence (Y_n) of nonzero elements of $\mathfrak{k}$ such that $Y_n \to 0$ and $\exp(Y_n) \in H$. Choose a norm $|\ |$ on $\mathfrak{k}$ and put $X_n := \frac{1}{|Y_n|}Y_n$. Possibly passing to a subsequence, we may assume that the sequence X_n converges to an element $X \in \mathfrak{k}$. Then $|X| = 1$, so in particular $X \neq 0$, but putting $t_n = |Y_n|$, claim 1 implies that $\exp(tX) \in H$ for all $t \in \mathbb{R}$ and hence $X \in \mathfrak{h}$. This is a contradiction, so claim 3 follows.

Now consider the map $\varphi : \mathfrak{h} \times \mathfrak{k} \to G$ defined by $\varphi(X, Y) := \exp(X)\exp(Y)$. This evidently has invertible tangent map in $(0, 0)$, so there are neighborhoods V and W of zero in $\mathfrak{h}$ respectively $\mathfrak{k}$ such that φ induces a diffeomorphism from $V \times W$ onto an open neighborhood U of e in G. By claim 3, we may shrink W in such a way that $\exp(W) \cap H = \{e\}$.

Since $V \subset \mathfrak{h}$ we have $\exp(V) \subset U \cap H$. Conversely, a point $x \in U \cap H$ can be uniquely written as $\exp(X)\exp(Y)$ for $X \in V$ and $Y \in W$. But then $\exp(-X) \in H$ and hence $\exp(Y) = \exp(-X)x \in H$. By construction, this implies $Y = 0$, so φ restricts to a bijection between $V \times \{0\}$ and $U \cap H$. Thus, $(U, u := (\varphi|_{V\times W})^{-1})$ is a submanifold chart for H defined locally around $e \in G$. For $h \in H$ we then obtain a submanifold chart defined locally around h as $(\lambda_h(U), u \circ \lambda_{h^{-1}})$. □

Closedness of a subgroup is often very easy to verify. A typical example of this situation is the *center* $Z(G)$ of a Lie group G. By definition,

$$Z(G) = \{g \in G : gh = hg \quad \forall h \in G\},$$

and this is evidently a subgroup of G. For fixed h, the map $g \mapsto g^{-1}h^{-1}gh$ is smooth and hence continuous. The preimage of e under this map is $\{g \in G : gh = hg\}$, so this set is closed. This represents $Z(G)$ as an intersection of closed subsets of G. Hence $Z(G)$ is a closed subgroup and thus a Lie subgroup of G.

There are lots of variations of this construction. For example, for any subset $A \subset G$ we can consider the *centralizer* of A in G, which is defined as

$$Z_G(A) := \{g \in G : ga = ag \quad \forall a \in A\}.$$

As before, one shows that this is a closed subgroup of G.

Using these, we can formulate the following result that in particular clarifies the consequences of injectivity of the derivative of a homomorphism between Lie groups.

Proposition 1.11. *Let $\varphi : G \to H$ be a homomorphism of Lie groups with derivative $\varphi' : \mathfrak{g} \to \mathfrak{h}$. Then the kernel* $\ker(\varphi) \subset G$ *is a normal Lie subgroup of G, whose Lie algebra coincides with* $\ker(\varphi') \subset \mathfrak{g}$.

In particular, if φ' is injective, then $\ker(\varphi)$ *is a discrete subgroup of G, which in addition is contained in the centralizer $Z(G_0)$ of the connected component of the identity of G. So if in addition G is connected* $\ker(\varphi)$ *is contained in the center $Z(G)$ of G.*

Proof. By definition $\ker(\varphi) = \varphi^{-1}(\{e\})$, so continuity of φ readily implies that it is a closed subset of G, while simple algebra implies that $\ker(\varphi)$ is a normal subgroup of G. So by Theorem 1.11, $\ker(\varphi)$ is Lie subgroup and its Lie algebra is formed by all $X \in \mathfrak{g}$ such that $\exp(tX) \in \ker(\varphi)$ for all t. By Theorem 1.9 this is equivalent to $\exp(t\varphi'(X)) = e$ for all t and hence to $\varphi'(X) = 0$, so the first claim follows.

If φ' is injective, then the Lie subgroup $\ker(\varphi) \subset G$ thus has Lie algebra $\{0\}$. So claim 3 in the proof of Theorem 1.11 in this case says that there is a neighborhood W of 0 in $\mathfrak{g}$ such that $\exp(W) \cap \ker(\varphi) = \{e\}$. For $g \in \ker(\varphi)$, $U_g := \lambda_g(\exp(W)) \subset G$ thus is open and evidently $U_g \cap \ker(\varphi) = \{g\}$. Hence we see that the induced topology on $\ker(\varphi)$ is discrete. Finally, for $X \in \mathfrak{g}$ and $g \in \ker(\varphi)$ consider the smooth curve $\exp(tX)g\exp(tX)^{-1}$. Since $\ker(\varphi)$ is normal, this must have values in $\ker(\varphi)$ so by discreteness it must be constantly equal to g. Thus we have $\exp(tX)g = g\exp(tX)$, so g also commutes with all elements of the subgroup generated by all elements of the form $\exp(X)$. By Theorem 1.9, this subgroup coincides with G_0, so $g \in Z(G_0)$, which coincides with $Z(G)$ if G is connected. □

1.12. Example: Matrix groups. These are probably the most important examples of Lie groups.

Definition 1.12. A *matrix group* is a closed subgroup of the group $GL(n, \mathbb{R})$ for some $n \in \mathbb{N}$.

By Theorem 1.11, any matrix group is a submanifold of $GL(n,\mathbb{R})$ (and hence of $M_n(\mathbb{R})$) and in particular a Lie group. For a matrix group $H \subset GL(n,\mathbb{R})$, the proof of Theorem 1.11 gives us two equivalent descriptions of the tangent space $\mathfrak{h}$ at the identity. Since the inclusion $i : H \hookrightarrow G$ is a Lie group homomorphism, the inclusion $\mathfrak{h} \hookrightarrow M_n(\mathbb{R})$ is a Lie algebra homomorphism. Using Example 1.4 we conclude that for any matrix group the Lie bracket on $\mathfrak{h} \subset M_n(\mathbb{R})$ is given by the commutator of matrices. By part (1) of Theorem 1.9 and Example 1.8 we next see that for any matrix group the exponential map is given by the matrix exponential. Finally, since the conjugation in H is just the restriction of the conjugation in G, we see from Example 1.10 that in any matrix group the adjoint representation is given by the conjugation.

Let us discuss some concrete examples of matrix groups. First one can take matrices of special form. Consider the subset $B(n,\mathbb{K}) \subset GL(n,\mathbb{K})$ consisting of those invertible matrices which are upper triangular, i.e. for which all entries below the main diagonal are zero. This evidently is a closed subset and one easily verifies that it is a subgroup. Thus, $B(n,\mathbb{K})$ is a Lie subgroup of $GL(n,\mathbb{K})$. Likewise, one can consider the subset $N(n,\mathbb{K}) \subset B(n,\mathbb{K})$ of those matrices for which in addition all entries on the main diagonal are equal to 1. This also is a closed subgroup of $GL(n,\mathbb{K})$ and thus a Lie group. The Lie algebras $\mathfrak{b}(n,\mathbb{K}) \supset \mathfrak{n}(n,\mathbb{K})$ of these groups clearly are the spaces of upper triangular and strictly upper triangular matrices, respectively. Also, invertible diagonal matrices form a closed subgroup of $GL(n,\mathbb{K})$. Similarly, one can deal with more general restrictions on the form of matrices, like block-diagonal or block-upper-triangular ones.

Second, let $b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a fixed bilinear map, and consider

$$H := \{A \in GL(n,\mathbb{R}) : b(Av, Aw) = b(v,w) \quad \forall w, v \in \mathbb{R}^n\}.$$

For fixed v and w, the map $A \mapsto b(Av, Aw) - b(v,w)$ is continuous, and hence the subset of invertible matrices satisfying the equation is closed. As an intersection of closed subsets of $GL(n,\mathbb{R})$, the subset H is closed, and one immediately verifies that it is a subgroup. To determine the Lie algebra $\mathfrak{h}$ of H, suppose that $c : \mathbb{R} \to GL(n,\mathbb{R})$ is a smooth curve with values in H. Differentiating the equation $b(c(t)v, c(t)w) = b(v,w)$ at $t = 0$ we see that $b(c'(0)v, w) + b(v, c'(0)w) = 0$, so $X := c'(0)$ is *skew symmetric* with respect to b.

Conversely, suppose that $X \in M_n(\mathbb{R})$ is skew symmetric with respect to b. Then we claim that $\exp(X) \in H$, which together with the above implies that $\mathfrak{h} = \{X \in M_n(\mathbb{R}) : b(Xv, w) + b(v, Xw) = 0\}$. As a bilinear map, b is continuous and thus for fixed v and w we get $b(\exp(X)v, \exp(X)w) = \sum_{i,j} \frac{1}{i!j!} b(X^i v, X^j w)$, and the sum converges absolutely. Collecting the terms with $i + j = k > 0$, we obtain

$$\tfrac{1}{k!} \textstyle\sum_{\ell=0}^{k} \binom{k}{\ell} b(X^{k-\ell}v, X^\ell w) = \tfrac{1}{k!} b(X^k v, w) \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} = 0,$$

and the claim follows.

More specifically, consider the case that b is the standard inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^n$. Then $H = O(n)$, the group of orthogonal $n \times n$-matrices, and $A \in H$ is equivalent to $A^t = A^{-1}$. The Lie algebra $\mathfrak{h} =: \mathfrak{o}(n)$ consists of all matrices X such that $0 = \langle Xv, w\rangle + \langle v, Xw\rangle$, which is equivalent to $X^t = -X$. Hence we obtain matrices which are skew symmetric in the usual sense. In particular, $\dim(O(n)) = \dim(\mathfrak{o}(n)) = \frac{n(n-1)}{2}$. Note that for an orthogonal matrix the columns form an orthonormal basis of $\mathbb{R}^n$. In particular, $O(n)$ is a bounded and hence compact subset of $M_n(R)$. Hence any closed subgroup of $O(n)$ is a compact Lie group.

As a slight variation, one may consider a non-degenerate symmetric bilinear form of signature (p,q) with $p+q=n$. (Recall from linear algebra that this form is essentially unique.) This gives rise to the Lie group $O(p,q)$, called the *(pseudo)-orthogonal group* of signature (p,q). If both p and q are nonzero, then $O(p,q)$ is not compact, since the unit sphere of an inner product of indefinite signature is unbounded. To obtain an explicit realization of $O(p,q)$, let $\mathbb{I}_{p,q}$ be the diagonal $n\times n$-matrix, whose first p entries are equal to 1, while the other entries are equal to -1. Then $(v,w)\mapsto\langle v,\mathbb{I}_{p,q}w\rangle$ is non-degenerate of signature (p,q). Using this form, we see that $A\in O(p,q)$ if and only if $A^t\mathbb{I}_{p,q}A=\mathbb{I}_{p,q}$. Since $\mathbb{I}_{p,q}^t=\mathbb{I}_{p,q}$ we see that the Lie algebra $\mathfrak{o}(p,q)$ consists of all matrices X such that $\mathbb{I}_{p,q}X$ is skew symmetric. Hence $\dim(O(p,q))=\dim(O(p+q))=\frac{(p+q)(p+q-1)}{2}$.

For $A\in O(p,q)$ the equation $A^t\mathbb{I}_{p,q}A=\mathbb{I}_{p,q}$ implies that $\det(A)^2=1$, so $\det(A)=\pm1$. Hence we obtain a closed subgroup $SO(p,q):=\{A:\det(A)=1\}\subset O(p,q)$. Evidently, any continuous curve in $O(p,q)$ which starts at the identity has to remain in $SO(p,q)$, so wee see that the connected component $O_0(p,q)$ of the identity in $O(p,q)$ has to be contained in $SO(p,q)$. We shall see later that $SO(n)$ coincides with the connected component $O_0(n)$.

There are two ways to generalize this to the complex world. From linear algebra, one knows that there is a unique (up to isomorphism) non degenerate symmetric bilinear form b on $\mathbb{C}^n$. As above, we obtain the Lie group $O(n,\mathbb{C})$ of complex orthogonal matrices and the subgroup $SO(n,\mathbb{C})$. As in the real case, $A\in O(n,\mathbb{C})$ if and only if $A^t=A^{-1}$, but the complex orthogonal groups are not compact. These are examples of *complex Lie groups*, i.e. they are complex manifolds (with holomorphic chart changes) such that multiplication and inversion are holomorphic maps. We will not go into this subject during this course.

The more common complex generalization of orthogonal groups are provided by using Hermitian inner products. Let $\langle\ ,\ \rangle$ be the standard Hermitian inner product on $\mathbb{C}^n$. This leads to the Lie group $U(n)$, the *unitary group*, whose elements can be alternatively characterized by $A^*=A^{-1}$. Here A^* is the conjugate transpose of A. For $n=1$, this reads as $\bar a=a^{-1}$ and hence $a\bar a=|a|^2=1$, so we recover the group $U(1)$ as defined in 1.1. The Lie algebra $\mathfrak{u}(n)$ of $U(n)$ consists of all matrices $X\in M_n(\mathbb{C})$ which are *skew Hermitian*, i.e. satisfy $X^*=-X$. Note that $\mathfrak{u}(n)$ is *not* a complex vector space, since for example $a\,\mathrm{id}$ in $\mathfrak{u}(n)$ if and only if $a\in i\mathbb{R}\subset\mathbb{C}$. Since a skew Hermitian $n\times n$-matrix is determined by $\frac{n(n-1)}{2}$ arbitrary complex numbers (the entries above the main diagonal) and n real numbers (the purely imaginary diagonal entries) we see that $\dim_{\mathbb{R}}(U(n))=\dim_{\mathbb{R}}(\mathfrak{u}(n))=n^2$.

Considering $(z,w)\mapsto\langle z,\mathbb{I}_{p,q}w\rangle$, we obtain a Hermitian form of signature (p,q), which is again essentially unique. This gives rise to the group $U(p,q)$, which is called the *(pseudo)-unitary group* of signature (p,q). A matrix A lies in $U(p,q)$ if $A^*\mathbb{I}_{p,q}A=\mathbb{I}_{p,q}$. Since $\mathbb{I}_{p,q}^*=\mathbb{I}_{p,q}$, the Lie algebra $\mathfrak{u}(p,q)$ of $U(p,q)$ is formed by all matrices X such that $\mathbb{I}_{p,q}X$ is skew Hermitian. Similarly to the real case, the group $U(n)$ is compact, while for $p,q\neq0$ the group $U(p,q)$ is non-compact. Similarly as before we have $\dim(U(p,q))=\dim(U(p+q))$.

For a matrix $A\in U(p,q)$ we obtain $\det(A)\det(A^*)=|\det(A)|^2=1$. Indeed, the determinant defines a smooth homomorphism $U(p,q)\to U(1)$. The kernel of this homomorphism is the closed normal subgroup $SU(p,q)\subset U(p,q)$, the *special unitary group* of signature (p,q). Of course, we have $\dim_{\mathbb{R}}(SU(p,q))=(p+q)^2-1$.

Let us finally mention, the similar to bilinear forms one can also deal with bilinear operations on a vector space. As a simple example, consider an arbitrary Lie algebra

$(\mathfrak{g}, [\ ,\ ])$. Since $\mathfrak{g}$ is a vector space, we have the Lie group $GL(\mathfrak{g})$ of linear automorphisms of $\mathfrak{g}$. Now we can consider the subset

$$\mathrm{Aut}(\mathfrak{g}) := \{A \in GL(\mathfrak{g}) : [A(X), A(Y)] = A([X,Y]) \quad \forall X, Y \in \mathfrak{g}\}.$$

This is immediately seen to be a subgroup of $GL(\mathfrak{g})$ called the *automorphism group* of the Lie algebra $\mathfrak{g}$. Moreover, for fixed $X, Y \in \mathfrak{g}$ the set of all $A \in GL(\mathfrak{g})$ such that $[A(X), A(Y)] = A([X,Y])$ is evidently closed. Hence $\mathrm{Aut}(\mathfrak{g})$ is a closed subgroup of $GL(\mathfrak{g})$ and thus a Lie group.

1.13. Initial submanifolds. For some purposes, the concept of a submanifold is too restrictive. The simplest example of this situation actually comes from Lie groups. Consider the group $G := U(1) \times U(1) = \{(z,w) \in \mathbb{C}^2 : |z| = |w| = 1\}$. The Lie algebra of this Lie group is $\mathbb{R}^2$ with the trivial bracket and the exponential map is given by $(t,s) \mapsto (e^{it}, e^{is})$. Apart from the trivial cases $\{0\}$ and $\mathbb{R}^2$, a Lie subalgebra of $\mathbb{R}^2$ is simply a one dimensional subspace. The coordinate axes correspond to the two factors $U(1)$ in the product. For any other one dimensional subspace, we can choose a generator of the form $(1, \alpha)$ with $\alpha \in \mathbb{R}_{>0}$. The image of this subalgebra under the exponential map is $H_\alpha := \{(e^{it}, e^{i\alpha t}) : t \in \mathbb{R}\}$, which is a subgroup of G. However, the nature of this subgroup strongly depends on α. Note first that $(e^{it}, e^{i\alpha t}) = (e^{is}, e^{i\alpha s})$ if and only if $s = t + 2k\pi$ and $\alpha s = \alpha t + 2\ell\pi$ for some $k, \ell \in \mathbb{Z}$. Inserting the first equation into the second, we see that existence of a solution with $s \neq t$ implies $\alpha k = \ell$ and hence $\alpha = \frac{\ell}{k} \in \mathbb{Q}$.

Writing $\alpha = \frac{p}{q}$ with $p, q \in \mathbb{N}$ without common factors, one immediately verifies that $e^{it} \mapsto (e^{iqt}, e^{ipt})$ induces a bijection $U(1) \to H_\alpha$. This bijection is evidently continuous and hence a homeomorphism by compactness of $U(1)$. Since H_α is compact it is closed in G and thus a Lie subgroup by Theorem 1.11.

For irrational α, the evident map $\mathbb{R} \to H_\alpha$ is injective, so as a group H_α is isomorphic to $\mathbb{R}$. It turns out that this line winds densely around the torus. In particular, there is a countable dense subset $A \subset U(1)$ such that $H_\alpha \cap U(1) \times \{1\} = A \times \{1\}$. This implies that the topology on $\mathbb{R}$ induced from this inclusion is different from the usual topology: Any neighborhood of $(1,1)$ contains infinitely many points from $A \times \{1\}$. Hence in the induced topology on $\mathbb{R}$, any neighborhood of 0 contains arbitrarily large points. Also, H_α cannot be a submanifold of G.

This example also shows how to obtain a solution to the problem: In an appropriate chart around $(1,1)$, the subset H_α (for irrational α) consists of countably many parallel line segments. While these segments come arbitrarily close to each other and hence inherit a strange topology, they are “disjoint” from the point of view of a smooth curve, which can never move from one segment to another. This motivates the following definitions:

Definition 1.13. (1) Let M be a smooth manifold, $A \subset M$ any subset and $x_0 \in A$ a point. The we define $C_{x_0}(A)$ to be the set of all $x \in A$ such that there is a smooth path $c : I \to M$ with values in A, which joins x_0 to x.

(2) Let M be a smooth manifold of dimension n. Then a subset $N \subset M$ is called an *initial submanifold* of M of dimension m if for any $x \in N$ there exists a chart (U, u) for M with $x \in U$ and $u(x) = 0$ such that $u(C_x(U \cap N)) = u(U) \cap \mathbb{R}^m \times \{0\}$.

For an initial submanifold $N \subset M$, one can take the subset N “out of M” and endow it with a new topology. To distinguish this in the notation, let us write $N^{\#}$ for this set and $i : N^{\#} \to M$ for the obvious inclusion map. Now declare $V \subset N^{\#}$ to be open if $u(i(V) \cap C_x(U \cap N))$ is open in $\mathbb{R}^m \times \{0\}$ for a family of adapted charts as in the definition. Then one can use the restrictions of adapted charts as charts on $N^\#$ and different charts obtained in that way are compatible by construction, so we obtain a smooth atlas. The resulting topology is finer than the one on N and thus Hausdorff, and if it is also second countable, then we have obtained the structure of a smooth manifold on $N^\#$ for which $i : N \hookrightarrow M$ is smooth. By construction i even is an injective immersion (see [**AnaMf**, Section1.19]).

Suppose now that X is any manifold and $f : X \to M$ is a smooth map such that $f(X) \subset N$. Then there is a unique map $f^\# : X \to N^\#$ such that $f = i \circ f^\#$. From the construction above, we readily conclude that $f^\#$ is smooth and conversely for a smooth map $g^\# : X \to N^\#$ also $i \circ g^\#$ is smooth. This universal property implies that the smooth structure on $N^\#$ is uniquely determined and it explains the name "initial submanifold".

The difference between initial submanifolds and true submanifolds really lies in the topology only: Let $N \subset M$ be an initial submanifold and $i : N \to M$ the injective immersion as constructed above, and suppose that i is a homeomorphism onto its image. Then for an adapted chart (U, u) around $x \in N$ the set $C_x(U \cap N)$ is open in the topology induced from M, so there is an open subset $V \subset M$ such that $C_x(U \cap N) = V \cap N$. But then $(U \cap V, u|_{U \cap V})$ is a submanifold chart for N at x, so $N \subset M$ is a true submanifold.

Conversely, one can show that for an injective immersion $i : N \to M$ which has the above universal property, the image $i(N) \subset M$ is an initial submanifold. Proofs for these facts and more details about initial submanifolds can be found in [**Michor**, Section 2].

1.14. Virtual Lie subgroups. Now for a Lie group G, one defines a *virtual Lie subgroup* of G as the image of an injective smooth homomorphism from some Lie group H to G. For injective homomorphism $i : H \to G$, Proposition 1.11 shows that $\ker(i') = \{0\}$, so $i' = T_e i : \mathfrak{h} \to \mathfrak{g}$ is injective. Since $i = \lambda_{i(h)} \circ i \circ \lambda_{h^{-1}}$, we conclude that $T_h i$ is injective for each $h \in H$, so i is an injective immersion. We shall see in the next chapter that $i(H)$ even is an initial submanifold of G, see Section 2.3. We will often suppress the injective map $i' : \mathfrak{h} \to \mathfrak{g}$, and simply view $\mathfrak{h}$ as subspace of $\mathfrak{g}$. Since i' is a Lie algebra homomorphism, $\mathfrak{h} \subset \mathfrak{g}$ is a *Lie subalgebra*, i.e. a subspace that is closed under the Lie bracket.

As a further example of the Lie group – Lie algebra correspondence, let us discuss the Lie algebraic characterization of normal subgroups.

Definition 1.14. Let $\mathfrak{g}$ be a Lie algebra. An *ideal* in $\mathfrak{g}$ is a linear subspace $\mathfrak{h} \subset \mathfrak{g}$ such that $[X, Y] \in \mathfrak{h}$ for all $X \in \mathfrak{h}$ and all $Y \in \mathfrak{g}$. (This implies that there is an induced Lie algebra structure on the quotient $\mathfrak{g}/\mathfrak{h}$.)

Proposition 1.14. *Let $i : H \to G$ be a connected virtual Lie subgroup of a connected Lie group G. Then $i(H)$ is a normal subgroup of G if and only if $\mathfrak{h}$ is an ideal in $\mathfrak{g}$.*

Proof. Note first that $i(H)$ is normal in G if and only if $\operatorname{conj}_g(i(h)) \in i(H)$ for all $h \in H$. If this condition is satisfied, then for $X \in \mathfrak{h}$ and $t \in \mathbb{R}$ we get $\operatorname{conj}_g(\exp(ti'(X))) = \exp(t \operatorname{Ad}(g)(i'(X))) \in i(H)$, and differentiating this at $t = 0$ we conclude that $\operatorname{Ad}(g)(i'(\mathfrak{h})) \subset i'(\mathfrak{h})$ for all $g \in G$. Putting $g = \exp(tY)$ for $Y \in \mathfrak{g}$ and differentiating again at $t = 0$, we get $\operatorname{ad}(Y)(i'(\mathfrak{h})) \subset i'(\mathfrak{h})$ for all $Y \in \mathfrak{g}$. This exactly means that $i'(\mathfrak{h})$ is an ideal in $\mathfrak{g}$.

Conversely, if $i'(\mathfrak{h})$ is an ideal in $\mathfrak{g}$, then $\operatorname{ad}(Y)(i'(\mathfrak{h})) \subset i'(\mathfrak{h})$ for all $Y \in \mathfrak{g}$. Then also $\operatorname{Ad}(\exp(Y)) = e^{\operatorname{ad}(Y)}$ preserves the subspace $i'(\mathfrak{h})$. Since G is connected and Ad is a homomorphism, part (2) of Theorem 1.9 implies that $\mathrm{Ad}(g)(i'(\mathfrak{h})) \subset i'(\mathfrak{h})$ for all $g \in G$. Hence $\exp(\mathrm{Ad}(g)(i'(X))) = \mathrm{conj}_g(i(\exp(X))) \in i(H)$ for all $X \in \mathfrak{h}$. But since H is connected and conj_g is a homomorphism, this implies $\mathrm{conj}_g(i(H)) \subset i(H)$ by part (2) of Theorem 1.9. $\square$

Actions and homogeneous spaces

1.15. Actions and related notions. A *left action* of a Lie group G on a set X is a map $\ell : G \times X \to X$ such that $\ell(e, x) = x$ and $\ell(g, \ell(h, x)) = \ell(gh, x)$ for all $g, h \in G$ and all $x \in X$. Fixing $g \in G$, we obtain the map $\ell_g : X \to X$ and fixing $x \in X$, we obtain the map $\ell^x : G \to X$, which are defined by $\ell_g(x) = \ell^x(g) = \ell(g, x)$.

A *right action* of a Lie group G on a set X is a map $r : X \times G \to X$ such that $r(x, e) = x$ and $r(r(x, g), h) = r(x, gh)$ for all $g, h \in G$ and all $x \in X$. Similarly as above, there are the partial maps $r^g : X \to X$ and $r_x : G \to X$.

