Projective compactness

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IHP, Paris, December 10, 2014
This is a survey talk based on joint work with A. Rod Gover (Auckland).
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The model case for all this is hyperbolic space with the sphere as its boundary at infinity. The Klein model of hyperbolic space suggests an interpretation this model in terms of projective differential geometry. This admits curved analogs via reduction of projective holonomy.
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These examples motivate the general definition of projective compactness, which involves an additional parameter called the *order* of projective compactness. This is a concept for affine connections on the interior of a manifold with boundary and, via the Levi–Civita connection, for pseudo–Riemannian metrics.
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The concept of projective compactness

Examples from holonomy reductions

More on projectively compact metrics
The most important classical example of a boundary at infinity is provided by hyperbolic space. In the simplest picture, *hyperbolic space* $\mathcal{H}^{n+1}$ of dimension $n+1$ is realized as the open unit ball $B \subset \mathbb{R}^{n+1}$ endowed with the Riemannian metric $g := \frac{1}{(1-r^2)^2} ds^2$, which has constant negative sectional curvature. Here $r$ is the Euclidean distance to the origin and $ds^2$ is the flat Riemannian metric on $\mathbb{R}^{n+1}$. 
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- Viewed from outside, however, it is very natural to attach the unit sphere $S^n$ as a “boundary at infinity” to $\mathcal{H}^{n+1}$.

The group of isometries of $\mathcal{H}^{n+1}$ is $SO_0(n + 1, 1)$, which coincides with the group of conformal isometries of the round metric on the boundary sphere. Taking into account only the action of this group, it is however hard to see, how the boundary is attached to the interior.
One interpretation of this situation is that the conformal class on $S^n$ is obtained from conformal rescalings of the hyperbolic metric which extend to the boundary. Generalizing this, one obtains R. Penrose’s concept of conformal compactness. Here we want to take a different route based on the Klein model of hyperbolic space.
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- View $S^{n+1}$ as the space of rays in $\mathbb{R}^{n+2}$, hence homogeneous under $\tilde{G} := SL(n + 2, \mathbb{R})$, the group of orientation preserving projective transformations of $S^{n+1}$. 
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- View $S^{n+1}$ as the space of rays in $\mathbb{R}^{n+2}$, hence homogeneous under $\tilde{G} := SL(n + 2, \mathbb{R})$, the group of orientation preserving projective transformations of $S^{n+1}$.
- Break this symmetry by introducing a Lorentzian inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{n+2}$, and put $G := SO_0(n + 1, 1)$. 
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- Break this symmetry by introducing a Lorentzian inner product $\langle , \rangle$ on $\mathbb{R}^{n+2}$, and put $G := SO_0(n + 1, 1)$.
- Then $S^{n+1}$ splits into 5 orbits for $G$, future pointing negative rays can be identified with $\mathcal{H}^{n+1}$, and future pointing null–rays with the boundary sphere $S^n$. 
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- The geometric structures on $\mathcal{H}^{n+1}$ and $S^n$ both arise from this picture.
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This example admits “curved analogs” in the setting of projective differential geometry. Recall that \textit{projective equivalence} of two torsion–free linear connections $\nabla$ and $\hat{\nabla}$ on the tangent bundle of a smooth manifold $N$ can be equivalently defined as

\[\hat{\nabla}_\xi \eta = \nabla_\xi \eta + \Upsilon(\xi) \eta + \Upsilon(\eta) \xi,\]

We will briefly write the second condition as $\hat{\nabla} = \nabla + \Upsilon$. The concept of projective compactness is motivated by studying the form of projective modifications which extend to the boundary in the example of $H^{n+1}$ (which corresponds to $\alpha = 2$) and a similar example related to the inclusion of the equator into a hemisphere (which corresponds to $\alpha = 1$.}
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- same geodesics up to parametrization
- existence of $\Upsilon \in \Omega^1(M)$ such that for all $\xi, \eta \in \mathfrak{X}(M)$ one has $\hat{\nabla}_\xi \eta = \nabla_\xi \eta + \Upsilon(\xi) \eta + \Upsilon(\eta) \xi$. 
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- same geodesics up to parametrization
- existence of $\gamma \in \Omega^1(M)$ such that for all $\xi, \eta \in \mathfrak{X}(M)$ one has $\hat{\nabla}_\xi \eta = \nabla_\xi \eta + \gamma(\xi) \eta + \gamma(\eta) \xi$.

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**Definition**

Let $\overline{M}$ be a smooth manifold with boundary $\partial M$ and interior $M$ and let $\alpha$ be a positive real number.
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Remarks

- This condition is independent of the defining function.
- Changing the parameter $\alpha$ corresponds to replacing $\rho$ by a power of $\rho$, so $\alpha$ cannot be eliminated.
- The concept automatically extends to pseudo–Riemannian metrics on $M$, by requiring the Levi–Civita connection to be projectively compact.
For $\nabla$ being projectively compact it is necessary that the projective structure $[\nabla]$ admits a smooth extension to $\bar{M}$. This can be characterized equivalently as the fact that in local charts the tracefree parts of the connection coefficients admit a smooth extension to the boundary.
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If $\nabla$ is special i.e. preserves a volume density, then it admits non–zero parallel densities of any projective weight and there is the following characterization of projective compactness which can be rephrased as uniform volume growth:
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**Proposition**

