Projective compactness

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- The model case for all this is hyperbolic space with the sphere as its boundary at infinity. The Klein model of hyperbolic space suggests an interpretation this model in terms of projective differential geometry. This admits curved analogs via reduction of projective holonomy.
- These examples motivate the general definition of projective compactness, which involves an additional parameter called the *order* of projective compactness. This is a concept for affine connections on the interior of a manifold with boundary and, via the Levi–Civita connection, for pseudo–Riemannian metrics.

Contents

1. The concept of projective compactness
2. Examples from holonomy reductions
3. More on projectively compact metrics

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The group of isometries of $\mathcal{H}^{n+1}$ is $SO_0(n+1,1)$, which coincides with the group of conformal isometries of the round metric on the boundary sphere. Taking into account only the action of this group, it is however hard to see, how the boundary is attached to the interior.

One interpretation of this situation is that the conformal class on S^n is obtained from conformal rescalings of the hyperbolic metric which extend to the boundary. Generalizing this, one obtains R. Penrose's concept of conformal compactness. Here we want to take a different route based on the Klein model of hyperbolic space.

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- Break this symmetry by introducing a Lorentzian inner product $\langle \ , \ \rangle$ on $\mathbb{R}^{n+2}$, and put $G := SO_0(n+1, 1)$.

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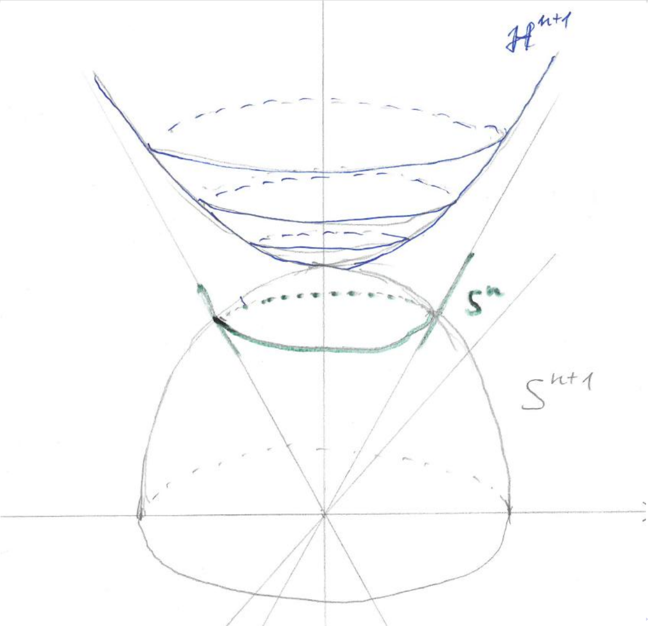
- Then S^{n+1} splits into 5 orbits for G, future pointing negative rays can be identified with $\mathcal{H}^{n+1}$, and future pointing null–rays with the boundary sphere S^n.
- The geometric structures on $\mathcal{H}^{n+1}$ and S^n both arise from this picture.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- same geodesics up to parametrization
- existence of $\Upsilon \in \Omega^1(M)$ such that for all $\xi, \eta \in \mathfrak{X}(M)$ one has $\hat{\nabla}_\xi \eta = \nabla_\xi \eta + \Upsilon(\xi)\eta + \Upsilon(\eta)\xi$.

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We will briefly write the second condition as $\hat{\nabla} = \nabla + \Upsilon$.
The concept of projective compactness is motivated by studying the form of projective modifications which extend to the boundary in the example of $\mathcal{H}^{n+1}$ (which corresponds to $\alpha = 2$) and a similar example related to the inclusion of the equator into a hemisphere (which corresponds to $\alpha = 1$.)

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A torsion free linear connection∇ on TM is called *projectively compact of order* α if for each $x \in \partial M$ there is a local defining function ρ for ∂M defined on an open neighborhood U of x in $\overline{M}$ such that the linear connection $\hat{\nabla} := \nabla + \frac{d\rho}{\alpha\rho}$ on $U \cap M$ admits a smooth extension to all of U.

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- This condition is independent of the defining function.
- Changing the parameter α corresponds to replacing ρ by a power of ρ, so α cannot be eliminated.
- The concept automatically extends to pseudo–Riemannian metrics on M, by requiring the Levi–Civita connection to be projectively compact.

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Proposition

If ∇ is special and $[\nabla]$ admits an extension to $\overline{M}$, then ∇ is projectively compact of order α if and only if a non–zero density $\sigma \in \Gamma(\mathcal{E}(\alpha))$ such that $\nabla\sigma = 0$ extends by zero to a defining density for ∂M.

Some further basic results

- Since $[\nabla]$ extends to $\overline{M}$, the hypersurface ∂M inherits the *projective second fundamental form*. This is a conformal class of (possibly degenerate) bundle metrics on $T\partial M$. In the projectively compact case, it can be described in terms of curvature asymptotics of ∇.

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Motivated by the last result, one usually considers projective compactness of orders ≤ 2.

Structure

1 The concept of projective compactness

2 Examples from holonomy reductions

3 More on projectively compact metrics

A curved analog of the Klein model of hyperbolic space is given by a projective manifold $(N, [\nabla])$ endowed with a bundle metric h of signature $(p, q+1)$ on the projective standard tractor bundle $\mathcal{T}$, which is parallel for the canonical tractor connection. From the BGG–machinery and the general theory of holonomy reductions one deduces:

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- A conformal structure of signature $(p-1, q)$ on N_0.

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Looking at the closure of N_+, one obtains a projective manifold with boundary and a Ricci flat connection in the interior, which is projectively compact of order one.

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If ∇ is the Levi–Civita connection of a Ricci–flat metric on M (and projectively compact of order one), then in addition to the induced projective structure on ∂M, one obtains a parallel bundle metric on the standard tractor bundle for this structure, with consequences as discussed before.

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The nature of this asymptotic form depends on α. If $\alpha \neq 2$, this form works only for specific defining functions. For $\alpha = 2$, the asymptotic form for one choice of ρ implies that it is available (with the same C and a conformally rescaled h) for any defining function.

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- One always obtains an asymptotic form as above with constant C.
- The tensor field h in the asymptotic form represents the projective second fundamental form. This implies that g satisfies an asymptotic form of the Einstein equation.
- One obtains an explicit description of the standard tractor bundle and –connection for the conformal structure on ∂M in terms of projective tractors on M.

Finally, there is a vast generalization of the result on projective compactness of non–Ricci flat Einstein metrics discussed before:

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Then g is projectively compact of order $\alpha = 2$.