When there is no risk of confusion, we will denote actions simply by dots: A left action will be written as $(g, x) \mapsto g \cdot x$. Then the defining properties read as $e \cdot x = x$ and $g \cdot (h \cdot x) = gh \cdot x$. Likewise, writing a right action as $(x, g) \mapsto x \cdot g$, the defining properties read as $x \cdot e = x$ and $(x \cdot g) \cdot h = x \cdot gh$.

For a left action ℓ of G on X we have $\ell_g \circ \ell_h = \ell_{gh}$ which together with $\ell_e = \mathrm{id}_X$ implies that the maps ℓ_g and $\ell_{g^{-1}}$ are inverse to each other. Hence we can view $g \mapsto \ell_g$ as a map from G to the set $Bij(X)$ of bijective maps $X \to X$. The bijections form a group under compositions, and the defining properties exactly mean that ℓ defines a group homomorphism $G \to Bij(X)$. For the case of a right action, we obtain $r^{g^{-1}} = (r^g)^{-1}$, but for a right action we get $r^{gh} = r^h \circ r^g$, so we obtain an *anti-homomorphism* $G \to Bij(X)$ in this case.

It is no problem to convert between left and right actions. Given a left action ℓ, we can define a right action r by $r^g := \ell_{g^{-1}}$ and vice versa. Nonetheless it will often be important to distinguish between the two kinds of actions.

Given a left action ℓ of G on X and a point $x \in X$, we define the *orbit* $G \cdot x$ of x as $G \cdot x = \mathrm{im}(\ell^x) = \{g \cdot x : g \in G\}$. One should think about an action as a way to use elements of G to move points in X around, and the orbit of x consists of those points that can be reached from x. If $x, y \in X$ are two points such that $G \cdot x \cap G \cdot y \neq \emptyset$, then we find $g, h \in G$ such that $g \cdot x = h \cdot y$ and hence $x = g^{-1}h \cdot y$ and $\tilde{g} \cdot x = \tilde{g}g^{-1}h \cdot y$. Thus we obtain $G \cdot x \subset G \cdot y$ and by symmetry the two orbits coincide. Hence lying in the same orbit is an equivalence relation and X is the disjoint union of the G-orbits. The set of equivalence classes is called the *orbit space* of the action and is denoted by X/G. An action is called *transitive* if there is only one orbit. Equivalently, this means that for each pair x, y of points in X, there is an element $g \in G$ such that $y = g \cdot x$. The notions of orbits and transitivity are defined analogously for right actions.

Consider again a left action ℓ of G on X and a point $x \in X$. Then we define the *stabilizer* or the *isotropy subgroup* G_x of x by $G_x := \{g \in G : g \cdot x = x\}$, so this consists of all elements of G which do not move x. Evidently, G_x is a subgroup of G. For $y \in G \cdot x$ consider the stabilizer G_y. If $y = h \cdot x$, then $h \cdot y = y$ is equivalent to $hg \cdot x = g \cdot x$ and hence to $g^{-1}hg \cdot x = x$. Otherwise put, we have $G_{g \cdot x} = gG_xg^{-1}$ and hence the isotropy subgroups of points in the same orbit are conjugated in G.

By definition, the map $\ell^x : G \to X$ induces a surjection $\ell^x : G \to G \cdot x$. Now for $g, h \in G$ we have $g \cdot x = h \cdot x$ if and only if $g^{-1}h \cdot x = x$, i.e. $g^{-1}h \in G_x$. But the latter condition exactly means that $h \in gG_x$ and hence $hG_x = gG_x$, i.e. g and h represent the same coset in G/G_x. Hence we conclude that ℓ^x induces a bijection between the space G/G_x of left cosets and the orbit $G \cdot x$. In particular, starting from a transitive action of G on X, we obtain a bijection between X and G/G_x for any fixed point $x \in X$. This can be used to identify X with the space G/G_x, which naturally carries a topology and, as we shall see soon, under weak assumptions is a smooth manifold. All this works similarly for right actions.

If the space X carries some additional structure, then we can require compatibility conditions for that structure. For example, if X is a topological space, then we can define a *continuous left action* of G on X by in addition requiring that $\ell : G \times X \to X$ is continuous. This immediately implies that the maps ℓ_g and ℓ^x are continuous for all $g \in G$ and $x \in X$. Then ℓ_g even is a homeomorphism, since its inverse is $\ell_{g^{-1}}$, and we can consider the action as a homomorphism from G to the group of homeomorphisms of X. Since we have an obvious surjection $X \to X/G$ the topology of X induces a topology on the orbit space X/G. However, even for very nice X and G the space X/G can be very badly behaved.

On the other hand, if X is Hausdorff, then continuity of ℓ^x implies that the isotropy subgroup G_x is a closed subgroup of G and hence a Lie subgroup by Theorem 1.11. In this case, we obtain a continuous bijection between G/G_x and the orbit $G \cdot x \subset X$. However, this is not a homeomorphism in general. As an example, consider the subgroup $H_\alpha \subset U(1) \times U(1)$ from 1.13 for irrational α. Multiplying from the left defines a continuous (even smooth) left action of H_α on the torus $U(1) \times U(1)$. For this action each isotropy group is trivial so in particular the map $H_\alpha \to H_\alpha \cdot e$ as constructed above is simply the inclusion of $H_\alpha \cong \mathbb{R}$ into $U(1) \times U(1)$, which is not a homeomorphism by 1.13. As in 1.13 one sees that each H_α-orbit is dense in $U(1) \times U(1)$. Thus the orbit spaces carries the indiscrete topology, i.e. the only open subsets are the empty set and the whole space.

Similarly, if M is a smooth manifold, then we define a *smooth left action* of G on M by in addition requiring the ℓ is smooth. Then each ℓ_g is a diffeomorphism. Continuous and smooth right actions are defined similarly as for left actions.

1.16. Homogeneous spaces. As we have seen above, any orbit of a G-action looks like the space G/H of left cosets for a subgroup $H \subset G$. The obvious surjection $G \to G/H$ can be used to endow this space with a topology. The preimage of the base point $o := eH \in G/H$ under this map is the subgroup H, so if one requires the topology on G/H to be Hausdorff, one must assume that the subgroup $H \subset G$ is closed. For a closed subgroup $H \subset G$, we call the space G/H of left cosets the *homogeneous space* of G with respect to H. Notice that the left translation by $g \in G$ induces a continuous map $\ell_g : G/H \to G/H$.

Recall from [**AnaMf**, Section 1.18] that a smooth map $f : M \to N$ between two manifolds is called a *submersion* if each tangent map of f is surjective. In particular, $\dim(M) \geq \dim(N)$ and one shows that in appropriate charts f has the form $(x^1, \dots, x^m) \mapsto (x^1, \dots, x^n)$. In particular, locally a surjective submersion admits smooth sections, i.e. a smooth map σ such that $f \circ \sigma = \mathrm{id}$. This implies that a surjective submersion $f : M \to N$ has a universal property. Namely, a map φ from N to any manifold Z is smooth if and only if $\varphi \circ f : M \to Z$ is smooth.

THEOREM 1.16. *Let G be a Lie group and $H \subset G$ a closed subgroup. Then there is a unique structure of a smooth manifold on the homogeneous space G/H such that the natural map $p : G \to G/H$ is a submersion. In particular,* $\dim(G/H) = \dim(G) - \dim(H)$*.*

PROOF. From Theorem 1.11 we know that H is a Lie subgroup of G and we denote by $\mathfrak{h} \subset \mathfrak{g}$ the corresponding Lie subalgebra. Choose a linear subspace $\mathfrak{k} \subset \mathfrak{g}$ which is complementary to the subspace $\mathfrak{h}$. Consider the map $\varphi : \mathfrak{k} \times \mathfrak{h} \to G$ defined by $\varphi(X,Y) := \exp(X)\exp(Y)$. From the proof of Theorem 1.11 we can find open neighborhoods W_1 of 0 in $\mathfrak{k}$ and V of 0 in $\mathfrak{h}$ such that $\exp(W_1) \cap H = \{e\}$ and such that φ restricts to a diffeomorphism from $W_1 \times V$ onto an open neighborhood U' of e in G.

Next, $(X_1, X_2) \mapsto \exp(X_1)^{-1}\exp(X_2)$ defines a continuous map $\mathfrak{k} \times \mathfrak{k} \to G$, so the preimage of U' under this map is an open neighborhood of $(0, 0)$. Thus we can find an open neighborhood W_2 of 0 in $\mathfrak{k}$ such that $\exp(X_1)^{-1}\exp(X_2) \in U'$ for $X_1, X_2 \in W_2$, and we consider the neighborhood $W := W_1 \cap W_2$ of 0 in $\mathfrak{k}$.

Now we claim that the smooth map $f : \mathfrak{k} \times H \to G$ defined by $f(X,h) := \exp(X)h$ restricts to a diffeomorphism from $W \times H$ onto an open neighborhood U of e in G. If $\exp(X_1)h_1 = \exp(X_2)h_2$, then $h_1h_2^{-1} = \exp(X_1)^{-1}\exp(X_2) \in H \cap U' = \exp(V)$. Hence there is an element $Y \in V$ such that $\exp(Y) = \exp(X_1)^{-1}\exp(X_2)$ and thus $\exp(X_1)\exp(Y) = \exp(X_2)$. But this means $\varphi(X_1,Y) = \varphi(X_2, 0)$ and hence $Y = 0$ and $X_1 = X_2$, which also implies $h_1 = h_2$. Hence the restriction of f to $W \times H$ is injective.

Next, we have $f(X, \exp(Y)) = \varphi(X,Y)$, so locally around $W \times \{e\}$ we can write f as $\varphi \circ (\mathrm{id}, \exp^{-1})$. In particular, the tangent map $T_{(X,e)}f$ is a linear isomorphism for each $X \in W$. Since $f \circ (\mathrm{id}, \rho^h) = \rho^h \circ f$, we see that all tangent maps of f are invertible, so f is a local diffeomorphism everywhere. Together with injectivity, this implies that $U := f(W \times H)$ is open and $f : W \times H \to U$ is a diffeomorphism.

Consider the natural map $p : G \to G/H$ and the subset $p(U) \subset G/H$. For $g \in U$ and $h \in H$, we by construction have $gh \in U$, so $p^{-1}(p(U)) = U$ and hence $p(U)$ is open in the quotient topology on G/H. Now define $\psi : W \to p(U)$ by $\psi(X) := p(\exp(X))$. If $\psi(X_1) = \psi(X_2)$, then $\exp(X_1)H = \exp(X_2)H$, which implies $X_1 = X_2$ by injectivity of f on $W \times H$. Surjectivity of f immediately implies that ψ is surjective, and ψ is continuous by construction. If $W' \subset W$ is open, then $p^{-1}(\psi(W')) = f(W' \times H)$ is open in G, and hence $\psi : W \to p(U)$ is a homeomorphism.

For $g \in G$ put $U_g := p(\lambda_g(U)) = \{g\exp(X)H : X \in W\} \subset G/H$ and define $u_g : U_g \to W$ by $u_g := \psi^{-1} \circ \ell_{g^{-1}}$, where $\ell_{g^{-1}}(g'H) = g^{-1}g'H$. We claim, that (U_g, u_g) is a smooth atlas for G/H. Suppose that $U_g \cap U_{g'} \neq \emptyset$. Then

$$(u_{g'} \circ u_g^{-1})(X) = u_{g'}(g\exp(X)H) = \psi^{-1}((g')^{-1}g\exp(X)H) = \mathrm{pr}_1(f^{-1}((g')^{-1}g\exp(X)))$$

an by construction $(g')^{-1}g\exp(X) \in U$. Hence the transition function $u_{g'} \circ (u_g)^{-1}$ is a restriction of the smooth map $\mathrm{pr}_1 \circ f^{-1} \circ \lambda_{(g')^{-1}g} \circ \exp$ and thus it is smooth. □

We have already observed that we can define a map $\ell : G \times G/H \to G/H$ by $\ell(g, g'H) := gg'H$. Otherwise put, $\ell \circ (\mathrm{id} \times p) = p \circ \mu$, where $p : G \to G/H$ is the canonical map. Since p is a surjective submersion, so is $\mathrm{id} \times p$ and hence ℓ is smooth by the universal property of surjective submersions discussed above. Clearly, ℓ defines a left action, so G acts smoothly on G/H and by construction the isotropy group of the base point $o = eH$ is $H \subset G$.

On the other hand, restricting the multiplication map to $G \times H$, we obtain a right action of H on G, and by definition the orbit space of this action is the homogeneous space G/H. This action has the additional property that it is *free*. In general, this (very strong) condition means that if $x \cdot h = x$ for one $x \in X$, then $h = e$. Here it is evidently satisfied by the properties of the multiplication.

The proof of the theorem gives us quite a bit more than just an atlas for G/H. For the subsets $U_g \subset G/H$ constructed in the proof, we can define a smooth map $\sigma_g : U_g \to G$ by $\sigma_g(x) := g\exp(u_g(x))$. By construction, this has values in $p^{-1}(U_g)$ and $p \circ \sigma_g = \mathrm{id}_{U_g}$. This can now be used to construct a smooth map $\varphi_g : p^{-1}(U_g) \to U_g \times H$ by $\varphi_g(y) := (p(y), \sigma_g(p(y))^{-1}y)$. This even is a diffeomorphism, since the inverse is given by $(x,h) \mapsto \sigma_g(x)h$. If $g, g' \in G$ are such that $V := U_g \cap U_{g'} \neq \emptyset$, then we obtain a transition function $\varphi_{g'} \circ \varphi_g^{-1} : V \times H \to V \times H$. By construction, this must be of the form $(x,h) \mapsto (x, \alpha(x,h))$ for some smooth function $\alpha : V \times H \to H$. From the formulae above, one reads off that $\alpha(x,h) = (\sigma_{g'}(x)^{-1}\sigma_g(x))h$ and $x \mapsto \sigma_{g'}(x)^{-1}\sigma_g(x)$ is a smooth map $V \to H$. We will later say that (U_g, φ_g) is a *principal fiber bundle atlas*, showing that $p : G \to G/H$ is a principal fiber bundle with structure group H.

1.17. Quotients of Lie groups. As a first application of Theorem 1.16, consider the case that $H \subset G$ is a closed normal subgroup. Then basic algebra implies that the multiplication on G induces a well defined multiplication on G/H via $(gH)(g'H) = (gg')H$, which makes G/H into a group. Denoting this multiplication by $\underline{\mu} : G/H \times G/H \to G/H$, we of course get $p \circ \mu = \underline{\mu} \circ (p \times p)$. Since p is a surjective submersion, the same is true for $p \times p$, so the universal property for surjective submersions implies that $\underline{\mu}$ is smooth. Hence the quotient G/H canonically is a Lie group and $p : G \to G/H$ is a homomorphism of Lie groups. Proposition 1.14 shows that the Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ actually is an ideal, so the quotient $\mathfrak{g}/\mathfrak{h}$ inherits a Lie algebra structure from $\mathfrak{g}$. Clearly $p' = T_e p$ induces a surjection $\mathfrak{g} \to T_{eH}G/H$ with kernel $\mathfrak{h}$ which is a Lie algebra homomorphism. Thus the Lie algebra of G/H is $\mathfrak{g}/\mathfrak{h}$ with the induced structure.

Using this, we can prove the following result that also clarifies the significance of surjectivity of the derivative of a homomorphism.

PROPOSITION 1.17. *Let $\varphi : G \to H$ be a smooth homomorphism of Lie groups with kernel $\ker(\varphi) \subset G$ and derivative $\varphi' : \mathfrak{g} \to \mathfrak{h}$.*

(1) We can write φ as the composition of the inclusion of $G/\ker(\varphi)$ as a virtual Lie subgroup into H with the canonical quotient projection $p : G \to G/\ker(\varphi)$.

(2) If φ' is surjective and H is connected, then φ induces an isomorphism of Lie groups $G/\ker(\varphi) \cong H$.

PROOF. (1) From Proposition 1.11, we know that $\ker(\varphi) \subset G$ is a closed normal subgroup, so $G/\ker(\varphi)$ is a Lie group and we have the canonical quotient homomorphism $p : G \to G/\ker(\varphi)$. Elementary algebra shows that φ induces and injective homomorphism $\underline{\varphi} : G/\ker(\varphi) \to H$ such that $\underline{\varphi} \circ p = \varphi$. Since p is a surjective submersion, $\underline{\varphi}$ is smooth and thus the inclusion of a virtual Lie subgroup.

(2) Since φ' is surjective, part (1) Theorem 1.9 shows that $\exp(X) \in \varphi(G)$ for all $X \in \mathfrak{h}$. Since $\varphi(G) \subset H$ is a subgroup, part (2) of that theorem implies that φ is surjective. This implies that map $\underline{\varphi} : G/\ker(\varphi) \to H$ from (1) is a bijective smooth homomorphism, and since $\ker(p') = \ker(\varphi')$, also is a linear isomorphism. But this immediately implies that all tangent maps of $\underline{\varphi}$ are linear isomorphisms, so $\underline{\varphi}^{-1}$ is smooth by the inverse function theorem and the claim follows. □

1.18. Further applications. An important line of application of Theorem 1.16 to make a broad variety of sets into smooth manifolds. For example, let $\mathbb{R}P^n$ be the set of lines through zero in $\mathbb{R}^{n+1}$. The group $G := GL(n+1, \mathbb{R})$ acts on this set by $A \cdot \ell = A(\ell)$. Let e_1 be the first vector in the standard basis of $\mathbb{R}^{n+1}$ and let ℓ_0 be the line generated by this vector. Given any line ℓ, choose a nonzero vector $v \in \ell$ and extend it to a basis of $\mathbb{R}^{n+1}$. Taking the resulting vectors as the columns of a matrix A, we obtain $A(e_1) = v$, and hence $A(\ell_0) = \ell$. The isotropy group G_{ℓ_0} is the subgroup of all invertible matrices whose first column is a multiple of e_1, so this is evidently closed in G. Hence we can identify $\mathbb{R}P^n$ with the manifold G/G_{ℓ_0}.

Replacing G by $O(n+1)$ we can use the same argument by taking a unit vector in ℓ and extending it to an orthonormal basis of $\mathbb{R}^{n+1}$. So we see that also $O(n+1)$ acts transitively on $\mathbb{R}P^n$. An orthogonal matrix which fixes ℓ_0 must map e_1 to $\pm e_1$, and any e_i for $i > 1$ to a vector orthogonal to e_1. Hence the isotropy group of ℓ_0 consists of all matrices of the form $\begin{pmatrix} \pm 1 & 0 \\ 0 & A \end{pmatrix}$ and one immediately sees that $A \in O(n)$. Hence we can view $\mathbb{R}P^n$ as $O(n+1)/(O(1) \times O(n))$ which in particular implies that $\mathbb{R}P^n$ is compact since $O(n+1)$ is compact.

This can be easily generalized to the so-called *Grassman manifolds*. By definition $Gr_k(\mathbb{R}^n)$ is the set of all k-dimensional subspaces of $\mathbb{R}^n$, so $\mathbb{R}P^n = Gr_1(\mathbb{R}^{n+1})$. Again $G = GL(n, \mathbb{R})$ acts on this space by $A \cdot V := A(V)$. Take V_0 to be the subspace spanned by the first k vectors in the standard basis. For an arbitrary subspace V, choose a basis $\{v_1, \dots, v_k\}$, extend it to a basis of $\mathbb{R}^n$ and use the resulting elements as the columns of a matrix A. Then $A(e_i) = v_i$ for $i = 1, \dots, k$ and hence $A(V_0) = V$. The isotropy subgroup of V_0 consists of all block matrices of the form $\begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$ with blocks of size k and $n-k$ so it is closed. Hence we can make $Gr_k(\mathbb{R}^n)$ into a smooth manifold, by identifying it with G/G_{V_0}.

Replacing bases by orthonormal bases, we can apply the same argument with $O(n)$ replacing $GL(n, \mathbb{R})$. An orthogonal matrix which preserves the subspace V_0 also preserves its orthocomplement, so the isotropy group of V_0 consists of block diagonal matrices and is isomorphic to $O(k) \times O(n-k)$. Hence we can also view $Gr_k(\mathbb{R}^n)$ as $O(n)/(O(k) \times O(n-k))$, which shows that $Gr_k(\mathbb{R}^n)$ is a compact manifold.

Next consider the set $\mathcal{G}_n$ of positive definite inner products on $\mathbb{R}^n$. An element of $\mathcal{G}_n$ by definition is a symmetric bilinear map $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that $g(x, x) > 0$ for all $x \neq 0$. The group $G := GL(n, \mathbb{R})$ acts on $\mathcal{G}_n$ by $A \cdot g(x, y) := g(A^{-1}(x), A^{-1}(y))$. (Checking that this defines a left action is a good exercise.) Let $\langle\ ,\ \rangle$ be the standard inner product. Applying the Gram–Schmidt procedure with respect to g to any basis of $\mathbb{R}^n$, we obtain a basis $\{v_1, \dots, v_n\}$ of $\mathbb{R}^n$ such that $g(v_i, v_j) = \delta_{ij}$. The matrix A which has these vectors as columns is invertible. For $x \in \mathbb{R}^n$ we then by construction have $A^{-1}(x) = A^{-1}(\sum x_i v_i) = \sum x_i e_i$ and therefore

$$\langle A^{-1}(x), A^{-1}(y) \rangle = \sum x_i y_i = g(\sum x_i v_i, \sum y_j v_j) = g(x, y).$$

Thus we see that $g = A \cdot \langle\ ,\ \rangle$ so G acts transitively on $\mathcal{G}_n$. The isotropy group of the standard inner product by definition is the closed subgroup $O(n) \subset GL(n, \mathbb{R})$, and hence we may identify $\mathcal{G}_n$ with the smooth manifold $GL(n, \mathbb{R})/O(n)$. (Of course, there is a simpler way to identify $\mathcal{G}_n$ as a smooth manifold, since it can be viewed as the space of all positive definite symmetric $n \times n$-matrices, which is an open subspace of the vector space of all symmetric matrices.)

We can also use homogeneous spaces to clarify connectedness properties:

LEMMA 1.18. *Let G be a Lie group and $H \subset G$ a closed subgroup such that the homogeneous space G/H is connected. Then the inclusion of H into G induces a surjective homomorphism $H/H_0 \to G/G_0$ between the component groups.*

PROOF. The inclusion $i : H \to G$ is a homomorphism and connectedness of H_0 implies that $i(H_0) \subset G_0$. Hence we get an induced homomorphism $H/H_0 \to G/G_0$ and to see that surjective, we have to show that any connected component of G contains an element of H.

Let $g_0 \in G$ be any element and consider $g_0 H \in G/H$. Since the homogeneous space is connected, we can find a continuous curve $c : [0, 1] \to G/H$ such that $c(0) = g_0 H$ and $c(1) = eH$. By compactness of $[0, 1]$ we can find a subdivision $0 = t_0 < t_1 < \cdots < t_n = 1$

such that $c([t_i, t_{i+1}])$ is contained in one of the charts constructed in the proof of Theorem 1.16. Let U_0 be the first of these charts and let $\sigma_0 : U_0 \to p^{-1}(U_0)$ be the corresponding section. Then there is a unique element $h_0 \in H$ such that $g_0 = \sigma_0(g_0H)h_0$ and we define $\tilde{c} : [0, t_1] \to G$ by $\tilde{c}(t) = \sigma_0(c(t))h_0$. Denoting the next chart and section by U_1 and σ_1, there is a unique element $h_1 \in H$ such that $\sigma_0(c(t_1))h_0 = \sigma_1(c(t_1))h_1$. Using this, we define $\tilde{c}$ on $[t_1, t_2]$ as $\sigma_1(c(t))h_1$, thus obtaining a continuous lift $\tilde{c}$ of c on $[0, t_2]$. Proceeding in that way we obtain a continuous lift $\tilde{c} : [0, 1] \to G$ of c. By construction $\tilde{c}(0) = g_0$ and $p(\tilde{c}(1)) = eH$, so $\tilde{c}(1) \in H$. Hence there is an element of H in the same connected component as g_0. □

Using this, we can now clarify the connectedness properties of several classical groups.

Proposition 1.18. *For $n \geq 1$ the groups $SL(n, \mathbb{R})$, $SO(n)$, $GL(n, \mathbb{C})$, $SL(n, \mathbb{C})$, $U(n)$, and $SU(n)$ are connected, while the groups $GL(n, \mathbb{R})$ and $O(n)$ have exactly two connected components.*

Proof. Consider a matrix $A \in GL(n, \mathbb{R})$ with columns $v_i \in \mathbb{R}^n$. Then $\{v_1, \dots, v_n\}$ is a basis of $\mathbb{R}^n$, so we can apply the Gram–Schmidt orthonormalization procedure. The resulting orthonormal basis $\{w_1, \dots, w_n\}$ is obtained in such a way that each w_j is a linear combination of the v_i with $i \leq j$ and the coefficient of v_j is positive. Letting $C = (w_1 \dots w_n)$ be the matrix with columns w_j, this reads as $C = AN$, where N is an upper-triangular matrix with positive entries on the main diagonal. But then for each $t \in [0, 1]$ also $tN + (1-t)\mathbb{I}$ is upper triangular with positive entries on the main diagonal, so $\det(tN + (1-t)\mathbb{I}) > 0$ for all t. Hence $t \mapsto A(tN + (1-t)\mathbb{I})$ defines a continuous path in $GL(n, \mathbb{R})$ joining A to C, so $O(n)$ meets each connected component of $GL(n, \mathbb{R})$. Since $GL(n, \mathbb{R})$ evidently has at least two connected components (corresponding to $\det(A) > 0$ and $\det(A) < 0$), the claim on $GL(n, \mathbb{R})$ follows if we prove that $O(n)$ has only two components.