If $\nabla$ is special and $[\nabla]$ admits an extension to $\overline{M}$, then $\nabla$ is projectively compact of order $\alpha$ if and only if a non–zero density $\sigma \in \Gamma(E(\alpha))$ such that $\nabla \sigma = 0$ extends by zero to a defining density for $\partial M$. 
Some further basic results

- Since $[\nabla]$ extends to $\overline{M}$, the hypersurface $\partial M$ inherits the *projective second fundamental form*. This is a conformal class of (possibly degenerate) bundle metrics on $T\partial M$. In the projectively compact case, it can be described in terms of curvature asymptotics of $\nabla$. 
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- If $\nabla$ is projectively compact of order $\alpha \leq 2$, then $\partial M$ is at infinity in the sense the geodesics for $\nabla$ do not reach the boundary in finite time.
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Motivated by the last result, one usually considers projective compactness of orders $\leq 2$. 
Structure

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2. Examples from holonomy reductions
3. More on projectively compact metrics
A curved analog of the Klein model of hyperbolic space is given by a projective manifold \((N, [\nabla])\) endowed with a bundle metric \(h\) of signature \((p, q + 1)\) on the projective standard tractor bundle \(\mathcal{T}\), which is parallel for the canonical tractor connection. From the BGG–machinery and the general theory of holonomy reductions one deduces:
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- An Einstein metric on \(N_+\) (resp. \(N_-\)) of signature \((p, q)\) (resp. \((p - 1, q + 1)\)), whose Levi–Civita connection \(\nabla\) is in the (restriction of the) projective class.
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- A conformal structure of signature \((p - 1, q)\) on \(N_0\).
In this situation, denoting by $\overline{M}$ the closure of $N_+$, which consists of $M := N_+$ and $\partial M \subset N_0$, the Levi–Civita connection $\nabla$ on $TM$ is projectively compact of order $\alpha = 2$. 
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**Theorem**

Suppose that for $\overline{M} = M \cup \partial M$, one has a linear connection $\nabla$ on $TM$ such that

1. $\nabla$ preserves a volume density
2. $\nabla$ has non–degenerate, parallel Ricci–tensor

Then $\nabla$ is projectively compact of order $\alpha = 2$ and one obtains a holonomy reduction as discussed above.
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- The section $I$ is equivalent to a density $\sigma \in \Gamma(E(1))$ which satisfies a projectively invariant second order PDE.
- A “curved orbit decomposition” $N = N^+ \cup N_0 \cup N^-$ according to the sign of $\sigma$ into open subsets $N^\pm$ and a separating embedded hypersurface $N_0$.
- Ricci flat connections $\nabla$ in the projective class on $N^\pm$.
- The hypersurface $N_0$ is totally geodesic and thus inherits a projective structure.
- Looking at the closure of $N^+$, one obtains a projective manifold with boundary and a Ricci flat connection in the interior, which is projectively compact of order one.
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Theorem

Conversely, suppose that for $\overline{M} = M \cup \partial M$, one has a linear connection $\nabla$ on $TM$ such that

- $\nabla$ preserves a volume density
- $\nabla$ is Ricci–flat

Then $\nabla$ is projectively compact of order $\alpha = 1$ and one obtains a holonomy reduction as discussed above.

If $\nabla$ is the Levi–Civita connection of a Ricci–flat metric on $M$ (and projectively compact of order one), then in addition to the induced projective structure on $\partial M$, one obtains a parallel bundle metric on the standard tractor bundle for this structure, with consequences as discussed before.
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Structure

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Here the relation to asymptotic forms is of particular interest. For fixed $\alpha$, the asymptotic form in question is that, locally around each boundary point, we have

$$g = C \rho^2 \rho^4/\alpha + h \rho^2/\alpha,$$

where $\rho$ is a defining function for $\partial M$, $h$ is smooth up to the boundary and non-degenerate on $T \partial M$, and $C$ is a smooth function asymptotic to a non-zero constant on $\partial M$. 

Proposition

If $\alpha \leq 2$ and $2\alpha \in \mathbb{Z}$, then a metric having such an asymptotic form is projectively compact of order $\alpha$.

The nature of this asymptotic form depends on $\alpha$. If $\alpha \neq 2$, this form works only for specific defining functions. For $\alpha = 2$, the asymptotic form for one choice of $\rho$ implies that it is available (with the same $C$ and a conformally rescaled $h$) for any defining function.
Here the relation to asymptotic forms is of particular interest. For fixed $\alpha$, the asymptotic form in question is that, locally around each boundary point, we have

$$g = C \frac{d\rho^2}{\rho^{4/\alpha}} + \frac{h}{\rho^{2/\alpha}},$$

where $\rho$ is a defining function for $\partial M$, $h$ is smooth up to the boundary and non–degenerate on $T\partial M$, and $C$ is a smooth function asymptotic to a non–zero constant on $\partial M$. 

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Finally, there is a vast generalization of the result on projective compactness of non–Ricci flat Einstein metrics discussed before:

\[ \text{Theorem} \]

Suppose that for \( M = M \cup \partial M \), one has a pseudo–Riemannian metric \( g \) on \( M \) with Levi–Civita connection \( \nabla \) such that the projective class \( [\nabla] \) extends to \( M \) but \( \nabla \) itself does not extend to any open neighborhood of a boundary point. If the scalar curvature of \( g \) is bounded away from zero, then \( g \) is projectively compact of order \( \alpha = 2 \).
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