If we start with $A \in SL(n, \mathbb{R})$, then $a(t) := \det(tN + (1-t)\mathbb{I}) > 0$ for all $t \in [0, 1]$ and $\det(C) = \pm 1$ implies $C \in SO(n)$. This also shows that $\det(N) = 1$ and hence $a(0) = a(1) = 1$. Since $a(t) > 0$ for all t, also $a(t)^{-1/n}$ is continuous and replacing our curve by $t \mapsto a(t)^{-1/n}A(tN + (1-t)\mathbb{I})$ is a continuous curve that connects A to C and stays in $SL(n, \mathbb{R})$. Hence connectedness of $SL(n, \mathbb{R})$ follows from connectedness of $SO(n)$.

For $A \in O(n)$ with $n \geq 2$ and $x \in \mathbb{R}^n$ we have $|Ax| = |x|$, so the action of $O(n)$ on $\mathbb{R}^n$ restricts to a smooth action on the unit sphere S^{n-1}. Since any unit vector can be extended to a positively oriented orthonormal basis, we see that even the subgroup $SO(n) \subset O(n)$ acts transitively on S^{n-1}. The stabilizer of the first vector in the standard basis consists of all matrices of the block form $\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$ and this lies in $O(n)$ (respectively $SO(n)$) if and only if $A \in O(n-1)$ (respectively $SO(n-1)$). Hence we conclude that both $O(n)/O(n-1)$ and $SO(n)/SO(n-1)$ are homeomorphic to S^{n-1}. For $n \geq 2$, S^{n-1} is connected, so by the lemma $O(n)$ has as most as many connected components as $O(n-1)$. Inductively, this implies that for $n \geq 2$, $O(n)$ has as most as many connected components as $O(1) = \{1, -1\} \subset \mathbb{R}$, and the results in the real case follow.

In the complex case, we can use the Gram–Schmidt procedure with respect to the standard Hermitian inner product on $\mathbb{C}^n$ in the same way. This shows that $U(n)$ meets any connected component of $GL(n, \mathbb{C})$, and $SU(n)$ meets any connected component of $SL(n, \mathbb{C})$. Finally, we have seen in 1.12 that the determinant defines a smooth homomorphism $U(n) \to U(1)$ with kernel $SU(n)$. This implies that $U(n)/SU(n)$ is homomorphic to the connected group $U(1) \cong S^1$, and all results in the complex case follow from connectedness of $SU(n)$. The standard action of $SU(n)$ on $\mathbb{C}^n$ restricts to an action on the unit sphere S^{2n-1} which is transitive for $n \geq 2$. The isotropy group of the first vector in the standard basis is isomorphic to $SU(n-1)$. As before, connectedness of $SU(n)$ for $n \geq 2$ follows from connectedness of $SU(1) = \{1\} \subset \mathbb{C}$. □

Remark 1.18. As we have seen in Section 1.12 invertible upper triangular matrices form a closed subgroup $B(n,\mathbb{R}) \subset GL(n,\mathbb{R})$. Any entry on the main diagonal of such a matrix must be non-zero, and hence is either positive or negative. Mapping $A = (a_{ij}) \in B(n,\mathbb{R})$ to $(\frac{a_{11}}{|a_{11}|}, \dots, \frac{a_{nn}}{|a_{nn}|})$ clearly defines a homomorphism $B(n,\mathbb{R}) \to O(1)^n$. The kernel of this homomorphism is a closed subgroup $\mathcal{P}_n \subset B(n,\mathbb{R})$ which consists of all upper-triangular matrices with positive entries on the main diagonal. Now the first step in the proof of Proposition 1.18 shows that any matrix $A \in GL(n,\mathbb{R})$ can be written as CP for $C \in O(n)$ and $P \in \mathcal{P}_n$ and if $A \in SL(n,\mathbb{R})$ then $C \in SO(n)$.

A complex version of this is obtained in a very similar way. Invertible upper triangular matrices with positive real entries on the main diagonal form a closed subgroup $\mathcal{P}_n^{\mathbb{C}} \subset GL(n,\mathbb{C})$ and any matrix $A \in GL(n,\mathbb{C})$ can be written as CP with $C \in U(n)$ and $P \in \mathcal{P}_n^{\mathbb{C}}$, and for $A \in SL(n,\mathbb{C})$, we get $C \in SU(n)$ (and also $\det(P) = 1$).

In each of the cases, the two subgroups involved in the construction intersect only in $\mathbb{I}$. Since $C_1P_1 = C_2P_2$ implies $C_2^{-1}C_1 = P_2P_1^{-1}$, this implies that the decomposition is unique in each case. Moreover, we have noted in Section 1.12 that $O(n) \subset GL(n,\mathbb{R})$, $SO(n) \subset SL(n,\mathbb{R})$, $U(n) \subset GL(n,\mathbb{C})$ and $SU(n) \subset SL(n,\mathbb{C})$ are compact subgroups. One can actually shows that in each of these cases, one actually obtains a *maximal compact subgroup*. The main property of the other factors is that they always have trivial topology, the map $(t,P) \mapsto tP + (1-t)\mathbb{I}$ in each case defines a homotopy from the identity to the constant map $\mathbb{I}$.

An analogous decomposition is available for general semi-simple Lie groups under the name *Iwasawa decomposition*. Any such group turns out to contain a maximal compact subgroup K and one gets a decomposition with one factor from K and one factor from a subgroup which is contractible, i.e. has trivial topology. Notice that even when starting with $\mathbb{K} = \mathbb{C}$, things happen in the setting of real Lie groups, since already $U(n)$ and $SU(n)$ are not complex Lie group. In the language of algebraic topology, one actually shows that the maximal compact subgroup in a semi-simple Lie group is a deformation retract, which in particular implies that any semi-simple Lie group is homotopy equivalent to its maximal compact subgroup.

The second part of the proof actually shows that $SO(n) \to S^{n-1}$ is a locally trivial fiber bundle with fiber $SO(n-1)$, while $SU(n) \to S^{2n-1}$ is a locally trivial fiber bundle with fiber $SU(n-1)$. There is a general result in algebraic topology that locally trivial fiber bundles over sufficiently nice spaces are so-called fibrations. Any fibration gives rise to a long exact sequence of homotopy groups, and the proof contains a direct argument for a very simple consequence of this long exact sequence. The long exact sequence associated to the two above fibrations can be used to deduce more information about the homotopy groups of the classical matrix groups that show up in Proposition 1.18. This is also done by an inductive procedure which need starting points. For this, one shows directly that, as a manifold, $SU(2) \cong S^3$, which implies information on its homotopy groups. On the other hand, one can use either the adjoint representation of $SU(2)$ or an interpretation via quaternions to shows that there is a two-fold covering $SU(2) \to SO(3)$, which gives information on the homotopy groups of $SO(3)$.

CHAPTER 2

The Frobenius theorem and existence results

To complete the basic understanding of the correspondence between Lie groups and Lie algebras, we have to prove some existence results. For example, we know that a Lie group leads to a Lie algebra, but we do not know yet how many Lie algebras are obtained in that way. Similarly, a homomorphism of Lie groups induces a homomorphism between their Lie algebras, which essentially determines the group homomorphism, but we do not know how many Lie algebra homomorphisms are obtained in that way. To prove existence results of that type, we use a a basic tool of differential geometry, the Frobenius theorem.

The Frobenius theorem

2.1. Distributions and integrability. The basic question answered by the Frobenius theorem is whether a family of linear subspaces in the tangent spaces of a smooth manifold M can be realized as the tangent spaces of a smooth submanifold. For the purpose of motivation let us consider the case of one-dimensional subspaces.

So let M be a smooth manifold and suppose that for each $x \in M$, we choose a 1-dimensional linear subspace $\ell_x \subset T_xM$. There is an obvious idea for what it should mean that ℓ_x depends smoothly on x: One requires that for each $x \in M$ there is an open neighborhood V of x in M and a smooth local vector field $\xi \in \mathfrak{X}(V)$ such that $\ell_y = \mathbb{R} \cdot \xi(y)$ for each $y \in V$. Now if $c : I \to M$ is an integral curve for ξ defined on some open interval $I \subset \mathbb{R}$, then $c'(t) = \xi(c(t)) \neq 0$ for all $t \in I$. Hence c is a regularly parametrized curve in M and, possibly shrinking I, $N := c(I)$ is an embedded submanifold in M such that $T_yN = \ell_y$ for each $y \in N$.

We can actually get a better result. By the flow box theorem [**AnaMf**, Theorem 2.9], we can find a local chart (U, u) for M with $U \subset V$ such that $\xi|_U = \frac{\partial}{\partial u^1}$. Now for $y \in U$, we have the affine line $u(y) + \mathbb{R}e_1$ in $\mathbb{R}^n$, whose image under u^{-1} is a smooth submanifold of M whose tangent spaces are the lines ℓ_z. Hence U is actually decomposed into a union of 1-dimensional submanifolds whose tangent spaces are the distinguished lines in the tangent spaces of M.

In higher dimensions things cannot be as easy. Simple examples show that for two vector fields ξ and η which are linearly independent in each point, the planes spanned by their values cannot be realized as tangent planes of a submanifold in general, see [**AnaMf**, Section 2.4]. The crucial additional ingredient coming into the game is the behavior of the Lie bracket of the two fields. After these considerations we can collect the basic notions to deal with the higher dimensional situation:

Definition 2.1. (1) For a smooth manifold M of dimension n, a *distribution* E of rank k on M is given by a k-dimensional subspace $E_x \subset T_xM$ for each $x \in M$. (This should not be confused with distributions in the sense of generalized functions.)

(2) A (smooth) *section* of distribution $E \subset TM$ is a vector field $\xi \in \mathfrak{X}(M)$ such that $\xi(x) \in E_x$ for all $x \in M$. A *local section* of E on an open subset U is a local vector field $\xi \in \mathfrak{X}(U)$ such that $\xi(x) \in E_x$ for all $x \in U$.

(3) The distribution $E \subset TM$ is called *smooth* or a if it can be locally spanned by smooth sections. This means that for each $x \in M$ there is an open neighborhood U of x in M and their are local sections $\xi_1, \dots, \xi_k$ of E on U such that for each $y \in U$ the vectors $\xi_1(y), \dots, \xi_k(y)$ form a basis for E_y. Such a collection of sections is then called a *local frame* for the distribution E.

Smooth distributions are also called *vector subbundles* of TM.

(4) A distribution $E \subset TM$ is called *involutive* if for any two local sections ξ and η of E also the Lie bracket $[\xi, \eta]$ is a section of E.

(5) A distribution $E \subset TM$ is called *integrable* if for each $x \in M$ there is a smooth immersed submanifold $N \subset M$ which contains x such that for each $y \in N$ we have $T_yN = E_y \subset T_yM$. Such an immersed submanifold is called an *integral submanifold* for E.

Observe that locally around each point $x \in N$, an immersed submanifold is an embedded submanifold, so one could also use embedded submanifolds in the definition of integrability. However, as we shall see below, integral submanifolds that are not embedded play an important role. Observe also that involutivity of a distribution can be checked effectively: Suppose that $\{\xi_1, \dots, \xi_k\}$ is a local frame for the distribution U. Then one can compute the Lie brackets $[\xi_i, \xi_j]$ of two elements of the frame and check whether they are again sections of the distribution. It this is the case, then by definition, for two sections ξ and η, there are smooth functions f_i and g_i such that $\xi = \sum_{i=1}^k f_i\xi_i$ and $\eta = \sum_{j=1}^k g_j\xi_j$. Hence we obtain $[\xi, \eta] = \sum_{i,j}[f_i\xi_i, g_j\xi_j]$ and expanding the brackets, we see that one obtains a linear combination (with smooth coefficients) of the fields ξ_i and the brackets $[\xi_i, \xi_j]$, and hence again a section. Thus a distribution is involutive, provided that locally around each point there is a local frame $\{\xi_i\}$ for which all brackets $[\xi_i, \xi_j]$ are again sections of the distribution.

Our above considerations then say that any smooth distribution of rank 1 is integrable. We can also see that for higher rank involutivity is a necessary condition for integrability. Suppose that $E \subset TM$ is integrable, $x \in M$ is a point, and $N \subset M$ is an integral manifold with $x \in N$. Given a local section ξ of E defined one some neighborhood U of x, we may replace N by $N \cap U$ to assume that ξ is defined on all of N. For $y \in N$, we then get $\xi(y) \in E_y = T_yN$, so ξ restricts on N to a vector field $\underline{\xi} \in \mathfrak{X}(N)$. This can be expressed as $\underline{\xi} \sim_i \xi$, where $i : N \to M$ is the inclusion. Given a second local section η, we similarly get $\underline{\eta} \in \mathfrak{X}(N)$ with $\underline{\eta} \sim_i \eta$, and Lemma 1.5 implies $[\underline{\xi}, \underline{\eta}] \sim_i [\xi, \eta]$. In particular $[\xi, \eta](x)$ lies in the image of T_xi which is E_x. Since x was arbitrary, we see that $[\xi, \eta]$ is a local section of E, which proves the claim.

The Frobenius theorem states that for smooth distributions also the converse is true, i.e. involutive smooth distributions are integrable. To prepare for this, we recall the characterization of coordinate vector fields on a smooth manifold, see [**AnaMf**, Section 2.11]. If there is no risk of confusion, we will write ∂_i for the coordinate vector field $\frac{\partial}{\partial u^i}$ determined by a chart (U, u). The coordinate vector fields act on a smooth function $f \in C^\infty(M, \mathbb{R})$ simply via the partial derivatives of the local coordinate representation $f \circ u^{-1}$ of f. Symmetry of the second partials then shows that $[\partial_i, \partial_j] = 0$ for all i, j. Indeed, this is all that is special about coordinate vector fields:

Lemma 2.1. *Let V be an open subset of a smooth manifold M of dimension n, and let $\xi_1, \dots, \xi_n \in \mathfrak{X}(V)$ be local vector fields such that $[\xi_i, \xi_j] = 0$ for all i, j and such that for each $y \in V$ the elements $\xi_1(y), \dots, \xi_n(y) \in T_yM$ form a basis.*

Then for any point $x \in V$ there is a chart (U, u) for M with $x \in U \subset V$ such that $\xi_i|_U = \frac{\partial}{\partial u^i}$ for all $i = 1, \dots, n$.

Sketch of proof. (see [**AnaMf**, Corollary 2.11] for details) Fixing a point x, one considers the map $(t^1,\dots,t^n)\mapsto(\mathrm{Fl}^{\xi_1}_{t^1}\circ\dots\circ\mathrm{Fl}^{\xi_n}_{t^n})(x)$, which is well defined and smooth on some open neighborhood W of 0 in $\mathbb{R}^n$. As shown in [**AnaMf**, Theorem 2.11], vanishing of the Lie bracket of two vector fields implies that their flows commute wherever they are defined. Thus the order of composition of the flows plays no role. Putting $\mathrm{Fl}^{\xi_i}_{t_i}$ to the leftmost place, one easily concludes that $\frac{\partial\varphi}{\partial t^i}(t)=\xi_i(\varphi(t))$. In particular, $T_t\varphi$ is invertible for all $t\in W$, so shrinking W we may assume that φ is a diffeomorphism from W onto an open neighborhood U of x in M. Putting $u:=\varphi^{-1}:U\to W$ the chart (U,u) does the job. $\square$

2.2. The Frobenius theorem. We are ready to prove the so-called "local version" of the Frobenius theorem:

Theorem 2.2. *Let M be a smooth manifold of dimension n and let $E\subset TM$ be a smooth involutive distribution of rank k. Then for each $x\in M$, there exists a local chart (U,u) for M with $x\in U$ such that $u(U)=V\times W\subset\mathbb{R}^n$ for open subsets $V\subset\mathbb{R}^k$ and $W\subset\mathbb{R}^{n-k}$ and for each $a\in W$ the subset $u^{-1}(V\times\{a\})\subset M$ is an integral manifold for the distribution E. In particular, any involutive distribution is integrable.*

Proof. Choose local smooth sections $\xi_1,\dots,\xi_k$ of E such that $\{\xi_1(x),\dots,\xi_k(x)\}$ is a basis for E_x, as well as an arbitrary local chart $(\tilde U,\tilde u)$ with $x\in\tilde U$ and $\tilde u(x)=0$. Putting $\partial_i:=\frac{\partial}{\partial\tilde u^i}$, we consider the local coordinate expressions for the vector fields ξ_j. This defines smooth functions $f^i_j:\tilde U\to\mathbb{R}$ for $i=1,\dots,n$ and $j=1,\dots,k$ such that $\xi_j=\sum_i f^i_j\partial_i$. Consider the $n\times k$-matrix $(f^i_j(y))$ for $y\in\tilde U$. Since the vectors $\xi_i(x)$ are linearly independent, the matrix $(f^i_j(x))$ has rank k and hence has k linearly independent rows. Renumbering the coordinates, we may assume that the first k rows are linearly independent. Then the top $k\times k$-submatrix has nonzero determinant in x. By continuity, this is true locally around x, and possibly shrinking $\tilde U$, we may assume that it holds on all of $\tilde U$.

For $y\in\tilde U$ let $(g^i_j(y))$ the $k\times k$-matrix which is inverse to the first k rows of $(f^i_j(y))$. Since the inversion in $GL(k,\mathbb{R})$ is smooth, each of the g^i_j defines a smooth function on $\tilde U$. For $i=1,\dots,k$ we now put $\eta_i:=\sum_j g^j_i\xi_j$. These are local smooth sections of E and since the matrix $(g^i_j(y))$ is always invertible, their values span E_y for each $y\in\tilde U$. Expanding η_i in the basis ∂_ℓ we obtain

$$\eta_i=\textstyle\sum_j g^j_i\xi_j=\sum_{j,\ell}g^j_if^\ell_j\partial_\ell=\partial_i+\sum_{\ell>k}h^\ell_i\partial_\ell, \tag{2.1}$$

for certain smooth functions h^ℓ_i on $\tilde U$.

We claim that the sections η_i of E satisfy $[\eta_i,\eta_j]=0$ for all i,j. Since the distribution E is involutive, we know that $[\eta_i,\eta_j]$ is a section of E, so there must be smooth functions c^ℓ_{ij} such that $[\eta_i,\eta_j]=\sum_\ell c^\ell_{ij}\eta_\ell$. Applying equation (2.1) to the right hand side, we see that $[\eta_i,\eta_j]$ is the sum of $\sum_{\ell=1}^k c^\ell_{ij}\partial_\ell$ and some linear combination of the ∂_ℓ for $\ell>k$. On the other hand, inserting (2.1) for η_i and η_j, we see that that $[\eta_i,\eta_j]$ must be a linear combination of the ∂_ℓ for $\ell>k$ only. This is only possible if all c^ℓ_{ij} vanish identically.

The vectors $T_x\tilde u\cdot\eta_1(x),\dots,T_x\tilde u\cdot\eta_k(x)$ generate a k-dimensional subspace in $\mathbb{R}^n$ and changing $\tilde u$ by some linear isomorphism, we may assume that this is $\mathbb{R}^k\subset\mathbb{R}^n$. Now we can find open neighborhoods V in $\mathbb{R}^k$ and $W\in\mathbb{R}^{n-k}$ of zero such that

$$\varphi(t^1,\dots,t^k,a):=(\mathrm{Fl}^{\eta_1}_{t^1}\circ\dots\circ\mathrm{Fl}^{\eta_k}_{t^k})(\tilde u^{-1}(a))$$

makes sense for all $t = (t^1, \dots, t^k) \in V$ and all $a \in W$. As in the proof of Lemma 2.1 we see that $\frac{\partial \varphi}{\partial t^i}(t, a) = \eta_i(\varphi(t, a))$ for $i = 1, \dots, k$. On the other hand, $\varphi(0, a) = \tilde{u}^{-1}(a)$, which by construction easily implies that $T_{(0,0)}\varphi$ is invertible. Possibly shrinking V and W, we may assume that φ is a diffeomorphism onto an open neighborhood U of x in M. But then we have already seen above that $\varphi^{-1}(V \times \{a\})$ is an integral submanifold, so putting $u := \varphi^{-1}$ we obtain a chart with the required properties. □

Charts of the type constructed in the theorem are called *distinguished charts*, and the integral submanifolds $u^{-1}(V \times \{a\})$ are called *plaques.* Now suppose that $i : N \to M$ is an integral submanifold for E and consider the intersection $i(N) \cap U$ for a distinguished chart (U, u). Then $i^{-1}(U)$ is open in N and hence a union of at most countably many connected components $\mathcal{O}_\alpha$, each of which is open in N. Now by construction the forms du^j for $j = k+1, \dots, n$ satisfy $i^*du^j = 0$, so by connectedness the functions $u^j \circ i$ are constant on of the $\mathcal{O}_\alpha$. Thus $i(\mathcal{O}_\alpha)$ has to be contained in one plaque $\mathcal{P}$. Now $\mathcal{P} \subset M$ is an embedded submanifold and $i|_{\mathcal{O}_\alpha} : \mathcal{O}_\alpha \to \mathcal{P}$ is an injective immersion between manifolds of the same dimension. Thus it is a local diffeomorphism, so $i(\mathcal{O}_\alpha)$ is open in $\mathcal{P}$ and hence also an embedded submanifold in M. But this directly shows that the distinguished charts identify $i(N)$ as an initial submanifold of M.

These considerations in particular show that for two integral submanifolds that intersect in a point, the intersection is open in both the manifolds. Using this, one proves that for a point x in M, one can make the union of all connected integral submanifolds that contain x into a connected initial integral submanifold $\mathcal{B}_x$, see [**Lee**, Lemma 19.22]. This is called the *leaf through* x and by maximality, any connected integral submanifold that has non-trivial intersection with $\mathcal{B}_x$ is contained in $\mathcal{B}_x$.

The leaves associated to an integrable distribution E decomposes M into a union of k-dimensional initial submanifolds. This decomposition is called the *foliation* of M defined by the involutive distribution E. The existence of maximal connected integral submanifolds for an involutive distribution or the fact that they form a foliation of M is often referred to as the global version of the Frobenius theorem. See [**Michor**, Chapter 3] or [**Lee**, Chapter 19] for more information. Michor's book also discusses the extension of the Frobenius theorem to distributions of non-constant rank.

Existence of subgroups and homomorphisms

2.3. Existence of virtual Lie subgroups. As a first application of the Frobenius theorem, we can show that for a Lie group G with Lie algebra $\mathfrak{g}$, any Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ corresponds to a virtual Lie subgroup of G.

THEOREM 2.3. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra. Then there is a unique connected virtual Lie subgroup $H \to G$ whose tangent space at the identity is $\mathfrak{h}$. This virtual Lie subgroup is an initial submanifold of G.*

PROOF. For $g \in G$ we define $E_g := \{\xi \in T_gG : T_g\lambda_{g^{-1}} \cdot \xi \in \mathfrak{h}\}$, so $E \subset TG$ corresponds to $\mathfrak{h} \subset \mathfrak{g}$ under the left trivialization. Choosing a basis $\{X_1, \dots, X_k\}$ of $\mathfrak{h}$, the tangent vectors $L_{X_1}(g), \dots, L_{X_k}(g)$ form a basis of E_g. Morover, $[L_{X_i}, L_{X_j}] = L_{[X_i, X_j]}$, which is a section of E since $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra. Thus E is a smooth involutive distribution.

Let $\mathcal{B}_e$ the leave of the corresponding foliation through $e \in G$. Then this is a connected initial submanifold of G. By construction, the distribution E is left invariant, i.e. for all $g, h \in G$ we have $T_h\lambda_g(E_h) = E_{gh}$. This implies that for $g \in G$ the subset $\lambda_g(\mathcal{B}_e)$ coincides with the leaf $\mathcal{B}_g$. If $g \in \mathcal{B}_e$, then clearly $\mathcal{B}_g = \mathcal{B}_e$ which means that the subset $\mathcal{B}_e \subset G$ is closed under the multiplication. The equation $\lambda_g(\mathcal{B}_e) = \mathcal{B}_e$ also implies that $g^{-1} \in \mathcal{B}_e$, so $\mathcal{B}_e \subset G$ is a subgroup. By restriction, the multiplication is smooth as a map $\mathcal{B}_e \times \mathcal{B}_e \to G$. Since $\mathcal{B}_e$ is an initial submanifold, smoothness as a map $\mathcal{B}_e \times \mathcal{B}_e \to \mathcal{B}_e$ follows. Hence we have found a connected virtual Lie subgroup of G corresponding to $\mathfrak{h}$, which is an initial submanifold.

Conversely, suppose that $H \to G$ is a connected virtual Lie subgroup such that $T_eH = \mathfrak{h} \subset \mathfrak{g}$. Then for $g \in H$ we have $T_gH = T_e\lambda_g(T_eH) = E_g$, so H is an integral submanifold for the distribution E. Since $e \in H$, we see that H must be contained in $\mathcal{B}_e$. On the other hand, H evidently contains $\exp(\mathfrak{h}) \subset \mathcal{B}_e$. Part (2) of Theorem 1.9 shows that $\exp(\mathfrak{h})$ generates the connected Lie group $\mathcal{B}_e$, so $H = \mathcal{B}_e$ follows. □

With the help of a result from the structure theory of Lie algebras, we also get a proof of what is called Lie's third fundamental theorem. The result on Lie algebras we need is called Ado's theorem. It states that any finite dimensional Lie algebra $\mathfrak{g}$ admits a representation on a finite dimensional vector space V, which is injective as a homomorphism $\mathfrak{g} \to L(V, V)$. (Such representations are usually called faithful.) Otherwise put, $\mathfrak{g}$ is isomorphic to a Lie subalgebra of $L(V, V)$. While Ado's theorem is usually considered as difficult, there is a short and fairly simple proof, see [**Neretin**].

Corollary 2.3 (Lie's third fundamental theorem). *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Then there is a Lie group G with Lie algebra $\mathfrak{g}$.*

Proof. By Ado's theorem discussed above, we may view $\mathfrak{g}$ as a Lie subalgebra of $L(V, V)$. By the Theorem, there is a virtual Lie subgroup $G \to GL(V)$ corresponding to $\mathfrak{g} \subset L(V, V)$. □

Remark 2.3. (1) The result even says that for a Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(V)$, one can find a Lie group with Lie algebra $\mathfrak{g}$ as a virtual Lie subgroup of $GL(V)$. It is not true however, that any connected finite dimensional Lie group is isomorphic to a virtual Lie subgroup of some group of the form $GL(V)$.

(2) There are simpler proofs of Lie's third fundamental theorem available, see for example [**Knapp**, Appendix B], but they all need a certain amount of the structure theory of Lie algebras.

2.4. Groups with isomorphic Lie algebras. Suppose that $\varphi : G \to H$ is a homomorphism between connected Lie groups such that $\varphi' : \mathfrak{g} \to \mathfrak{h}$ is an isomorphism. Then Propositions 1.11 and 1.17 tell us that $\ker(\varphi)$ is a discrete normal subgroup of G that is contained in the center $Z(G)$ and that φ induces an isomorphism $G/\ker(\varphi) \to H$. There is more that we can prove about φ, however, and this provides a connection to (algebraic) topology and in particular to covering maps.

Recall that a continuous map $p : E \to X$ between two topological spaces is called a *covering map* if and only if each point $x \in X$ has an open neighborhood U such that $p^{-1}(U)$ is a disjoint union of open subsets V_α of E and for each α the restriction $p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism. Such a subset U is called *trivializing* for the covering p. We will mainly deal with the case that E and X are smooth manifolds, p is smooth, and the restriction of p to each V_α is a diffeomorphism. The standard example of a covering map (which has this additional property) is the map $\mathbb{R} \to U(1)$ given by $t \mapsto e^{it}$.

Proposition 2.4. *Let $\varphi : G \to H$ be a smooth homomorphism between connected Lie groups whose derivative $\varphi' : \mathfrak{g} \to \mathfrak{h}$ is a linear isomorphism. Then $\varphi : G \to H$ is a local diffeomorphism and a covering map.*

Proof. Differentiating the equation $\varphi\circ\lambda_g=\lambda_{\varphi_g}\circ\varphi$ we immediately see that bijectivity of φ' implies bijectivity of $T_g\varphi$, so φ is a local diffeomorphism. To prove that φ is a covering map, we may use Proposition 1.17 to replace $\varphi:G\to H$ by $p:G\to G/\ker(\varphi)$. Since $\ker(\varphi)$ is discrete, there is an open neighborhood U of e in G such that $U\cap\ker(\varphi)=\{e\}$. By continuity of μ and ν, there is an open neighborhood V of e in G such that for $g,h\in V$ we always have $h^{-1}g\in U$, so in particular $V\subset U$. Put $V_g=\rho^g(V)$ for $g\in G$. Suppose that $g,h\in V$ and $g_1,g_2\in\ker(\varphi)$ are such that $gg_1=hg_2$. Then $h^{-1}g=g_2g_1^{-1}\in U\cap\ker(\varphi)$, so $g_1=g_2$ and $g=h$. On the one hand, this shows that $p|_V:V\to p(V)$ is bijective. On the other hand, it shows that for $g_1\neq g_2$ the subsets V_{g_1} and V_{g_2} are disjoint. Hence we see that $p^{-1}(p(V))$ is the disjoint union of the open sets V_g for $g\in\ker(\varphi)$. By construction, the restriction of p to each of the sets V_g for $g\in\ker(\varphi)$ is a bijective local diffeomorphism and hence a diffeomorphism.

For an arbitrary element $g\in G$ we can now use $p(V_g)$ as a neighborhood of $g\ker(\varphi)$ in $G/\ker(\varphi)$ which is trivializing for p. □

2.5. Covering groups and existence of homomorphisms. We now move to the question of existence of smooth homomorphisms between Lie groups with prescribed derivative. One can immediately see that this is not a trivial question: The covering map $t\mapsto e^{it}$ can be viewed as a group homomorphism $(\mathbb{R},+)\to U(1)$, whose derivative is an isomorphism of Lie algebras. There cannot exist a group homomorphism $U(1)\to\mathbb{R}$, however, whose derivative is the inverse of this isomorphism of Lie algebras. The point is that the image of such a homomorphism would be a non trivial compact subgroup of $\mathbb{R}$ and such a subgroup does not exist. The problem actually is that the identity on $\mathbb{R}$ does not descend to $U(1)\cong\mathbb{R}/\mathbb{Z}$.

There are two ways out of this problem. One is not to ask for homomorphisms but only for local homomorphisms.

Definition 2.5. Let G and H be Lie groups. A *local homomorphism* from G to H is given by an open neighborhood U of e in G and a smooth map $\varphi:U\to H$ such that $\varphi(e)=e$ and $\varphi(gh)=\varphi(g)\varphi(h)$ whenever g, h, and gh all lie in U.

Note that for a local homomorphism $\varphi:G\supset U\to H$ one also has the derivative $\varphi':\mathfrak{g}\to\mathfrak{h}$. The proof of Proposition 1.5 applies without changes to local homomorphisms, so φ' is a homomorphism of Lie algebras.

The other possibility is to restrict to a subclass of groups for which such covering phenomena are impossible: Via the notion of the fundamental group, covering maps are closely related to algebraic topology. Recall that a path connected topological space X is called *simply connected*, if for any continuous map $f:S^1\to X$ there is a continuous map $H:S^1\times[0,1]\to X$ and a point $x_0\in X$ such that $H(z,0)=f(z)$ and $H(z,1)=x_0$ for all $z\in S^1$. This means that any closed continuous curve in X can be continuously deformed to a constant curve. For example, the spheres S^n are simply connected for $n>1$ but S^1 is not simply connected. A fundamental result of algebraic topology says that if X is a simply connected space and $p:E\to X$ is a covering map such that E is path connected, then p is a homeomorphism.

Using this, we can now state:

Theorem 2.5. *Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ and let $f:\mathfrak{g}\to\mathfrak{h}$ be a homomorphism of Lie algebras. Then there exists a local homomorphism $\varphi:G\supset U\to H$ such that $\varphi'=f$. If G is simply connected, then there even is a homomorphism $\varphi:G\to H$ with this property.*

PROOF. Consider the Lie group $G\times H$. From 1.4 we know that its Lie algebra is $\mathfrak{g}\times\mathfrak{h}$ with the pointwise operations. Consider the subset $\mathfrak{k} := \{(X, f(X)) : X \in \mathfrak{g}\} \subset \mathfrak{g}\times\mathfrak{h}$, i.e. the graph of the Lie algebra homomorphism f. Since f is a linear map, this is a linear subspace of $\mathfrak{g}\times\mathfrak{h}$, and since f is a homomorphism we get

$$[(X, f(X)),(Y, f(Y))] = ([X, Y],[f(X), f(Y)]) = ([X, Y], f([X, Y])).$$

Thus $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}\times\mathfrak{h}$ and by Theorem 2.3 we obtain a virtual Lie subgroup $K \to G\times H$ corresponding to $\mathfrak{k}$. The two projections define smooth homomorphisms from $G\times H$ to G and H. Restricting the first projection to K, we obtain a smooth homomorphism $\pi : K \to G$. By construction $\pi'(X, f(X)) = X$, so this is a linear isomorphism $\mathfrak{k}\to\mathfrak{g}$. By Proposition 2.4, π is a covering and a local diffeomorphism from K onto G_0. Take open neighborhoods V and U of e in K respectively G such that π restricts to a diffeomorphism $V \to U$, and define $\varphi : U \to H$ as $\varphi := \mathrm{pr}_2\circ(\pi|_V)^{-1}$. Since π is a homomorphism, one immediately concludes that φ is a local homomorphism and by construction φ' is given by $X \mapsto (X, f(X)) \mapsto f(X)$.

If G is simply connected, then $G = G_0$ and the covering $\pi : K \to G$ must be a homeomorphism and a local diffeomorphism. Hence π is an isomorphism of Lie groups, and $\varphi := \mathrm{pr}_2\circ\pi^{-1}$ is the required homomorphism. □

2.6. Classification of Lie groups. We can now give a complete description of all Lie groups which have a given Lie algebra $\mathfrak{g}$. This assumes Lie's third fundamental theorem from 2.3 for which we have not given a complete proof. Without assuming this result, we obtain the classification provided that there is at least one Lie group G with Lie algebra $\mathfrak{g}$.

As a background, we need a fundamental result on coverings: In algebraic topology one shows that for any topological space X which satisfies certain connectedness hypotheses (which are satisfied for smooth manifolds), there is a covering map $p : \tilde{X} \to X$ such that $\tilde{X}$ is simply connected. This is called the *universal covering* of X, and it has several important properties: On the one hand, for every covering map $\pi : E \to X$ with path connected E, there is a covering map $q : \tilde{X} \to E$ such that $p = \pi\circ q$. Thus the universal covering covers all connected coverings. This also implies that the universal covering is uniquely determined up to isomorphism. On the other hand, if Y is a simply connected space and $f : Y \to X$ is a continuous map, then there is a continuous lift $\tilde{f} : Y \to \tilde{X}$ (i.e. $p\circ\tilde{f} = f$) which is uniquely determined by its value in one point.

Finally, suppose that we have given a covering map $p : E \to M$ of a smooth manifold M. Let us further suppose that the covering has at most countably many sheets, i.e. for any open subset U which is trivializing for p, the preimage $p^{-1}(U)$ consists of at most countably many subsets V_α. Then the fact that M is separable and metrizable implies that E has the same properties. Further we can take an atlas for M consisting of charts (U, u) such that $U \subset M$ is trivializing for p. Then we can use $u\circ p|_{V_\alpha} : V_\alpha \to u(U)$ as charts for E. This makes E into a smooth manifold in such a way that p is smooth and $p|_{V_\alpha}$ is a diffeomorphism for each α.

THEOREM 2.6. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Then there is a unique (up to isomorphism) simply connected Lie group $\tilde{G}$ with Lie algebra $\mathfrak{g}$. Any other Lie group G with Lie algebra (isomorphic to) $\mathfrak{g}$ is isomorphic to the quotient of $\tilde{G}$ by a discrete normal subgroup $H \subset \tilde{G}$ which is contained in the center $Z(\tilde{G})$.*

PROOF. By Theorem 2.3 there is a connected Lie group G with Lie algebra $\mathfrak{g}$. From above we know that there is a covering map $p : \tilde{G} \to G$ such that $\tilde{G}$ is simply connected. Further, we can make $\tilde{G}$ into a smooth manifold in such a way that p is a local diffeomorphism. Fix an element $\tilde{e} \in \tilde{G}$ such that $p(\tilde{e}) = e$.

Since $\tilde{G}$ is simply connected, so is $\tilde{G} \times \tilde{G}$. For the continuous map $\mu \circ (p \times p) : \tilde{G} \times \tilde{G} \to G$ we therefore find a unique continuous lift $\tilde{\mu} : \tilde{G} \times \tilde{G} \to \tilde{G}$ such that $\tilde{\mu}(\tilde{e}, \tilde{e}) = \tilde{e}$. Likewise, there is a unique lift $\tilde{\nu} : \tilde{G} \to \tilde{G}$ of $\nu \circ p : \tilde{G} \to G$ such that $\tilde{\nu}(\tilde{e}) = \tilde{e}$. Restricting to small open subsets, one easily shows that smoothness of μ and ν implies smoothness of $\tilde{\mu}$ and $\tilde{\nu}$. Now $\tilde{\mu} \circ (\mathrm{id} \times \tilde{\mu})$ and $\tilde{\mu} \circ (\tilde{\mu} \times \mathrm{id})$ lift the same map $\tilde{G} \times \tilde{G} \times \tilde{G} \to G$, and have the same value on $(\tilde{e}, \tilde{e}, \tilde{e})$. Hence this two maps coincide which means that $\tilde{\mu}$ defines an associative multiplication on $\tilde{G}$. The other group axioms are verified similarly to show that $(\tilde{G}, \tilde{\mu}, \tilde{\nu})$ is a Lie group.

If $\hat{G}$ is any connected Lie group with Lie algebra (isomorphic to) $\mathfrak{g}$, then by Theorem 2.5 there is a homomorphism $\varphi : \tilde{G} \to \hat{G}$ whose derivative is the identity (respectively the given isomorphism). By Proposition 2.4, φ is a covering map and induces an isomorphism between $\hat{G}$ and a quotient of $\tilde{G}$ as required. Finally, if $\hat{G}$ also is simply connected, then the local diffeomorphism φ must be a homeomorphism and thus an isomorphism of Lie groups. □

Example 2.6. (1) We have noted in the end of 1.18 that $SU(2)$ is diffeomorphic to S^3 and that there is a fibration $SU(n) \to S^{2n-1}$ with fiber $SU(n-1)$ for $n \geq 3$. Thus $SU(2)$ is simply connected and the long exact homotopy sequence of the fibration inductively implies that $SU(n)$ is simply connected for all $n \geq 2$. Thus we have found the simply connected Lie group with Lie algebra $\mathfrak{su}(n)$.

To determine how many Lie groups with Lie algebra $\mathfrak{su}(n)$ exist, it therefore remains to determine the center of $SU(n)$. If A and B are commuting matrices in $SU(n)$, then A must map any eigenspace of B to itself. Given any complex line $\ell \subset \mathbb{C}^n$, we can choose an orthogonal line $\tilde{\ell}$, and consider the map B which acts by multiplication by i on ℓ by multiplication by $-i$ on $\tilde{\ell}$ and as the identity on $(\ell \oplus \tilde{\ell})^\perp$. This defines an element of $SU(n)$, so we see that any $A \in Z(SU(n))$ must map the line ℓ to itself. Since this works for any line ℓ, any $A \in Z(SU(n))$ must be a complex multiple of the identity. Since $\det(z\,\mathrm{id}) = z^n$, we see that $Z(SU(n))$ is isomorphic to the group $\mathbb{Z}_n$ of nth roots of unity. Hence connected Lie groups with Lie algebra $\mathfrak{su}(n)$ correspond to subgroups of $\mathbb{Z}_n$ and hence to natural numbers dividing n.

As a specific example, consider $SU(2) \cong S^3$. Since $Z(SU(2)) \cong \mathbb{Z}_2$, there are (up to isomorphism) only two Lie groups with Lie algebra $\mathfrak{su}(2)$, namely $SU(2)$ and $SU(2)/\{\pm\,\mathrm{id}\}$. It turns out that the latter group is isomorphic to $SO(3)$ and the isomorphism is induced by the adjoint representation $\mathrm{Ad} : SU(2) \to GL(\mathfrak{su}(2)) \cong GL(3, \mathbb{R})$.

(2) For the group $SO(n)$ with $n \geq 3$ the situation is a bit more complicated. We have noted above the the universal covering of $SO(3)$ is $SU(2)$ and the kernel of the covering homomorphism is a two element subgroup. This shows that the fundamental group of $SO(3)$ is isomorphic to $\mathbb{Z}_2$ and using the fibration $SO(n) \to S^{n-1}$ with fiber $SO(n-1)$ constructed in 1.18 one shows that the same is true for $SO(n)$ with $n \geq 4$.

The simply connected Lie group with Lie algebra $\mathfrak{so}(n)$ is called the *spin group* $\mathrm{Spin}(n)$. There is always a covering homomorphism $\mathrm{Spin}(n) \to SO(n)$ with kernel $\mathbb{Z}_2$. The spin groups play an important role in various parts of differential geometry and theoretical physics. For $n \leq 6$, the spin groups are isomorphic to other classical Lie groups, but for $n > 6$ one needs an independent construction using Clifford algebras.

Similarly as in (1), one shows that the center of $SO(n)$ can only consist of multiples of the identity, so it is trivial for odd n and isomorphic to $\mathbb{Z}_2$ for even n. Hence for odd

n, any Lie group with Lie algebra $\mathfrak{so}(n)$ is isomorphic to Spin(n) or to $SO(n)$. For even n, the center of Spin(n) has four elements, and it is either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ depending on whether n is a multiple of 4 or not. Hence there are either three or four connected Lie groups with Lie algebra $\mathfrak{so}(n)$ for even n.

Maurer–Cartan form and fundamental theorem of calculus

We will next discuss an analog of basic calculus for functions with values in a Lie group. A central role in these considerations is played by the Maurer–Cartan form, which is an alternative way to view the left trivialization of the tangent bundle of a Lie group.

2.7. The Maurer–Cartan form. Several elements of the usual theory of differential forms easily generalize to forms with values in a finite dimensional vector space V. Given a smooth manifold M, one defines a V-valued k-form as a map φ which associates to each $x \in M$ a k-linear, alternating map $\varphi(x) : (T_xM)^k \to V$. This map is required to be smooth in the obvious sense, i.e. for vector fields $\xi_1, \dots, \xi_k$ on M the function $\varphi(\xi_1, \dots, \xi_k) : M \to V$ should be smooth. The space of all V-valued k-forms is denoted by $\Omega^k(M, V)$. It is a vector space and a module over $C^\infty(M, \mathbb{R})$ under pointwise operations. The usual proof (see [**AnaMf**, Lemma 3.3]) shows that elements of $\Omega^k(M, V)$ can be characterized as those k-linear, alternating maps $\mathfrak{X}(M)^k \to C^\infty(M, V)$ which are linear over $C^\infty(M, \mathbb{R})$ in one (and hence in any) entry. Next, for a smooth map $f : M \to N$ and a form $\varphi \in \Omega^k(N, V)$, one defines $f^*\varphi \in \Omega^k(M, V)$ by

$$(f^*\varphi)(x)(\xi_1, \dots, \xi_k) := \varphi(f(x))(T_xf \cdot \xi_1, \dots, T_xf \cdot \xi_k).$$

The exterior derivative generalizes to an operator $d : \Omega^k(M, V) \to \Omega^{k+1}(M, V)$: Choosing a baisis $\{v_i\}$ for V, we can write $\varphi \in \Omega^k(M, V)$ as $\varphi = \sum \varphi_i v_i$ with $\varphi_i \in \Omega^k(M)$ and then define $d\varphi := \sum (d\varphi_i)v_i$. One immediately verifies that this is independent of the choice of the basis. Standard properties of the exterior derivative then imply that the identities $d(f^*\varphi) = f^*(d\varphi)$ and $d^2 = d \circ d = 0$ generalize to V-valued forms. Recall further that the action of vector fields on real-valued functions generalizes to V-valued functions (again by decomposing with respect to a basis), see [**AnaMf**, Section 2.2]. Hence the construction implies that the "global formula" for the exterior derivative (formula (3.17) in [**AnaMf**]) continues to hold for V-valued forms. In particular, for $\varphi \in \Omega^1(M, V)$ and $\xi_0, \xi_1 \in \mathfrak{X}(M)$, we have

$$d\varphi(\xi_0, \xi_1) = \xi_0 \cdot \varphi(\xi_1) - \xi_1 \cdot \varphi(\xi_0) - \varphi([\xi_0, \xi_1]). \tag{2.2}$$

Now let G be a Lie group. Then from 1.3 we know the left trivialization $TG \to G \times \mathfrak{g}$ of the tangent bundle of G. We can reformulate this as a canonical $\mathfrak{g}$-valued one form, which is called the (left) Maurer Cartan form:

Definition 2.7. Let G be a Lie Group with Lie algebra $\mathfrak{g}$. Then we define the left Maurer Cartan form $\omega \in \Omega^1(G, \mathfrak{g})$ by $\omega(g)(\xi) := T_g\lambda_{g^{-1}} \cdot \xi$.

It follows immediately from 1.3 that ω is indeed smooth. The basic properties of the Maurer–Cartan form are now easy to prove:

Proposition 2.7. *For any Lie group G with Lie algebra $\mathfrak{g}$, the left Maurer–Cartan form $\omega \in \Omega^1(G, \mathfrak{g})$ has the following properties:*

(1) $(\lambda_g)^*\omega = \omega$ for all $g \in G$.
(2) $(\rho^g)^*\omega = \mathrm{Ad}(g^{-1}) \circ \omega$ for all $g \in G$.
(3) For the left invariant vector field L_X generated by $X \in \mathfrak{g}$, we have $\omega(L_X) = X$.
(4) For each $g \in G$, the map $\omega(g) : T_gG \to \mathfrak{g}$ is a linear isomorphism.

(5) ("Maurer–Cartan equation") For vector fields $\xi, \eta \in \mathfrak{X}(G)$ we have

$$d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)] = 0,$$

where the bracket is in $\mathfrak{g}$.

PROOF. By definition, $\omega(h) = T_h\lambda_{h^{-1}}$, which implies (3) and (4). Next,

$$((\lambda_g)^*\omega)(h) = \omega(gh) \circ T_h\lambda_g = T_h\lambda_{h^{-1}} \circ T_{gh}\lambda_{g^{-1}} \circ T_h\lambda_g = \omega(h),$$

and (1) follows. Likewise,

$$((\rho^g)^*\omega)(h) = \omega(hg) \circ T_h\rho^g = T_g\lambda_{g^{-1}} \circ T_{hg}\lambda_{h^{-1}} \circ T_h\rho^g = T_g\lambda_{g^{-1}} \circ T_e\rho^g \circ T_h\lambda_{h^{-1}},$$

and (2) follows since $T_g\lambda_{g^{-1}} \circ T_e\rho^g = \mathrm{Ad}(g^{-1})$.

For part (5), we observe that the value of $d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)]$ in a point $g \in G$ depends only on $\xi(g)$ and $\eta(g)$. Since the values of left invariant vector fields span each tangent space, it suffices to check the equation for $\xi = L_X$ and $\eta = L_Y$ for $X, Y \in \mathfrak{g}$. But then since $\omega(L_X)$ and $\omega(L_Y)$ are constant, formula (2.2) implies that

$$d\omega(L_X, L_Y) = -\omega([L_X, L_Y]) = -\omega(L_{[X,Y]}) = -[X, Y],$$

and the result follows. □

2.8. The left logarithmic derivative. Now we can generalize calculus to functions with values in a Lie group:

DEFINITION 2.8. Let G be a Lie group with Lie algebra $\mathfrak{g}$, let M be a smooth manifold and $f : M \to G$ a smooth function. Then the *left logarithmic derivative* or the *Darboux derivative* $\delta f \in \Omega^1(M, \mathfrak{g})$ is defined by $\delta f(x)(\xi) = T_{f(x)}\lambda_{(f(x))^{-1}} \cdot T_x f \cdot \xi \in T_eG = \mathfrak{g}$ for $x \in M$ and $\xi \in T_xM$. Equivalently, $\delta f := f^*\omega$, where $\omega \in \Omega^1(G, \mathfrak{g})$ is the left Maurer Cartan form of G.

One may think of the left logarithmic derivative as an analog of the exterior derivative $d : C^\infty(M, \mathbb{R}) \to \Omega^1(M)$ for smooth real valued function. The name logarithmic derivative comes from the special case of the Lie group $(\mathbb{R}_{>0}, \cdot)$: Suppose that $f : M \to \mathbb{R}_{>0}$ is smooth. Viewing f as a real valued function, we have $df \in \Omega^1(M)$, which is given by $df(x)(\xi) = T_xf \cdot \xi \in T_{f(x)}\mathbb{R} = \mathbb{R}$. Since left translations in $\mathbb{R}_{>0}$ are (the restrictions of) linear maps, we see from the explicit formula that $\delta f \in \Omega^1(M, \mathbb{R}) = \Omega^1(M)$ is given by $\delta f(x)(\xi) = \frac{1}{f(x)} T_xf \cdot \xi$ and hence $\delta f = \frac{df}{f} = d(\log(f))$.

It is easy to prove some basic properties of the logarithmic derivative:

PROPOSITION 2.8. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $f, g : M \to G$ be smooth functions.*

(1) For the pointwise product of f and g we have

$$\delta(fg)(x) = \delta g(x) + \mathrm{Ad}(g(x)^{-1})(\delta f(x)).$$

(2) For the pointwise inverse $\nu \circ f$ we obtain

$$\delta(\nu \circ f)(x) = -\mathrm{Ad}(f(x))(\delta f(x)).$$

PROOF. (1) The pointwise product is given by $\mu \circ (f, g)$, so by Lemma 1.2 we obtain

$$T_x(fg) = T_{g(x)}\lambda_{f(x)} \circ T_xg + T_{f(x)}\rho^{g(x)} \circ T_xf.$$

To obtain $\delta(fg)(x)$, we have to compose this from the left with $T_{f(x)g(x)}\lambda_{g(x)^{-1}f(x)^{-1}}$. For the first summand, this simply gives $T_{g(x)}\lambda_{g(x)^{-1}} \circ T_xg = \delta g(x)$. In the second summand, we can commute the right translation with the left translation to obtain $\mathrm{Ad}(g(x)^{-1}) \circ T_{f(x)}\lambda_{f(x)^{-1}} \circ T_xf = \mathrm{Ad}(g(x)^{-1}) \circ \delta f(x)$.

(2) This immediately follows from (1) using that $\delta(f(\nu \circ f)) = \delta(e) = 0$. □

One can explicitly compute the logarithmic derivative of the exponential mapping $\exp : \mathfrak{g} \to G$, see [**Michor**, 4.27]. This can be used to characterize the elements $X \in \mathfrak{g}$ for which $T_X \exp$ is invertible, and hence exp is a local diffeomorphism around X. This is the case if and only if no eigenvalue of $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ is of the form $2\pi i k$ for some $k \in \mathbb{Z}$. Finally, the computation of $\delta \exp$ also leads to a proof for the Baker–Campbell–Hausdorff formula, which locally around $(0,0) \in \mathfrak{g} \times \mathfrak{g}$ computes the map $(X, Y) \mapsto \exp^{-1}(\exp(X)\exp(Y))$, see [**Michor**, 4.29]. The remarkable fact about this formula is that it only involves iterated Lie brackets of X and Y, so it shows that the Lie bracket determines the multiplication on an open neighborhood of e in G.

2.9. The fundamental theorem of calculus. To finish this chapter, we want to prove a characterization of the elements of $\Omega^1(M, \mathfrak{g})$ which are of the form δf for a smooth function $f : M \to G$. This generalizes the fundamental theorem of one-variable calculus.

We first observe that there is a simple necessary condition for solvability of the equation $\delta f = \varphi$ for given $\varphi \in \Omega^1(M, \mathfrak{g})$. By definition, $\delta f = f^*\omega$, and hence $d\delta f = f^* d\omega$. Using this, we compute

$$d\delta f(\xi, \eta) + [\delta f(\xi), \delta f(\eta)] = d\omega(T_x f \cdot \xi, T_x f \cdot \eta) + [\omega(T_x f \cdot \xi), \omega(T_x f \cdot \eta)],$$

so this vanishes by the Maurer–Cartan equation, see Theorem 2.7. Hence we see that φ has to satisfy the Maurer Cartan equation in order that it can be of the form δf. As we shall see below, this locally is the only restriction on φ.

Before we can prove our main result, we need a criterion for the involutivity of certain distributions which is often useful.

LEMMA 2.9. *Let M be a smooth manifold and let $E \subset TM$ be a smooth distribution. Suppose that there is a one form φ on M with values in a finite dimensional vector space V such that $E_x = \ker(\varphi(x))$ for all $x \in M$.*

Then E is involutive if and only if $d\varphi(\xi, \eta) = 0$ for all vector fields $\xi, \eta \in \mathfrak{X}(M)$ such that $\varphi(\xi) = \varphi(\eta) = 0$.

PROOF. By definition, a vector field ξ is a section of E if and only if $\varphi(\xi) = 0$. For such sections ξ and η, we then have $d\varphi(\xi, \eta) = -\varphi([\xi, \eta])$, so this vanishes if and only if $[\xi, \eta]$ is a section of E. □

Using this, we can now prove:

THEOREM 2.9. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let M be a connected smooth manifold.*

(1) Let $f_1, f_2 : M \to G$ be smooth maps such that $\delta f_1 = \delta f_2 \in \Omega^1(M, \mathfrak{g})$. Then there is an element $g \in G$ such that $f_2 = \lambda_g \circ f_1$.

(2) Let $\varphi \in \Omega^1(M, \mathfrak{g})$ be a form which satisfies the Maurer–Cartan equation $0 = d\varphi(\xi, \eta) + [\varphi(\xi), \varphi(\eta)]$ for all $\xi, \eta \in \mathfrak{X}(M)$. Then for each point $x \in M$ there is an open subset $U \subset M$ with $x \in U$ and a smooth function $f : U \to G$ such that $\delta f = \varphi|_U$.

PROOF. (1) Consider the function $h := f_1(\nu \circ f_2) : M \to G$. By Proposition 2.8 we obtain

$$\delta h(x) = \delta(\nu \circ f_2)(x) + \mathrm{Ad}(f_2(x)) \circ \delta f_1(x) = \mathrm{Ad}(f_2(x))(\delta f_1(x) - \delta f_2(x)) = 0.$$

Since the Maurer–Cartan form is injective in each point, this implies that $T_x h = 0$ for all $x \in M$. Since M is connected, h must be constantly equal to some element $g \in G$, and by definition this means $f_1(x) = g f_2(x)$ for all $x \in M$.

(2) Consider the product $M \times G$ and the distribution $E \subset TM \times TG$ defined by $E_{(x,g)} = \{(\xi, \eta) : \varphi(x)(\xi) = \omega(g)(\eta)\}$. For a chart on M with coordinate vector fields ∂_i we obtain smooth sections $(x, g) \mapsto (\partial_i(x), L_{\varphi(\partial_i(x))}(g))$ of E. For $(\xi, \eta) \in E$, we must have $\eta = L_{\varphi(\xi)}(g)$, so we see that these sections form a basis of E wherever they are defined. Hence E is a smooth distribution of rank $\dim(M)$. Moreover, we can define $\Omega \in \Omega^1(M \times G, \mathfrak{g})$ by $\Omega(\xi, \eta) = \varphi(\xi) - \omega(\eta)$, and then evidently $E_{(x,g)} = \ker(\Omega(x, g))$. Otherwise put, $\Omega = \mathrm{pr}_1^*\varphi - \mathrm{pr}_2^*\omega$. Therefore, $d\Omega = \mathrm{pr}_1^* d\varphi - \mathrm{pr}_2^* d\omega$, which means that

$$d\Omega((\xi_1, \eta_1), (\xi_2, \eta_2)) = d\varphi(\xi_1, \xi_2) - d\omega(\eta_1, \eta_2).$$

Since both φ and ω satisfy the Maurer–Cartan equation, this vanishes if $\Omega(\xi_1, \eta_1) = \Omega(\xi_2, \eta_2) = 0$.

By Lemma 2.9, the distribution E is integrable, and for $x \in M$ we consider the leaf $\mathcal{B}_{(x,e)}$ of the corresponding foliation. Restricting the projections we obtain smooth maps from $\mathcal{B}_{(x,e)}$ to M and G. By construction the first projection induces a linear isomorphism $E_{(x,e)} \to T_x M$, so the first projection restricts to a local diffeomorphism around (x, e). Hence we find an open neighborhood U of x in M and a smooth map $\alpha : U \to \mathcal{B}_{(x,e)}$ which is inverse to the first projection. Composing this with the restriction of the second projection, we obtain a smooth map $f : U \to G$. Evidently, its tangent maps have to have the form $\xi \mapsto (\xi, \eta) \mapsto \eta$, where (ξ, η) is the unique element of E with first component ξ. But this implies that $\omega(\eta) = \varphi(\xi)$ and hence $\delta f = f^*\omega = \varphi$. □

Remark 2.9. (1) From this theorem one easily deduces a global result for $I = [a, b]$. Since this is one-dimensional, any $\mathfrak{g}$-valued one form on I satisfies the Maurer–Cartan equation, so any $\varphi \in \Omega^1(I, \mathfrak{g})$ is of the form δf for a smooth $f : I \to G$. Specializing to $G = (\mathbb{R}, +)$ this give the usual fundamental theorem of calculus, namely existence of an anti-derivative which is unique up to an additive constant.

(2) There is also a global version of the theorem. Given $\varphi \in \Omega^1(M, \mathfrak{g})$, one first has to define the monodromy of φ. Fix a point $x_0 \in M$ and consider a smooth closed curve $c : [0, 1] \to M$ with $c(0) = c(1) = x_0$. From part (1) of this remark, we see that there is a unique smooth curve $\tilde{c}$ in G with $\tilde{c}(0) = e$ such that $\delta\tilde{c} = c^*\varphi$, and we consider $\tilde{c}(1) \in G$. It turns out that for homotopic curves one obtains the same result and this induces a homomorphism from the fundamental group of M to G, called the *monodromy representation*. If φ satisfies the Maurer–Cartan equation, then existence of a global smooth function $f : M \to G$ with $\delta f = \varphi$ is equivalent to the fact that the monodromy representation is trivial. More details about this can be found in [**Sharpe**, 3.,§7].

(3) This theorem gives us an alternative proof for the existence of local homomorphisms with prescribed derivative: Suppose that G and H are Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ and that $\alpha : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of Lie algebras. Then for the Maurer Cartan form ω_G of G, we can consider $\alpha \circ \omega_G \in \Omega^1(G, \mathfrak{h})$. Since α is a homomorphism, one easily shows that this satisfies the Maurer–Cartan equation, so there is an open neighborhood U of e in G and a smooth map $f : U \to H$ such that $f(e) = e$ and $\delta f = \alpha \circ \omega_G$. For $g \in U$, one then considers $f \circ \lambda_g : \lambda_g^{-1}(U) \cap U \to H$. One easily verifies that this has the same logarithmic derivative as f, so it must be of the form $\lambda_h \circ f$. Looking at the values at e, we see that $h = f(g)$, so f is a local homomorphism.

One can also deduce existence of virtual Lie subgroups from the theorem, see [**Sharpe**, 3., Exercise 6.3].

CHAPTER 3

Compact Lie groups and representation theory

In this chapter we study compact Lie groups. A good part of the results is based on special properties of representations of compact Lie groups, so have have to develop the basics of representation theory on the way.

Basic representation theory

Representation theory is one of the fundamental parts of the theory of Lie groups and Lie algebras. Applying the theory to the adjoint representation also leads to structure theory for Lie algebras and Lie groups.

3.1. Representations of Lie groups. We have already met the notion of a representation of a Lie group G on a finite dimensional vector space V in 1.10. Equivalently to the definition as a smooth homomorphism $G \to GL(V)$ used there, one can also view a representation of G on V as a smooth left action (see 1.15) $\ell : G \times V \to V$ such that for each $g \in G$ the map $\ell_g : V \to V$ is linear. By a *complex representation* we mean a representation on a complex vector space such that each of the maps ℓ_g is complex linear. As in the case of actions, we will often denote representations by $(g, v) \mapsto g \cdot v$.

Given representations of G on V and W, a linear map $f : V \to W$ is called a *morphism* or *G–equivariant* if $f(g \cdot v) = g \cdot f(v)$ for all $g \in G$. An *isomorphism* of representations is a G–equivariant linear isomorphism $f : V \to W$. If such an isomorphism exists, then we say that V and W are isomorphic and write $V \cong W$.

Suppose that we have given representations of G on V_1 and V_2. Then there is an obvious representation on $V_1 \oplus V_2$ defined by $g \cdot (v_1, v_2) := (g \cdot v_1, g \cdot v_2)$. This construction is referred to as the *direct sum of representations*. Of course, the natural inclusion of the two summands into $V_1 \oplus V_2$ are G–equivariant. A representation V of G is called *decomposable* if it is isomorphic to a direct sum $V_1 \oplus V_2$ with $\dim(V_1), \dim(V_2) > 0$. If this is not the case, then the representation is called *indecomposable*.

Let V be a representation of G. A linear subspace $W \subset V$ is called *G–invariant* or a *subrepresentation* if $g \cdot w \in W$ for all $g \in G$ and $w \in W$. In that case, we obtain representations of G on W and on V/W, defined by restriction respectively by $g \cdot (v + W) := (g \cdot v) + W$. For any representation of G on V, the subspaces $\{0\}$ and V of V are evidently invariant. If these are the only invariant subspaces then the representation is called *irreducible*. Parts (2) and (3) of the following lemma, which are known as *Schur's lemma*, show that irreducible representations have nice properties.

LEMMA 3.1. *Let V and W be representations of G and let $\varphi : V \to W$ be a morphism.*
(1) $\ker(\varphi) \subset V$ and $\operatorname{im}(\varphi) \subset W$ are G–invariant subspaces and φ induces an isomorphism $V/\ker(\varphi) \to \operatorname{im}(\varphi)$ of representations.
(2) If V and W are irreducible, then φ is either zero or an isomorphism.
(3) If V is a complex irreducible representation and $W = V$, then φ is a complex multiple of the identity.

PROOF. (1) is evident from the definitions.
(2) Since V is irreducible, we must have $\ker(\varphi) = V$ and hence $\varphi = 0$, or $\ker(\varphi) = \{0\}$. In the second case, $\mathrm{im}(\varphi) \neq \{0\}$ and hence $\mathrm{im}(\varphi) = W$ by irreducibility of W. Then φ is an isomorphism by (1).
(3) Since V is complex and $\varphi : V \to V$ is complex linear, it must have at least one eigenvalue. The corresponding eigenspace is immediately seen to be G–invariant and thus must equal V by irreducibility. □

As a simple consequence we get a result on irreducible representations of commutative groups:

COROLLARY 3.1. *Let G be a commutative Lie group. Then each complex irreducible representation of G has dimension* 1.

PROOF. By commutativity, $\ell_g \circ \ell_h = \ell_h \circ \ell_g$ for all $g, h \in G$. This means that each of the maps ℓ_h is G–equivariant, and hence a multiple of the identity by part (3) of the theorem. Hence any subspace of the representation space V is G–invariant, so irreducibility is only possible if $\dim(V) = 1$. □

EXAMPLE 3.1. We want to show that there is a big difference between indecomposable and irreducible representations in general. Let $B_n \subset GL(n, \mathbb{R})$ be the subgroup of invertible upper triangular matrices. Since this subgroup is evidently closed, it is a Lie subgroup by Theorem 1.11, and its Lie algebra $\mathfrak{b}_n$ is the space of all upper triangular matrices.

The inclusion $B_n \hookrightarrow GL(n, \mathbb{R})$ defines a representation of B_n on $\mathbb{R}^n$. For each $k = 1, \dots, n-1$ the subspace $\mathbb{R}^k \subset \mathbb{R}^n$ spanned by the first k vectors in the standard basis of $\mathbb{R}^n$ is clearly B_n–invariant. In particular, this representation is not irreducible for $n > 1$. We claim however, that these are the only invariant subspaces. Suppose that $x = (x^1, \dots, x^n) \in \mathbb{R}^n$ is a nonzero element such that for some k we have $x^k \neq 0$ but $x^{k+1} = \dots = x^n = 0$. We will show that any B_n–invariant subspace V which contains x must also contain $\mathbb{R}^k \subset \mathbb{R}^n$, which implies the claim. Indeed, one immediately constructs a matrix $A \in B_n$ such that $Ax = e_k$, the kth vector in the standard basis, so $e_k \in V$. But for $i = 1, \dots, k-1$, there obviously is a matrix $A_i \in B_n$ such that $Ae_k = e_i - e_k$, so the result follows. In particular we see that two nonzero invariant subspaces of $\mathbb{R}^n$ always have nonzero intersection, so $\mathbb{R}^n$ is indecomposable as a representation of B_n.

Indeed, it can be shown that irreducible representations of B_n are automatically one–dimensional, and the action of elements of B_n on such a representation is only via the diagonal entries.

To obtain decomposability of a representation, one does not only need a nontrivial invariant subspace but also a complementary subspace, which is invariant, too. A representation of G is called *completely reducible* if any G–invariant subspace $W \subset V$ admits a G–invariant complement. If this is the case, then V is a direct sum of irreducible subrepresentations. Indeed, if V does not admit a non–trivial invariant subspace, then V itself is irreducible. If $W \subset V$ is a non–trivial invariant subspace, then there is an invariant subspace W' such that $V = W \oplus W'$. Since both W and W' have smaller dimension than V, the claim now follows by induction.

While complete reducibility is very difficult to verify in general, there is a simple condition on a representation, which ensures complete reducibility: A representation of G on V is called *unitary*, if there is a positive definite inner product $\langle \ , \ \rangle$ on V (Hermitian if V is complex) which is G–invariant in the sense that $\langle g \cdot v, g \cdot w \rangle = \langle v, w \rangle$ for all $g \in G$ and $v, w \in V$.

Proposition 3.1. *Any unitary representation is completely reducible.*

Proof. For any subspace $W \subset V$, the orthocomplement $W^\perp \subset V$ is a complementary subspace. Thus it suffices to show that $W^\perp$ is G–invariant if W is G–invariant. For $g \in G$, $v \in W^\perp$ and $w \in W$ we compute

$$\langle g \cdot v, w\rangle = \langle g^{-1} \cdot g \cdot v, g^{-1} \cdot w\rangle = \langle v, g^{-1} \cdot w\rangle = 0$$

since $g^{-1} \cdot w \in W$. By definition, this means that $g \cdot v \in W^\perp$. □

3.2. Representations of Lie algebras. We have met the concept of a representation of a Lie algebra $\mathfrak{g}$ on a finite dimensional vector space V in 1.10. Such a representation can be either viewed as a homomorphism $\mathfrak{g} \to L(V, V)$ (where the bracket on $L(V, V)$ is given by the commutator of linear maps) or as a bilinear map $\mathfrak{g} \times V \to V$ with an evident compatibility condition with the bracket. If the actual representation is clear from the context, we will denote it by $(X, v) \mapsto X \cdot v$.

The concepts of morphisms, isomorphisms, direct sums, invariant subspaces and quotients, indecomposability, irreducibility, and complete reducibility from 3.1 carry over to representations of Lie algebras without any change. Also, Lemma 3.1 continues to hold with the same proof.

The only notion which changes is the one of unitarity. From 1.12 it is clear that the right notion of unitarity is that V admits an inner product with respect to which any element of $\mathfrak{g}$ acts by a skew symmetric (respectively skew Hermitian in the complex case) linear map. Then it is evident that the orthocomplement of a $\mathfrak{g}$–invariant subspace is $\mathfrak{g}$–invariant, too, so unitary representations are again completely reducible.

For a Lie group G with Lie algebra $\mathfrak{g}$ and a representation $\varphi : G \to GL(V)$, we obtain the associated infinitesimal representation $\varphi' : \mathfrak{g} \to L(V, V)$. From Theorem 1.9 we see that for $X \in \mathfrak{g}$ we have $\varphi(\exp(tX)) = \exp(t\varphi'(X))$. Since evaluation in $v \in V$ is a linear map, we conclude that we can compute the infinitesimal representation as $X \cdot v = \frac{d}{dt}|_{t=0} \exp(tX) \cdot v$. If G is connected, then there is a perfect correspondence between all concepts for the two representations introduced so far:

Proposition 3.2. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and let $\varphi : G \to GL(V)$ be a finite dimensional representation with derivative $\varphi' : \mathfrak{g} \to L(V, V)$.*
(1) A subspace $W \subset V$ is G–invariant if and only if it is $\mathfrak{g}$–invariant. In particular, V is indecomposable, irreducible, or completely reducible as a representation of G if and only if it has the same property as a representation of $\mathfrak{g}$.
(2) V is unitary as a representation of G if and only if it is unitary as a representation of $\mathfrak{g}$.

Proof. (1) Suppose that $W \subset V$ is G–invariant. Then for $X \in \mathfrak{g}$ and $w \in W$ the smooth curve $t \mapsto \exp(tX) \cdot w$ has values in W. Hence its derivative $X \cdot w$ in $t = 0$ lies in W, too.

Conversely, assume that W is $\mathfrak{g}$–invariant. By Theorem 1.9 we have $\varphi(\exp(X)) = e^{\varphi'(X)} = \sum_{k=0}^{\infty} \frac{1}{k!}\varphi'(X)^k$. Since $\varphi'(X)(W) \subset W$, the same holds for $\varphi'(X)^k$, and since linear subspaces in finite dimensional vector spaces are automatically closed, we see that $\varphi(\exp(X))$ maps W to itself. Hence W is invariant under that action of all elements of $\exp(\mathfrak{g}) \subset G$ and thus also under the elements of the subgroup of G generated by this subset. Now the result follows from Theorem 1.9.
(2) If φ is unitary, then it has values in $O(V)$ respectively $U(V)$. Then φ' has values in $\mathfrak{o}(V)$ respectively $\mathfrak{u}(V)$, so it is unitary, too. Conversely, if φ' has values in $\mathfrak{o}(V)$ respectively $\mathfrak{u}(V)$, then $\varphi(\exp(X)) = \exp(\varphi'(X))$ for $X \in \mathfrak{g}$ shows that φ maps $\exp(\mathfrak{g})$

to $O(V)$ respectively $U(V)$. Then the same is true for the subgroup generated by $\exp(\mathfrak{g})$. □

3.3. Some structure theory. We can next apply the ideas of representation theory to the adjoint representation $\mathrm{ad} : \mathfrak{g} \to L(\mathfrak{g}, \mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. A subspace $\mathfrak{h} \subset \mathfrak{g}$ by definition is $\mathfrak{g}$–invariant if and only if $[X, Y] \in \mathfrak{h}$ for all $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$, i.e. if and only if $\mathfrak{h}$ is an ideal in $\mathfrak{g}$. We have met the direct sum of Lie algebras in 1.4. If $\mathfrak{g} = \mathfrak{h}' \oplus \mathfrak{h}''$, then by definition $\mathfrak{h}'$ and $\mathfrak{h}''$ are ideals in $\mathfrak{g}$. Hence indecomposability of the adjoint representation exactly means that $\mathfrak{g}$ is not a direct sum of two non–trivial ideals.

A Lie algebra $\mathfrak{g}$ is called *simple* if $\dim(\mathfrak{g}) > 1$ and $\mathfrak{g}$ does not have any ideals except $\{0\}$ and $\mathfrak{g}$. It is called *semisimple* if it is a direct sum of simple ideals. Finally, $\mathfrak{g}$ is called *reductive* if any ideal in $\mathfrak{g}$ admits a complementary ideal, i.e. if the adjoint representation is completely reducible. One defines a Lie group G to be simple, semisimple, or reductive, if its Lie algebra $\mathfrak{g}$ has that property.

We can now easily clarify some basic properties of these kinds of Lie algebras. For this, we need two more definitions: Let $\mathfrak{g}$ be any Lie algebra. Then we define the *center* of $\mathfrak{g}$ to $\mathfrak{z}(\mathfrak{g}) := \{X \in \mathfrak{g} : [X, Y] = 0 \quad \forall Y \in \mathfrak{g}\}$. Further, we define the *derived subalgebra* $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$ as the subspace of $\mathfrak{g}$ spanned by all elements of the form $[X, Y]$ with $X, Y \in \mathfrak{g}$. A moment of thought shows that both $\mathfrak{z}(\mathfrak{g})$ and $[\mathfrak{g}, \mathfrak{g}]$ are ideals in $\mathfrak{g}$ and clearly $\mathfrak{z}(\mathfrak{g})$ is Abelian.

THEOREM 3.3. *(1) If $\mathfrak{g}$ is semisimple then $\mathfrak{z}(\mathfrak{g}) = \{0\}$ and $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Moreover, if $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ is a decomposition into a direct sum of simple ideals, then any ideal in $\mathfrak{g}$ is the sum of some of the $\mathfrak{g}_i$. In particular, $\mathfrak{g}$ does not have a nonzero Abelian ideal.*
(2) If $\mathfrak{g}$ is reductive then $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$, and $[\mathfrak{g}, \mathfrak{g}]$ is semisimple.

PROOF. (1) In an Abelian Lie algebra, every linear subspace is an ideal. Now for a simple Lie algebra $\mathfrak{g}$, we must by definition have $\dim(\mathfrak{g}) > 1$, so $\mathfrak{g}$ cannot be Abelian. This implies that $\mathfrak{z}(\mathfrak{g}) \neq \mathfrak{g}$ and hence $\mathfrak{z}(\mathfrak{g}) = \{0\}$, and $[\mathfrak{g}, \mathfrak{g}] \neq \{0\}$ and hence $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ in the simple case.

Next, suppose that $\mathfrak{g}$ is semisimple and that $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ is a decomposition into a direct sum of simple ideals. The $[\mathfrak{g}_i, \mathfrak{g}_i] = \mathfrak{g}_i$ for all i immediately implies $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. On the other hand, suppose that $\mathfrak{h} \subset \mathfrak{g}$ is an ideal. Then $\mathfrak{h} \cap \mathfrak{g}_i$ is an ideal in $\mathfrak{g}_i$ for each i. Since $\mathfrak{g}_i$ is simple we must have either $\mathfrak{h} \cap \mathfrak{g}_i = \{0\}$ or $\mathfrak{h} \cap \mathfrak{g}_i = \mathfrak{g}_i$. Suppose that $\mathfrak{h} \cap \mathfrak{g}_i = \{0\}$ and that $X \in \mathfrak{h}$ decomposes as $X = X_1 + \cdots + X_k$ with $X_j \in \mathfrak{g}_j$. Then for any $Y_i \in \mathfrak{g}_i$ we get $[X, Y_i] \in \mathfrak{h} \cap \mathfrak{g}_i$ (since both subsets are ideals) so $[X, Y_i] = 0$ for any Y_i. But by definition $[X, Y_i] = [X_i, Y_i]$, so $X_i \in \mathfrak{z}(\mathfrak{g}_i) = \{0\}$. This shows that $\mathfrak{h}$ is the sum of those $\mathfrak{g}_j$ for which $\mathfrak{h} \cap \mathfrak{g}_j = \mathfrak{g}_j$. In particular, a semisimple Lie algebra does not have non–trivial Abelian ideals and hence $\mathfrak{z}(\mathfrak{g}) = \{0\}$.
(2) Since $\mathfrak{g}$ is reductive, there is an ideal $\mathfrak{h} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{h}$. It is an easy exercise to verify that any ideal in a reductive Lie algebra is reductive, too. Inductively, we can decompose $\mathfrak{h}$ as $\mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_k$, such that each $\mathfrak{h}_i$ does not admit any nontrivial ideals. But this means that $\mathfrak{h}_i$ either is simple or one–dimensional. In the latter case, one immediately concludes that $\mathfrak{h}_i \subset \mathfrak{z}(\mathfrak{g})$ which is a contradiction. This shows that $\mathfrak{h}$ is semisimple and hence $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$ and hence $[\mathfrak{g}, \mathfrak{g}] \supset \mathfrak{h}$. Since the other inclusion is obvious, the result follows. □

REMARK 3.3. (1) The proposition shows that semisimple Lie algebras are automatically reductive. On the other hand, we see that to understand reductive (and hence also semisimple) Lie algebras, it suffices to understand the simple ones. It turns out

that simple real and complex Lie algebras can be completely classified using linear algebra. In the complex case, there are the so–called classical series consisting of the Lie algebra $\mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{so}(n,\mathbb{C})$, and $\mathfrak{sp}(n,\mathbb{C})$, plus five exceptional simple Lie algebras, see [**Cap:Liealg**, chapter 3]. In the real case, the classification is more complicated but still manageable, see e.g. [**Knapp**].
(2) The opposite end of the spectrum of Lie algebras to the ones discussed here is formed by solvable Lie algebras. For a Lie algebra $\mathfrak{g}$, we define $\mathfrak{g}^{(1)} := \mathfrak{g}$, $\mathfrak{g}^{(2)} := [\mathfrak{g},\mathfrak{g}]$, and inductively, $\mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(i)},\mathfrak{g}^{(i)}]$. Then one obtains the *derived series* of $\mathfrak{g}$, namely $\mathfrak{g} \supset \mathfrak{g}^{(2)} \supset \dots \supset \mathfrak{g}^{(i)} \supset \mathfrak{g}^{(i+1)} \supset \dots$, and $\mathfrak{g}$ is called *solvable* if $\mathfrak{g}^{(k)} = \{0\}$ for some k. The typical example of a solvable Lie algebra is the Lie algebra $\mathfrak{b}_n$ of upper triangular matrices, see 3.1.

From the proposition we see that for a semisimple Lie algebra $\mathfrak{g}$ we have $\mathfrak{g}^{(i)} = \mathfrak{g}$ for all i, and hence the classes of semisimple and solvable Lie algebras are disjoint. Likewise, the classes of reductive and solvable Lie algebras intersect only in the Abelian ones.
(3) The property we have used as the definition of semisimplicity is usually viewed as the result of a theorem. The usual definition (which is equivalent) is the $\mathfrak{g}$ does not contain a non–trivial solvable ideal, see [**Cap:Liealg**, chapter 2].

The following result leads to many examples of reductive and semisimple Lie algebras:

PROPOSITION 3.3. *Let $\mathfrak{g} \subset M_n(\mathbb{R})$ be a Lie subalgebra which is closed under forming the transpose of matrices. Then $\mathfrak{g}$ is reductive.*

PROOF. One immediately verifies that $\langle X,Y\rangle := \operatorname{tr}(XY^t)$ is the standard inner product on $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ and hence restricts to a positive definite inner product on $\mathfrak{g}$. We claim that for an ideal $\mathfrak{h} \subset \mathfrak{g}$ also the orthocomplement $\mathfrak{h}^\perp$ is an ideal, which proves the result. Suppose that $X \in \mathfrak{h}^\perp$ and $Y \in \mathfrak{g}$. Then for $Z \in \mathfrak{h}$ we compute

$$\begin{aligned}\langle [X,Y],Z\rangle &= \operatorname{tr}(XYZ^t - YXZ^t) = \operatorname{tr}(X(YZ^t - Z^tY)) = \\ &= \operatorname{tr}(X(ZY^t - Y^tZ)^t) = -\langle X,[Y^t,Z]\rangle,\end{aligned}$$

which vanishes since $Y^t \in \mathfrak{g}$ and hence $[Y^t,Z] \in \mathfrak{h}$. □

This immediately shows that $\mathfrak{gl}(n,\mathbb{R})$, $\mathfrak{sl}(n,\mathbb{R})$, $\mathfrak{o}(n)$, and $\mathfrak{o}(p,q)$ are reductive Lie algebras. It is elementary to show that the center of $\mathfrak{gl}(n,\mathbb{R})$ consists of all multiples of the identity matrix, while for $\mathfrak{sl}(n,\mathbb{R})$ with $n \geq 2$ and $\mathfrak{o}(p,q)$ with $p+q \geq 3$ the center is trivial and hence these algebras are semisimple. Indeed, except for $\mathfrak{o}(p,q)$ with $p+q = 4$, they are even simple.

3.4. Compact groups. The first fundamental result about compact groups that we can prove is that their representations are automatically completely reducible, which also has strong consequences on their structure. This needs some background on integration.

First we observe that any Lie group G is an orientable manifold, see Section 4.3 of [**AnaMf**]. Indeed, to define a compatible orientation on each tangent space, choose a basis $\{X_1,\dots,X_n\}$ of the Lie algebra $\mathfrak{g}$ of G and then define the basis $\{L_{X_1}(g),\dots,L_{X_n}(g)\}$ of T_gG to be positively oriented for each $g \in G$. Recall further that the right objects for integration on n–dimensional oriented manifolds are n–forms, see Section 4.4 of [**AnaMf**]. An n–form $\varphi \in \Omega^n(M)$ smoothly assigns to each $x \in M$ an n–linear, alternating map $\varphi(x) : (T_xM)^n \to \mathbb{R}$. In local coordinates, such an n–form is determined by one smooth function, and integration is defined by decomposing an n–form into a sum of forms supported in one chart and then integrating the local coordinate expressions.

Returning to the case of a Lie group G, we can consider the space of n–linear alternating maps $\mathfrak{g}^n \to \mathbb{R}$, which is one–dimensional. Chose a nonzero element α in this space and define $\varphi \in \Omega^n(G)$ by

$$\varphi(g)(\xi_1, \dots, \xi_n) := \alpha(T\lambda_{g^{-1}} \cdot \xi_1, \dots, T\lambda_{g^{-1}} \cdot \xi_n).$$

The resulting n–form is *left invariant* in the sense that $(\lambda_g)^*\varphi = \varphi$ for all $g \in G$. If G is compact, then $\int_G \varphi$ is a well defined real number, and we can uniquely rescale α in such a way that the integral equals one. Let us write vol for the resulting n–form, which we call the *volume form* of the compact Lie group G.

Having this volume form at hand, we can integrate smooth functions on G. For $f : G \to \mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ we have $f\,\mathrm{vol} \in \Omega^n(G, \mathbb{K})$, and we will also write $\int_G f$ or $\int_G f(g)dg$ for $\int_G f\,\mathrm{vol} \in \mathbb{K}$. The fundamental property of this integral is again left invariance: By construction, for any $g \in G$ the left translation $\lambda_g : G \to G$ is an orientation preserving diffeomorphism, and hence $\int_G f = \int_G \lambda_g^*(f\,\mathrm{vol})$. But $\lambda_g^*(f\,\mathrm{vol}) = (f \circ \lambda_g)\lambda_g^*\,\mathrm{vol} = (f \circ \lambda_g)\,\mathrm{vol}$, and therefore $\int_G f \circ \lambda_g = \int_G f$ or equivalently $\int_G f(gh)dh = \int_G f(h)dh$ for all $g \in G$. The volume form leads to a measure on G, called the (left) *Haar measure* which can then be used to extend integration to continuous functions and further. All these extensions are still left invariant. It is a nice exercise to show that compactness of G implies that also $(\rho^g)^*\,\mathrm{vol} = \mathrm{vol}$ for all $g \in G$.

Using integration, we can prove the first fundamental result on compact Lie groups. To deal with $\mathbb{K} = \mathbb{R}$ and $\mathbb{C}$ simultaneously we we always use Hermitian forms in computations and define the conjugation on $\mathbb{R}$ to be the identity.

Theorem 3.4. *Let G be a compact Lie group. Then any finite dimensional representation of G is unitary and hence completely reducible.*

Proof. For a representation on V, choose an arbitrary positive definite Hermitian inner product on V, which we denote by $b(v, w)$. Then define $f_{v,w} : G \to \mathbb{K}$ by $f_{v,w}(g) := b(g^{-1} \cdot v, g^{-1} \cdot w)$. Define a map $\langle\ ,\ \rangle : V \times V \to \mathbb{K}$ by

$$\langle v, w\rangle := \int_G f_{v,w}.$$

By construction we have $f_{v_1+tv_2,w} = f_{v_1,w} + tf_{v_2,w}$ which implies that $\langle\ ,\ \rangle$ is linear in the first variable. Next, $f_{w,v} = \overline{f_{v,w}}$ implies that $\langle\ ,\ \rangle$ is a Hermitian form, and since $f_{v,v}(g) > 0$ for all $v \neq 0$ and all $g \in G$ we obtain a positive definite Hermitian inner product. Finally, by definition we get $f_{g^{-1}\cdot v, g^{-1}\cdot w}(h) = f_{v,w}(gh)$, which means that $f_{g^{-1}\cdot v, g^{-1}\cdot w} = f_{v,w} \circ \lambda_g$. Left invariance of the integral now shows that $\langle g^{-1} \cdot v, g^{-1} \cdot w\rangle = \langle v, w\rangle$ for all $v, w \in V$ and all $g \in G$. □

By Proposition 3.2, this carries over to representations of the Lie algebra $\mathfrak{g}$ of G, and applying it to the adjoint representation we get

Corollary 3.4. *Let G be a compact Lie group. Then the Lie algebra $\mathfrak{g}$ of G is reductive and hence $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$ with $[\mathfrak{g}, \mathfrak{g}]$ semisimple.*

3.5. Matrix coefficients and characters. For any representation $\varphi : G \to GL(V)$ of a Lie group G on a finite dimensional $\mathbb{K}$–vector space V, one defines the *character* of φ as the smooth function $\chi : G \to \mathbb{K}$ given by $\chi(g) := \mathrm{tr}(\varphi(g))$. Since $\varphi : G \to GL(V)$ is a homomorphism, we get $\varphi(ghg^{-1}) = \varphi(g)\varphi(h)\varphi(g)^{-1}$, which implies that $\chi(ghg^{-1}) = \chi(h)$, so the character is a *class function*. On the other hand, suppose that $f : V \to W$ is an isomorphism between two representations φ and ψ. Then $\psi(g)\circ f = f\circ\varphi(g)$ or $\psi(g) = f\circ\varphi(g)\circ f^{-1}$ which shows that isomorphic representations have the same character.

For a unitary representation on $(V, \langle\ ,\ \rangle)$, there is a larger class of associated smooth functions $G \to \mathbb{K}$. One defines a *matrix coefficient* to be a map of the form $g \mapsto \langle g\cdot v, w\rangle$ for $v, w \in V$. If $\{v_1, \dots, v_n\}$ is an orthonormal basis for V, then the matrix representation of $\varphi(g)$ is given by $(\langle g\cdot v_i, v_j\rangle)_{i,j}$, so we really obtain matrix entries in that way. Moreover, we clearly have $\chi(g) = \sum_i \langle g\cdot v_i, v_i\rangle$, so the character is a sum of matrix coefficients.

Of course, the matrix coefficients depend on the choice of the inner product. However, this dependence does not really matter: For a representation of a Lie group G on a vector space V, consider the dual space V^* and for $\alpha \in V^*$ define $g\cdot\alpha : V \to \mathbb{K}$ by $g\cdot\alpha(v) := \alpha(g^{-1}\cdot v)$. One immediately verifies that this defines a representation of G on V^*, called the *dual* of the given representation. Looking at the derivative, one sees that for Lie algebra representations the right definition is $(X\cdot\alpha)(v) := -\alpha(X\cdot v)$.

Given an inner product $\langle\ ,\ \rangle$ on V, we obtain a map $V \to V^*$ which sends $v \in V$ to the functional $\alpha(w) := \langle w, v\rangle$. The statement that the inner product is G–invariant is equivalent to the corresponding map $V \to V^*$ being G–equivariant. In the complex case, a similar statement holds, but the isomorphism $V \to V^*$ induced by a Hermitian inner product is conjugate linear rather than complex linear. If V is irreducible, then one easily verifies that V^* is irreducible, which by Schur's lemma implies that any two isomorphisms $V \to V^*$ are proportional. In particular, an irreducible representation V of G admits, up to multiples, at most one invariant inner product.

A general unitary representation can be written as an orthogonal direct sum of irreducible representations. This implies that each matrix coefficient is a linear combination of matrix coefficients of the irreducible components. In particular, the space of matrix coefficients of a unitary representation, viewed as a subspace of $C^\infty(G,\mathbb{K})$, is independent of all choices.

The basic fact about matrix coefficients for compact Lie groups is provided by the *Schur orthogonality relations*:

Proposition 3.5. *Let G be a compact Lie group, and let V and W be irreducible unitary representations of G (with both inner products denoted by $\langle\ ,\ \rangle$).*
(1) If V and W are non–isomorphic, then for any matrix coefficients φ of V and ψ of W, we have $\int_G \varphi\bar{\psi} = 0$.
(2) If V is complex and $W = V$, then for $v_1, v_2, w_1, w_2 \in V$ we have

$$\int_G \langle g\cdot v_1, v_2\rangle\overline{\langle g\cdot w_1, w_2\rangle}dg = \frac{1}{\dim(V)}\langle v_1, w_1\rangle\overline{\langle v_2, w_2\rangle}.$$

Proof. Take any linear map $f : V \to W$. For fixed elements $v \in V$ and $w \in W$, consider $\int_G \langle f(g^{-1}\cdot v), g^{-1}\cdot w\rangle dg$. This expression is evidently conjugate linear in w, so for fixed $v \in V$ we obtain an element $F(v) \in W$ such that

$$\langle F(v), w\rangle = \int_G \langle f(g^{-1}\cdot v), g^{-1}\cdot w\rangle dg$$

for all $w \in W$. Since the right hand side is also linear in v, this defines a linear map $F : V \to W$. Now

$$\langle h\cdot F(v), w\rangle = \langle F(v), h^{-1}\cdot w\rangle = \int_G \langle f(g^{-1}\cdot v), (hg)^{-1}\cdot w\rangle dg.$$

Using $g^{-1}\cdot v = (hg)^{-1}\cdot h\cdot v$, we conclude from invariance of the integral that $\langle h\cdot F(v), w\rangle = \langle F(h\cdot v), w\rangle$ for all $v\in V$, $w\in W$, and $h\in G$. Thus, $F: V\to W$ is a morphism of representations.
(1) Since V and W are non–isomorphic, we must have $F=0$ for any choice of f by Schur's lemma. Defining $f(v) := \langle v, v_1\rangle w_1$ for fixed $v_1\in V$ and $w_1\in W$, we obtain

$$0 = \int_G \langle g^{-1}\cdot v, v_1\rangle\langle w_1, g^{-1}\cdot w\rangle dg = \int_G \langle g\cdot w_1, w\rangle\overline{\langle g\cdot v_1, v\rangle}dg.$$

(2) Again by Schur's lemma, any choice of f has to lead to $F = a\,\mathrm{id}$ for some $a\in\mathbb{C}$. Hence for all $v, w\in V$ we get

$$(*) \qquad \int_G \langle f(g^{-1}\cdot v), g^{-1}\cdot w\rangle dg = a\langle v, w\rangle.$$

Taking an orthonormal basis $\{v_i\}$ of V, inserting $v=w=v_i$, and summing over all i, the right hand side of $(*)$ equals $a\dim(V)$. On the other hand, since then also $\{g^{-1}\cdot v_i\}$ is an orthonormal basis, the left hand side of $(*)$ equals $\int_G \mathrm{tr}(f) = \mathrm{tr}(f)$, which shows that $a = \frac{\mathrm{tr}(f)}{\dim(V)}$.

Similarly as in (1), we now put $f(v) := \langle v, v_1\rangle w_1$. Then $\mathrm{tr}(f) = \langle w_1, v_1\rangle$, so the left hand side of $(*)$ becomes $\frac{1}{\dim(V)}\langle w_1, v_1\rangle\langle v, w\rangle$. On the other hand, as in (1), the right hand side of $(*)$ evaluates to

$$\int_G \langle g\cdot w_1, w\rangle\overline{\langle g\cdot v_1, v\rangle}dg.$$

□

This has immediate strong consequences on characters:

Corollary 3.5. *Let G be a compact Lie group.*
(1) For complex irreducible representations V_1 and V_2 of G with characters χ_1 and χ_2 we have

$$\int_G \chi_1(g)\overline{\chi_2(g)}dg = \begin{cases} 0 & V_1\ncong V_2 \\ 1 & V_1\cong V_2\end{cases}$$

(2) Two complex representations of G are isomorphic if and only if they have the same character.

Proof. Part (1) follows easily from the theorem since $\chi(g) = \sum_i\langle g\cdot v_i, v_i\rangle$ for an orthonormal basis $\{v_i\}$.
(2) Let us first assume that V_1 and V_2 are complex irreducible representations with the same character χ. If $V_1\ncong V_2$, then by part (1) we have $\int_G |\chi|^2 = 0$ and hence $\chi = 0$. But this is impossible, since $\chi(e)$ is the dimension of the representation space.

Let V be a complex representation of G. By Theorem 3.4, V decomposes as a direct sum of irreducible representations and taking together isomorphic factors, we can write V as $V\cong (V_1)^{n_1}\oplus\cdots\oplus(V_k)^{n_k}$ for pairwise non–isomorphic complex irreducible representations V_i and positive integers n_i. This implies that the character of V is given by $\chi = n_1\chi_1+\cdots+n_k\chi_k$, where χ_j is the character of V_j. But by part (1) we can recover the representations V_j and the integers n_j since $n_j = \int_G \chi(g)\overline{\chi_j(g)}dg$. □

The number n_j which shows up in the proof of part (2) is called the *multiplicity* of the irreducible representation V_j in V.

Maximal tori

We continue our study of the structure of compact Lie groups by looking at maximal connected Abelian subgroups. As we shall see, such a subgroup is essentially unique and meets each conjugacy class. This has interesting consequences both for structure theory and for the theory of characters.

Let us describe the main results in this direction for the prototypical example $G = U(n)$, where they follow from linear algebra. The subset $T \subset G$ of unitary diagonal matrices is a connected commutative subgroup of G. Evidently, $T \cong U(1)^n$ so this is a torus. Since any unitary matrix can be unitarily diagonalized, any element of G is conjugate to an element of T. Any family of commuting unitary matrices can be simultaneously diagonalized unitarily, so any commutative subgroup of G is conjugate to a subgroup of T. In particular, T is a maximal commutative subgroup of G and any such subgroup is conjugate to T.

Since T is a commutative subgroup of G, there is no conjugation inside of T. Still the question of whether two elements of T are conjugate in G is of interest. Since conjugate matrices have the same eigenvalues (including multiplicities), two diagonal matrices can only be conjugate if there entries agree up to a permutation. But of course, any permutation σ of the vectors of the standard basis defines a unitary matrix $A_\sigma \in G$ so we see that any permutation of diagonal entries can be realized by a conjugation inside of G. This will identify the permutation group $\mathfrak{S}_n$ (together with its action on T) as the Weyl group associated to T.

3.6. We first have to clarify the structure of connected Abelian Lie groups. We write $\mathbb{T}^n$ for the n–dimensional torus, so $\mathbb{T}^n \cong U(1)^n \cong \mathbb{R}^n/\mathbb{Z}^n$.

Proposition 3.6. *Any connected Abelian Lie group is isomorphic to $\mathbb{R}^m \times \mathbb{T}^n$ for some $n, m \in \mathbb{N}$. In particular, any compact connected Abelian Lie group is a torus.*

Proof. Let G be connected and Abelian. By Proposition 1.5, the Lie algebra $\mathfrak{g}$ of G is Abelian. Take $X, Y \in \mathfrak{g}$ and consider $t \mapsto \exp(tX)\exp(tY)$. Since G is commutative, this is a one parameter subgroup, and its derivative at $t = 0$ is $X + Y$. In particular, $\exp(X)\exp(Y) = \exp(X+Y)$, so $\exp : \mathfrak{g} \to G$ is a homomorphism from $(\mathfrak{g}, +) \cong \mathbb{R}^{\dim(\mathfrak{g})}$ to G. The derivative $\exp' : \mathfrak{g} \to \mathfrak{g}$ is the identity by Theorem 1.8, so by Propositions 1.11 and 1.17, $K := \ker(\exp) \subset \mathfrak{g}$ is a discrete subgroup and exp induces an isomorphism $\mathfrak{g}/K \to G$. Let $V \subset \mathfrak{g}$ be the subspace spanned by K and let $W \subset \mathfrak{g}$ be a complementary subspace. Then evidently $\mathfrak{g}/K \cong (V/K) \times W$, so it suffices to show that V/K is a torus to complete the proof.

So we have to show that for a discrete subgroup $K \subset \mathbb{R}^n$ which spans $\mathbb{R}^n$, there is a basis $\{x_1, \dots, x_n\}$ such that K consists of all integral linear combinations of the x_i. Take an element $x_1 \in K$ of minimal norm. Then $\mathbb{Z}x_1 \subset K$ and we claim that $\mathbb{R}x_1 \cap K = \mathbb{Z}x_1$. Indeed, if $\lambda x_1 \in K$ for some $\lambda \in \mathbb{R} \setminus \mathbb{Z}$, we may assume without loss of generality that $\lambda > 0$. But then we find $k \in \mathbb{Z}$ such that $k < \lambda < k+1$ and $(\lambda - k)x_1 \in K$ would be an element of smaller norm than x_1. In particular, this proves the result if $n = 1$.

Proceeding inductively, assume that $K \subset \mathbb{R}^n$ for $n \geq 2$ and that the result has been proved for subgroups of $\mathbb{R}^{n-1}$. For an element $x_1 \in K$ as above, we can form $\mathbb{R}^n/\mathbb{R}x_1 \cong \mathbb{R}^{n-1}$ and obtain an embedding of $K/\mathbb{Z}x_1$ into $\mathbb{R}^{n-1}$. By construction, the image is a subgroup that spans $\mathbb{R}^{n-1}$ and one easily shows it is again discrete. By induction hypothesis, it consists of all integral linear combinations of the elements of a basis for $\mathbb{R}^{n-1}$, and we choose preimages $x_2, \dots, x_n \in \mathbb{K}$ of the basis elements.

Then by construction $\{x_1, \dots, x_n\}$ is a basis for $\mathbb{R}^n$ and K contains all integral linear combinations of the basis elements.

Conversely, given $x \in K$, the element $x + \mathbb{R}x_1 \in \mathbb{R}^n/\mathbb{R}x_1$ can be written as an integral linear combination of our basis elements, which leads to a representation $x = \lambda x_1 + a_2x_2 + \dots + a_nx_n$ with $\lambda \in \mathbb{R}$ and $a_2, \dots, a_n \in \mathbb{Z}$. But this readily implies $\lambda x_1 \in K$ and hence $\lambda \in \mathbb{Z}$, which completes the argument. $\square$

Now let us consider a compact Lie group G. Then G contains connected commutative subgroups, for example one parameter subgroups. Hence it makes sense to look for connected Abelian subgroups $T \subset G$ which are maximal, i.e. not contained in larger subgroups having the same property. As we shall see below, such a subgroup must be isomorphic to $\mathbb{T}^k$ for some k, and hence is called a *maximal torus* in G.

Likewise, any one–dimensional subspace in the Lie algebra $\mathfrak{g}$ is an Abelian subalgebra, so we can also look for maximal commutative subalgebras of $\mathfrak{g}$, which evidently exist by dimensional considerations.

THEOREM 3.6. *Let G be a compact Lie group with Lie algebra $\mathfrak{g}$.*
(1) Any maximal connected Abelian subgroup T of G is closed and hence a Lie subgroup and isomorphic to a torus.
(2) The maximal tori in G are exactly the unique (virtual) Lie subgroups corresponding to maximal commutative Lie subalgebras of $\mathfrak{g}$.

PROOF. The main point in the proof is that for a connected Abelian subgroup $H \subset G$ also the closure $\bar{H}$ is a connected Abelian subgroup. Connectedness of $\bar{H}$ is clear, and for $a, b \in \bar{H}$ we find sequences (a_n) and (b_n) in H converging to a and b respectively. But then $ab = \lim a_nb_n = \lim b_na_n = ba \in \bar{H}$ and $a^{-1} = \lim a_n^{-1} \in \bar{H}$. From this, (1) follows immediately.

For (2), suppose that H is a connected Lie group whose Lie algebra $\mathfrak{h}$ is commutative. Then since ad is the derivative of Ad, connectedness of H implies that $\mathrm{Ad}(h) = \mathrm{id}_{\mathfrak{h}}$ for all $h \in H$. Since $\mathrm{Ad}(h)$ is the derivative of conj_h, this must be the identity, too, so H is commutative. Hence there is a direct correspondence between commutative Lie subalgebras and connected commutative Lie subgroups, from which (2) easily follows. $\square$

3.7. The Weyl group. Let G be a compact Lie group and let $T \subset G$ be a maximal torus. Then we define the *normalizer* $N := N_G(T)$ of T in G as $\{g \in G : gag^{-1} \in T \quad \forall a \in T\}$. This is evidently a subgroup of G and since T is a closed subset of G, also the subgroup N is closed and hence compact. By definition, $T \subset N$ is a normal subgroup, and $N \subset G$ is the maximal subgroup for which this is true. In particular, we can form the quotient group $W := N/T$, which is called the *Weyl group* of the torus T.

Let us look at the case $G = U(n)$ and $T \subset G$ is the torus of diagonal matrices. Let $D \in T$ have all eigenvalues different. If $A \in G$ is such that $ADA^{-1} = \tilde{D} \in T$, then also all eigenvalues of $\tilde{D}$ are different, and $AD = \tilde{D}A$. Hence A must map any eigenspace of D to an eigenspace of $\tilde{D}$, and hence each coordinate axis to some other coordinate axis. Thus we exactly recover permutation matrices as dicussed earlier and $W = N/T$ in this case is isomorphic to the permutation group $\mathfrak{S}_n$ of n elements. This acts on T by permuting the entries of diagonal matrices. Our next task is to prove that the group W is finite in general.

LEMMA 3.7. *Let $T \subset G$ be a maximal torus and let N be its normalizer in G. Then T coincides with the connected component of the identity of N, and $W = N/T$ is a finite group.*

Proof. Let $\mathfrak{t}$ and $\mathfrak{n}$ be the Lie algebras of T and N. Since T is a normal subgroup of N, the Lie algebra $\mathfrak{t}$ is an ideal in $\mathfrak{n}$. From 3.4 we know that there is an inner product on $\mathfrak{n}$, which is invariant under the adjoint action. Accordingly, we get $\mathfrak{n} = \mathfrak{t} \oplus \mathfrak{t}^\perp$. Since $\mathfrak{t}$ is an ideal in $\mathfrak{n}$, so is $\mathfrak{t}^\perp$ (see 3.4). If $\mathfrak{t}^\perp \neq \{0\}$ then we choose a one–dimensional subspace $\ell \subset \mathfrak{t}^\perp$ and see that $\mathfrak{t} \oplus \ell$ is an Abelian Lie subalgebra of $\mathfrak{n}$ and hence of $\mathfrak{g}$, which strictly contains $\mathfrak{t}$. This contradicts part (2) of Theorem 3.6, so we obtain $\mathfrak{n} = \mathfrak{t}$.

Since N and T have the same Lie algebra and T is connected, it must coincide with the connected component of the identity of N. Thus the quotient group $W = N/T$ is discrete and since N is a closed and hence compact subgroup of G, the group W must be finite. □

The group N acts on T by conjugation. Since T is commutative, elements of T act trivially, so we we obtain an induced left action $W \times T \to T$.

3.8. Conjugacy. To prove the fundamental results about maximal tori, we need some background on the mapping degree, see Section 4.10 of [**AnaMf**] for more information. Suppose that M and N are oriented compact smooth manifolds of dimension n, and that $f : M \to N$ is a smooth map. Choose a form $\alpha \in \Omega^n(N)$ such that $\int_N \alpha = 1$, consider the pullback $f^*\alpha \in \Omega^n(M)$ and define $\deg(f) := \int_M f^*\alpha \in \mathbb{R}$. This is well defined, since the integral of an n–form over a compact smooth n–manifold vanishes if and only if it is the exterior derivative of an $(n-1)$–form. Hence any other choice for α is of the form $\alpha + d\beta$ and $f^*(\alpha + d\beta) = f^*\alpha + d(f^*\beta)$ and the integral remains unchanged. (More formally this can be phrased as the fact that the action $f^* : H^n(N) \to H^n(M)$ between the top de–Rham cohomologies, which are isomorphic to $\mathbb{R}$, is given by multiplication by $\deg(f)$).

We only need two facts about the mapping degree: First if $f : M \to N$ is not surjective, then $\deg(f) = 0$. Indeed, in that case we can find an open subset $U \subset N$ which is disjoint to $f(M)$. Using an appropriate bump function, we can construct an n–form α on N with support contained in U and $\int_N \alpha = 1$. But then $f^*\alpha = 0$ and hence $\deg(f) = 0$.

On the other hand, suppose that $y \in N$ is a *regular value* for $f : M \to N$, i.e. for all $x \in f^{-1}(\{y\})$ the map $T_xf : T_xM \to T_yN$ is a linear isomorphism. In this case each such x has a neighborhood U_x on which f restricts to a diffeomorphism $U_x \to f(U_x)$, so in particular $U_x \cap f^{-1}(\{y\}) = \{x\}$. This shows that $f^{-1}(\{y\})$ is discrete in M but since it is clearly closed and thus compact. Hence $f^{-1}(\{y\})$ is finite and we number its elements as $x_1, \dots, x_k$ and define ϵ_i to be $+1$ if $T_{x_i}f$ is orientation preserving and -1 otherwise. Then we can find connected neighborhoods U_i of x_i and V of y such that f restricts to a diffeomorphism $U_i \to V$ for each $i = 1, \dots, k$. As above, we find $\alpha \in \Omega^n(N)$ with support contained in V and $\int_N \alpha = 1$. But then $\int_M f^*\alpha = \sum_{i=1}^k \int_{U_i} f^*\alpha$ and by diffeomorphism invariance of the integral we obtain $\deg(f) = \sum_{i=1}^k \epsilon_i$.

On the other hand, we need the notion of a generator of a Lie group. An element g of a Lie group G is called a *generator* if the subset $\{g^k : k \in \mathbb{Z}\}$ (i.e. the subgroup of G generated by g) is dense in G. From 3.6 we know that the closure of a commutative subgroup is commutative, so only Abelian Lie groups can have a generator. While it is easy to see that $\mathbb{R}^n$ does not admit a generator, we get the following result for tori:

Lemma 3.8. *For each $n \geq 0$ the torus $\mathbb{T}^n$ admits a generator. Indeed, the set of generators is dense in $\mathbb{T}^n$.*

Proof. Let us view $\mathbb{T}^n$ as $\mathbb{R}^n/\mathbb{Z}^n$, so the exponential map is just the canonical projection $\mathbb{R}^n \to \mathbb{T}^n$. We call a cube in $\mathbb{R}^n$ any subset of the form $\{x : |x^i - \xi^i| \leq \epsilon \quad \forall i\}$

for fixed $\xi \in \mathbb{R}^n$ and $\epsilon > 0$. Choose any cube C_0 contained in $[0,1]^n$ and let $\{U_k : k \in \mathbb{N}\}$ be a countable basis for the open subsets of $\mathbb{T}^n$. Suppose inductively, that we have already constructed cubes $C_0 \supset \cdots \supset C_{m-1}$ and numbers $N_1 < \cdots < N_{m-1}$ such that for each $x \in C_k$ we have $\exp(x)^{N_k} \in U_k$. Let 2ϵ be the length of the sides of C_{m-1}. Choose N_m such that $N_m 2\epsilon > 1$. Then $N_m C_{m-1}$ is a cube of side length > 1, which implies that $x \mapsto \exp(x)^{N_m}$ defines a continuous surjection $C_{m-1} \to \mathbb{T}^n$. Hence the preimage of U_m under this map is open and therefore contains a cube C_m. Thus we find cubes C_k for all $k \in \mathbb{N}$.

By compactness of $[0,1]^n$, the intersection $\cap_{k\in\mathbb{N}} C_k$ is non–empty and for an element x in there we have $\exp(x)^{N_k} \in U_k$ for all $k \in \mathbb{N}$. Since any open subset of T contains one of the sets U_k, the element $\exp(x)$ is a generator. □

Having the background at hand, we can now prove the main result about maximal tori:

Theorem 3.8. *Let G be a connected compact Lie group and $\mathbb{T} \subset G$ a maximal torus. Then any element of G is conjugate to an element of $\mathbb{T}$.*

Proof. Define a map $\varphi : G \times \mathbb{T} \to G$ by $(g,a) \mapsto gag^{-1}$. Since $\mathbb{T}$ is Abelian, we have $\varphi(gb,a) = \varphi(g,a)$ for all $g \in G$ and $a,b \in \mathbb{T}$. Thus there is a map $\psi : G/\mathbb{T} \times \mathbb{T} \to G$ such that $\varphi = \psi \circ (p \times \mathrm{id})$, which is smooth since $p : G \to G/\mathbb{T}$, and thus $p \times \mathrm{id}$, is a surjective submersion. Now $G/\mathbb{T} \times \mathbb{T}$ and G have the same dimension, so we can prove that the mapping degree of ψ is nonzero in order to show that ψ is surjective, which implies the claim.

Denoting by $\mathfrak{t} \subset \mathfrak{g}$ the Lie algebra of $\mathbb{T}$, we have $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{t}^\perp$ for some fixed ad–invariant inner product on $\mathfrak{g}$. Fix orientations on $\mathfrak{t}$ and $\mathfrak{t}^\perp$. This gives an orientation of $\mathfrak{g}$, so via left translations we obtain orientations of $\mathbb{T}$ and G. Since G and $\mathbb{T}$ are connected, and translating with e is the identity, all left and right translations as well as conjugations are orientation preserving diffeomorphisms.

The projection $p : G \to G/\mathbb{T}$ is a surjective submersion and $\ker(T_e p) = \mathfrak{t}$, so it induces a linear isomorphism $\mathfrak{t}^\perp \to T_o(G/\mathbb{T})$, where $o = e\mathbb{T}$. For $g \in G$, we obtain a linear isomorphism $T_o\ell_g : T_o(G/\mathbb{T}) \to T_{g\mathbb{T}}(G/\mathbb{T})$. Since $\hat{g}\mathbb{T} = g\mathbb{T}$ implies $\hat{g} = ga$ for some $a \in \mathbb{T}$ and $\mathbb{T}$ is connected, all these maps lead to the same orientation on $T_{g\mathbb{T}}(G/\mathbb{T})$. Hence we get an orientation on $G/\mathbb{T}$ such that all the maps ℓ_g are orientation preserving.

For $a \in \mathbb{T}$, the derivative $T_{(e,a)}\varphi : \mathfrak{g} \times T_a\mathbb{T} \to T_aG$ is given by $(X, R_Y(a)) \mapsto R_X(a) - L_X(a) + R_Y(a)$, which using Proposition 1.10 can be written as $R_{Y+X-\mathrm{Ad}(a)(X)}(a)$. By construction, the derivative $T_{(o,a)}\psi$ is given by the same formula when viewed as a map $\mathfrak{t}^\perp \times T_a\mathbb{T} \to T_aG$. Since $\mathrm{Ad}(a)(\mathfrak{t}) \subset \mathfrak{t}$, the map $\mathrm{Ad}(a)$ also preserves $\mathfrak{t}^\perp$, which shows that $\ker(T_{(o,a)}\psi) = \ker((\mathrm{id} - \mathrm{Ad}(a))|_{\mathfrak{t}^\perp})$.

We claim that for a generator a of $\mathbb{T}$, the map $\mathrm{id} - \mathrm{Ad}(a)$ restricts to an orientation preserving linear isomorphism on $\mathfrak{t}^\perp$. Indeed, if $Y \in \mathfrak{g}$ is such that $\mathrm{Ad}(a)(Y) = Y$, then of course $\mathrm{Ad}(a^k)(Y) = Y$ for all $k \in \mathbb{Z}$ and since $\{a^k : k \in \mathbb{Z}\}$ is dense in $\mathbb{T}$, we get $\mathrm{Ad}(b)(Y) = Y$ for all $b \in \mathbb{T}$. Putting $b = \exp(Z)$ we see that $[Y,Z] = 0$ for all $Z \in \mathfrak{t}$ and hence $Y \in \mathfrak{t}$ by maximality. Since $\mathbb{T}$ is connected, we must have $\mathrm{Ad}(a) \in SO(\mathfrak{t}^\perp)$. Now any such matrix can be brought to block diagonal forms which blocks of the form -1 ($+1$ is impossible since $\mathrm{id} - \mathrm{Ad}(a)$ is an isomorphism) or $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$. Since such a block contributes a factor 2 respectively $2(1-\cos\alpha)$ to the determinant of $\mathrm{id} - \mathrm{Ad}(a)$, the claim follows.

We have now shown that for each generator a of $\mathbb{T}$, the tangent map $T_{(o,a)}\psi$ is an orientation preserving linear isomorphism. By definition, $\psi \circ (\ell_g \times \mathrm{id}) = \mathrm{conj}_g \circ \psi$, which shows that for each $g \in G$ the tangent map $T_{(g\mathbb{T},a)}\psi : T_{g\mathbb{T}}(G/\mathbb{T}) \times T_a\mathbb{T} \to T_{gag^{-1}}G$ is an orientation preserving linear isomorphism. Now still for a generator a of $\mathbb{T}$ consider the preimage $\psi^{-1}(a)$. If $a = gbg^{-1}$ for some $g \in G$ and $b \in \mathbb{T}$, then $\mathrm{conj}_{g^{-1}}(a) \in \mathbb{T}$. Hence also $\mathrm{conj}_{g^{-1}}(a^k) \in \mathbb{T}$ for all $k \in \mathbb{Z}$ and by continuity $\mathrm{conj}_{g^{-1}}(T) \subset T$, which implies that $g \in N_G(T)$. But then also $b = \mathrm{conj}_{g^{-1}}(a)$ is a generator of T, so we see that for each point $(g\mathbb{T}, b) \in \psi^{-1}(a)$, the tangent map $T_{(g\mathbb{T},b)}\psi$ is an orientation preserving linear isomorphism, so a is a regular value and $\deg(\psi) > 0$. □

With a bit more care, one shows that actually $\deg(\psi)$ equals the order of the Weyl group W of $\mathbb{T}$.

3.9. Some applications. The basic conjugacy result from Theorem 3.8 has several immediate but surprisingly strong consequences for the structure of compact Lie groups:

COROLLARY 3.9. *Let G be a compact connected Lie group and $T \subset G$ a maximal torus.*
(1) Any connected Abelian subgroup of G is conjugate to a subgroup of T, and any maximal torus of G is conjugate to T.
(2) Any element of G lies in some maximal torus of G.
(3) The center $Z(G)$ is contained in any maximal torus of G.
(4) The exponential mapping $\exp : \mathfrak{g} \to G$ is surjective.
(5) For any $k \in \mathbb{Z} \setminus \{0\}$, the kth power map $g \mapsto g^k$ is surjective.

PROOF. (1) For a connected Abelian subgroup $S \subset G$ also the closure $\bar{S}$ is connected and Abelian. Since $\bar{S}$ is a torus, it has a generator a, and by the theorem we find an element $g \in G$ such that $\mathrm{conj}_g(a) \in T$. But then $\mathrm{conj}_g(a^k) \in T$ for all $k \in \mathbb{Z}$ and by continuity we get $\mathrm{conj}_g(\bar{S}) \subset T$. The statement on maximal tori is then evident.
(2) Given $g \in G$ we find an element $h \in G$ such that $\mathrm{conj}_h(g) \in T$. But then g lies in $\mathrm{conj}_{h^{-1}}(T)$, which evidently is a maximal torus in G.
(3) This is evident since for $g \in Z(G)$ and $h \in G$ we have $\mathrm{conj}_h(g) = g$.
(4) follows immediately from (2) since the exponential map for tori is surjective.
(5) This is clear from (4) since $\exp(X) = \exp(\frac{1}{k}X)^k$ for all $k \in \mathbb{Z} \setminus \{0\}$. □

Since any two maximal tori of G are conjugate, they in particular have the same dimension. This dimension is called the *rank* of the compact group G. It also follows easily that the Weyl groups associated to two maximal tori of G are isomorphic, so the Weyl group can be considered as an invariant of the compact Lie group G.

3.10. Remarks on representation theory and the Weyl character formula. Now we want to sketch applications of maximal tori to the theory of characters and hence to representation theory. Recall from Section 3.5 that the character χ of any finite dimensional representation is a class function, i.e. $\chi(ghg^{-1}) = \chi(h)$ for all $g, h \in G$. For a maximal torus $T \subset G$, part (1) of the corollary therefore implies that any such character is uniquely determined by its restriction to T. For a representation on a $\mathbb{K}$–vector space, this restriction is a smooth function $T \to \mathbb{K}$. The action of the Weyl group W of T on T is given by the restriction to T of the conjugation by an element $g \in N_G(T)$, so it follows that the restriction of any character to T is invariant under the action of W. With a bit more work, one can show that restriction to T actually defines a bijection between the set of continuous class functions on G and W–invariant continuous functions on T.

Any representation of G on V gives rise to a representation of T by restriction. As a representation of T, V is much easier to handle, since by Theorem 3.4 and Corollary

3.1 it is isomorphic to a direct sum of 1–dimensional representations. But evidently, the character of the representation of T on V is simply the restriction of the character of the G–representation. Using Corollary 3.5 we see from above that two complex representations of G are isomorphic if and only if their restrictions to T are isomorphic.

One can push this much further: A one-dimensional complex representation of T is given by a homomorphism $T \to \mathbb{C} \setminus \{0\}$. Such a homomorphism is uniquely determined by its derivative, which is a linear map $\mathfrak{t} \to \mathbb{C}$, i.e. an element of the dual $\mathfrak{t}^*$. This is a counterpart to the theory of weights for complex reductive Lie algebras. It turns out that an element of $\mathfrak{t}^*$ comes from a homomorphism $T \to \mathbb{C} \setminus \{0\}$ if and only if any element of ker(exp) is mapped to an integer multiple of $2\pi i$. Such functionals are called *analytically integral* and one shows that there is a basis of $\mathfrak{t}^*$ for which the analytically integral weights are exactly the linear combinations of the basis elements with integral coefficients. One denotes by $\mathfrak{t}^*_{\mathbb{R}}$ the real span of this basis.

The action of the (rather large) Weyl group W of T induces actions on $\mathfrak{t}$ and on $\mathfrak{t}^*_{\mathbb{R}}$ and the set of functionals obtained from a representation of G is invariant under this action. This invariance also refers to the multiplicity with which an irreducible representation of T occurs in the decomposition of V. To deal with this symmetry, one introduces the notion of *dominant* elements of $\mathfrak{t}^*_{\mathbb{R}}$. These elements form a closed cone $\mathcal{D}$ in $\mathfrak{t}^*_{\mathbb{R}}$ which meets each orbit of the Weyl group W and if the intersections of $\mathcal{D}$ with an orbit contains an interior point of $\mathcal{D}$, it consists of that point only. (So $\mathcal{D}$ is a fundamental domain for the action of W on $\mathfrak{t}^*_{\mathbb{R}}$.) This also gives rise to a total order on $\mathfrak{t}^*_{\mathbb{R}}$, so in particular for a representation V of G one can look at the largest functional that occurs. This is automatically dominant and integral and it is called the *highest weight* of V. It is rather easy to show that two irreducible representations of G with the same highest weight are isomorphic.

As mentioned above, there is also a Lie algebraic version of these notions in the setting of complex semisimple Lie algebras which is essentially equivalent to the setting of real compact Lie algebras. In many respects the resulting approach based on linear algebra is simpler than the approach via compact groups discussed here. There is a very important result, however, for which the two settings lead to very different proofs, which both are very interesting. This is the so-called Weyl character formula, which provides detailed information on the structure of irreducible representations and is one of the cornerstones of representation theory. Given a dominant integral functional λ, this formula describes what the character of an irreducible representation V with highest weight λ has to look like. (It turns out that such representations always exist, but this is not needed for the proof.)

The expression for the character obtained in the character formula is of relatively complicated nature and we will not discuss it here. However, in the setting of compact groups, there is a small but crucial input coming from integration theory. This is closely related to the ideas for the proof of Theorem 3.8, so we sketch it here. What one wants to exploit in the proof of the character formula is the fact that for an irreducible representation V of G the character χ_V satisfies $\int_G \chi_V \overline{\chi}_V = 1$. To combine this with the above considerations on a maximal torus T, one has to relate the integral over G to an integral over T, which is the content the *Weyl integration formula.*

This formula is based on the map $\psi : G/T \times T \to G$ from Theorem 3.8 which was defined by $\psi(gT, t) := gtg^{-1}$. As we have noted in 3.8, the mapping degree of ψ is given by $|W|$, which together with Fubini's theorem shows that for $f : G \to \mathbb{C}$, one gets $\int_G f \operatorname{vol}_G = \frac{1}{|W|} \int_T \int_{G/T} (f \circ \psi)\psi^* \operatorname{vol}_G$, where vol_G denotes the volume form of G from 3.4. To compute $\psi^* \operatorname{vol}_G$, one needs to compute the Jacobian of ψ, which is also essentially done in the proof of Theorem 3.8. The result is that in a point (gT, t), this Jacobian is given by $J(t) := \det((\mathbb{I} - \mathrm{Ad}(t^{-1}))|_{\mathfrak{t}^\perp})$. Since this depends only on t and not on gT, we see that it can be taken out of the integral over G/T, so for that part we only get an integral of the form $\int_{G/T} f(gtg^{-1})d(gT)$. In particular, if f is a class function, then this will simply give a constant that can then be taken out of the integral over T. Hence in the case of a class function (which we are interested in), we have successfully expressed the integral over G as an integral over T. A second important step is that $J(t)$ can be explicitly expressed using specific elements of $\mathfrak{t}^*$ coming from the adjoint representation of $\mathfrak{g}$, but we will not go into that.

The Peter–Weyl theorem

The final circle of ideas about compact groups we will discuss is centered around the set of matrix coefficients of finite dimensional complex representations.

3.11. Peter–Weyl theorem for matrix groups. Basically, the Peter–Weyl theorem can be thought of as a generalization of the theory of Fourier series. The first result in this direction is Weierstrass' theorem, which states that any real valued continuous function on $\mathbb{R}$ which is periodic with period 2π can be uniformly approximated by linear combinations of the functions $t \mapsto \sin(kt)$ and $t \mapsto \cos(kt)$ with $k \in \mathbb{Z}$. More conceptually and slightly simpler, we can look at continuous complex valued functions on the unit circle $S^1 \subset \mathbb{C}$. Then Weierstrass' result states that these can be uniformly approximated by functions of the form $\sum_{k=-N}^{N} a_k z^k$ with $a_k \in \mathbb{C}$.

To get a connection to representation theory, let us view S^1 as $U(1)$. Since this is a compact commutative group, we know from 3.1 and 3.4 that any finite dimensional representation of $U(1)$ is a direct sum of one–dimensional representations. A one dimensional representation simply is a homomorphism $\varphi : U(1) \to \mathbb{C} \setminus \{0\}$. The derivative of such a homomorphism is a homomorphism $\varphi' : i\mathbb{R} \to \mathbb{C}$ so it must be of the form $\varphi'(it) = tz_0$ for some element $z_0 \in \mathbb{C}$. For the commutative groups $U(1)$ and $\mathbb{C} \setminus \{0\}$, the exponential map is simply the usual exponential. Using $\varphi(\exp(it)) = \exp(\varphi'(it))$ for $t = 2\pi$ we see that $\exp(2\pi z_0) = 1$, which implies that $z_0 \in i\mathbb{Z}$. On the other hand, the map $z \mapsto z^k$ for $k \in \mathbb{Z}$ evidently defines a homomorphism $U(1) \to \mathbb{C} \setminus \{0\}$ with derivative $it \mapsto ikt$, so these exhaust all possible homomorphisms. Hence the matrix coefficients of one–dimensional representations of $U(1)$ are exactly the complex multiples of the functions $z \mapsto z^k$ and the matrix coefficients of general finite dimensional representations are exactly the expressions $z \mapsto \sum_{k=N_0}^{N_1} a_k z^k$ with $N_0, N_1 \in \mathbb{Z}$ and $a_k \in \mathbb{C}$.

Hence we can view Weierstrass' theorem as the statement that any complex valued continuous function on $U(1)$ can be uniformly approximated by matrix coefficients of finite dimensional complex representations.

The analogous statement for compact matrix groups can be proved using the abstract Stone–Weierstrass theorem from topology: Recall that for a compact space X, the vector space $C(X) = C(X, \mathbb{C})$ of complex valued continuous functions on X naturally is a complex Banach space under the norm $\|f\| := \sup_{x \in X} |f(x)|$. The abstract Stone–Weierstrass theorem deals with subalgebras $\mathcal{A} \subset C(X)$, i.e. linear subspaces which are also closed under the pointwise product of functions. The theorem states that a subalgebra $\mathcal{A} \subset C(X)$ is *dense*, i.e. the closure $\bar{\mathcal{A}}$ is all of $C(X)$ provided that

(i) all constant functions belong to $\mathcal{A}$
(ii) for $f \in \mathcal{A}$, the conjugate $\bar{f}$ lies in $\mathcal{A}$
(iii) $\mathcal{A}$ separates points, i.e. for $x, y \in X$ with $x \neq y$, there is an $f \in \mathcal{A}$ such that $f(x) \neq f(y)$

To apply this theorem, we need one more construction. From 3.5 we know that a finite dimensional representation of G on V gives rise to the dual representation on V^* which is defined by $(g \cdot \lambda)(v) := \lambda(g^{-1} \cdot v)$ for $\lambda \in V^*$ and $v \in V$. Having given two finite dimensional representations of G on V and W, we define a representation on $L(V, W)$ by $(g \cdot f)(v) = g \cdot (f(g^{-1} \cdot v))$. One immediately verifies that this indeed is a representation. The corresponding representation of the Lie algebra $\mathfrak{g}$ of G is determined by $(X \cdot f)(v) := X \cdot (f(v)) - f(X \cdot v)$.

Putting these two constructions together, we see that representations on V and W give rise to a representation on $V \otimes W := L(V^*, W)$, called the *tensor product* of the representations V and W. For $v \in V$ and $w \in W$ we define $v \otimes w \in L(V^*, W)$ by $(v \otimes w)(\lambda) := \lambda(v)w$. Then $(v, w) \mapsto v \otimes w$ defines a bilinear map $V \times W \to V \otimes W$. One easily verifies that for bases $\{v_1, \dots, v_n\}$ of V and $\{w_1, \dots, w_m\}$ of W, the set $\{v_i \otimes w_j\}$ forms a basis for $V \otimes W$. Using this, one easily verifies that for any vector space Z and bilinear map $\varphi : V \times W \to Z$ there is a unique linear map $\tilde{\varphi} : V \otimes W \to Z$ such that $\varphi(v, w) = \tilde{\varphi}(v \otimes w)$. This is called the universal property of the tensor product of the vector spaces V and W.

Now let us compute what the natural representation on $V \otimes W$ looks like. We compute

$$(g \cdot (v \otimes w))(\lambda) = g \cdot (v \otimes w(g^{-1} \cdot \lambda)) = \lambda(g \cdot v)(g \cdot w) = ((g \cdot v) \otimes (g \cdot w))(\lambda),$$

so the representation is characterized by $g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w)$. The corresponding infinitesimal representation is given by $X \cdot (v \otimes w) = (X \cdot v) \otimes w + v \otimes (X \cdot w)$.

Finally assume that V and W are endowed with Hermitian inner products, which we both denote by $\langle\ ,\ \rangle$. Then one immediately verifies that

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle := \langle v_1, v_2 \rangle \langle w_1, w_2 \rangle$$

defines a Hermitian inner product on $V \otimes W$. This has the property that for orthonormal bases $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_m\}$ the set $\{v_i \otimes w_j\}$ defines an orthonormal basis for $V \otimes W$. Now for the matrix coefficients of $V \otimes W$ we obtain

$$\langle g \cdot (v_1 \otimes w_1), v_2 \otimes w_2 \rangle = \langle g \cdot v_1, v_2 \rangle \langle g \cdot w_1, w_2 \rangle,$$

so this is the product of a matrix coefficient of V with a matrix coefficient of W. Using this we can now prove:

THEOREM 3.11 (Peter–Weyl theorem for matrix groups). *Let $G \subset GL(n, \mathbb{R})$ be a compact matrix group. Then the set $\mathcal{A}$ of all matrix coefficients of finite dimensional representations of G is dense in $C(G, \mathbb{C})$.*

PROOF. A linear combination of a matrix coefficient of V and a matrix coefficient of W is a matrix coefficient of $V \oplus W$, so $\mathcal{A} \subset C(G)$ is a linear subspace. The product of a matrix coefficient of V and one of W is a matrix coefficient of $V \otimes W$, so $\mathcal{A}$ even is a subalgebra. For the trivial representation $G \to \mathbb{C}^*$, $g \mapsto 1$, the matrix coefficients are the constant functions, so $\mathcal{A}$ contains constants.

For a representation of G on V, an invariant inner product $\langle\ ,\ \rangle$ gives a conjugate linear isomorphism $V \to V^*$. This immediately implies that the conjugate of a matrix coefficient of V is a matrix coefficient of V^*, so $\mathcal{A}$ is closed under conjugation. Finally, the inclusion $G \to GL(n, \mathbb{R}) \subset GL(n, \mathbb{C})$ defines a representation of G on $\mathbb{C}^n$. Taking an invariant Hermitian inner product on $\mathbb{C}^n$ and using matrices with respect to an orthonormal basis for this inner product, we obtain an injective homomorphism $G \to U(n)$. The matrix entries of the image are matrix coefficients of the corresponding representation on $\mathbb{C}^n$, which implies that $\mathcal{A}$ separates points. Now the result follows from the Stone–Weierstrass theorem. $\square$

3.12. The general result. The proof of the Peter–Weyl theorem for general compact groups needs a deeper result from functional analysis, namely the so–called Hilbert–Schmidt theorem. Consider a compact topological space X and the Banach space $C(X) = C(X, \mathbb{C})$ as before. Choosing a Borel measure on X, one can integrate continuous functions and then form spaces like $L^p(X)$. In particular, one has the Hilbert space $L^2(X)$. Since on the compact space X any continuous function is square integrable, we have an inclusion $C(X) \hookrightarrow L^2(X)$, which is bounded for the usual norms on the two spaces.

We have to study a class of linear operators which are defined analogously to matrices in finite dimensions. Namely, consider a continuous function $K : X \times X \to \mathbb{C}$ (called the *kernel* of the operator) and define an operator T on functions by $Tf(x) := \int_X K(x,y)f(y)dy$. If f is continuous, then so is Tf, so we can interpret T as an (evidently bounded) linear operator on $C(X)$. However, much more is true. It is easy to show that for $f \in L^2(X)$ the function Tf is continuous and even the family $\{Tf : f \in L^2(X), \|f\|_2 \leq 1\}$ is equicontinuous. Therefore we can interpret T as a compact linear operator $L^2(X) \to C(X)$ and via the inclusion also as a compact operator on $L^2(X)$. If one in addition assumes that K is *Hermitian*, i.e. the $K(y,x) = \overline{K(x,y)}$, then T is self adjoint as an operator on the Hilbert space $L^2(X)$, so one can apply the spectral theorem.

Notice that for an eigenfunction f of T, we by definition have $Tf = \lambda f$ for some number λ, so since T has values in $C(X)$, eigenfunctions are automatically continuous. Now the Hilbert–Schmidt theorem states the following:

- Any eigenvalue of T is real and the set of eigenvalues is finite or countable with zero as the only possible accumulation point.
- Ordering the eigenvalues as $|\lambda_1| \geq |\lambda_2| \geq \dots$ one has $\sum_j |\lambda_j|^2 < \infty$, so T is a *Hilbert–Schmidt operator*. In particular, any eigenspace of T is finite dimensional.
- Any continuous function in the image of T can be uniformly approximated by a linear combination of eigenfunctions.

Using this background, we can now obtain the general Peter–Weyl theorem:

Theorem 3.12 (Peter–Weyl). *Let G be a compact Lie group. The the space of matrix coefficients of finite dimensional representations of G is dense in $C(G)$.*

Proof. We have to show that for each continuous function $f_0 : G \to \mathbb{C}$ we can find a linear combination of matrix coefficients which is uniformly close to f_0.
Claim: For any $\epsilon > 0$, there is an open neighborhood U of e in G such that for $g, h \in G$ such that $g^{-1}h \in U$ we have $|f_0(g) - f_0(h)| < \epsilon$.

For each $x \in G$ we find a neighborhood U_x of x in G such that $|f_0(y) - f_0(x)| < \frac{\epsilon}{2}$ for all $y \in U_x$. Then $\lambda_{x^{-1}}(U_x)$ is a neighborhood of e in G and by continuity of multiplication we find a neighborhood V_x of e in G such that $ab \in \lambda_{x^{-1}}(U_x)$ for all $a, b \in V_x$. Then $\lambda_x(V_x)$ is a neighborhood of x in G. By compactness, we find finitely many points $g_1, \dots, g_N \in G$ such that $G = \lambda_{g_1}(V_{g_1}) \cup \dots \cup \lambda_{g_N}(V_{g_N})$ and we consider the neighborhood $U := V_{g_1} \cap \dots \cap V_{g_N}$ of e in G. If $g, h \in G$ are such that $g^{-1}h \in U$, then there is an index $j \in \{1, \dots, N\}$ such that $g \in \lambda_{g_j}(V_{g_j})$, i.e. $g_j^{-1}g \in V_{g_j}$. Since $g^{-1}h \in V \subset V_{g_j}$ we conclude that $g_j^{-1}h \in \lambda_{g_j^{-1}}(U_{g_j})$, i.e. $h \in U_{g_j}$. By construction $V_{g_j} \subset \lambda_{g_j^{-1}}(U_{g_j})$, so we also have $g \in U_{g_j}$. But then

$$|f_0(g) - f_0(h)| \leq |f_0(g) - f_0(g_j)| + |f_0(g_j) - f_0(h)| < \epsilon,$$

and the claim follows.

Using a partition of unity, we can construct a smooth function $\mu : G \to \mathbb{R}$ with support contained in U such that $\mu(g) \geq 0$ for all $g \in G$ and $\mu(e) > 0$. Without loss of generality, we may assume that $\mu(g) = \mu(g^{-1})$ for all $g \in G$. Now $\int_G \mu(x)dx > 0$, so rescaling μ we may assume that $\int_G \mu(x)dx = 1$. Define $K : G \times G \to \mathbb{C}$ by $K(x,y) := \mu(x^{-1}y)$. This is continuous, real valued and symmetric and thus Hermitian, and we consider the corresponding integral operator $Tf(x) := \int_G K(x,y)f(y)dy$ on $C(G)$. By construction $Tf_0(x) = \int_G \mu(x^{-1}y)f_0(y)dy$. The product $\mu(x^{-1}y)f_0(y)$ vanishes unless $x^{-1}y \in U$ and if $x^{-1}y \in U$ then $|f_0(y) - f_0(x)| < \epsilon$. Hence we conclude that

$$|\mu(x^{-1}y)f_0(y) - \mu(x^{-1}y)f_0(x)| < \epsilon\mu(x^{-1}y)$$

for all $x, y \in G$. Now $\int_G \mu(x^{-1}y)f_0(y)dy = Tf_0(x)$ by definition, while

$$\int_G \mu(x^{-1}y)f_0(x)dy = f_0(x)\int_G \mu(x^{-1}y)dy = f_0(x).$$

In the same way, integrating the right hand side we obtain ϵ, so we conclude that $|Tf_0(x) - f_0(x)| < \epsilon$, so f_0 is uniformly close to an element in the image of T. By the Hilbert–Schmidt theorem, any element of this image can be uniformly approximated by linear combinations eigenfunctions of T, so we can complete the proof by relating eigenfunctions of T to matrix coefficients.

Consider the map $G \times C(G) \to C(G)$, $(g, f) \mapsto g \cdot f$, which is defined by $(g \cdot f)(h) := f(g^{-1}h)$. One immediately verifies that this defines a left action of G on the space $C(G)$. For fixed $f \in C(G)$ consider a neighborhood U of e in G as constructed for f_0 above. For $\hat{g} \in \lambda_g(\nu(U))$ and $\hat{f} \in C(G)$ with $\|\hat{f} - f\| < \epsilon$, we then compute

$$|\hat{f}(\hat{g}^{-1}h) - f(g^{-1}h)| \leq |\hat{f}(\hat{g}^{-1}h) - f(\hat{g}^{-1}h)| + |f(\hat{g}^{-1}h) - f(g^{-1}h)| < 2\epsilon,$$

since by construction $(g^{-1}h)(\hat{g}^{-1}h)^{-1} = g^{-1}\hat{g} \in \nu(U)$ and hence $(\hat{g}^{-1}h)(g^{-1}h)^{-1} \in U$. But this says that we have actually defined a continuous left action of G on the topological space $C(G)$, and of course for $g \in G$ the map $f \mapsto g \cdot f$ is linear.

Observe that the kernel K of the operator $T : C(G) \to C(G)$ from above is defined by $K(x,y) = \mu(x^{-1}y)$ for $x, y \in G$. In particular, this shows that $K(gx, gy) = K(x,y)$ for all $g, x, y \in G$. Using this, we compute

$$\begin{aligned}(g \cdot Tf)(x) = Tf(g^{-1}x) = \int_G K(g^{-1}x, y)f(y)dy = \\ = \int_G K(x, gy)f(y)dy = \int_G K(x,y)f(g^{-1}y)dy = T(g \cdot f)(x),\end{aligned}$$

where we have used invariance of the integral in the last but one step. But this shows that $T : C(G) \to C(G)$ is equivariant for the G–action on $C(G)$. If $f \in C(G)$ is an eigenfunction with eigenvalue λ, then we get $T(g \cdot f) = g \cdot (Tf) = g \cdot (\lambda f) = \lambda(g \cdot f)$, so any eigenspace of T is a G–invariant subspace. Any such eigenspace V is finite dimensional by the Hilbert–Schmidt theorem, so we obtain a finite dimensional continuous representation of G on V. But this is a smooth representation of G by Corollary 1.8. Fix an invariant inner product $\langle\ ,\ \rangle$ on V. Since the map $f \mapsto f(e)$ is linear, we find an element $w \in V$ such that $\langle f, w\rangle = f(e)$. But then the corresponding matrix coefficient is given by $g \mapsto \langle g \cdot f, w\rangle = (g \cdot f)(e) = f(g^{-1})$.

Putting together what we have proved so far, we see that f_0 is uniformly close to a linear combination of functions of the form $\varphi \circ \nu$, where φ is a matrix coefficient of a finite dimensional representation of G. Applying this to $f_0 \circ \nu$, the result follows. □

As an immediate corollary, we can show that any compact Lie group is isomorphic to a matrix group:

Corollary 3.12. *Any compact Lie group G is isomorphic to a closed subgroup of a unitary group $U(N)$ for sufficiently large N.*

Proof. By the theorem, the matrix coefficients of finite dimensional representations of G form a dense subset of $C(G)$. If $g \in G$ is such that $\varphi(g) = \varphi(e)$ for all matrix coefficients, then the same property holds for all $f \in C(G)$, so $g = e$. This implies that for each $e \neq g \in G$ there is a representation $\varphi_g : G \to GL(V)$ such that $\varphi_g(g) \neq \mathrm{id}$. Choose an arbitrary element $g_1 \in G$ and consider the representation φ_{g_1}. Then $\ker(\varphi_{g_1})$ is a closed subgroup of G and hence a compact Lie group. If this group is nontrivial, then choose $e \neq g_2 \in \ker(\varphi_{g_1})$ and consider $\varphi_{g_1} \oplus \varphi_{g_2}$. The kernel of this representation is a proper subgroup of $\ker(\varphi_1)$ so it must either have smaller dimension or the number of connected components (which is always finite by compactness) must be smaller. But this shows that in finitely many steps we obtain a representation $\varphi : G \to GL(V)$ which is injective. Taking an invariant inner product on V and matrices with respect to an orthonormal basis, we obtain an injective homomorphism $\varphi : G \to U(\dim(V))$. Since G is compact, this must be an isomorphism onto a closed subgroup. □

3.13. Infinite dimensional representations. Up to now, we have only considered representations of Lie groups on finite dimensional vector spaces, but for some purposes this concept is too restrictive. Let us illustrate this on two examples: On the one hand, many non–compact groups do not admit finite dimensional unitary representations. Consider the group $G := SL(2,\mathbb{R})$ and suppose that we have a homomorphism $\varphi : G \to U(n)$ for some $n \in \mathbb{N}$. For the matrix $E := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ in the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ we get $\exp(tE) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. For $t \neq 0$ all these matrices are conjugate, since they all have 1 as an eigenvalue with multiplicity 2 but are not diagonalizable. Hence the elements $\varphi(\exp(tE)) = \exp(t\varphi'(E))$ for $t \neq 0$ are conjugate in $U(n)$. However, the conjugacy class of h in $U(n)$ is the image of $U(n)$ under the continuous map $g \mapsto ghg^{-1}$, and hence it is compact and therefore closed in $U(n)$. But this implies that $\varphi(\exp(tE))$ must be in the same conjugacy class for all $t \in \mathbb{R}$, which implies that $\varphi(\exp(tE)) = \mathrm{id}$ for all $t \in \mathbb{R}$. Hence $\varphi'(E) = 0$ so the kernel of φ' is an ideal in $\mathfrak{g}$ which contains E. It is a trivial exercise to show that such an ideal must be all of $\mathfrak{g}$, and thus φ is trivial. On the other hand, there are many nontrivial unitary representations of G on infinite dimensional Hilbert spaces. The simplest example of such a representation is provided by the *left regular representation* of G. Using a nonzero 3–linear alternating map $\mathfrak{g}^3 \to \mathbb{R}$, one obtains as in 3.4, a left invariant 3–form on G. Via this, one obtains a left invariant integral $\int_G : C_c^\infty(G,\mathbb{C}) \to \mathbb{C}$ defined on compactly supported smooth functions. Then one defines $L^2(G)$ to be the completion of $C_c^\infty(G,\mathbb{C})$ with respect to the norm $\|f\|_2 := \int_G |f(g)|^2 dg$. Then $L^2(G)$ is a Hilbert space and one defines a representation of G on $L^2(G)$ by $(g \cdot f)(x) = f(g^{-1}x)$, i.e. $g \cdot f = f \circ \lambda_{g^{-1}}$. Invariance of the integral immediately implies that this representation is unitary.

The second example shows that, more drastically, there are Lie groups which do not have any interesting finite dimensional representations. The example we discuss is the so called *Heisenberg group* $\mathcal{H}$. Consider the group of all 3×3 matrices of the form $\begin{pmatrix}1 & a & c\\ 0 & 1 & b\\ 0 & 0 & 1\end{pmatrix}$. Let us write this matrix as $(\binom{a}{b}, c)\in\mathbb{R}^2\oplus\mathbb{R}$. In this picture, the product is given by

$$(\begin{pmatrix}a\\ b\end{pmatrix}, c)\cdot(\begin{pmatrix}u\\ v\end{pmatrix}, w) = (\begin{pmatrix}a+u\\ b+v\end{pmatrix}, c+w+av).$$

In partiuclar, $(\binom{a}{b}, c)^{-1} = (\binom{-a}{-b}, ab-c)$. The elements of the form $(\binom{0}{0}, c)$ form a subgroup which is contained in the center. In particular, the elements which in addtion have $c\in\mathbb{Z}$ form a normal subgroup and $\mathcal{H}$ is defined as the quotient group. Denoting elements in the quotient group as $(\binom{a}{b}, c+\mathbb{Z})$ the product is given as above, but computing modulo $\mathbb{Z}$ in the second factor.

The elements of the form $(\binom{0}{0}, c+\mathbb{Z})$ form a subgroup in $\mathcal{H}$ which is contained in the center $Z(\mathcal{H})$ and isomorphic to $\mathbb{R}/\mathbb{Z}\cong U(1)$. It is easy to see that this subgroup coincides with $Z(\mathcal{H})$. Mapping $(\binom{a}{b}, c+\mathbb{Z})$ to $\binom{a}{b}$ induces an isomorphism $\mathcal{H}/Z(\mathcal{H})\to\mathbb{R}^2$. The group $\mathcal{H}$ itself is not commutative, however. For example,

$$(\begin{pmatrix}a\\ 0\end{pmatrix}, 0)\cdot(\begin{pmatrix}0\\ 1\end{pmatrix}, 0)\cdot(\begin{pmatrix}-a\\ 0\end{pmatrix}, 0)\cdot(\begin{pmatrix}0\\ -1\end{pmatrix}, 0) = (\begin{pmatrix}0\\ 0\end{pmatrix}, 2a+\mathbb{Z}),$$

which shows that any element of $Z(\mathcal{H})$ can be written as a commutator. A moment of thought shows that $Z(\mathcal{H}) = [\mathcal{H},\mathcal{H}]$.

Now suppose that $\varphi:\mathcal{H}\to GL(V)$ is a finite dimensional complex representation. We can restrict the representation to the subgroup $Z(\mathcal{H}) = U(1)$ and then decompose into irreducibles. This means that there are finitely many different elements $k_1,\dots,k_n\in\mathbb{Z}$ and V splits as $V = V_1\oplus\dots\oplus V_n$ such that $z\in U(1)$ acts on V_i by multiplication by z^{k_i}. But since $U(1)$ is contained in the center of $\mathcal{H}$, each of the subspaces V_i is $\mathcal{H}$–invariant. Let us restrict to one the components V_i. Then $\varphi_i(z)\in GL(V_i)$ can be written as a commutator since z is a commutator in $\mathcal{H}$. But this implies that $\det(\varphi_i(z)) = 1$ for all z which is only possible if $k_i = 0$. Hence any finite dimensional representation of $\mathcal{H}$ is trivial on the subgroup $U(1)$ and factorizes through the commutative quotient $\mathbb{R}^2$. On the other hand, one shows that the group $\mathcal{H}$ has injective unitary representations on a separable Hilbert space.

For infinite dimensional representations, purely algebraic concepts are not sufficient, so one has to work with topological vector spaces. This can be done on various levels of generality and for our purposes the amount of generality is of minor importance. We will restrict our attention to representations on Banach spaces, but everything we do carries over to complete locally convex vector spaces. The main property we need from a space V is that for a compact Lie group G endowed with its volume form as in 3.4, there is a well defined integral for V–valued continuous functions on G. This has the property that if $f:G\to V$ has values in some ball, then also $\int_G f(g)dg$ lies in that ball. Of course, this integral still is left invariant, i.e. $\int_G f(hg)dg = \int_G f(g)dg$ for all $h\in G$.

Given a Lie group G and a Banach space V, one defines a *representation* of G on V as a continuous left action $G\times V\to V$ such that for each $g\in G$ the map $v\mapsto g\cdot v$ is linear. Of course, for a morphism between two representations, one does not only require equivariancy but also continuity.

In the proof of Theorem 3.12 we have seen that for any compact Lie group G, one obtains a representation in that sense on the Banach space $C(G,\mathbb{C})$ by defining $(g\cdot f)(h) := f(g^{-1}h)$. The same method can be used to define representations of a general Lie group G on various spaces of complex valued functions on G respectively on a homogeneous space G/H of G. A particularly important case is provided by Hilbert spaces of L^2–functions in which one automatically obtains unitary operators by invariance of the integral. To work with infinite dimensional representations often needs quite a lot of functional analysis.

Generalizing notions from finite dimensional representations to the infinite dimensional case, one usually has to replace "subspace" by "closed subspace". This is necessary for various reasons. For example, a subspace of a Banach space is complete if and only if it is closed, or for a subspace W of a Hilbert space H we have $H = W \oplus W^\perp$ if and only if W is closed, and so on. Hence one defines a representation to be indecomposable if it cannot be written as the direct sum of two closed invariant subspaces and to be irreducible if it does not admit non–trivial closed invariant subspaces. Likewise, complete reducibility is defined as the fact that any closed invariant subspace admits an invariant complement. Note that unitary representations on Hilbert spaces always have these property, so for unitary representations indecomposability is equivalent to irreducibility.

Even though there is a simple notion of smoothness in Banach spaces, one has to work with continuous rather than smooth representations, simply because many of the representations that come up canonically are not smooth. Consider for example the representation of a compact group G on $C(G,\mathbb{C})$ from above. The evaluation in e defines a continuous linear and hence smooth map $\mathrm{ev}_e : C(G,\mathbb{C}) \to \mathbb{C}$. Now fix a function $f \in C(G,\mathbb{C})$ and consider the map $g \mapsto \mathrm{ev}_e(g \cdot f)$. By definition, this is given by $g \mapsto f(g^{-1})$, so it equals $f \circ \nu$. Hence we see that the partial mapping $g \mapsto g \cdot f$ can only be smooth if f itself is smooth. Notice however, that the space $C^\infty(G,\mathbb{C})$ forms a dense subspace of $C(G,\mathbb{C})$.

This example also shows that passing from a representation of a Lie group G to a representation of its Lie algebra $\mathfrak{g}$ is much more subtle in infinite dimensions. The natural idea to obtain an action of the Lie algebra is by restricting representations to one–parameter subgroups and then differentiating. As we see from above, one parameter subgroups do not act differentiably in general, so the Lie algebra cannot act on the whole representation space. However, in the special case above, we do obtain a representation of $\mathfrak{g}$ on the dense subspace $C^\infty(G,\mathbb{C}) \subset C(G,\mathbb{C})$. This representation is explicitly given by

$$(X \cdot f)(g) = \tfrac{d}{dt}|_{t=0}(\exp(tX) \cdot f)(g) = \tfrac{d}{dt}|_{t=0}f(\exp(-tX)g) = (R_{-X} \cdot f)(g),$$

so we have to differentiate functions using right invariant vector fields. The fact that this is a representation of $\mathfrak{g}$ (on an infinite dimensional space) follows immediately from the fact that $[R_{-X}, R_{-Y}] = R_{-[X,Y]}$, see 1.6.

It turns out that general infinite dimensional representations behave similarly. Starting from a representation of G, one obtains a representation of the Lie algebra $\mathfrak{g}$ on a dense subspace. This can be used to analyze infinite dimensional representations using Lie algebra techniques.

3.14. Infinite dimensional representations of compact Lie groups. As a final topic in this chapter, we want to use the Peter–Weyl theorem to show that for compact Lie groups infinite dimensional representations are not much more difficult to handle than finite dimensional ones.

Definition 3.14. Consider a continuous representation of a Lie group G on a complex Banach space V.

(1) An element $v \in V$ is called *G–finite* if it is contained in a finite dimensional G–invariant subspace of V. The linear subspace of V spanned by all G–finite vectors is denoted by V^{fin}.
(2) For a finite dimensional complex irreducible representation W of G, we define the *W–isotypical component* $V_W \subset V$ to as the space of all $v \in V$ for which there is a morphism $\varphi : W \to V$ with $v \in \operatorname{im}(\varphi)$.

Note that for finite dimensional W, any linear map $W \to V$ is automatically continuous. Since morphisms from W to V form a vector space, we see immediately that the isotypical component V_W is a linear subspace of V. For $v \in V_W$, the image of the corresponding morphism φ is a finite dimensional G–invariant subspace of V containing v, so $V_W \subset V^{fin}$.

In the special case of a compact Lie group G, there also is a converse to this result. For $v \in V^{fin}$ there is a finite dimensional G–invariant subspace $W \subset V$ containing v. By Theorem 3.4, W splits into a direct sum of irreducible representations, which implies that v can be written as a linear combination of vectors contained in some isotypical component. This implies that for compact G we get $V^{fin} = \oplus_{W\in\hat{G}} V_W$, where $\hat{G}$ denotes the set of all isomorphism classes of finite dimensional irreducible representations of G and we consider the algebraic direct sum, i.e. the subspace spanned by all finite linear combinations of elements of the subspaces V_W.

THEOREM 3.14. *For any representation of a compact Lie group G on a complex Banach space V, the subspace $V^{fin} \subset V$ of G–finite vectors is dense in V.*

PROOF. The main tool for this proof is an extension of the action of G on V to an action of $C(G, \mathbb{C})$. This technique is referred to as "smearing out the action" by physicists. So let $f : G \to \mathbb{C}$ be a continuous function and for $v \in V$ consider the map $G \to V$ defined by $h \mapsto f(h)(h \cdot v)$. Evidently, this is continuous, so we can define $T_f(v) := \int_G f(h)h \cdot vdh \in V$.

Given $v_0 \in V$, continuity of the action of G on V implies that we can find a neighborhood U of e in G, such that $\|g\cdot v_0 - v_0\| < \frac{\epsilon}{2}$ for all $g \in U$. As in the proof of Theorem 3.12 we find a smooth function $\mu : G \to \mathbb{R}$ with non–negative values and support contained in U such that $\int_G \mu(h)dh = 1$. As in that proof, we next get the estimate that $\|\mu(h)h \cdot v_0 - \mu(h)v_0\| < \frac{\epsilon}{2}\mu(h)$ and integrating both sides shows that $\|T_\mu(v_0) - v_0\| < \frac{\epsilon}{2}$.

On the other hand, one easily verifies that $g \cdot T_f(v) = \int_G f(h)gh \cdot vdh$, which by invariance of the integral equals $\int_G f(g^{-1}h)h \cdot vdh = T_{g\cdot f}(v)$. In particular, if $f \in C(G, \mathbb{C})^{fin}$, then we have $T_f(v) \in V^{fin}$. Now certainly any linear combination of matrix coefficients lies in $C(G, \mathbb{C})^{fin}$, so by the Peter–Weyl theorem 3.12 we find a function $f \in C(G, \mathbb{C})^{fin}$ arbitrarily close to μ. Since $g \mapsto \|g \cdot v_0\|$ is continuous and G is compact, there is an $M \in \mathbb{R}$ such that $\|g \cdot v_0\| < M$ for all $g \in G$, and we choose $f \in C(G, \mathbb{C})^{fin}$ in such a way that $\|f - \mu\| < \frac{\epsilon}{2M}$. Then $f(h)h \cdot v_0 - \mu(h)h \cdot v_0$ lies in the ball of radius $\frac{\epsilon}{2}$ around zero for all $h \in G$, so integrating shows that $\|T_f(v_0) - T_\mu(v_0)\| < \frac{\epsilon}{2}$. Together with the above, we see that $T_f(v_0) \in V^{fin}$ is ϵ–close to v_0. □

COROLLARY 3.14. *For a compact Lie group G, any irreducible representation on a Banach space is finite dimensional.*

PROOF. This evidently follows from the theorem since for an infinite dimensional irreducible representation all isotypical components and hence the space of finite vectors would be zero. □

Remark 3.14. (1) The tools introduced in the proof of the theorem are very important in representation theory of general Lie groups. If G is a general Lie group, then for representations of G on V smearing out the action cannot be done with continuous functions, but rather one uses L^1–functions. These form a Banach space $L^1(G)$, which can be made into a Banach algebra by using the *convolution*, defined by $(f_1 * f_2)(g) := \int_G f_1(h)f_2(h^{-1}g)dh$. The above construction then makes V into a module over the convolution algebra. The convolution algebra has no identity element (which would be the delta distribution in e) but one can approximate the identity by L^1–functions, which is the technique used in the proof.
(2) The theorem itself is also very useful for representations of general Lie groups. Starting with a general Lie group G, one often has a nice maximal compact subgroup $K \subset G$. Prototypical examples are $SU(n) \subset SL(n, \mathbb{C})$, and $SO(n) \subset SL(n, \mathbb{R})$. Given a representation of G on V, one may first analyze V as a representation of K. By the theorem, this mainly amounts to understanding the isotypical components, which in turn means finding the *multiplicities* of finite dimensional irreducible representations W in V, i.e. the dimensions of the spaces of G–equivariant maps from W to V.

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